Partially hyperbolic geodesic flows

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Abstract

We construct a category of examples of partially hyperbolic geodesic flows which are not Anosov, changing the metric of a compact locally symmetric space of nonconstant negative curvature. We also show that candidates for such example as the product metric and locally symmetric spaces of nonpositive curvature with rank bigger than one are not partially hyperbolic. We also prove that if a metric of nonpositive curvature is not a Riemannian product and its geodesic flow is partially hyperbolic, then its rank is one. Other obstructions to partial hyperbolicity of a geodesic flow are also analyzed

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1 Introduction

The theory of hyperbolic dynamics has been one of the extremely successful stories in dynamical systems. Originated by studying dynamical properties of geodesic flows on manifolds with negative curvature [An] and geometrical properties of homoclinic points [Sm], hyperbolicity is the cornerstone of uniform and robust chaotic dynamics; it characterizes the structural stable systems; it provides the structure underlying the presence of homoclinic points; a large category of rich dynamics are hyperbolic (geodesic flows in negative curvature, billiards with negative curvature, linear automorphisms, some mechanical systems, etc.); the hyperbolic theory has been fruitful in developing a geometrical approach to dynamical systems; and, under the assumption of hyperbolicity one obtains a satisfactory (complete) description of the dynamics of the system from a topological and statistical point of view. Moreover, hyperbolicity has provided paradigms or models of behavior that can be expected to be obtained in specific problems.

Nevertheless, hyperbolicity was soon realized to be a property less universal than it was initially thought: it was shown that there are open sets in the space of dynamics which are nonhyperbolic. To overcome these difficulties, the theory moved in different directions; one being to develop weaker or relaxed forms of hyperbolicity, hoping to include a larger class of dynamics.

There is an easy way to relax hyperbolicity, called partial hyperbolicity, which allows the tangent bundle to split into Df-invariant subbundles $TM = E^s \oplus E^c \oplus E^u$, such that the behavior of vectors in E^s, E^u is similar to the hyperbolic case, but vectors in E^c may be neutral for the action of the tangent map. This notion arose in a natural way in the context of time one maps of Anosov flows, frame flows or group extensions. See [BP], [Sh], [M1], [BD], [BV] for examples of these systems and [HP], [PS] for an overview.

However, and differently to hyperbolic ones, partially hyperbolic systems where unknown in the context of geodesic flows induced by Riemannian metrics. As far as we know, the way to produce partially hyperbolic systems in discrete dynamics are the following: time-one maps of Anosov flows, skew-products over hyperbolic dynamics, products and derived of Anosov deformations (DA). The two last approaches can be adapted to flows.

Our work shows that one is able to deform a specific metric that provides an Anosov geodesic flow to get a partially hyperbolic geodesic flow. This is done inspired by the Mañé's DA construction of a partially hyperbolic diffeomorphism [M1].

We prove the following theorems:

Theorem 1.1. *There is a Riemannian metric such that its geodesic flow is partially hyperbolic but not Anosov.*

Actually, we prove:

Theorem 5.17. *For some compact locally symmetric space (M, g) whose sectional curvature takes values in the whole interval $[-1, -\frac{1}{4}]$, there is a metric g^* in M such that its geodesic flow is partially hyperbolic but not Anosov.*

Remark 1.2. The theorem works for the compact Kahler manifold of constant holomorphic curvature -1 [G], and also for the quaternionic Kahler locally symmetric spaces of negative curvature. Both these locally symmetric spaces are even-dimensional. In section 7.3 we will see that there are no partially hyperbolic geodesic flows for Riemannian metrics in odd dimensional manifolds, by an idea of Contreras [Co2].

Remark 1.3. A classical Mañé theorem [M3] says that if, for a geodesic flow of a Riemannian manifold there is an invariant Lagrangian subbundle, then this Riemannian manifold does not have conjugate points. The existence of a partially hyperbolic non-Anosov geodesic flow implies that this theorem does not generalize to the case of invariant isotropic subbundles.

The next two corollaries are given by the persistence of quasi-elliptic nondegenerate periodic orbits.

Corollary 5.19. *There is an open set $\mathcal{U}$ of metrics in the set of metrics of M such that for $g \in \mathcal{U}$, the geodesic flow of g is partially hyperbolic but not Anosov, for (M, g) as in the previous theorem. There is also an open set $\mathcal{U}'$ of metrics such that for $g \in \mathcal{U}'$, the geodesic flow of g is partially hyperbolic non-Anosov and with conjugate points.*

Corollary 5.21. *There is an open set $\mathcal{V}$ of Hamiltonians in the set of Hamiltonians of (TM, ω_{TM}), near geodesic Hamiltonians, such that for $h \in \mathcal{U}$, the Hamiltonian flow of h is partially hyperbolic but not Anosov.*

For Hamiltonians it is easy to construct one that is partially hyperbolic, just create a Hamiltonian flow that is a suspension flow. But suspensions are not close to geodesic flows.

We also show that product metrics of Anosov geodesic flows are not examples with the partially hyperbolic property:

Theorem 3.3. *If (M_1, g_1) and (M_2, g_2) are Riemannian manifolds such that the geodesic flow of at least one of them is Anosov, then the geodesic flow of $(M_1 \times M_2, g_1 + g_2)$ is not partially hyperbolic.*

For compact locally symmetric spaces of nonpositive curvature the following holds:

Theorem 6.9. *If the geodesic flow of a compact locally symmetric space of nonpositive curvature is partially hyperbolic, then is curvature takes values on the whole interval $[-1, -\frac{1}{4}]$.*

The proof of theorem 3.3 and theorem 6.9 imply the following:

Theorem 7.2. *If (M, g) is a compact Riemannian manifold with nonpositive sectional curvature and partially hyperbolic geodesic flow then (M, g) has rank one.*

And these hypothesis above imply also that:

Theorem 7.3. *If* (M^n, g) *is a Riemannian manifold with partially hyperbolic geodesic flow then* n *is even, and if* $n \equiv 2 \mod 4$, *then* $\dim E^s = 1$ *or* $n - 1$.

Remark 1.4. If the locally symmetric space has nonconstant negative curvature, then its curvature takes values on the whole interval $[-1, -\frac{1}{4}]$ [H].

Remark 1.5. Of course, if we multiply the metric by a constant, the Anosov or the partially hyperbolic splitting remain the same, but the curvature does not. So, we consider the maximal sectional curvature of the locally symmetric space to be -1, which is true after multiplication of the metric by a constant.

Roughly speaking, the strategy of the construction of theorem 1.1 is done following the next steps:

1. It is chosen a metric whose geodesic flow is Anosov and whose hyperbolic invariant splitting is of the form $T(SM) = E^{ss} \oplus E^s \oplus < X > \oplus E^u \oplus E^{uu}$ (section 4);
2. we take a closed geodesic γ without self-intersections (section 5.2);
3. we change the metric in a tubular neighborhood of γ in M, such that along γ the strong subbundles (E^{ss} and E^{uu}) remain invariant and the weak subbundles disapear, becoming a central subbundle with no hyperbolic behavior (section 5.3):
4. outside the tubular neighborhood of γ, the dynamics remains hyperbolic;
5. we show that for the geodesics that intersect the tubular neighborhood the cones associated to the extremal subbundles (E^{ss} and E^{uu}) are preserved (sections 5.3.3, 5.3.4, 5.3.5, 5.3.6); we prove that for vectors in the unstable cones there is expansion, and for vectors inside the stable cones there is contraction, under the action of the derivative of the new geodesic flow (section 5.3.7)

We would like to recall that in the symplectic context, the existence of a dominated splitting with two subbundles of equal dimension implies hyperbolicity. This was observe first by Newhouse for surfaces maps [Ne], latter by Mane in any dimension [M2], by Ruggiero in the context of geodesic flows [R1] and Contreras for symplectic and contact flows [Co1]. We want to point out that thse results do not contradict ours: the splitting that the opens set of examples shows has more than two subbundles.

There are partially hyperbolic Σ-geodesic flows, defined over a distribution $\Sigma \subsetneq TM$ which arise in the study of the dynamics of free particles in a system with constrains (see [CKO]). However, if the distribution is involutive then the leaves of the distribution have negative curvature, and we are again in the Anosov geodesic flows case.

The article is organized as follows:

In the second section of the article, we introduce basic results about the geodesic flow, partial hyperbolicity and the equivalent property of the proper invariance of cone fields [P], [HP].

In the third section we prove that product metrics are not examples of partially hyperbolic non-Anosov geodesic flows.

In the fourth section we introduce properties and the classification of locally symmetric spaces of negative curvature which are the natural candidates to deform into partially hyperbolic non-Anosov geodesic flows.

In the fifth section we show that the deformed metric has a partial hyperbolic non-Anosov geodesic flow. We give a proof of the proper invariance of the strong cones based on the calculation of the variation of the opening of the cones of an appropiate cone field, and then we prove the exponential expansion or contration for vectors in the strong unstable and stable cones.

In the sixth section we show that compact locally symmetric spaces of nonpositive sectional curvature are not partially hyperbolic, except the spaces of nonconstant negative sectional curvature, which are the candidates for the deformation, since they have the property mentioned in the first item of the strategy above [E2],[E3], [J].

In the last section we show some obstructions to the existence of a partially hyperbolic geodesic flow. Obstructions for the rank of the manifold if the Riemannian manifold has nonpositive sectional curvature, and for the dimension of the Riemannian manifold and the dimension of the hyperbolic invariant subbundles in the general case.

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2 Preliminary definitions

In this section, we give some preliminary definitions. In the first subsection, the definitions are about geodesic flows. The basic reference for this subsection is the book by Paternain [P]. In the second subsection, we give the main definitions about partial hyperbolicity and the basic reference is the survey by Hasselblat and Pesin [HP].

2.1 Geodesic flows

A Riemannian manifold (M, g) is a C^∞-manifold with an Euclidean inner product g_x in each T_xM which varies smoothly with respect to $x \in M$. So a Riemannian metric is a smooth section $g : M \to Symm_2^+(TM)$, where $Symm_2^+(TM)$ is the set of positive definite bilinear and symmetric forms in TM. Along the article we will consider the topology of the space of metrics of a manifold M to be the C^2-topology on the space of these sections.

The geodesic flow of the metric g is the flow

$$\phi_t : TM \to TM : (x, v) \to (\gamma_{(x,v)}(t), \gamma'_{(x,v)}(t)),$$

such that $\gamma_{(x,v)}$ is the geodesic for the metric g with initial conditions $\gamma_{(x,v)}(0) = x$ and $\gamma'_{(x,v)}(0) = v$. Since the speed of the geodesics is constant, we can consider the flow restricted to $SM := \{(x, v) \in TM : g_x(v, v) = 1\}$.

Definition 2.1. $\pi_V : V(TM) \to TM$,which is called the vertical subbundle, is the bundle whose fiber at $\theta \in T_xM$, $V(\theta)$, is given by $V(\theta) = ker(d_\theta\pi)$, where $\pi : TM \to M$ is the canonical projection of the tangent bundle.

Definition 2.2. $K : T(TM) \to TM$, which is called the connection map associated to the metric g, is defined as follows: given $\xi \in T_\theta TM$ let $z : (-\epsilon, \epsilon) \to TM$ be an adapted curve to ξ; let $\alpha : (-\epsilon, \epsilon) \to M : t \to \pi_M \circ z(t)$, and Z the vector field along α such that $z(t) = (\alpha(t), Z(t))$; then $K_\theta(\xi) := (\nabla_{\alpha'}Z)(0)$. $\pi_H : H(TM) \to TM$, the horizontal subbundle, is given by $H(\theta) := ker(K_\theta)$.

Some properties of H and V are:

1. $H(\theta) \cap V(\theta) = 0$,
2. $d_\theta\pi$ and K_θ give identifications of $H(\theta)$ and $V(\theta)$ with T_xM,
3. $T_\theta TM = H(\theta) \oplus V(\theta)$.

The geodesic vector $G : TM \to T(TM)$ in this decomposition $H(\theta) \oplus V(\theta) \approx T_xM \oplus T_xM$ is given by $(v, 0)$.

The decomposition in horizontal and vertical subbundles allows us to define the Sasaki metric on TM:

$$\begin{aligned}\widehat{g}_\theta(\xi, \eta) &:= g_x(d_\theta\pi(\xi), d_\theta\pi(\eta)) + g_x(K_\theta(\xi), K_\theta(\eta)) \\ &= g_x(\xi_h, \eta_h) + g_x(\xi_v, \eta_v)\end{aligned}$$

for ξ and $\eta \in T_\theta TM$, with $\xi = (\xi_h, \xi_v)$ and $\eta = (\eta_h, \eta_v)$ in the decomposition $T_\theta TM = H(\theta) \oplus V(\theta)$, with ξ_h and $\eta_h \in T_xM \cong H(\theta)$, ξ_v and $\eta_v \in T_xM \cong V(\theta)$.

It also allows us to define a symplectic 2-form and a almost complex structure $\widetilde{J}$ on TM and a contact form on SM:

$$
\begin{aligned}
\Omega_\theta(\xi,\eta) &:= g_x(d_\theta\pi(\xi), K_\theta(\eta)) - g_x(K_\theta(\xi), d_\theta\pi(\eta)) \\
&= g_x(\xi_h,\eta_v) - g_x(\eta_h,\xi_v), \\
\widetilde{J}(\xi_h,\xi_v) &:= (-\xi_v,\xi_h), \\
\alpha_\theta(\xi) &:= \widehat{g}_\theta(\xi, G(\theta)) = g_x(d_\theta\pi(\xi), v) = g_x(\xi_h, v).
\end{aligned}
$$

Since the geodesic flow leaves SM invariant, and we can define a contact form on SM such that its Reeb vector field is the geodesic vector field, $S(\theta) := ker\alpha_\theta$ is an invariant subbundle for the geodesic flow, and $\mathbb{R}\cdot G \oplus S = T(SM)$.

The derivative of the geodesic flow is related to the Jacobi fields of the metric that generates the flow.

Definition 2.3. A Jacobi field along a geodesic γ_θ, $\theta = (x,v)$ is a vector field obtained by a variation of the geodesic γ_θ through geodesics:

$$\zeta(t) := \frac{\partial}{\partial s}|_{s=0}\pi \circ \phi_t(z(s)),$$

where $z(0) = \theta$, $z'(0) = \xi$ and $z(s) = (\alpha(s), Z(s))$.

It satisfies the following equation:

$$\zeta'' + R(\gamma_\theta', \zeta)\gamma_\theta' = 0.$$

Its initial conditions are:

$$\zeta(0) = \frac{\partial}{\partial s}|_{s=0}\pi \circ z(s) = d_\theta\pi\xi = \xi_h,$$

$$
\begin{aligned}
\zeta'(0) &= \frac{D}{dt}\frac{\partial}{\partial s}|_{t=0,s=0}\pi\circ\phi_t(z(s)) = \frac{D}{\partial s}\frac{\partial}{\partial t}|_{s=0,t=0}\pi\circ\phi_t(z(s)) \\
&= \frac{D}{\partial s}|_{s=0}Z(s) = K_\theta\xi = \xi_v.
\end{aligned}
$$

The derivative of a geodesic flow is: $d_\theta\phi_t(\xi) = (\zeta_\xi(t), \zeta_\xi'(t))$.

Remark 2.4. We define the curvature tensor $R : \Gamma(TM)\times\Gamma(TM)\times\Gamma(TM) \to \Gamma(TM)$ as in do Carmo's book [Ca]:

$$R(X,Y)Z := \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z.$$

2.2 Partial hyperbolicity

Definition 2.5. A partially hyperbolic flow $\phi_t : M \to M$ in the manifold M generated by the vector field $X : M \to TM$ is a flow such that its quotient bundle $TM/\langle X \rangle$ have an invariant splitting $TM/\langle X \rangle = E^s \oplus E^c \oplus E^u$ such that these subbundles are non trivial and with the following properties:

$$d\phi_t(x)(E^s(x)) = E^s(\phi_t(x)), d\phi_t(x)(E^c(x)) = E^c(\phi_t(x)), d\phi_t(x)(E^u(x)) = E^u(\phi_t(x)),$$

$$||d\phi_t(x)|_{E^s}|| \leq C\exp(t\lambda), ||d\phi_{-t}(x)|_{E^u}|| \leq C\exp(t\lambda),$$
$$C\exp(t\mu) \leq ||d\phi_t(x)|_{E^c}|| \leq C\exp(-t\mu),$$

for $\lambda < \mu < 0 < C$.

Definition 2.6. A splitting $E \oplus F$ of the quotient bundle $TM/\langle X \rangle$ is called a dominated splitting if:

$$d\phi_t(x)(E(x)) = E(\phi_t(x)), d\phi_t(x)(F(x)) = F(\phi_t(x)),$$

$$||d\phi_t(x)|_{E(x)}|| \cdot ||d\phi_{-t}(\phi_t(x))|_{F(\phi_t(x))}|| < C\exp(-t\lambda)$$

for some constants C and $\lambda > 0$.

2.2.1 Partial hyperbolicity and cone fields

There is a criterion useful for verifying partial hyperbolicity, called the cone criterion:

Given $x \in M$, a subspace $E \subset T_xM$ and a number δ, we define the cone at x centered around E with angle δ as

$$C(x, E(x), \delta) = \{v \in T_xM : \angle(v, E) < \delta\},$$

where $\angle(v, E)$ is the angle that the vector $v \in T_xM$ makes with its own projection to the subspace $E \subset T_xM$.

One flow is partially hyperbolic if there are $\delta > 0$, some time $T > 0$, and two continuous cone families $C(x, E_1(x), \delta)$ and $C(x, E_2(x), \delta)$ such that:

$$d_x\phi_{-t}(C(x, E_1(x), \delta)) \subsetneq C(x, E_1(\phi_{-t}(x)), \delta),$$

$$d_x\phi_t(C(x, E_2(x), \delta)) \subsetneq C(x, E_2(\phi_t(x)), \delta),$$

$$\|d_x\phi_t\xi_1\| < K\exp(t\lambda), \|d_x\phi_{-t}\xi_2\| < K\exp(t\lambda),$$

for $\xi_1 \in C(x, E_1(x), \delta)$, $\xi_2 \in C(x, E_2(\phi_t(x)), \delta)$, some constants $K > 0$, $\lambda < 0$ and all $t > 0$.

2.2.2 Partial hyperbolicity and angle cone variation

To know if there is proper invariance of cones we need to check the following inequality:

$$\frac{d}{dt}\frac{g(Pr_E v, Pr_E v)}{g(v,v)} > 0 \tag{1}$$

for $v \in \partial C(x, E(x), \delta) := \{v \in T_xM : \angle(v, E(x)) = \delta\}$, where $Pr_E : TM \to E$ is the orthogonal projection to E, where $\pi_E : E \to M$ is a vector subbundle of TM.

Remark 2.7. The quantity on the left side of the inequality equals twice the square of the cosine of the angle between v and the subspace E, so if it increases along the flow, then the cone field is properly invariant. Since the quantity above is the angle between vectors in the cone, and since we need to calculate this quantity only at the boundary of the cones of a cone field, we call the derivative above the angle cone variation. This calculation is inspired by the calculations in [W], although we do not use quadratic forms here.

The proper invariance of the cones by the derivative of the geodesic flow implies the existence of a dominated splitting. For the exponential expansion or contraction in the unstable and stable directions, respectively, we only need to check exponential expansion or contraction inside the unstable and stable cones, respectively.

Lemma 2.8. *For a fixed $\delta > 0$, and a fixed subbundle $E \to M$, $E(x) \subset T_xM$, if inequality (1) holds for $v \in \partial C(x, E(x), \delta)$, then the cone field is proper invariant for the geodesic flow.*

Proof. Let $c \in (1, 2)$ be such that $2\cos^2(\delta) = c$. Then

$$\begin{aligned} C(x, E(x), \delta) &= \left\{ \frac{g(Pr_E v, Pr_E v)}{g(v,v)} \geq c \right\}, \\ \partial C(x, E(x), \delta) &= \left\{ \frac{g(Pr_E v, Pr_E v)}{g(v,v)} = c \right\}. \end{aligned}$$

Notice that the quantity on the left side of (1) is the same for v and for kv for every $k > 0$. Then we can calculate for v such that $g(v, v) = 1$. Define

$$\partial_1 C(x, E(x), \delta) = \{g(Pr_E v, Pr_E v) = c, g(v, v) = 1\}.$$

Then the set of vectors in the boundary of the cones of the cone field is compact, which implies the derivative is bounded away from zero:

$$\frac{d}{dt}\frac{g(Pr_E v, Pr_E v)}{g(v,v)} \geq a > 0.$$

Its imediate consequence is that the cone field is properly invariant for the flow of X. $\square$

3 The geodesic flow of a product metric is not partially hyperbolic

Now, we are going to show that some simple candidates for partially hyperbolic geodesic flows are not partially hyperbolic. In particular, we are going to prove that product metrics are not Anosov or partially hyperbolic.

A natural candidate for symplectic partially hyperbolic dynamics is the following: take any hyperbolic symplectic action $\Phi : \mathbb{R} \to Sp(E, \omega)$, $\pi : E \to B$ a symplectic bundle with ω as its symplectic 2-form, one can produce another symplectic action $\Phi^* : \mathbb{R} \to Sp(E, \omega) \oplus Sp(B \times \mathbb{R}^2, \omega_0) : t \to \Phi(t) \oplus Id$. The symplectic flow associated with this symplectic $\mathbb{R}$-action is partially hyperbolic with a central direction of dimension 2. In the case of geodesic flows this construction does not work.

Suppose we have a Riemannian manifold (M, g) whose geodesic flow is Anosov. Then, we can say:

Theorem 3.1. *The product Riemannian manifold* $(M \times \mathbb{T}^n, g + g_0)$ *where* $(\mathbb{T}^n, g_0)$ *is* $\mathbb{T}^n$ *with its canonical flat metric, is not partially hyperbolic.*

Proof. Observe that $\{x\} \times \mathbb{T}^n$ is a totally geodesic submanifold of $(M \times \mathbb{T}^n, g + g_0)$. So, its second fundamental form is identically zero. Since the metric in $\mathbb{T}^n$ is flat this implies that:

$$R(\gamma'_{(x,y,0,v)}, (0, w))\gamma'_{(x,y,0,v)} = 0.$$

For a product metric in $(M_1 \times M_2, g_1 + g_2)$, let us say R is the curvature tensor of the product Riemannian manifold with the product metric, K its curvature, R^1 the curvature tensor of the Riemannian manifold M_1. Then the following properties hold:

i. $R(X, Y, Z, W) = R^1(X, Y, Z, W)$, for X, Y, Z, W tangent to M_1, because of the Gauss' equation and the fact that the second fundamental form is zero [Ca];

ii. $R(X, Y, Z, N)$, for X, Y, Z tangent to M_1 and N tangent to M_2, because of Codazzi's equation and the fact that the second fundamental form is zero [Ca];

iii. $R(X, N, X, \widehat{N}) = 0$, for X, Y tangent to M_1 and $N, \widehat{N}$ tangent to M_2, because $K(X, N) = 0$ [Ca].

Then, for a submanifold $\{x\} \times \mathbb{T}^n$ with the flat metric:

$$R(\gamma'_{(x,y,0,v)}, \cdot)\gamma'_{(x,y,0,v)} \equiv 0.$$

So, the derivative of the geodesic flow along geodesics in $\{x\} \times \mathbb{T}^n$ does not have any exponential contration or expansion. So, there is no partially hyperbolic splitting for its geodesic flow. $\square$

Now, suppose we have two Riemannian manifolds with Anosov geodesic flows: (M_1, g_1) and (M_2, g_2).

Theorem 3.2. *The geodesic flow of the Riemannian manifold* $(M_1 \times M_2, g_1 + g_2)$ *is not Anosov.*

Proof. The proof that this geodesic flow is not Anosov is easy. It is a classical result that $(x_0, \gamma_{(y,v)}(t))$ and $(\gamma_{(x,u)}(t), y_0)$ are geodesics of the product metric, $x_0 \in M_1$, $y_0 \in M_2$, $u \in T_xM_1$, $v \in T_yM_2$, $\gamma_{(x,u)}(0) = x$ and $\gamma'_{(x,u)}(0) = u$, $\gamma_{(y,v)}(0) = y$ and $\gamma'_{(y,v)}(0) = v$. So, we choose x_0 and $x_1 \in M_1$ close enough, and $(x_0, \gamma_{(y,v)}(t))$ and $(x_1, \gamma_{(y,v)}(t))$ are two geodesics with initial conditions $(x_0, y, 0, v)$ and $(x_1, y, 0, v)$. Let $dist$ be the distance function for the Sasaki metric of $S(M_1 \times M_2)$ and $dist_1$ be the distance function for the Sasaki metric of SM_1. The geodesic flow is not expansive, because $dist(\phi_t(x_0, y, 0, v), \phi_t(x_1, y, 0, v) = dist_1((x_0, 0), (x_1, 0))$: if x_0 and x_1 are close enough, $dist_1((x_0, 0), (x_1, 0)) < \epsilon$, for any $\epsilon > 0$, then the geodesic flow is not Anosov. □

Theorem 3.3. *The geodesic flow of the product metric of a product manifold of two Riemannian manifolds with Anosov geodesic flows is not partially hyperbolic.*

Proof. Take local coordinates for the geodesic flow of the product metric. Let $x \in M_1$, $y \in M_2$, $u \in T_xM_1$, $v \in T_yM_2$, and let $\gamma_{(x,y,u,v)}(t)$ be the geodesic with initial conditions $\gamma_{(x,y,u,v)}(0) = (x, y)$ and $\gamma'_{(x,y,u,v)}(0) = (u, v)$. Since the product metric is a sum of the two metrics, we have that $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$, the natural projection from the product manifold to M_i, is a isometric submersion. So $\gamma_{(x,y,u,v)}(t) = (\gamma_{(x,u)}(t), \gamma_{(y,v)}(t))$.

Let us construct an orthonormal basis of parallel vector fields for $\gamma_{(x,y,u,v)}(t)$. Suppose $g^1_x(u, u) = 1$ and $g_y(v, v) = 1$. So, to have (x, y, u, v) in the unitary tangent bundle of $M_1 \times M_2$ we take $(x, y, \alpha u, \beta v)$, and

$$g_{(x,y)}((\alpha u, \beta v), (\alpha u, \beta v)) = \alpha^2 g^1_x(u, u) + \beta^2 g_y(v, v) = \alpha^2 + \beta^2 = 1.$$

Then

$$\gamma_{(x,y,\alpha u,\beta v)}(t) = (\gamma_{(x,\alpha u)}(t), \gamma_{(y,\beta v)}(t)), \gamma'_{(x,y,\alpha u,\beta v)}(t) = (\alpha\gamma'_{(x,u)}(t), \beta\gamma'_{(y,v)}(t)).$$

Take E_i, $i = 2, \ldots, dim(M_1)$, an orthogonal frame of parallel vector fields along the geodesic $\gamma_{(x,u)}$. Take F_j, $j = 2, \ldots, dim(M_2)$, an orthogonal frame of parallel vector fields along the geodesic $\gamma_{(y,v)}$.

Notice that along the geodesic $\gamma_{(x,y,\alpha u,\beta v)}$, since its componentes are $\gamma_{(x,\alpha u)}$ and $\gamma_{(y,\beta v)}$, the following holds:

$$g^1_{\gamma_{(x,\alpha u)}(t)}(\gamma'_{(x,\alpha u)}(t), \gamma'_{(x,\alpha u)}(t)) = \alpha^2, g^2_{\gamma_{(y,\beta v)}(t)}(\gamma'_{(y,\beta v)}(t), \gamma'_{(y,\beta v)}(t)) = \beta^2,$$

so the proportion (α, β) is preserved along the geodesic.

So $\{(\alpha\gamma'_{(x,u)}(t), \beta\gamma'_{(y,v)}(t)), (\beta\gamma'_{(x,u)}(t), -\alpha\gamma'_{(y,v)}(t)), (E_i(t), 0), (0, F_j(t))\}_{i,j}$ is an orthonormal frame of parallel vector fields along the geodesic $\gamma_{(x,y,\alpha u,\beta v)}(t)$.

The fact that the second fundamental form of the submanifolds $\{p\} \times M_2$ and $M_1 \times \{q\}$ is zero, together with Gauss and Codazzi equations, imply that:

$$\begin{aligned}
R((u_1,0),(u_2,0),(u_3,0),(u_4,0)) &= R^1(u_1,u_2,u_3,u_4),\\
R((0,v_1),(0,v_2),(0,v_3),(0,v_4)) &= R^2(v_1,v_2,v_3,v_4),\\
R((u_1,0),(u_2,0),(u_3,0),(0,v_1)) &= 0,\\
R((0,v_1),(0,v_2),(0,v_3),(u_1,0)) &= 0.
\end{aligned}$$

Also the fact that the curvature is zero for planes generated by one vector tangent to M_1 and another tangent to M_2 implies:

$$R((u_1,0),(0,v_1),(u_2,0),(0,v_2)) = 0.$$

All these equations imply that along the geodesic $\gamma_{(x,y,\alpha u,\beta v)}(t)$:

$$R(\gamma'_{(x,y,\alpha u,\beta v)},(E_i,0),\gamma'_{(x,y,\alpha u,\beta v)},(E_k,0)) = \alpha^2 R^1(\gamma'_{(x,u)},E_i,\gamma'_{(x,u)},E_k),$$

$$R(\gamma'_{(x,y,\alpha u,\beta v)},(0,F_j),\gamma'_{(x,y,\alpha u,\beta v)},(0,F_l)) = \beta^2 R^2(\gamma'_{(y,v)},F_j,\gamma'_{(y,v)},F_l),$$

$$R(\gamma'_{(x,y,\alpha u,\beta v)},(E_i,0),\gamma'_{(x,y,\alpha u,\beta v)},(0,F_j)) = 0.$$

Now, we are going to write the system of Jacobi fields. If we have $\zeta(t) = \sum_{i=2} f_i U_i$, then $\zeta''(t) = \sum_{i=2} f_i'' U_i$ and

$$0 = \sum_{j=2}(f_j'' + \sum_{i=2} f_i R(\gamma',U_i,\gamma',U_j))U_j.$$

So, it can be written as:

$$\begin{bmatrix} f \\ f' \end{bmatrix}' = \begin{bmatrix} 0 & I \\ -K & 0 \end{bmatrix} \begin{bmatrix} f \\ f' \end{bmatrix}$$

where $K_{ij} = R(\gamma',U_i,\gamma',U_j)$.

In the case of the product metric we have:

$$\begin{bmatrix} f \\ f' \end{bmatrix}' = \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -\alpha^2 K^1 & 0 & 0 & 0 \\ 0 & -\beta^2 K^2 & 0 & 0 \end{bmatrix} \begin{bmatrix} f \\ f' \end{bmatrix}.$$

With a change in the order of the basis of parallel vector fields we have:

$$F' = \begin{bmatrix} 0 & I & 0 & 0 \\ -\alpha^2 K^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -\beta^2 K^2 & 0 \end{bmatrix} F.$$

So the systems decouples and the solutions are given imediately by the solutions for M_1 and M_2.

Now suppose the geodesic flow of the product metric is partially hyperbolic with splitting $E^s \oplus E^c \oplus E^u$, $dimE^s = p$, $dimE^u = q$. So the geodesic flow of each metric g_1 and g_2 is partially hyperbolic, each geodesic flow inherits a partially hyperbolic splitting:

$$E_1^s \oplus E_1^c \oplus E_1^u,$$

along geodesics in $M_1 \times \{y\}$ ($\beta = 0$), such that $E_1^s \oplus E_1^u \subset T_x M_1 \oplus \{0\} \subset T_x M_1 \oplus T_y M_2$, and

$$E_2^s \oplus E_2^c \oplus E_2^u,$$

along geodesics in $\{x\} \times M_2$ ($\alpha = 0$), such that $E_2^s \oplus E_2^u \subset \{0\} \oplus T_y M_2 \subset T_x M_1 \oplus T_y M_2$.

For geodesics of the product metric which have $\alpha \neq 0 \neq \beta$, we get a splitting into five invariant subbundles $E_1^s \oplus E_2^s \oplus E^c \oplus E_1^u \oplus E_2^u$, without the domination, since α and β multiply the Lyapunov exponents of each subbundle. Since we already have an splitting, E^s and E^u are necessarily one of a combination of subbundles of E_1^s and E_2^s, E_1^u and E_2^u, respectively:

$$E^s \in \{E \oplus F : E \subset E_1^s, F \subset E_2^s, dimE + dimF = p\},$$

$$E^u \in \{E \oplus F : E \subset E_1^u, F \subset E_2^u, dimE + dimF = q\}.$$

So there is no way to go from the case $\alpha = 0$ to $\beta = 0$ without breaking the continuity of the splitting, because one cannot go from the case dim $E = 0$, when $\beta = 0$, to dim $F = 0$, when $\alpha = 0$ continuously.

□

4 Anosov geodesic flow with many invariant subbundles

In this section, we introduce the metric which we are going to change to produce the example of a partially hyperbolic and non-Anosov geodesic flow.

The candidate for the deformation is a compact locally symmetric space which is a quotient of the symmetric space of nonconstant negative curvature $M := G/K$ by a cocompact lattice Γ [Bo].

Cartan classified the symmetric spaces of negative curvature (see [H], [He]). They are:

i. the hyperbolic space $\mathbb{R}H^n$ of constant curvature $-c^2$, which is the canonical space form of negative constant curvature;

ii. the hyperbolic space $\mathbb{C}H^n$ of curvature $-4c^2 \leq K \leq -c^2$, which is the canonical Kahler hyperbolic space of constant negative holomorphic curvature $-4c^2$ [G];

iii. the hyperbolic space $\mathbb{H}H^n$ of curvature $-4c^2 \leq K \leq -c^2$, which is the canonical quaternionic Kahler symmetric space of negative curvature [Be], [Wo];

iv. the hyperbolic space CaH^2 of curvature $-4c^2 \leq K \leq -c^2$, which is the canonical hyperbolic symmetric space of the octonions of constant negative curvature.

Theirs geodesic flows are all Anosov, but the geodesic flow of the first one has not more than the two invariant subbundles, the stable and the unstable, which can not be decomposed in other subbundles. The others have more invariant subbundles, as in the first item of the strategy written in section 1. So, the metrics which are the candidates to produce a partially hyperbolic geodesic flow which is not Anosov are the metrics in items [ii.], [iii.] and [iv.]. Through the article we are going to consider $c = \frac{1}{2}$.

For these type of metrics we need the following properties to hold:

i. For all $v \in T_xM$, the subspace $\{w \in T_xM : K(v,w) = -1\}$ is parallel; another way to say this is that the derivative of the projection to this subspace of T_xM along geodesics is zero;

ii. For closed geodesics $\gamma : [0,T] \to M$, $(\gamma(0), \gamma'(0)) = (\gamma(T), \gamma'(T))$, the parallel translation from $\gamma(0)$ to $\gamma(T)$ along γ of these subspaces $\{w \in T_xM : K(v,w) = -1\}$ and $\{w \in T_xM : K(v,w) = -\frac{1}{4}\}$, where $v = \gamma'(0)$, preserves orientation.

The examples that satisfy the properties above are:

i. compact Kahler manifolds of negative holomorphic curvature -1 (see [G]),

ii. compact locally symmetric quaternionic Kahler manifolds of negative curvature (see [Be]).

4.1 Subspaces of $S(SM)$ and SM

Since the candidate has nonconstant negative curvature, then its sectional curvature, up to multiplication of the metric by a constant, has planes of setional curvature -1 and planes of sectional curvature $-\frac{1}{4}$. Actually, every vector $v \in TM$ is in a plane with curvature -1 and in another with curvature $-\frac{1}{4}$.

We define

$$\begin{aligned} A(x,v) &:= \{w \in T_xM : K(v,w) = -1\}, &(2)\\ B(x,v) &:= \{w \in T_xM : K(v,w) = -\frac{1}{4}\}. &(3)\end{aligned}$$

If we restrict the derivative of the geodesic flow to the subbundle $S(SM) = ker\alpha \to SM$, where α is the contact form on SM, then $S(x,v) = \widehat{H}(x,v) \oplus \widehat{V}(x,v)$, and $\widehat{H}(x,v)$ and $\widehat{V}(x,v)$ are identified with $\{v\}^\perp = A(x,v) \oplus B(x,v) \subset T_xM$, $(x,v) \in SM$. The subbundles A and B are invariant by parallel translation along the geodesic with initial conditions (x,v).

Lemma 4.1. *The geodesic flow of the symmetric spaces of nonconstant negative curvature induces a hyperbolic splitting of the contact structure defined on SM: $S(SM) = E^{ss} \oplus E^s \oplus E^u \oplus E^{uu}$.*

Proof. We can define the subbundles $P^u_K(v), P^s_K(v) \subset T_{(x,v)}SM$, $K = A, B$ such that

$$d\phi_t(v)P^u_K(v) = P^u_K(\phi_t(v)), d\phi_t(v)P^s_K(v) = P^s_K(\phi_t(v)),$$

$$P^u_K(v) = \{(w, \alpha_K w) \in S(x,v) : w \in K(x,v)\},$$
$$P^s_K(v) = \{(w, -\alpha_K w) \in S(x,v) : w \in K(x,v)\}.$$

where $\alpha_A = 1$ and $\alpha_B = \frac{1}{2}$.

This invariant subbundles are exactly the subbundles of the decomposition in the first item of the strategy stated in the introduction:

$$E^{uu}(x,v) = P^u_A(x,v),$$
$$E^{ss}(x,v) = P^s_A(x,v),$$
$$E^s(x,v) = P^s_B(x,v),$$
$$E^u(x,v) = P^u_B(x,v).$$

Following proposition 6.4, they are invariant subbundles and the splitting is dominated: Jacobi fields in E^{uu} and E^{ss} contract for the past and the future, respectively, at rate e^{-t} and Jacobi fields in E^u and E^s contract for the past and the future, respectively, at rate $e^{-t/2}$. □

4.1.1 Angle cone variation for the Anosov flow with many subbundles

Let us calculate the proper invariance of the cones in the case of the geodesic flow of the compact locally symmetric Riemannian manifold of nonconstant negative sectional curvature.

We use the following family of trajectories for the system:

$$q(t,u) = \pi \circ \phi_t(z(u)),$$

$q(t,u), |u| < \epsilon$.
The Jacobi field is given by

$$\xi = \frac{dq}{du}|_{u=0}, \eta = \frac{Dv}{du}|_{u=0} = \frac{D}{du}|_{u=0}\frac{dq}{dt}.$$

So the following equations hold:

$$\frac{D\xi}{dt} = \eta, \frac{D\eta}{dt} = -R(v,\xi)v.$$

The quantity (1), which in this case is

$$\frac{g(Pr_A(\xi+\eta), Pr_A(\xi+\eta))}{g(\xi,\xi)+g(\eta,\eta)},$$

indicates twice the square of the cosine of the angle between the vector $(\xi,\eta) \in T_{(x,v)}SM$ and its projection to $P^u_A(x,v)$. The cone in this case is

$$C(v, P^u_A(x,v), \delta) = \{(\xi,\eta) \in T_{(x,v)}SM : \frac{\widehat{g}(Pr_{P^u_A(v)}(\xi,\eta), Pr_{P^u_A(v)}(\xi,\eta))}{\widehat{g}((\xi,\eta),(\xi,\eta))} \geq c\},$$

where $\widehat{g}$ is the Sasaki metric and $c = 2\cos^2\delta$. So, it is the same to prove that the cone fields are properly invariant or to prove that the cosine of this angle increases under the action of the derivative of the geodesic flow, for vector in the boundary of the cone fields, or $(\xi,\eta) \in T_\theta SM$ such that

$$\frac{g(Pr_A(\xi+\eta), Pr_A(\xi+\eta))}{g(\xi,\xi)+g(\eta,\eta)} = c \in (1,2).$$

Remember that if (M,g) is locally symmetric then:

$$\frac{d}{dt}g(u,v) = g\left(\frac{Du}{dt}, v\right) + g\left(u, \frac{Dv}{dt}\right),$$

$$\frac{D}{dt}Pr_A\xi = Pr_A\frac{D}{dt}\xi.$$

Let us call, to simplify the equations,

$$\xi_A := Pr_A\xi, \xi_B := Pr_B\xi,$$

$$\xi'_A = Pr_A\frac{D}{dt}\xi, \xi'_B = Pr_B\frac{D}{dt}\xi,$$

and remember that $\xi = \xi_A + \xi_B$. Then, for

$$\frac{g(\xi_A+\eta_A, \xi_A+\eta_A)}{g(\xi,\xi)+g(\eta,\eta)} = c \in (1,2),$$

the following holds:

$$\begin{aligned}\frac{d}{dt}\frac{g(\xi_A+\eta_A, \xi_A+\eta_A)}{g(\xi,\xi)+g(\eta,\eta)} &= 2\frac{g(\xi_A+\eta_A, \eta_A-(R(v,\xi)v)_A)}{g(\xi,\xi)+g(\eta,\eta)} \\ &- 2\frac{g(\xi_A+\eta_A, \xi_A+\eta_A)}{(g(\xi,\xi)+g(\eta,\eta))^2}(g(\xi,\eta)-R(v,\xi,v,\eta)).\end{aligned}$$

But for the locally symmetric metric of negative curvature, the following holds:

$$R(v,\xi)v = -\frac{1}{4}\xi_B - \xi_A.$$

So, we have:

$$\frac{d}{dt}\frac{g(\xi_A+\eta_A,\xi_A+\eta_A)}{g(\xi,\xi)+g(\eta,\eta)} = 2\frac{g(\xi_A+\eta_A,\xi_A+\eta_A)}{(g(\xi,\xi)+g(\eta,\eta))^2}(g(\xi,\xi)+g(\eta,\eta)-g(\xi,\eta)-$$
$$g(\xi_A,\eta_A)-\frac{1}{4}g(\xi_B,\eta_B)) = 2\frac{g(\xi_A+\eta_A,\xi_A+\eta_A)}{(g(\xi,\xi)+g(\eta,\eta))^2}(g(\xi_A,\xi_A)+g(\xi_B,\xi_B)+g(\eta_A,\eta_A)+$$
$$+g(\eta_B,\eta_B)-g(\xi_A,\eta_A)-g(\xi_B,\eta_B)-g(\xi_A,\eta_A)-\frac{1}{4}g(\xi_B,\eta_B)) = 2\frac{g(\xi_A+\eta_A,\xi_A+\eta_A)}{(g(\xi,\xi)+g(\eta,\eta))^2}$$
$$(g(\xi_A,\xi_A)-2g(\xi_A,\eta_A)+g(\eta_A,\eta_A)+g(\xi_B,\xi_B)-\frac{5}{4}g(\xi_B,\eta_B)+g(\eta_B,\eta_B)) =$$
$$2\frac{g(\xi_A+\eta_A,\xi_A+\eta_A)}{(g(\xi,\xi)+g(\eta,\eta))^2}(g(\xi_A-\eta_A,\xi_A-\eta_A)+g(\xi_B-\frac{5}{8}\eta_B,\xi_B-\frac{5}{8}\eta_B)+\frac{39}{64}g(\eta_B,\eta_B)).$$

Since the derivative is the same if (ξ,η) is multiplied by a scalar, we consider (ξ,η) such that $g(\xi,\xi)+g(\eta,\eta)=1$, and such that they are in the boundary of the cones of the cone field of opening c. This is a compact set and the derivative for this values of (ξ,η) is far away from zero. This means that the cones are properly invariant under the action of the derivative of the geodesic flow.

To get the exponential growth, we need to calculate:

$$\begin{aligned}\frac{d}{dt}g(\xi_A+\eta_A,\xi_A+\eta_A) &= 2g(\xi_A+\eta_A,\eta_A-(R(v,\xi)v)_A)\\ &= 2g(\xi_A+\eta_A,\xi_A+\eta_A).\end{aligned}$$

This implies that the vectors inside the cone grow at the rate of e^t.

4.1.2 Orientability of A and B

Recall that a Kahler manifold is a triple (M,J,ω), such that $J:TM\to TM$ is a integrable complex map with $J^2=-Id_{TM}$, and ω is a J-compactible symplectic form. In the case of negative holomorphic curvature -1, $A(x,v)=\mathbb{R}\cdot Jv$ and $B(x,v)$ has a basis of the form $(e_1,Je_1,\dots,e_k,Je_k)$. If γ is a closed geodesic then the parallel transport along γ sends Jv to Jv and sends $(e_1,Je_1,\dots,e_k,Je_k)$ to $(\tilde{e}_1,J\tilde{e}_1,\dots,\tilde{e}_k,J\tilde{e}_k)$, which have the same orientation.

In the Kahler quaternionic case, instead of one map J, there are three maps I, J, K, such that $I^2=J^2=K^2=-Id_{TM}$, $IJ=-JI$, $K=IJ$ [Be], [Wo]. In this case, $A(x,v)$ has as its basis (Iv,Jv,Kv). The three maps are not parallel, but the orthogonal projection to A is parallel. Also, $Q(v)=Iv\wedge Jv\wedge Kv$ is parallel, so along closed geodesics the orientation of $A(x,v)$ is preserved [Gr]. For the same reason, Q being parallel, $B(x,v)$ has its orientation preserved along closed geodesics.

We need orientability to ensure that we can define normal coordinates along closed geodesics as we do in next section.

5 The partially hyperbolic non-Anosov example

In the first subsection we give a more detailed strategy for the deformation of the metric introduced in the previous section.

In the second subsection we give some definitions and we introduce the deformation of the original metric whose geodesic flow is partially hyperbolic and non-Anosov.

In the subsections 5.3.3, 5.3.4, 5.3.5 we show that the new geodesic flow preserves a strong stable and a strong unstable cone fields. We first show that along the closed geodesic γ the strong stable and strong unstable cones are properly invariant under the action of the derivative of the deformed geodesic flow. Then, we show that for geodesics which are close to $(v_0, 0, 0, \ldots, 0)$ the strong stable and strong unstable cones are properly invariant too (subsections 5.3.3 and 5.3.4). Then we show that for geodesics that cross the neighborhood of the deformation of the compact locally symmetric metric the strong stable and strong unsable cones are not properly invariant, but we manage to control the lack of this property in such a way that, after crossing the neighborhood, and inside the region where the metric remains the same, proper invariance is obtained (subsection 5.3.5). Then we prove that there is expansion for the vectors in the strong unstable cones, and contraction for the vectors in the strong stable cones (subsection 5.3.7).

In subsection 5.4 we state the main theorem and some of its corollaries.

Remark 5.1. We only need to show the strong unstable cone is properly invariant, because this garantees that we have one unstable subbundle E^u invariant under the flow. For the same reasons there is a properly invariant unstable subbundle for the inverse of the flow, which is the stable subbundle, since geodesic flows are reversible flows.

5.1 The strategy to construct the example

The strategy of the construction of the example of theorem 1.1 is done following the next steps:

1. It is chosen a metric whose geodesic flow is Anosov and whose hyperbolic invariant splitting is of the form $T(SM) = E^{ss} \oplus E^s \oplus < X > \oplus E^u \oplus E^{uu}$ (section 4);

2. we take a closed geodesic γ without self-intersections (section 5.2);

3. we change the metric in a tubular neighborhood of γ in M, such that along γ the strong subbundles (E^{ss} and E^{uu}) remain invariant and the weak subbundles disapear, becoming a central subbundle with no hyperbolic behavior (section 5.3):

3.1. to accomplish the non-hyperbolicity we change the metric in such a way that the directions of small curvature become directions of zero curvature (section 5.3);

3.2. to obtain that the strong subbundles remain the same along γ_0 we change it in a way that the directions of larger curvature (E^{ss} and E^{uu}) remain (section 5.3);

4. outside the tubular neighborhood of γ, the dynamics remains hyperbolic;

5. we show that for the geodesics that intersect the tubular neighborhood the cones associated to the extremal subbundles (E^{ss} and E^{uu}) are preserved (sections 5.3.3, 5.3.4, 5.3.5. 5.3.6):

5.1. first, we verify that for orbits of the geodesic flow which are close to γ_0 ('parallel' region) the cones associated with the extremal subbundles are preserved (sections 5.3.3, 5.3.4);

5.2. second, we verify that for orbits of the geodesic flow which are 'transversal' to γ_0 ('transversal' region) we can control the angle cone variation for the cones associated with the extremal subbundles with its own axis under the action of the derivative of the geodesic flow (section 5.3.5);

5.3. then, we prove that the time spent in the 'transversal' region is as small as we need in comparison to the time spent outside it (section 5.3.5);

6. we prove that for vectors in the unstable cones there is expansion, and for vectors inside the stable cones there is contraction, under the action of the derivative of the new geodesic flow (section 5.3.7).

5.2 The new metric g^* and its properties

Let us call (M^n, g) a compact locally symmetric space of nonconstant negative curvature of dimension n, introduced in section 4.

Let us fix a closed prime geodesic $\gamma : [0,T] \to M^n$, with $\gamma(0) = \gamma(T)$ and $\gamma'(0) = \gamma'(T)$, without self-intersections. This is the closed geodesic which we use to construct the tubular neighborhood where we change the metric g. There is always a geodesic with these properties in a compact Riemannian manifold [Kl].

Definition 5.2. Let us call $B(\gamma, \epsilon) = \Psi(U(\epsilon))$ a tubular neighborhood of the geodesic γ constructed as follows:

We introduce normal coordinates along this geodesic. Take an orthonormal basis of vector fields $\{e_0(t) := \gamma'(t), e_1(t), \ldots, e_{n-1}(t)\}$ in $T_{\gamma(t)}M$, such that $\{e_1(t), \ldots, e_r(t)\}$ is a basis for $A(\gamma(t), \gamma'(t))$, and $\{e_{r+1}(t), \ldots, e_{n-1}(t)\}$ is a basis for $B(\gamma(t), \gamma'(t))$. This is possible because the parallel transport preserves orientation and M is orientable. $\Psi : [0,T] \times (-\epsilon_0, \epsilon_0)^{2n-1} \to M : (t,x) \to \exp_{\gamma(t)}(x_1 e_1(t) + x_2 e_2(t) + \ldots + x_{n-1} e_{n-1}(t))$ with ϵ_0 less than the injectivity radius, so $\Psi|_U$ is a diffeomorphism, with $U = [0,T] \times (-\epsilon_0, \epsilon_0)^{n-1}$. We define $U(\epsilon) := [0,T] \times (-\epsilon_0, \epsilon_0)^{n-1}$.

Definition 5.3. The set of vectors $\{(x,v) \in SM : x \in B(\gamma,\epsilon), |v_i| < \theta, i = 1,\dots,n-1\}$ is called the set of θ-parallel vectors to γ, the set $\{(x,v) \in SM : x \in B(\gamma,\epsilon), |v_i| \geq \theta,$ for some $i = 1,\dots,n-1\}$ is called the set of θ-transversal vectors to γ. If $(x,v) \in SM$ belongs to the set of θ-parallel vectors for all θ, then we call it a parallel vector to γ. Notice that $\{(x,v)$ is θ-parallel to $\gamma\} \cup \{(x,v)$ is θ-transversal to $\gamma\} = B(\gamma,\epsilon)$.

Let $g_{ij}(t,x)$ denote the components of the metric in this tubular neighborhood of γ where Ψ is defined. We define a new Riemannian metric g^* as:

$$g^*_{00}(t,x) := g_{00}(t,x) + \alpha(t,x),$$

$$\alpha(t,x) := \sum_{i,j=1}^{n-1} \Phi_{ij}(t,x) x_i x_j,$$

$$g^*_{ij}(t,x) := g_{ij}(t,x), (i,j) \neq (0,0),$$

with $\Phi_{ij} : [0,T] \times (-\epsilon_0,\epsilon_0)^{n-1} \to \mathbb{R}$, where each Φ_{ij} is a bump function. This kind of deformation allows us to change the curvature (change the second derivative), as γ and the parallel transport along γ (the metric up to its first derivative) remain the same. This becomes clear if we look to the formulas of the metric, the parallel transport and the curvature with respect to a coordinate system.

For this new metric g^*, the coordinates along γ are:

$$g^{*ij}(t,0) = g^{ij}(t,0), 0 \leq i,j \leq n-1,$$

$$g^*_{ij}(t,0) = g_{ij}(t,0), 0 \leq i,j \leq n-1,$$

$$\partial_k g^{*ij}(t,0) = \partial_k g^{ij}(t,0), 0 \leq i,j,k \leq n-1,$$

$$\partial_k g^*_{ij}(t,0) = \partial_k g_{ij}(t,0), 0 \leq i,j,k \leq n-1.$$

These equalities imply that the closed geodesic γ still is a closed geodesic for g^*. We are going to use the following deformation:

$$\alpha(t,x) = \sum_{k=r+1}^{n-1} x_k^2 \Phi_k(x),$$

The first property we need for the function $\alpha : U \to \mathbb{R}$ is that $\Phi_k(t,0) = -\frac{1}{4}$. The Φ_k are going to be products of bump functions define on a tubular neighborhood of γ of radius $\epsilon < \epsilon_0$. We need to change ϵ along the proof, so we can say that this functions Φ_k are going to be ϵ-parameter families of functions. For some ϵ small enough the new metric g^* is going to be partially hyperbolic.

Now we are going to state other properties that are going to help us prove the proper invariance of the cones under the action of the derivative of the geodesic flow.

First, to simplify the problem, we try to perturb the curvature only in the direction of the subspace generated by $\frac{\partial}{\partial x_k}, k = r+1, \dots, n-1$, at least for some geodesics. This is impossible, but we can construct a bump function such that, as $\epsilon \to 0$, only the term $\partial^2_{x_k x_k}, k = r+1, \dots, n-1$ perturbs the curvature.

Second, let us construct

$$\Phi_k(t,x) = \frac{1}{4}\phi_{k,1}(x_1)\phi_{k,2}(x_2)\phi_{k,3}(x_3)\dots\phi_{k,2n-1}(x_{2n-1}),$$

ϕ_i bump functions themselves. So, the second property is that Φ_k does not depend on t.

Third, let us define $\phi_{k,1}, \dots \phi_{k,n-1}$, except $\phi_{k,k}$, with support on $[-\epsilon, \epsilon]$, such that $\phi_{k,i}(0) = 1$, $\phi_{k,i}(\pm\epsilon) = 0$, with $\epsilon < \epsilon_0$, and $\phi_{k,k}$ with support on $[-\epsilon^2, \epsilon^2]$, $\phi_{k,k}(0) = -1$ and $\phi_{k,k}(\pm\epsilon^2) = 0$. This ensures that the only second order partial derivative of α that does not goes to 0 as $\epsilon \to 0$ is $\partial^2_{k,k}\alpha$. Moreover, α is C^1-close to the constant zero function. Since x_k^2 is of order ϵ^4, we can say that α is of order ϵ^4, $d\alpha$ is of order ϵ^2 and $d^2\alpha$ is of order 1, so that $d^2\alpha$ is limited, with limitation independent of ϵ.

Lemma 5.4. *For $\alpha : U \to \mathbb{R} : (t,x) \to \sum_{k=r+1}^{n-1} x_k^2 \Phi_k(t,x)$, the following inequalities are satisfied:*

i. $|\alpha(t,x)| \leq M_0\epsilon^4$,

ii. $|\partial_{x_j}\alpha(t,x)| \leq M_0\epsilon^2$,

iii. $|\partial^2_{x_i x_j}\alpha(t,x)| \leq M_0\epsilon$, *if* $i \neq j$, *or if* $i \leq r$, *or* $j \leq r$,

iv. $|\partial^2_{x_k x_k}\alpha(t,x)| \leq M_0$, M_0 *independent of* ϵ, $k = r+1, \dots, n-1$.

Proof. Item i. $|\alpha(x)| \leq \frac{1}{4}\epsilon^4$. Item ii.: $|\partial_{x_j}\alpha(x)| \leq \frac{1}{4}\epsilon^4 2\epsilon^{-2}$. Item iii.: $|\partial^2_{x_j x_i}\alpha(x)| \leq \frac{n}{4}\epsilon^4 4\epsilon^{-2}$ if $j \neq i$. Item iv.: $|\partial^2_{x_k x_k}\alpha| \leq \frac{1}{4}\epsilon^4 3\epsilon^{-4} \leq 1$. □

Lemma 5.5. *For every $\delta > 0$ there is a bump function ϕ, such that its minimum value is at $x = 0$, $\phi(\pm\epsilon^2) = 0$, and $F(\phi)(x) := x^2\phi''(x) + 4x\phi'(x) + 2\phi(x) \in [(-2-\delta)F(\phi)(0), (2+\delta)F(\phi)(0)]$.*

Proof. To prove the lemma, first we construct a C^2 function ϕ such that the property stated in the lemma holds for $\frac{\delta}{2}$. Then, there will be a C^∞ function ϕ such that it holds for δ. To construct this C^2 function is easy. We define the following function φ_τ, continuous and piecewise-C^1 in $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$:

. $\varphi_\tau(0) = \varphi_\tau(1) = \varphi(\frac{1}{2}) = 0$,

. $\varphi'_\tau(x) = \frac{h_\tau}{\tau}$, if $x \in (0,\tau) \cup (1-\tau, 1)$,

. $\varphi'_\tau(x) = -\frac{h_\tau}{\tau}$, if $x \in (\frac{1}{2}-\tau, \frac{1}{2}+\tau)$,

. $\varphi_\tau(x) = h_\tau$ for $x \in (\tau, \frac{1}{2} - \tau)$, $\varphi_\tau(x) = -h_\tau$, for $x \in (\frac{1}{2} - \tau, 1 - \tau)$.

Then we define ϕ_τ such that $\phi_\tau(1) = 0$, $\phi'_\tau(0) = \phi'_\tau(1) = 0$ and $\phi''_\tau = \varphi_\tau$. Then ϕ_τ is C^2 and $\phi_\tau(0) = -\frac{h_\tau}{4}(1 - 2\tau)$. We use the fact that it holds for $\tau = 0$ and then show that it holds for τ small enough.

For $\tau = 0$, ϕ_0 is not C^2 but this is not a problem. For ϕ_0, we have that $F(\phi_0)(x) = (-\frac{1}{2} + 6x^2)h_0$ for $x \in (0, \frac{1}{2})$ and $F(\phi_0)(x) = (-1 + 6x + 6x^2)h_0$ for $x \in (\frac{1}{2}, 1)$. Then it is simple to see that $F(\phi)(0) = -\frac{h_0}{2}$ and $F(\phi_0)(x) \in [-h_0, h_0]$. Then, for ϕ_0 we have that $F(\phi_0)(x) \in [-2F(\phi)(0), 2F(\phi)(0)]$. So, why does it holds for ϕ_τ, with τ small enough?

First, we notice that the first term of $F(\phi_\tau)$ is the only one that does not varies continuosly as τ varies. The other two do vary continuosly because ϕ_τ is C^1-close to ϕ_0. So we have to analyse only $x^2\phi''_\tau(x)$. But $\phi''_\tau(x) \in [-h_\tau, h_\tau]$, which implies $\phi''_\tau(x) \in [-\frac{1}{1-2\tau}h_0, \frac{1}{1-2\tau}h_0]$. Then $x^2\phi''_\tau(x) \in [-\frac{1}{1-2\tau}x^2h_0, \frac{1}{1-2\tau}x^2h_0]$. This, in turn, implies that $F(\phi_\tau)(x) \in [-\frac{2}{1-2\tau}F(\phi_0)(0) - \delta'(\tau), \frac{2}{1-2\tau}F(\phi)(0) + \delta'(\tau)] = [-2F(\phi_\tau)(0) - \delta'(\tau), 2F(\phi_\tau)(0) + \delta'(\tau)]$. Then, for τ small enough, the lemma holds for a C^2 ϕ. This implies it holds for a C^∞ ϕ.

Our bump functions are defined in an interval of lenght ϵ^2, so let us notice that if the lemma holds for ϕ with support in $[0,1]$, then it holds for ϕ^λ such that $\phi^\lambda(x) := \phi(\lambda x)$. It holds also if ϕ is multiplied by a constant.

□

Remark 5.6. Previous lemma says that if the curvature is changed by $\frac{1}{4}$ along the closed geodesic γ, then the curvature is deformed by $\pm\frac{1}{2}$ in the weak directions of the splitting of the geodesic flow, so the curvature for the strong directions is still greater than in the other directions. This explains in a rough way why the geodesic flow still preserves the strong directions.

5.3 Partial hyperbolicity of the geodesic flow of g^*

To prove that the geodesic flow of the new metric g^* is partially hyperbolic we are going to define the strong stable and strong unstable cones of the geodesic flow of g^*.

Definition 5.7. The strong unstable and strong stable cone fields for g^* are:

$$C^u(v, c) := \left\{ (\xi, \eta) \in S(x, v) : \frac{g^*(\xi_A + \eta_A, \xi_A + \eta_A)}{g^*(\xi, \xi) + g^*(\eta, \eta)} \geq c \right\},$$

$$C^s(v, c) := \left\{ (\xi, \eta) \in S(x, v) : \frac{g^*(\xi_A - \eta_A, \xi_A - \eta_A)}{g^*(\xi, \xi) + g^*(\eta, \eta)} \geq c \right\}.$$

for a real number $c \in (1, 2)$, and $v \in T_xM$, $g^*_x(v, v) = 1$.

Remark 5.8. Notice that the cone field defined above coincides with the cone field associated with g outside the region of the deformation of the metric g.

Remark 5.9. Remember that

$$\xi_A := Pr_A\xi, \xi_B := Pr_B\xi,$$

$$\xi'_A = Pr_A\frac{D}{dt}\xi, \xi'_B = Pr_B\frac{D}{dt}\xi.$$

We also define

$$\xi_{A'} = \left(\frac{D}{dt}Pr_A\right)\xi, \xi_{B'} = \left(\frac{D}{dt}Pr_B\right)\xi,$$

because for g^* the subspaces A and B are not parallel.

Proposition 5.10. *The geodesic flow of g^* preserves the strong unstable cone field $C^u(v,c)$ and the strong stable cone field $C^s(v,c)$.*

We only prove proper invariance of the strong unstable cone (see remark 5.1). We divide the proof in several steps, but first, in the next subsection, we prove that along the geodesic γ, the geodesic flow of g^* is partially hyperbolic but not hyperbolic.

5.3.1 Along γ the geodesic flow of g^* is not hyperbolic

By a corollary from Eberlein's article [E1]:

Corollary 3.4 [E1]. *If the geodesic flow is Anosov, then the following holds: Let any γ be a unit speed geodesic, and $E(t)$ any non-zero perpendicular parallel vector field along γ, then the sectional curvature $K(\gamma', E)(t) < 0$ for some real number t.*

For the geodesic flow of the new metric g^*, $E(t)$ is a non-zero perpendicular parallel vector field along γ, and $K(\gamma', E)(t) = 0$, then the geodesic flow of the metric g^* is not Anosov.

Lemma 5.11. *If $\Phi_k(t,0) = -\frac{1}{4}$ then, following Eberlein's criterion, the geodesic flow of g^* is not Anosov.*

Proof. The coordinates of the curvature tensor in this neighborhood are:

$$R_{ijkl} = -\frac{1}{2}(\partial^2_{ik}g_{jl} + \partial^2_{jl}g_{ik} - \partial^2_{il}g_{jk} - \partial^2_{jk}g_{il}) - \Gamma^T_{ik}g^{-1}\Gamma_{jl} + \Gamma^T_{il}g^{-1}\Gamma_{jk}, \tag{4}$$

where $\Gamma_{ik} := [\Gamma_{j,ik}]_j$ and $\Gamma_{j,ik} := \frac{1}{2}(\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ik})$.

So, at γ, the curvature tensor is:

$$\begin{aligned} R^*_{ijkl}(t,0) = R_{ijkl}(t,0) \;&-\; \frac{1}{2}(\delta_{j+l,0}\partial^2_{ik}\alpha(t,0) + \delta_{i+k,0}\partial^2_{jl}\alpha(t,0) \\ &-\; \delta_{j+k,0}\partial^2_{il}\alpha(t,0) - \delta_{i+l,0}\partial^2_{jk}\alpha(t,0)), \end{aligned}$$

and

$$R^*_{0j0l}(t,0) = R_{0j0l}(t,0) - \frac{1}{2}(\partial^2_{jl}\alpha(t,0)).$$

Then, along γ:

$$R^*_{0i0j}(t,0) = R_{0i0j}(t,0), i \neq j, i,j = 2,\dots,n-1,$$

$$\begin{aligned} R^*_{0k0k}(t,0) &= R_{0k0k}(t,0) - \frac{1}{2}(\partial^2_{kk}\alpha(t,0)) \\ &= R_{0k0k}(t,0) - \Phi_k(t,0). \end{aligned}$$

For the initial metric and $k = r+1,\dots,2n-1$:

$$R_{0k0k}(t,0) = g_{00}(t,0)g_{kk}(t,0)K(\gamma'(t), e_k(t)) = -\frac{1}{4}.$$

So, if $\Phi_k(t,0) = -\frac{1}{4}$, then $R^*_{0k0k}(t,0) = 0$. Then, Eberlein's corollary applies, and the geodesic flow of g^* is not Anosov. □

5.3.2 Along γ the geodesic flow of g^* is partially hyperbolic

We are going to show that the strong unstable cone field of the geodesic flow of section 4 still works for the geodesic flow of the new metric g^* along γ.

Lemma 5.12. *For the new metric g^* and along the geodesic γ there is an invariant splitting $S(t) = E^{ss} \oplus E^c \oplus E^{uu}$, such that $E^{ss} = E^{ss}_g$, $E^c = E^s_g \oplus E^u_g$, $E^{uu} = E^{uu}_g$, where E^σ_g are the subbundles of the hyperbolic invariant splitting of the geodesic flow of the original metric g, $\sigma = uu, u, s, ss$, and $S(t)$ is the contact structure of S^*M along $(\gamma(t), \gamma'(t))$.*

Proof. The new metric g^* has the same coordinates as g along the closed geodesic γ, and observe that it has the same Christoffel symbols along γ. This implies that g^* has the same parallel transport as g along γ. So, if $\{E_0(t) = \gamma'(t), E_1(t),\dots,E_r(t), E_{r+1}(t),\dots,E_{n-1}(t)\}$ is a orthonormal basis of parallel vector fields in $T_{\gamma(t)}M$, $\zeta(t) = \sum_{i=0}^{2n-1} f_i(t)E_i(t)$ are Jacobi fields along γ if they are the solutions of the following equation:

$$\begin{aligned} 0 &= \zeta''(t) + R^*(\gamma'(t), \zeta(t))\gamma'(t) \\ &= \sum_{i,j=0}^{2n-1} (f_i''(t) + R^*(E_0, E_j, E_0, E_i)(t) f_j(t))E_i(t) \end{aligned}$$

$$\Rightarrow 0 = f_i''(t) + \sum_{j=1}^{2n-1} R^*(E_0, E_j, E_0, E_i)(t) f_i(t), i = 0,\dots,2n-1$$

$$\Rightarrow \begin{bmatrix} f(t) \\ f'(t) \end{bmatrix}' = \begin{bmatrix} 0 & I \\ -K^*(t) & 0 \end{bmatrix} \begin{bmatrix} f(t) \\ f'(t) \end{bmatrix},$$

$$K^*_{ij}(t) := R^*(E_0, E_j, E_0, E_i)(t).$$

Along γ we have:

$$K^*(t) = \begin{bmatrix} -Id_r & 0 \\ 0 & 0 \end{bmatrix}.$$

The hyperbolic subbundles are E^{uu}, spanned by $(e^t e_i(t), e^t e_i(t))$, $i = 1, \ldots, r$ and E^{ss}, spanned by $(e^{-t} e_i(t), -e^{-t} e_i(t))$, $i = 1, \ldots, r$ and E^{ss}, the same as for the metric g. And there is a central direction spanned by the Jacobi fields related to the curvature $K(\gamma'(t), E_k(t))$, $E_k(t)$ and $tE_k(t)$, for $k = r+1, \ldots, 2n-1$. This implies we have a central bundle E^c along the geodesic γ. Notice that $\{e_k(t)\}_{k=r+1}^{2n-1}$ and $\{E_k(t)\}_{k=r+1}^{2n-1}$ generate the same subspace of $T_{\gamma(t)}M$, invariant by parallel transport because it is orthogonal to $\gamma'(t)$ and $A(\gamma(t), \gamma'(t))$. Then $E^c = E^s_g \oplus E^u_g$. $\square$

5.3.3 Preservation of the cone field for parallel vectors

Now we do the same calculations of section 4 for the geodesic flow of the new metric g^*. To prove the partial hyperbolicity of this new flow, we divided the set of vectors whose geodesics cross the neighborhood where we change the original metric. First we verify the proper invariance of the cone field for parallel vectors (see definition in the beginning of section 5.3).

By the formula of the bump function Φ_k we have that, as ϵ goes to zero, the partial derivatives of second order of α which do not involve the direction of $\frac{\partial}{\partial x_k}$ go to zero. The only one that does not shrink is $\partial^2_{k,k}\Phi_k$.

So, the following holds:

$$R^*_{010k} \approx R_{010k}, k = 2, \ldots, n-1,$$

$$R^*_{0k0k} \approx R_{0k0k} - \frac{1}{2}\partial^2_{k,k}\alpha$$

If $v = (v_0, 0, \ldots, 0)$ then:

$$R^*_{v\xi v\eta} \approx R_{v\xi v\eta} - \frac{1}{2}\partial^2_{\xi\eta}\alpha v_0^2$$

$$\approx R_{v\xi v\eta} - \frac{1}{2}\sum_{k=r+1}^{n-1} \partial^2_{kk}\alpha v_0^2 \xi_k \eta_k.$$

When we use the symbol $\approx$ we mean that the difference between the left side and the right side is of order ϵ. It depends on the size of $|\alpha|, |\partial\alpha|, |\partial^2_{ij}\alpha|, i \neq j$, and the size of $supp(\Phi_i), i = r+1, \ldots, n-1$.

Lemma 5.13. *For parallel vectors the angle cone variation is positive (the cone closes).*

Proof. We begin by approximating the angle cone variation at parallel vectors with respect to the derivative of the geodesic flow by an expression that is better to work with. This expression is equal to the one for the geodesic flow of g except for the term related to the second derivative of α and ξ_k, η_k, k related to the central direction along γ:

$$\frac{d}{dt}\frac{g^*(\xi_A+\eta_A,\xi_A+\eta_A)}{g^*(\xi,\xi)+g^*(\eta,\eta)} - 2\frac{g^*(\xi_A+\eta_A,\xi_A+\eta_A)}{(g^*(\xi,\xi)+g^*(\eta,\eta))^2}\left(\frac{5}{8}g^*(\xi-\eta,\xi-\eta)\right.$$
$$\left.+\frac{3}{8}g^*(\xi,\xi) - \frac{3}{4}g^*(\xi_A,\eta_A) + \frac{1}{2}\sum_{k=r+1}^{n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k + \frac{3}{8}g^*(\eta,\eta)\right)$$
$$= 2\frac{g^*(\xi_A+\eta_A,\xi_A+\eta_A)}{(g^*(\xi,\xi)+g^*(\eta,\eta))^2}\left(\left(\frac{g^*(\xi_A+\eta_A,\xi_{A'}+\eta_{A'})}{g(\xi_A+\eta_A,\xi_A+\eta_A)}\right)(g^*(\xi,\xi)\right.$$
$$+g^*(\eta,\eta)) - \left(\frac{g^*(\xi_A+\eta_A,\xi_A+R^*(v,\xi)v)}{g^*(\xi_A+\eta_A,\xi_A+\eta_A)}\right)(g^*(\xi,\xi)+g^*(\eta,\eta)) +$$
$$\left.\frac{1}{4}g^*(\xi,\eta) + \frac{3}{4}g^*(\xi_A,\eta_A) - \frac{1}{2}\sum_{k=r+1}^{n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k + R^*(v,\xi,v,\eta)\right).$$

We define as $\xi_{A'}$ the covariant derivative of the projection to A applied to ξ: $(\nabla^* Pr_A)\xi$. If c is the opening of the cone and $g^*(\xi,\xi)+g^*(\eta,\eta)=1$, because the derivative does not depend on the norm of the (ξ,η), the equation above is:

$$= 2C(C^{-1}(g^*(\xi_A+\eta_A,\xi_{A'}+\eta_{A'})) - C^{-1}(g^*(\xi_A+\eta_A,\xi_A+R^*(v,\xi)v))$$
$$+\frac{1}{4}g^*(\xi,\eta) + \frac{3}{4}g^*(\xi_A,\eta_A) - \frac{1}{2}\sum_{k=r+1}^{n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k + R^*(v,\xi,v,\eta)).$$

Then:

$$\begin{aligned}|g^*(\xi_A+\eta_A,\xi_A+R^*(v,\xi)v)| &\leq |g^*(\xi_A+\eta_A,R^*(v,\xi)v-R(v,\xi)v)|\\ &+ |g^*(\xi_A+\eta_A,\xi_A+R(v,\xi)v)|\,.\end{aligned}$$

Since $|g^*(\xi_A+\eta_A,\xi_A+R(v,\xi)v)|$ depends on $|\alpha|$, and $|g^*(\xi_A+\eta_A,R^*(v,\xi)v-R(v,\xi)v)|$ depends on $|\alpha|$, $|\partial\alpha|$, and $|\partial^2_{j\xi}\alpha|$, $j=1,\ldots,r$ and these terms are limited by $M\epsilon$, we can say that, for some big enough M_1 independent of ϵ:

$$|g^*(\xi_A+\eta_A,\xi_A+R^*(v,\xi)v)| \leq |g^*(\xi_A+\eta_A,R^*(v,\xi)v-R(v,\xi)v)| +$$
$$|g^*(\xi_A+\eta_A,\xi_A+R(v,\xi)v)| \leq M_1\epsilon.$$

For the same reasons:

$$|g^*(\xi_A+\eta_A,\xi_{A'}+\eta_{A'})| \leq M_0 \left\|g^*-g\right\|_{C^1}(|\xi|^*+|\eta|^*) \leq M_1\epsilon.$$

$$\left|\frac{1}{4}g^*(\xi,\eta)+\frac{3}{4}g^*(\xi_A,\eta_A)-\frac{1}{2}\sum_{k=r+1}^{n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k+R^*(v,\xi,v,\eta)\right| \leq M_1\epsilon.$$

Suppose M_1 sufficiently big to be the same in the three inequalities above. So we have:

$$\left|\frac{d}{dt}\frac{g^*(\xi_A+\eta_A,\xi_A+\eta_A)}{g^*(\xi,\xi)+g^*(\eta,\eta)}-2\frac{g^*(\xi_A+\eta_A,\xi_A+\eta_A)}{(g^*(\xi,\xi)+g^*(\eta,\eta))^2}\left(\frac{5}{8}g^*(\xi-\eta,\xi-\eta)+\right.\right.$$
$$\left.\left.\frac{3}{8}g^*(\xi,\xi)-\frac{3}{4}g^*(\xi_A,\eta_A)+\frac{1}{2}\sum_{k=r+1}^{n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k+\frac{3}{8}g^*(\eta,\eta)\right)\right|$$
$$\leq 2c(3M_1)\epsilon = M_2\epsilon.$$

Let us analyse the following expression over the initial closed geodesic:

$$\left(\frac{3}{8}g^*(\xi,\xi)-\frac{3}{4}g^*(\xi_A,\eta_A)+\frac{1}{2}\sum_{k=r+1}^{n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k+\frac{3}{8}g^*(\eta,\eta)\right)=$$

$$\frac{3}{8}(\xi_1^2+\xi_2^2+\ldots+\xi_{n-1}^2+\eta_1^2+\eta_2^2+\ldots+\eta_{n-1}^2-2\sum_{k=1}^{r}\xi_k\eta_k+\frac{4}{3}\sum_{k=r+1}^{n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k).$$

The expression $\xi_1^2+\eta_1^2+\xi_2^2+\eta_2^2+\ldots+\xi_{n-1}^2+\eta_{n-1}^2-2\xi_1\eta_1-\ldots-2\xi_r\eta_r+\frac{4}{3}\sum_{k=r+1}^{n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k$ equals $\sum_{i=1}^{r}(\xi_i-\eta_i)^2+\sum_{i=r+1}^{n-1}\xi_i^2-\frac{2}{3}\xi_i\eta_i+\eta_i^2=\sum_{i=1}^{r}(\xi_i-\eta_i)^2+\sum_{i=r+1}^{n-1}(\xi_i-\frac{1}{3}\eta_i)^2+\frac{8}{9}\eta_i^2$ which is positive in the border of the cone with opening c. This implies that along the closed geodesic γ the cone is preserved, but that we already knew. We need to prove the positivity of the derivative along the other geodesics of the flow. So, we need the following:

$$inf_{a\in[-1-\frac{\delta}{2},1+\frac{\delta}{2}]}inf\{\sum_{k=r+1}^{n-1}\xi_k^2-\frac{4a}{3}\xi_k\eta_k+\eta_k^2\} \geq L(a,b)>0,$$

for any (ξ,η) in the boundary of the cone with opening $c\in[a,b]\subset(1,2)$.

Because g^* is a C^∞ metric, and its coordinates along γ are δ_{ij}, if the neighborhood of γ is sufficiently small, if ϵ is small enough, we can conclude:

$$inf_{x\in supp(\alpha)} inf\{(g^*(\xi,\xi) - 2g^*(\xi_A,\eta_A) + \frac{4}{3}\sum_{k=r+1}^{n-1} \partial^2_{kk}\alpha v_0^2\xi_k\eta_k + g^*(\eta,\eta))\} \geq \frac{1}{2}L(a,b) > 0.$$

So:

$$inf_{x\in supp(\alpha)} inf\{\frac{3}{8}g^*(\xi,\xi) - \frac{3}{4}g^*(\xi_A,\eta_A) + \frac{1}{2}\sum_{k=r+1}^{n-1} \partial^2_{kk}\alpha v_0^2\xi_k\eta_k + \frac{3}{8}g^*(\eta,\eta)\} \geq$$
$$L'(a,b) = \frac{3}{16}L(a,b) > 0.$$

This implies that, if $\epsilon < \frac{1}{2M_2}L'(a,b)$, for (ξ,η) in the boundary of the cone with opening $c \in [a,b] \subset (1,2)$, and for $v = (v_0, 0, \ldots, 0)$, then the derivative of equation (1) is positive. □

5.3.4 Extension of the cone property to θ-parallel vectors

Now we are going to show that this derivative is positive not only for parallel vectors $(v = (v_0, 0, \ldots, 0))$, but for θ-parallel vectors.

Lemma 5.14. *For θ-parallel vectors the angle cone variation is positive (the cone closes)*

Proof.

$$R^*(v,\xi,v,\eta) - R(v,\xi,v,\eta) \approx -\frac{1}{2}\sum_{k=r+1}^{n-1} \partial^2_{kk}\alpha(v_k^2\xi_0\eta_0 + v_0^2\xi_k\eta_k - v_0v_k(\xi_0\eta_k + \xi_k\eta_0)).$$

This is so because (4) implies the following relation:

$$R^*_{ijkl} - R_{ijkl} \approx -\frac{1}{2}(\partial^2_{ik}\Delta g_{jl} + \partial^2_{jl}\Delta g_{ik} - \partial^2_{il}\Delta g_{jk} - \partial^2_{jk}\Delta g_{il}), \tag{5}$$

where $\approx$ means that the rest of the equation depends on α and $\partial\alpha$, and $\Delta g_{ij} := g^*_{ij} - g_{ij}$. So we can say that:

$$\left| R^*(v,\xi,v,\eta) - R(v,\xi,v,\eta) + \frac{1}{2}\sum_{k=r+1}^{n-1} \partial^2_{kk}\alpha v_0^2\xi_k\eta_k \right| \leq M_1\epsilon + M_0|\theta|(\|\xi\|^*\|\eta\|^*).$$

So, for the derivative we have:

$$\left|\frac{d}{dt}\frac{g^*(\xi_A+\eta_A,\xi_A+\eta_A)}{g^*(\xi,\xi)+g^*(\eta,\eta)} - 2\frac{g^*(\xi_A+\eta_A,\xi_A+\eta_A)}{(g^*(\xi,\xi)+g^*(\eta,\eta))^2}\left(\frac{5}{8}g^*(\xi-\eta,\xi-\eta)+\right.\right.$$
$$\left.\left.\frac{3}{8}g^*(\xi,\xi) - \frac{3}{4}g^*(\xi_A,\eta_A) + \frac{1}{2}\sum_{k=r+1}^{n-1}\partial^2_{kk}\alpha v_0^2\xi_k\eta_k + \frac{3}{8}g^*(\eta,\eta)\right)\right|$$
$$\leq M_2\epsilon + M_0|\theta|(\|\xi\|^*\|\eta\|^*).$$

So, if we calculate for (ξ,η) in $g^*(\xi,\xi)+g^*(\eta,\eta)=1$, we have that if $|\theta| < \frac{1}{4M_0}L'(a,b)$ and $\epsilon < \frac{1}{2M_2}L'(a,b)$, then:

$$\frac{d}{dt}\frac{g^*(\xi_A+\eta_A,\xi_A+\eta_A)}{g^*(\xi,\xi)+g^*(\eta,\eta)} \geq \frac{1}{2}L'(a,b) > 0$$

Then we conclude that, in the band $\{(x,v)$ is θ-parallel to $\gamma\}$ the cones are properly invariant for the geodesic flow.

□

5.3.5 The control of the cones for θ-transversal vectors

For vectors that are not θ-close to $(1,0,\ldots,0)$, that are θ-transversal to γ, we do not have preservation of the cones. But this is not at all a problem if we choose an ϵ small enough such that the cone with openning b stays inside the cone with opening a. This is possible because α is C^1 close to zero, the second derivative of α is limited and this limitation does not depend on ϵ. So:

$$\frac{d}{dt}\frac{g^*(\xi_A+\eta_A,\xi_A+\eta_A)}{g^*(\xi,\xi)+g^*(\eta,\eta)} \geq M$$

As ϵ goes to 0, the support of the deformation of the metric shrinks. As it shrinks, the time that the geodesics take to cross this neighborhood of the geodesic γ goes to zero. So, as we can control the time which these geodesics spend inside the neighborhood, we choose an ϵ such that the cone with opening b stays inside the cone of openning a.

Let us be more precise:

Lemma 5.15. *The time which θ-transversal geodesics cross the neighborhood of the deformation of the metric g is comparable to ϵ.*

Proof. To see that the time spent is comparable to ϵ we need to express the geodesic vector field in Fermi coordinates of the neighborhood. We can use Fermi coordinates now because we don't need the coordinates in the whole neighborhood of the closed geodesic γ in this case. The maps $d\pi$ and K are:

$$d\pi\xi = (\xi_0,\xi_1,\ldots,\xi_{2n-1}),$$

$$K\xi = \left(\xi_{2n+k} + \sum_{i,j=0}^{2n-1} \Gamma_{ij}^{*k} v_i \xi_j\right)_{k=0}^{2n-1}.$$

So, the pre-image of $(v, 0)$ by the map $(d\pi, K)$ is:

$$\left(v_0, v_1, \ldots, v_{2n-1}, -\sum_{i,j=0}^{2n-1} \Gamma_{ij}^{*0} v_i v_j, -\sum_{i,j=0}^{2n-1} \Gamma_{ij}^{*1} v_i v_j, \ldots, -\sum_{i,j=0}^{2n-1} \Gamma_{ij}^{*2n-1} v_i v_j\right).$$

Since g^* is C^∞ and along the geodesic γ, $\Gamma_{ij}^{*k} = 0$, then, if ϵ is sufficiently small, the geodesic vector field is approximately $(v_0, v_1, \ldots, v_{2n-1}, 0, 0, \ldots, 0)$.

Since the second part of the geodesic vector field is small as ϵ is small, we can say that geodesics such that $|v_i| \geq \theta$ for some $i = 1, \ldots, 2n-1$ cross the neighborhood in at most $\frac{\epsilon}{\theta}$, and they leave the neighborhood at least $\frac{\theta}{2}$ far from $(1, 0, \ldots, 0)$, or, better said, outside the set $\{v \in S^*M : |v_i| < \frac{\theta}{2}, i = 1, 2, \ldots, 2n-1\}$.

□

5.3.6 Proof of proposition 5.10

Recall the statement of proposition 5.10:

Proposition 5.10. *The geodesic flow of g^* preserves the strong unstable cone field $C^u(v, c)$ and the strong stable cone field $C^s(v, c)$, for some $c \in (1, 2)$ and some ϵ small enough.*

Proof. First, take an orbit of the geodesic flow of g^*. If it never crosses the region of the deformation, where g^* equals the original metric g, then the cone field is preserved. If it crosses the region of deformation, then it takes some time T' inside this region. If it is θ-parallel to the geodesic γ, then it preserves the cone field (lemma 5.14). If it turns, after this time T', into a θ-transversal geodesic, then it spends $T' + k\epsilon$ time inside this region (lemma 5.15), and then it leaves it and spend some time outside it. As the set of the orbits which leave this region is a compact set, the infimum is positive. Let us say they spend at least T_ϵ outside the neighborhood. As ϵ goes to zero, T_ϵ does not goes to zero. If it did, we could get a sequence of geodesics outside $\{v \in S^*M : |v_i| < \frac{\theta}{2}, i = 1, 2, \ldots, 2n-1\}$ which would spend very little time outside the neighborhood of γ before enter it again. So, in the limit, there would be a contradiction with the unicity of the solutions of the ordinary differential equations of the geodesic flow. So the time spent outside the neighborhood of γ is bounded from below - let us say it is bounded from below by T. This means that we can choose ϵ so that the quotient between the time spent inside and the time spent outside of the neighborhood of γ is as small as we want. As small as it is necessary for the preservation of the strong unstable and strong stable cones. So, the orbit spends some time $k\epsilon$ where there is a little expansion of the angle of the cone field, then spends time at least T in the region where there's contraction of the angle of the cone field.

Outside the neighborhood of the deformation the following holds:

$$\frac{d}{dt}\frac{(g^*(\xi_A+\eta_A,\xi_A+\eta_A)}{g^*(\xi,\xi)+g^*(\eta,\eta)}=\frac{d}{dt}\frac{g(\xi_A+\eta_A,\xi_A+\eta_A)}{g(\xi,\xi)+g(\eta,\eta)}\geq\frac{3}{8}c(2-c),$$

for (ξ,η) in the boundary of the cone of openning c. So, for cones with border in $[a,b]$, we have:

$$\frac{d}{dt}\frac{g^*(\xi_A+\eta_A,\xi_A+\eta_A)}{g^*(\xi,\xi)+g^*(\eta,\eta)^*}\geq\frac{3}{8}b(2-b).$$

So we choose a' such that $|a'-b|<\frac{3}{16}b(2-b)T$. This ensures that outside the neighborhood the geodesic flow sends the cone with opening a' inside the cone with opening B in time $\frac{T}{2}$. For ϵ sufficiently small, with the inferior limit of the derivative not depending on ϵ, the cone with opening b is not sent outside the cone with opening a'.

So, we have preservation of the cone field, although there is a region where the cone field is not properly invariant, because the orbits of length T of the geodesic flow cross this region in an interval of time as small as we want. So the preservation of the cone field holds because after that it takes an interval of length $\frac{T}{2}$ for the cones to be properly contained.

So, there is an invariant subbundle E^u. The same happens for the stable invariant direction E^s, because for a geodesic flow the 'past' of the orbit of $v\in S^*M$ is the future of the orbit of $-v$. □

5.3.7 Exponential growth of the Jacobi fields

So, the strong unstable cone is preserved by the new geodesic flow. By symmetry, or by the reversibility of geodesic flows, the strong stable cone is preserved too. But preservation of these cones only proves that there are invariant subbundles with domination. We have to show that there is exponential growth along these strong directions.

Proposition 5.16. *For the geodesic flow of g^* there is exponential expansion of vectors in $C^u(v,c)$.*

Proof. Outside the neighborhood of γ where we deform the metric, the following holds:

$$\begin{aligned}\frac{d}{dt}g^*(\xi_A+\eta_A,\xi_A+\eta_A) &= \frac{d}{dt}g(\xi_A+\eta_A,\xi_A+\eta_A)\\ &= 2(g(\xi_A+\eta_A,\eta_A-R(v,\xi)v)\\ &= 2g(\xi_A+\eta_A,\xi_A+\eta_A)\\ &= 2g^*(\xi_A+\eta_A,\xi_A+\eta_A).\end{aligned}$$

For vectors $v\in\{v\in S^*M : v\ \theta\text{-close to }(1,0,\ldots,0)\}$:

$$\begin{aligned}
&\frac{d}{dt}g^*(\xi_A+\eta_A,\xi_A+\eta_A) = 2g^*(\xi_A+\eta_A,\eta_A+\xi_{A'}+\eta_{A'}-R^*(v,\xi)v)\\
&\geq 2g^*(\xi_A+\eta_A,\eta_A-R^*(v,\xi)v)-L\epsilon(|\xi|^*+|\eta|^*))\\
&\geq 2(1-2L\epsilon)g^*(\xi_A+\eta_A,\xi_A+\eta_A).
\end{aligned}$$

So for ϵ sufficiently small we have exponential growth in this case. Now, in the case of v 'transversal' to γ:

$$\begin{aligned}
&\frac{d}{dt}g^*(\xi_A+\eta_A,\xi_A+\eta_A) = 2g^*(\xi_A+\eta_A,\eta_A+\xi_{A'}+\eta_{A'}-R^*(v,\xi)v)\\
&\geq Kg^*(\xi_A+\eta_A,\xi_A+\eta_A),
\end{aligned}$$

for some $K\in\mathbb{R}$ which does not depend on ϵ.

So, if we take any geodesic $c:[0,T]\to M$, we have that it takes only ϵ inside the neighborhood, and 'transversal' to γ'. So, if we call $f(t):=(g^*(\xi,Jv)+g^*(\eta,Jv))^2$, we have that $f'(t)\approx 2f(t)$ for time $T-\epsilon$ and $f'(t)\geq Kf(t)$ for time ϵ, at most. This implies:

$$\begin{aligned}
\int_0^T(\log f)'(s)ds &\geq 2(T-\epsilon)+K\epsilon = 2T+(K-2)\epsilon \Rightarrow\\
\log f(T)-\log f(0) &\geq 2T+(K-2)\epsilon \Rightarrow f(T)\geq f(0)e^{(2T+(K-2)\epsilon)}.
\end{aligned}$$

So for ϵ sufficiently small, we have that f grows exponentially for the (ξ,η) inside the unstable cone we have exponential growth. □

5.4 Conclusion

So, we proved the proper invariance of the unstable and stable cones. And we proved the exponential expansion or contraction respectively, in the previous subsection. Then we conclude:

Theorem 5.17. *For (M,g) Kahler manifold of negative holomorphic curvature -1 or quaternion Kahler locally symmetric space of negative curvature, there is a metric g^* in M such that its geodesic flow is partially hyperbolic but not Anosov.*

Remark 5.18. A classical Mañé theorem [M3] says that if, for a geodesic flow of a Riemannian manifold (M,g), there is an invariant Lagrangian subbundle of $S(SM)$, then this Riemannian manifold does not have conjugate points. The existence of a partially hyperbolic non-Anosov geodesic flow implies that this theorem does not generalize to the case of invariant isotropic subbundles.

Corollary 5.19. *There is an open set $\mathcal{U}'$ of metrics in the set of metrics of (M,g) such that for $g^*\in\mathcal{U}$, the geodesic flow of g^* is partially hyperbolic but not Anosov. Moreover, for all metrics g^* in an open subset $\mathcal{U}'\subset\mathcal{U}$, (M,g^*) has conjugate points.*

Proof. We can make the closed geodesic γ, which is a geodesic for both metrics g and g^*, a quasi-elliptic nondegenerate closed geodesic. The linearized Poincare map of a quasi-elliptic nondegenerate orbit has eigenvalues on the unit circle but they are different than one. We only need to multiply the bump function by a constant greater but sufficiently close to 1 such that the geodesic flow remains partially hyperbolic. Since quasi-elliptic nondegenerate closed geodesics are persistent, there is an open neighborhood $\mathcal{U}$ of g^* in the set of metrics of M such that all metrics in this open set are partially hyperbolic, and are far away from the set of Anosov metrics. Also, the existence of a quasi-elliptic closed geodesic implies that the metric has conjugate points, which is an open condition. So, for $g^* \in \mathcal{U}' \subset \mathcal{U}$ the geodesic flow of g^* is partially hyperbolic and g^* has conjugate points. □

Remark 5.20. Ruggiero [R2] proved that the C^2-interior of the set of metrics with no conjugate points is the set of metrics whose geodesic flow is Anosov. For partially hyperbolic geodesic flows we have an C^2-open set of metrics with conjugate points.

Corollary 5.21. *There is an open set $\mathcal{V}$ of Hamiltonians in the set of Hamiltonians of (TM, ω), near geodesic Hamiltonians, such that for $h \in \mathcal{U}$, the Hamiltonian flow of h is partially hyperbolic but not Anosov.*

Proof. For the same reasons of the previous corollary there is an open set of Hamiltonians with the same property, near geodesic Hamiltonians. □

6 Symmetric spaces of nonpositive curvature

In this section we give a brief introdution of the subject of symmetric and locally symmetric spaces [E2],[E3],[J], and prove that compact locally symmetric spaces of nonpositive curvature are partially hyperbolic only if their sectional curvature takes values on the whole interval $[-1, -\frac{1}{4}]$.

Definition 6.1. A simply connected Riemannian manifold is called symmetric if for every $x \in M$ there is an isometry $\sigma_x : M \to M$ such that

$$\sigma_x(x) = x, d\sigma_x(x) = -id_{T_xM}.$$

The property of being symmetric is equivalent to:

- $\nabla R \equiv 0$,
- if $X(t)$, $Y(t)$ and $Z(t)$ are parallel vector fields along $\gamma(t)$, then $R(X(t), Y(t))Z(t)$ is also a parallel vector field along $\gamma(t)$.

Remark 6.2. A symmetric riemannian manifold is geodesically complete and every two points can be connected by a geodesic.

Definition 6.3. A complete Riemannian manifold with $\nabla R \equiv 0$ is called locally symmetric.

Each locally symmetric space N is the quotient of a simply connected symmetric space M and a group Γ acting on M discretely, without fixed points, and isometrically, such that $N = M/\Gamma$.

Proposition 6.4. *Let N be a locally symmetric space, $p \in N$, $v \in T_pN$, c geodesic such that $c(0) = p$, $c'(0) = v$, there are $v_1, \ldots, v_{n-1}$ an orthogonal basis of eigenvectors of $R_{c'(0)}$ orthogonal to v with eigenvalues $\rho_1, \ldots, \rho_{n-1}$, and parallel vector fields $v_1(t), \ldots, v_{n-1}(t)$ along c such that $v_i(0) = v_i$. Then the Jacobi fields along c are linear combinations of the following Jacobi fields*

$$c_{\rho_j}(t)v_j(t) \text{ and } s_{\rho_j}(t)v_j(t),$$

where

$$c_\rho(t) := \cos\sqrt{\rho}t, \rho > 0, c_\rho(t) := \cosh\sqrt{-\rho}t, \rho < 0, c_\rho(t) := 1, \rho = 0,$$

$$s_\rho(t) := \frac{1}{\sqrt{\rho}}\sin\sqrt{\rho}t, \rho > 0, s_\rho(t) := \frac{1}{\sqrt{-\rho}}\sinh\sqrt{-\rho}t, \rho < 0, s_\rho(t) := t, \rho = 0.$$

The proof of the proposition relies on the two facts: $R_v : T_pN \to T_pN : w \to R(v,w)v$ is a self-adjoint map and the curvature tensor is parallel.

Definition 6.5. Let $\mathfrak{g}$ be the algebra of Killing fields on the symmetric space M, $p \in M$. Define

$$\mathfrak{k} := \{X \in \mathfrak{g} : X(p) = 0\},$$
$$\mathfrak{p} := \{X \in \mathfrak{g} : \nabla X(p) = 0\}.$$

For these subspaces of $\mathfrak{g}$, $\mathfrak{k} \oplus \mathfrak{p} = \mathfrak{g}$ and $\mathfrak{k} \cap \mathfrak{p} = \{0\}$, and T_pM identifies with $\mathfrak{p}$.

Definition 6.6. Given $p \in M$, we define the involution $\phi_p(g) : G \to G : g \to \sigma_p \circ g \circ \sigma_p$. Then, we obtain $\theta_p : d\phi_p : \mathfrak{g} \to \mathfrak{g}$. Since $\theta_p^2 = id$ and θ_p preserves the lie brackets, the properties of this subspaces of $\mathfrak{g}$ are:

i. $\theta_{p|\mathfrak{k}} = id$,

ii. $\theta_{p|\mathfrak{p}} = -id$,

iii. $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$,

Proposition 6.7. *With the identification $T_pM \cong \mathfrak{p}$ the curvature tensor of M satisfies*

$$R(X,Y)Z(p) = [X,[Y,Z]](p)$$

for all $X, Y, Z \in \mathfrak{p}$. In particular, $R(X,Y)X(p) = -(ad_X)^2(Y)(p)$.

Remark 6.8. We are going to consider only symmetric spaces with nonpositive sectional curvature.

Fix a maximal Abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. Let Λ denote the set of roots determined by $\mathfrak{a}$, and

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Lambda} \mathfrak{g}_\alpha.$$

$\mathfrak{g}_\alpha = \{w \in \mathfrak{g} : (adX)w = \alpha(X)w\}$, $\alpha : \mathfrak{a} \to \mathbb{R}$ is a one-form.

Define a corresponding decomposition for each $\alpha \in \Lambda$, $\mathfrak{k}_\alpha = (id + \theta)\mathfrak{g}_\alpha$ and $\mathfrak{p}_\alpha = (id - \theta)\mathfrak{g}_\alpha$. Then:

i. $id + \theta : \mathfrak{g}_\alpha \to \mathfrak{k}_\alpha$ and $id - \theta : \mathfrak{g}_\alpha \to \mathfrak{p}_\alpha$ are isomorphisms,

ii. $\mathfrak{p}_\alpha = \mathfrak{p}_{-\alpha}$, $\mathfrak{k}_\alpha = \mathfrak{k}_{-\alpha}$, and $\mathfrak{p}_\alpha \oplus \mathfrak{k}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$,

iii. $\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \Lambda} \mathfrak{p}_\alpha$, $\mathfrak{k} = \mathfrak{k}_0 + \sum_{\alpha \in \Lambda} \mathfrak{k}_\alpha$, where $\mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{k}$.

For $X \in \mathfrak{a}$ we have that, along the geodesic c in M with initial conditions $c(0) = p$, $c'(0) = X$, the Jacobi fields are linear combinations of the following Jacobi fields:

$$c_{-\alpha(X)^2}(t)v_j(t) \text{ and } s_{-\alpha(X)^2}(t)v_j(t).$$

So, we define for a vector $X \in \mathfrak{a}$, and for α such that $\alpha(X) \neq 0$, the invariant subspaces $P^u_\alpha(X), P^s_{\alpha(X)} \subset T_{(p,X)}SM$ such that

$$d\phi_t(X)P^u_\alpha(X) = P^u_\alpha(\phi_t(X)), d\phi_t(X)P^s_\alpha(X) = P^s_\alpha(\phi_t(X)),$$

$$P^u_\alpha(X) = \{(w, |\alpha(X)|w) \in \mathfrak{p} : w \in \mathfrak{p}_\alpha\}, P^s_\alpha(X) = \{(w, -|\alpha(X)|w) \in \mathfrak{p} : w \in \mathfrak{p}_\alpha\}.$$

Along the same lines of the proof that product metrics are not partially hyperbolic:

Theorem 6.9. *If the geodesic flow of a compact locally symmetric space of nonpositive curvature is partially hyperbolic, then is curvature takes values on the whole interval* $[-1, -\frac{1}{4}]$.

Proof. If the locally symmetric space N has a partially hyperbolic geodesic flow, then the symmetric space M such that $N = M/\Gamma$ has a partially hyperbolic geodesic flow.

Fix $x \in M$ and consider $v \in S_xM$. Let $\mathfrak{a}$ be the maximal Abelian subspace of $\mathfrak{g}$ in x such that $X \in \mathfrak{a}$.

Suppose $dim(\mathfrak{a}) \geq 2$. If the geodesic flow of the symmetric space M is partially hyperbolic, then there is a splitting into invariant subbundles:

$$S(SM) = E^s \oplus E^c \oplus E^u.$$

This decomposition and the curvature tensor formula imply that

$$E^u(x, v) = \{(\xi, \eta) \in T_{(x,v)}SM : (\xi, \eta) \in P^u_{\alpha_i}(v)\},$$

$$E^s(x,v) = \{(\xi,\eta) \in T_{(x,v)}SM : (\xi,\eta) \in P^s_{\alpha_i}(v)\},$$

$i = 1,\ldots,k$, $|\alpha_1| > |\alpha_2| > \ldots > |\alpha_k|$, such that if $\beta \neq \alpha_i$, $\forall i = 1,\ldots,k$, then $\beta(v) < \alpha_i(v)$, $\forall i = 1,\ldots,k$.

Now we pick (x,v') such that $\alpha_1(v') = 0$, for some i. Then:

$$E^u(x,v') = \{(\xi,\eta) \in T_{(x,v')}SM : (\xi,\eta) \in P_{\beta_j}\},$$

$$E^s(x,v') = \{(\xi,\eta) \in T_{(x,v')}SM : (\xi,\eta) \in P_{\beta_j}\},$$

for some $\beta_j \in \Lambda$, $j = 1,\ldots,k'$, $|\beta_1| > |\beta_2| > \ldots > |\beta_{k'}|$. Notice that $\alpha_1(v') = 0$ implies $\beta_j \neq \alpha_1$, $\forall j = 1,\ldots,k'$. As in the proof of the product metric, there is no way to go from one decomposition to the other continuously. So, there are no Abelian subspaces with dimension greater than one, and the symmetric space of nonpositive curvature has rank one. If dimension of the Abelian subspaces is one then the symmetric space has negative curvature, which implies by the classification of Heintze [H] that it is a Kahler hyperbolic space, or quaternionic hyperbolic space, or the hyperbolic space over the Cayley numbers. □

7 Further results and questions

This section is about the obstructions to have a partially hyperbolic geodesic flow and some questions we hope to adress in our next works about the subject of partially hyperbolic geodesic flows.

There is an obstruction to partial hyperbolicity if one add the hypothesis of nonpositive sectional curvature in the Riemannian manifold: the rank of the Riemannian manifold.

Definition 7.1. Let (M,g) be a Riemannian manifold of nonpositive sectional curvature. Then, for $v \in T_xM$, $rank(v) := \dim \mathcal{J}^c(v)$, where $\mathcal{J}^c(v)$ is the set of parallel Jacobi fields along the geodesic γ, such that $\gamma(0) = x$, $\gamma'(0) = v$. The rank of M is $rank(M) := inf_{v \in T_xM} rank(v)$ [Ba1], [E2], [E3].

Theorem 7.2. *If M is a compact Riemannian manifold with nonpositive curvature such that its geodesic flow is partially hyperbolic, then M has rank one.*

Proof. By theorem 3.3, M has to be irreducible. By the rank rigidity theorem of Ballmann [Ba2] and Burns-Spatzier [BS], if M is irreducible, has nonpositive curvature, and rank bigger than one, then it is a locally symmetric space of rank bigger than one. Then, by theorem 6.9, its geodesic flow is not partially hyperbolic. □

Another obstruction is the dimension of the Riemannian manifold, and also the dimension of the extremal subbundles of the partially hyperbolic splitting. We use the following result in Steenrod's classical book:

Theorem 27.18 [St]. *Let S^n be the n-dimensional sphere. Then, it does not admit a continuous field of tangent k-planes if n is even or if $n \equiv 1 \mod 4$ and $2 \leq k \leq n-2$.*

So, we can state the following:

Theorem 7.3. *If (M^n, g) is a Riemannian manifold with partially hyperbolic geodesic flow then n is even, and if $n \equiv 2 \mod 4$, then $\dim E^s = 1, n-2$ or $n-1$.*

Proof. First, let $E^s \oplus E^c \oplus E^u$ be the splitting of $S(SM)$, the contact structure on SM, and $\widetilde{J} : T(TM) \to T(TM)$ the almost complex structure, which leaves $S(SM)$ invariant. The orthogonal projection of $E^s \oplus \widetilde{J}E^s$ to the horizontal subbundle H has dimension equal to $\dim(E^s)$.Then, for a fixed $x \in M$,

$$X(v) := d_{(x,v)}\pi \cdot proj_{H(x,v)}(E^s(x,v) \oplus \widetilde{J}E^s(x,v))$$

is a continuous tangent field of $(\dim E^s)$-planes on S_xM, which is a $(n-1)$-dimensional sphere. So we apply theorem 6 to conclude that n is even, and if $n \equiv 2 \mod 4$, then $\dim E^s = 1, n-2$ or $n-1$ (The last case, where $\dim E^s = n-1$, is the case of trivial central bundle). □

Remark 7.4. The idea that partial hyperbolicity of the geodesic flow implies odd dimension of the Riemannian manifold is due to Gonzalo Contreras, who communicated an idea of the proof of this fact to the second author of this article.

There are some questions that we hope to adress in the future:

Question 1. Is there a partially hyperbolic non-Anosov geodesic flow with nonpositive sectional curvature?

We know that it does imply that the Riemannian manifold has rank one, and so it is transitive. If there is not such an example, this would imply that we need some positive curvature to get a partially hyperbolic geodesic flow which is not Anosov, as in the example we built above.

Question 2. Is the example above which we constructed in section 5 transitive? Is it ergodic?

It would be natural to ask this questions, from the dynamical systems point of view, to know which systems have these properties, in the set of diffeomorphisms, flows and geodesic flows.

Question 3. Is there an Riemannian manifold (M, g) with partially hyperbolic geodesic flow such that one of the invariant subbundles E^s and E^u contain Jacobi fields such that $J(t) = 0$ to some $t \in \mathbb{R}$? This would imply that there are conjugate points associated to them.

Mañé's theorem [M3], which says that an invariant Lagrangian subbundle implies lack of conjugate points, does not hold if the hypothesis is changed to existence of an invariant isotropic subbundle. But our result about obstructions for the dimensions of the manifold (M, g) and the dimensions of the strong invariant subbundles E^s and E^u does not prove that there are no conjugate points associated with stable or unstable Jacobi fields of partially hyperbolic geodesic flows.

References

[An] D. V. Anosov, Geodesic flows on closed Riemann manifolds with negative curvature, Proceedings of the Steklov Institute of Mathematics, No. 90 American Mathematical Society Providence, R.I. (1969)

[Ba1] Werner Ballmann, Lectures on spaces of nonpositive curvature, DMV Seminar, vol. 25, Birkhauser, Boston, 1995.

[Ba2] Werner Ballmann, Nonpositively curved manifolds of higher rank, Ann. of Math. 122 (1985), 597-609.

[Be] Arthur L. Besse, Einstein manifolds, Classics in mathematics, Springer-Verlag (1987).

[BD] C. Bonatti, L. J. Diaz, Persistence of transitive diffeomorphisms, *Annals of Math* **143** (1995), 367-396.

[Bo] A. Borel, Compact Clifford-Klein forms of symmetric spaces, Topology 2 (1963) 111-122.

[BP] M. I. Brin, Ja. B. Pesin; Flows of frames on manifolds of negative curvature. (Russian) *Uspehi Mat. Nauk* 28 (1973), no. 4(172), 209–210.

[BS] Keith Burns and Ralf Spatzier, Manifolds of nonpositive curvature and their buildings, Publ. Math. IHES 65 (1987), 35-59.

[BV] C. Bonatti, M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.* **115** (2000), 157–193.

[Ca] Manfredo do Carmo, Geometria Riemanniana, Instituto Nacional de Matemática Pura e Aplicada, Rio de Janeiro (1988).

[CKO] Castro, H. M. A.; Kobayashi, M. H.; Oliva, W. M. Partially hyperbolic Σ-geodesic flows. Special issue in celebration of Jack K. Hale's 70th birthday, Part 3 (Atlanta, GA/Lisbon, 1998). J. Differential Equations 169 (2001), no. 1, 142-168.

[Co1] Gonzalo Contreras, Partially hyperbolic geodesic flows are Anosov, C. R. Acad. Sci. Paris, Ser. I 334 (2002) 585-590.

[Co2] Gonzalo Contreras, private communication.

[DoP] Victor Donnay and Charles Pugh, Finite horizon Riemann structures and ergodicity, Ergodic Theory Dynam. Systems 24 (2004), no. 1, 89-106.

[E1] Patrick Eberlein, When is a geodesic flow of Anosov type? I, Journal of Differential Geometry, 8 (1973), 437-463.

[E2] Patrick Eberlein, Structure of manifolds of nonpositive curvature, Lecture Notes in Mathematics , vol. 1156 , Springer Verlag , (1985) , 86-153.

[E3] Patrick Eberlein, Geometry of Nonpositively Curved Manifolds, Chicago Lectures in Mathematics.

[G] William M. Goldman, Complex hyperbolic geometry, Claredon Press, Oxford, 1999.

[Gr] Alfred Gray, A note on manifolds whose holonomy group is a subgroup of Sp(n).Sp(1), Michigan Mathematical Journal, vol.16, issue 2 (1969), 125-128.

[H] Ernst Heintze, On Homogeneous Manifolds of Negative Curvature, Mathematische Annalen. Volume 211, Number 1, 23-34.

[He] Sigurdur Helgason, Differential geometry, Lie groups, and symmetric spaces, Pure and Applied Mathematics, Academic Press, New York-London (1978).

[HK] Boris Hasselblatt, Anatole Katok, Introduction to the modern theory of dynamical systems, Cambridge University Press, Cambridge, 1995.

[HP] Boris Hasselblatt, Yakov Pesin, Partially Hyperbolic Dynamical Systems, Handbook of Dynamical Systems, vo. 1B, Elsevier, North-Holland (2006).

[J] Jurgen Jost, Riemannian Geometry and Geometric Analysis. Universitext. Springer-Verlag

[Kl] W. Klingenberg, Lectures on closed geodesics, Springer-Verlag, 1978.

[KN] S. Kobayashi, K. Nomizu, Foundations of differential geometry, Vol. II, Interscience Tracts in Pure and Applied Mathematics, No. 15 Vol. II Interscience Publishers John Wiley and Sons, Inc., New York-London-Sydney 1969.

[M1] Ricardo Mañé, Contributions to the stability conjecture, Topology 17 (1978), no. 4, 383-396.

[M2] Ricardo Mañé, Oseledec's theorem from the generic viewpoint, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 1269-1276, PWN, Warsaw, 1984.

[M3] Ricardo Mañé, On a theorem of Klingenberg, Dynamical systems and bifurcation theory, Proc. Meet., Rio de Janeiro/Braz. 1985, Pitman Res. Notes Math. Ser. 160, 319-345 (1987).

[Ne] Sheldon E. Newhouse, Quasi-elliptic periodic points in conservative dynamical systems. American Journal of Mathematics 99 (1977), no. 5, 1061-1087.

[P] Gabriel Paternain, Geodesic flows Progress in mathematics, Birkhauser, Boston, 1999.

[PS] E. R. Pujals, M. Sambarino, Topics on homoclinic bifurcation, dominated splitting, robust transitivity and related results, Handbook of dynamical systems vol 1B, Elsevier (2005) 327-378.

[R1] Rafael Ruggiero, Persistently expansive geodesic flows, Comm. Math. Phys. 140 (1991), no. 1, 203-215.

[R2] Rafael Ruggiero, On the creation of conjugate points, Mathematische Zeitschrift, 208, 41-55, Springer-Verlag (1991).

[Sh] M. Shub, Topologically transitive diffeomorphism of $\mathbb{T}^4$, in *Symposium on Differential Equations and Dynamical Systems (University of Warwick, 1968/69)*, pp. 39-40. Lecture Notes in Math.. 206. Springer-Verlag, Berlin-NewYork, 1971.

[Sm] Steven Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc. 73 (1967), 747-817.

[St] Norman Steenrod, Topology of fiber bundles, Princeton landmarks in Mathematics, Princeton University Press (1999).

[W] Maciej Wojtjowski, Magnetic flows and Gaussian thermostats on manifolds of negative curvature, Fundamenta Mathematicae 163 (2000), no. 2, 177-191.

[Wo] Joseph A. Wolf, Complex homogeneous contact manifolds and quaternionic symmetric spaces, Journal of Mathematics and Mechanics, vol. 14, No. 6 (1965), 1033-1047.