COMPUTATION OF THE JORDAN NORMAL FORM OF A MATRIX USING VERSAL DEFORMATIONS

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Abstract. Numerical problem of finding multiple eigenvalues of real and complex nonsymmetric matrices is considered. Using versal deformation theory, explicit formulae for approximations of matrices having nonderogatory multiple eigenvalues in the vicinity of a given matrix $\mathbf{A}$ are derived. These formulae provide local information on the geometry of a set of matrices having a given nonderogatory Jordan structure, distance to the nearest matrix with a multiple eigenvalue of given multiplicity, values of multiple eigenvalues, and corresponding Jordan chains (generalized eigenvectors). The formulae use only eigenvalues and eigenvectors of the initial matrix $\mathbf{A}$, which typically has only simple eigenvalues. Both the case of matrix families (matrices smoothly dependent on several parameters) and the case of real or complex matrix spaces are studied. Several examples showing simplicity and efficiency of the suggested method are given.

Key words. Jordan normal form, multiple eigenvalue, generalized eigenvector, matrix family, versal deformation, bifurcation diagram

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1. Introduction. Computation of the Jordan normal form (JNF) of a square nonsymmetric matrix $\mathbf{A}$ is needed in many branches of mathematics, mechanics, and physics. Due to its great importance this problem was intensively studied during the last century. It is well known that a generic (typical) matrix has only simple eigenvalues, which means that its JNF is a diagonal matrix [1, 2]. The problem of finding the JNF of a generic matrix has been well studied both from theoretical and practical (numerical) sides; see [24]. Nevertheless, many interesting and important phenomena, associated with qualitative changes in the dynamics of mechanical systems [17, 18, 22], stability optimization [3, 14, 20], and eigenvalue behavior under matrix perturbations [4, 21, 23], are connected with degenerate matrices, i.e., matrices having multiple eigenvalues.

In the presence of multiple eigenvalues the numerical problem of calculation of the JNF becomes very complicated due to its instability, since arbitrarily small perturbations (caused, for example, by round-off errors) destroy the nongeneric structure. This means that we should consider a matrix given with some accuracy, thus, substituting the matrix $\mathbf{A}$ by its neighbor in the matrix space. Such formulation leads to several important problems. The first problem left open by Wilkinson [25, 26] is to find a distance from a given matrix $\mathbf{A}$ to the nearest nondiagonalizable (degenerate) matrix $\widehat{\mathbf{A}}$. This problem can be reformulated in the extended form:

- to find a distance (with respect to a given norm) from a matrix $\mathbf{A}$ to the nearest matrix $\widehat{\mathbf{A}}$ having a given Jordan structure;
- to find a matrix $\widehat{\mathbf{A}}$ with the least generic Jordan structure in a given neighborhood of the matrix $\mathbf{A}$ (the least generic Jordan structure means that a set of matrices with this Jordan structure has the highest codimension).

Here we assume that two matrices have the same Jordan structure, if their JNFs differ only by values of different eigenvalues. Codimensions of sets of matrices having the same Jordan structure were studied by Arnold [1, 2]. According to Arnold, matrices $\widehat{\mathbf{A}}$ with the least generic Jordan structure have the most powerful influence on the

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eigenvalue behavior in the neighborhood of the matrix $\mathbf{A}$. Therefore, the above stated problems represent the key point in the numerical analysis of multiple eigenvalues.

Several authors addressed the problem of finding bounds on the distance to the nearest degenerate matrix; see [5, 6, 9, 15, 25, 26] and more references therein. A numerical method for finding the JNF of a degenerate matrix given with some accuracy was proposed in [12, 13]. This method is based on making a number of successive transformations of the matrix and deleting small elements whenever appropriate. Nevertheless, this procedure doesn't provide the least generic JNF, and it is restricted to the use of the Frobenius matrix norm (for example, it can not be used in the case, when different entries of a matrix are given with different accuracy). Another method based on singular value decompositions was given in [19]. This method is simpler, but it has the same properties. More references of JNF computation methods can be found in [8].

It is understood now that the problem of calculation of the nondiagonal JNF needs the singularity theory approach based on the use of versal deformations (locally generic families of matrices). Versal deformations were introduced by Arnold [1, 2] in order to stabilize the JNF transformation, but no numerical methods were suggested. Fairgrieve [9] succeeded in using the idea of versality for construction of a stable numerical procedure, which calculates matrices of a given Jordan structure. Though this method does not provide the nearest matrix of a given Jordan structure, it shows the usefulness of the singularity theory for the numerical eigenvalue problem. In [7] versal deformations were used for improvement of an existing numerical program in a similar problem of finding the Kronecker normal form of a matrix pencil. Application of the singularity theory to computation of a distance from a matrix $\mathbf{A}$ to the nearest matrix with a double eigenvalue [15] showed that this numerical problem is very complex even in the case of a double eigenvalue.

This paper is intended to prove that the versal deformation theory can be an independent powerful tool for analysis of multiple eigenvalues. It is shown that using versal deformations we can avoid a number of transformations and singular value decompositions (used in most algorithms). Instead of this, explicit approximate formulae describing the geometry of a set of matrices with given Jordan structure in the vicinity of a given matrix $\mathbf{A}$ are derived. These formulae use only information on eigenvalues and eigenvectors of the matrix $\mathbf{A}$. Since the matrix $\mathbf{A}$ is typically nondegenerate, i.e., has only simple eigenvalues, its eigenvalues and eigenvectors can be obtained by means of standart codes. As a result, we obtain approximations of a distance from $\mathbf{A}$ to the nearest matrix having a given Jordan structure, its multiple eigenvalues, and the Jordan chains (generalized eigenvectors). The method does not depend on the norm used in the matrix space. Moreover, it allows analyzing multiple eigenvalues of matrices dependent on several parameters (matrix families), where we can not vary matrix elements independently. Approximations derived in the paper have the accuracy $O(\varepsilon^2)$, where ε is the distance from $\mathbf{A}$ to a matrix we are looking for. All these properties make the suggested approach useful and attractive for applications.

In this paper the case of a nonderogatory JNF (every eigenvalue has one corresponding Jordan block) is considered. Derogatory Jordan structures require further investigation and new ideas. Difficulties appearing in the derogatory case are associated with the lack of uniqueness of the transformation to a versal deformation and will be discussed in the conclusion.

The paper is organized as follows. In section 2 some concepts of the singularity theory are introduced and the general idea of the paper is described. Sections 3 and 4 give approximate formulae for matrices having a prescribed nonderogatory Jordan structure in the neighborhood of a given generic (nondegenerate) complex or real matrix $\mathbf{A}$. In section 5 the same problem is studied in the case, when the initial matrix $\mathbf{A}$ is degenerate. Conclusion discusses extension of the method to the derogatory case and possibilities for its application to specific types of matrices or matrix pencils.

In the paper matrices are denoted by bold capital letters, vectors take the form of bold small letters, and scalars are represented by small italic characters.

2. Bifurcation diagram. Let us consider an $m \times m$ complex (real) nonsymmetric matrix $\mathbf{A}$ holomorphically (smoothly) dependent on a vector of complex (real) parameters $\mathbf{p} = (p_1, \ldots, p_n)$. The space of all $m \times m$ matrices can be considered as a special case, where all entries of $\mathbf{A}$ are independent parameters.

Matrices $\mathbf{A}$ and $\mathbf{B}$ are said to have the same Jordan structure if there exists one-to-one correspondence between different eigenvalues of $\mathbf{A}$ and $\mathbf{B}$ such that sizes of Jordan blocks in the JNFs of $\mathbf{A}$ and $\mathbf{B}$ are equal for corresponding eigenvalues (JNFs of the matrices $\mathbf{A}$ and $\mathbf{B}$ differ only by values of their eigenvalues). For example, among the following matrices

$$\mathbf{A} = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{2.1}$$

the matrices $\mathbf{A}$ and $\mathbf{B}$ have the same Jordan structure, while the matrix $\mathbf{C}$ has a different Jordan structure. In the generic case a set of values of the vector $\mathbf{p}$ corresponding to the matrices $\mathbf{A}(\mathbf{p})$ having the same Jordan structure represents a smooth submanifold of the parameter space [1, 2]. This submanifold is called a *stratum* (also called a *colored orbit* of a matrix). A *bifurcation diagram* of the matrix family $\mathbf{A}(\mathbf{p})$ is a partition (stratification) of the parameter space according to the Jordan structure of the matrix $\mathbf{A}(\mathbf{p})$ [1, 2]. Generic (typical) matrices $\mathbf{A}$ form a stratum of zero codimension consisting of nondegenerate matrices having only simple eigenvalues. The bifurcation diagram has singularities at boundary points of the strata (boundary of a stratum consists of strata having higher codimensions). Due to singularities the bifurcation diagram has rather complicated structure.

In this paper we consider matrices having a nonderogatory multiple eigenvalue λ (there is one eigenvector and, hence, one Jordan block corresponding to λ). The strata corresponding to such matrices are denoted by λ^d, where d is the algebraic multiplicity of the multiple eigenvalue λ. If the matrix $\mathbf{A}$ has several multiple nonderogatory eigenvalues $\lambda_1, \ldots, \lambda_s$ with algebraic multiplicities $d_1, \ldots, d_s$, then the corresponding stratum is denoted by $\lambda_1^{d_1} \cdots \lambda_s^{d_s}$. In the case of real matrix families we will differ strata determined by real eigenvalues and complex conjugate pairs of eigenvalues by using notations α^d and $(\alpha \pm i\omega)^d$ respectively.

Geometry of the stratification can be illustrated by the following example

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} 2p_1 & p_2 \\ p_3 & 0 \end{pmatrix}, \quad \mathbf{p} = (p_1, p_2, p_3). \tag{2.2}$$

The bifurcation diagram consists of the following strata: α^2 (the cone surface $p_1^2 + p_2 p_3 = 0$ except the origin), $\alpha\alpha$ (the origin), and the other part of the parameter space corresponding to matrices $\mathbf{A}(\mathbf{p})$ with distinct eigenvalues (interior and exterior of the cone); see Figure 2.1. Here $\alpha\alpha$ denotes the stratum corresponding to a derogatory

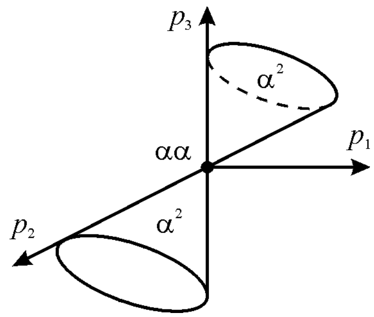

FIG. 2.1. *Geometry of the bifurcation diagram.*

double eigenvalue (with two corresponding eigenvectors). The strata α^2 and $\alpha\alpha$ are associated with a double eigenvalue, but differ by a number of the Jordan blocks (equal to the number of eigenvectors). Clearly, the strata α^2 and $\alpha\alpha$ are smooth submanifolds of codimension 1 and 3 respectively.

A matrix with multiple eigenvalues can be made nondegenerate by an arbitrarily small perturbation. That is why in numerical analysis we usually deal with nondegenerate matrices. A double nonderogatory eigenvalue represents the most generic type of multiple eigenvalues. The corresponding stratum λ^2 (α^2 in the real case) is a smooth hypersurface in the parameter space. Nonderogatory eigenvalues are the most generic (typical) between all types of eigenvalues of given multiplicity. Nonderogatory eigenvalues of multiplicity d determine the stratum λ^d of codimension $d-1$ (the stratum $\lambda_1^{d_1}\cdots\lambda_s^{d_s}$ has the codimension $d_1+\cdots+d_s-s$). In the real case the strata α^d and $(\alpha\pm i\omega)^d$ have the codimension $d-1$ and $2(d-1)$ respectively [1, 2, 10].

Let us describe an idea of the method used in this paper. The versal deformation theory allows studying different strata independently. From one side, this simplifies the analysis, since a stratum is a smooth manifold without singularities. From the other side, we get additional information about the Jordan structure of a matrix. Let $\widehat{\mathbf{p}}$ be a point on the stratum λ^d; see Figure 2.2. The neighborhood of $\widehat{\mathbf{p}}$ can be explored by means of a *versal deformation*. As a result, the stratum λ^d is described locally by the equation

$$\mathbf{A}'(\mathbf{q}'(\mathbf{p})) = \mathbf{J}_d, \tag{2.3}$$

where $\mathbf{A}'(\mathbf{q}')$ is a simple explicit family of $d\times d$ matrices (a block of the versal deformation); $\mathbf{q}'(\mathbf{p})$ is a vector smoothly dependent on $\mathbf{p}$; $\mathbf{J}_d$ represents the Jordan block of dimension d. Let a point $\mathbf{p}_0$ of the parameter space be given. Assuming that the point $\mathbf{p}_0$ is close to $\widehat{\mathbf{p}}$, we can write equation (2.3) in the form

$$\mathbf{A}'(\mathbf{q}'(\mathbf{p}_0)) + \Delta\mathbf{A}' + O(\|\mathbf{p}-\mathbf{p}_0\|^2) = \mathbf{J}_d, \quad \Delta\mathbf{A}' = \sum_{i=1}^{n} \frac{\partial \mathbf{A}'}{\partial p_i}(p_i - p_{0i}), \tag{2.4}$$

where $\|\mathbf{p}\| = (p_1\overline{p}_1+\cdots+p_n\overline{p}_n)^{1/2}$ is the norm in the parameter space; derivatives are calculated at $\mathbf{p}_0$. The matrices $\mathbf{A}'(\mathbf{q}'(\mathbf{p}_0))$ and $\partial\mathbf{A}'/\partial p_i$ can be found from the analysis of the versal deformation at $\mathbf{p}_0$. The latter represents the most nontrivial part of the study. Omitting $O(\|\mathbf{p}-\mathbf{p}_0\|^2)$ in (2.4), we obtain the equation determining a plane σ, which is the first order approximation of the stratum λ^d; see Figure 2.2. A distance from $\mathbf{p}_0$ to λ^d can be found approximately as a distance to σ. Further analysis

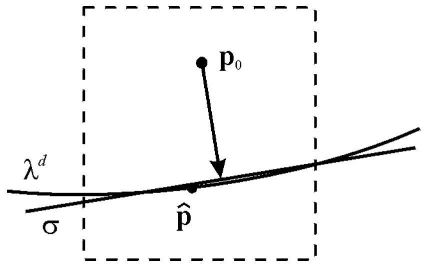

FIG. 2.2. *Approximation of the stratum λ^d from the point $\mathbf{p}_0$.*

of the versal deformation allows finding approximations of multiple eigenvalues and the Jordan chains (generalized eigenvectors) on the stratum. Note that all these approximations use only information at the point $\mathbf{p}_0$, typically corresponding to a nondegenerate matrix $\mathbf{A}(\mathbf{p}_0)$.

The described approach is similar to Newton's method for finding a root of a smooth function, where the approximation of the root is determined using the linear approximation of the function at the initial point. Hence, if $\varepsilon = \mathrm{dist}(\mathbf{p}_0, \lambda^d)$ is the distance from $\mathbf{p}_0$ to λ^d, then the proposed method gives the nearest point of λ^d with the accuracy $O(\varepsilon^2)$. Making k iterations, one finds a point of the stratum λ^d with the accuracy $O(\varepsilon^{2^k})$.

3. Nonderogatory eigenvalue of a complex matrix. In this section we study complex matrices $\mathbf{A}(\mathbf{p})$ holomorphically dependent on a vector of complex parameters $\mathbf{p}$. Let us consider a point $\widehat{\mathbf{p}}$ in the parameter space, where the matrix $\widehat{\mathbf{A}} = \mathbf{A}(\widehat{\mathbf{p}})$ has a nonderogatory eigenvalue $\widehat{\lambda}$ with algebraic multiplicity d. The corresponding Jordan chain $\widehat{\mathbf{u}}_1, \ldots, \widehat{\mathbf{u}}_d$ (the eigenvector and associated vectors, also called generalized eigenvectors) is determined by the equations

$$
\begin{aligned}
\widehat{\mathbf{A}}\widehat{\mathbf{u}}_1 &= \widehat{\lambda}\widehat{\mathbf{u}}_1, \\
\widehat{\mathbf{A}}\widehat{\mathbf{u}}_2 &= \widehat{\lambda}\widehat{\mathbf{u}}_2 + \widehat{\mathbf{u}}_1, \\
&\vdots \\
\widehat{\mathbf{A}}\widehat{\mathbf{u}}_d &= \widehat{\lambda}\widehat{\mathbf{u}}_d + \widehat{\mathbf{u}}_{d-1}.
\end{aligned}
\tag{3.1}
$$

These vectors form an $m \times d$ matrix $\widehat{\mathbf{U}} = [\widehat{\mathbf{u}}_1, \ldots, \widehat{\mathbf{u}}_d]$ satisfying equation

$$
\widehat{\mathbf{A}}\widehat{\mathbf{U}} - \widehat{\mathbf{U}}\widehat{\mathbf{J}} = 0, \quad \widehat{\mathbf{J}} = \begin{pmatrix} \widehat{\lambda} & 1 & & \\ & \widehat{\lambda} & \ddots & \\ & & \ddots & 1 \\ & & & \widehat{\lambda} \end{pmatrix},
\tag{3.2}
$$

where $\widehat{\mathbf{J}}$ is the Jordan block of dimension d. The left Jordan chain $\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_d$ corresponding to the eigenvalue $\widehat{\lambda}$ (the eigenvector and associated vectors of the transposed matrix $\widehat{\mathbf{A}}^T$) is determined by the equations

$$
\begin{aligned}
\widehat{\mathbf{v}}_1^T \widehat{\mathbf{A}} &= \widehat{\lambda}\widehat{\mathbf{v}}_1^T, \\
\widehat{\mathbf{v}}_2^T \widehat{\mathbf{A}} &= \widehat{\lambda}\widehat{\mathbf{v}}_2^T + \widehat{\mathbf{v}}_1^T, \\
&\vdots \\
\widehat{\mathbf{v}}_d^T \widehat{\mathbf{A}} &= \widehat{\lambda}\widehat{\mathbf{v}}_d^T + \widehat{\mathbf{v}}_{d-1}^T,
\end{aligned} \tag{3.3}
$$

$$
\widehat{\mathbf{v}}_1^T \widehat{\mathbf{u}}_d = 1,\ \widehat{\mathbf{v}}_i^T \widehat{\mathbf{u}}_d = 0,\ i = 2, \dots, d. \tag{3.4}
$$

Equalities (3.4) represent normalization conditions uniquely determining the vectors $\widehat{\mathbf{v}}_1, \dots, \widehat{\mathbf{v}}_d$ for given vectors $\widehat{\mathbf{u}}_1, \dots, \widehat{\mathbf{u}}_d$. System (3.3), (3.4) can be written in the equivalent matrix form

$$
\widehat{\mathbf{V}}^T \widehat{\mathbf{A}} - \widehat{\mathbf{J}}\widehat{\mathbf{V}}^T = 0, \quad \widehat{\mathbf{V}}^T \widehat{\mathbf{U}} = \mathbf{I}, \quad \widehat{\mathbf{V}} = [\widehat{\mathbf{v}}_d, \dots, \widehat{\mathbf{v}}_1], \tag{3.5}
$$

where $\mathbf{I}$ is the identity matrix. Note that the matrix $\widehat{\mathbf{V}}$ consists of the vectors $\widehat{\mathbf{v}}_1, \dots, \widehat{\mathbf{v}}_d$ listed in the reversed order.

Let us consider a point $\mathbf{p}_0$ in the vicinity of $\widehat{\mathbf{p}}$. In this section we consider the generic case, when the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ has simple eigenvalues $\lambda_1, \dots, \lambda_d$ with the corresponding eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_d$ such that $\lambda_1, \dots, \lambda_d \to \widehat{\lambda}$ as $\mathbf{p}_0 \to \widehat{\mathbf{p}}$. The left eigenvectors corresponding to $\lambda_1, \dots, \lambda_d$ are denoted by $\mathbf{v}_1, \dots, \mathbf{v}_d$. The $m \times d$ matrices $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_d]$ and $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_d]$ satisfy the equations

$$
\mathbf{A}_0\mathbf{U} - \mathbf{U}\mathbf{J}_0 = 0, \quad \mathbf{V}^T\mathbf{A}_0 - \mathbf{J}_0\mathbf{V}^T = 0, \quad \mathbf{V}^T\mathbf{U} = \mathbf{I}, \tag{3.6}
$$

where the diagonal $d \times d$ matrix $\mathbf{J}_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_d)$. The last equality of (3.6) is the normalization condition, which uniquely defines $\mathbf{V}$ for given $\mathbf{U}$. Since the eigenvalue $\widehat{\lambda}$ is nonderogatory (has one eigenvector $\widehat{\mathbf{u}}_1$), every eigenvector $\mathbf{u}_i$ tends to $c_i\widehat{\mathbf{u}}_1$ as $\mathbf{p}_0 \to \widehat{\mathbf{p}}$ for some constant c_i [23].

The goal of this paper is to find the first order approximations (up to the terms $O(\|\mathbf{p} - \mathbf{p}_0\|^2)$) of the stratum λ^d in the parameter space, the nonderogatory multiple eigenvalue λ on the stratum, and the corresponding Jordan chain $\widehat{\mathbf{U}}$. We are going to find these approximations using only information at the point $\mathbf{p}_0$, where the matrix $\mathbf{A}_0$ has simple eigenvalues. To formulate the main result, let us introduce the following scalars and vectors of dimension n

$$
\begin{aligned}
&\lambda_0 = (\lambda_1 + \cdots + \lambda_d)/d, \quad \delta_i = \lambda_i - \lambda_0, \quad \xi_i = \left(\prod_{j=1,\ j\neq i}^{d} (\delta_i - \delta_j) \right)^{-1}, \\
&\mathbf{n}_i = \left(\mathbf{v}_i^T \frac{\partial \mathbf{A}}{\partial p_1}\mathbf{u}_i, \dots, \mathbf{v}_i^T \frac{\partial \mathbf{A}}{\partial p_n}\mathbf{u}_i \right), \quad i = 1, \dots, d, \quad \mathbf{n} = \sum_{i=1}^{d} \mathbf{n}_i/d,
\end{aligned} \tag{3.7}
$$

where derivatives are taken at $\mathbf{p}_0$. Let us define $d \times d$ matrices $\mathbf{K} = \operatorname{diag}(k_1, \dots, k_d)$, $\mathbf{Y}$, and $\mathbf{S}$, whose entries k_i, y_{ij}, and s_{ij}, $i, j = 1, \dots, d$, are

$$
\begin{aligned}
&k_i = 1/(\mathbf{u}_i^T \overline{\mathbf{u}}'), \quad \mathbf{u}' = \mathbf{u}_1 \Big/ \sqrt{\mathbf{u}_1^T \overline{\mathbf{u}}_1}, \quad y_{ij} = -\xi_i \delta_j^d, \\
&s_{ij} = (-1)^r \xi_i \sum_{\substack{1 \le i_1 < \cdots < i_r \le d \\ i_1, \dots, i_r \neq i}} \delta_{i_1} \cdots \delta_{i_r}, \quad r = d - j, \quad s_{id} = \xi_i.
\end{aligned} \tag{3.8}
$$

The matrix $\mathbf{S}$ can also be expressed as $\mathbf{S} = \mathbf{R}^{-1}$, where $\mathbf{R}$ is a $d \times d$ matrix with the components $r_{ij} = \delta_j^{i-1}$ (we assume that $\delta_j^0 = 0$ even if $\delta_j = 0$). The ith row and jth column of a matrix will be denoted by $\mathbf{S}_{<i>}$ and $\mathbf{S}^{<j>}$ respectively.

THEOREM 3.1. *Let $\lambda_1, \ldots, \lambda_d$ be simple eigenvalues of the matrix $\mathbf{A}_0$. Then the first order approximation of the stratum λ^d (where the eigenvalues $\lambda_1, \ldots, \lambda_d$ form a nonderogatory eigenvalue λ of multiplicity d) is given by the following system of $d-1$ linear equations*

$$
\begin{gathered}
q_j^0 + \nabla q_j (\mathbf{p} - \mathbf{p}_0)^T = 0, \quad j = 1, \ldots, d-1, \\
q_j^0 = \sum_{i=1}^{d} s_{ij} \delta_i^d, \quad \nabla q_j = \sum_{i=1}^{d} \frac{s_{ij}}{\xi_i} (\mathbf{n}_i - \mathbf{n}).
\end{gathered} \tag{3.9}
$$

The first order approximations of the multiple eigenvalue λ and the corresponding Jordan chain $\widehat{\mathbf{U}} = [\widehat{\mathbf{u}}_1, \ldots, \widehat{\mathbf{u}}_d]$ on the stratum are given by

$$
\lambda = \lambda_0 + \Delta\lambda, \quad \Delta\lambda = \mathbf{n}(\mathbf{p} - \mathbf{p}_0)^T, \tag{3.10}
$$

$$
\begin{gathered}
\widehat{\mathbf{U}} = \big(\mathbf{U}(\mathbf{K} + \mathbf{K}') + \mathbf{W}\big)\mathbf{S}, \\
\mathbf{W}^{<i>} = (\mathbf{A}_0 - \lambda_i \mathbf{I} - \overline{\mathbf{v}}_i \mathbf{v}_i^T)^{-1} (\mathbf{UKY} + \Delta\lambda \mathbf{UK} - \Delta\mathbf{AUK})^{<i>}, \\
\mathbf{K}' = \operatorname{diag}(k_1', \ldots, k_d'), \quad k_i' = -\mathbf{v}_i^T \mathbf{W} \mathbf{S}^{<d>} / \xi_i, \quad i = 1, \ldots, d, \\
\Delta\mathbf{A} = \sum_{i=1}^{n} \frac{\partial \mathbf{A}}{\partial p_i} (p_i - p_{0i}),
\end{gathered} \tag{3.11}
$$

where all derivatives are taken at $\mathbf{p}_0$.

Let us write equations (3.9) in the matrix form

$$
\mathbf{Q}(\mathbf{p} - \mathbf{p}_0)^T = \mathbf{x}, \quad \mathbf{Q} = \begin{bmatrix} \nabla q_1 \\ \vdots \\ \nabla q_{d-1} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} -q_1^0 \\ \vdots \\ -q_{d-1}^0 \end{bmatrix}, \tag{3.12}
$$

where $\mathbf{Q}$ is a $(d-1) \times n$ matrix; $\mathbf{x}$ is a vector of dimension $d-1$.

COROLLARY 3.2. *Under the conditions of Theorem 3.1 the first order approximation of the point $\mathbf{p}_{\min} \in \lambda^d$ nearest to $\mathbf{p}_0$ (a distance is measured by the Euclidean norm) has the form*

$$
\mathbf{p}_{\min} = \mathbf{p}_0 + \mathbf{x}^T (\overline{\mathbf{Q}} \mathbf{Q}^T)^{-1} \overline{\mathbf{Q}}. \tag{3.13}
$$

Similarly, Theorem 3.1 can be used for finding the nearest point $\mathbf{p}$, where the matrix $\mathbf{A}(\mathbf{p})$ has a special multiple eigenvalue, for example, nonderogatory multiple eigenvalue with the zero real part or absolute value equal to one (this is important in stability problems). Theorem 3.1 can also be used in the case of several different nonderogatory eigenvalues.

COROLLARY 3.3. *Let us consider a stratum $\lambda_1^{d_1} \cdots \lambda_s^{d_s}$, which is determined by matrices having s different nonderogatory eigenvalues $\widehat{\lambda}_1, \ldots, \widehat{\lambda}_s$ with multiplicities $d_1, \ldots, d_s$ respectively. Then the first order approximation of this stratum from the point $\mathbf{p}_0$ is the intersection of planes (3.9) calculated for all $\widehat{\lambda}_1, \ldots, \widehat{\lambda}_s$. The multiple eigenvalues and the corresponding Jordan chains are approximated by (3.10) and (3.11) for every $\widehat{\lambda}_i$.*

3.1. Entries of a matrix as independent parameters. If the space of all complex $m \times m$ matrices is considered, we can take all entries of $\mathbf{A}$ as independent parameters. In this case the matrix $\mathbf{A}$ can be used instead of the parameter vector $\mathbf{p}$. Then the derivative $\partial\mathbf{A}/\partial p_j$ reduces to a matrix with the unit on the (j,l)th place and zeros on the other places, which corresponds to the derivative of $\mathbf{A}$ with respect to its element a_{jl}. The product $\nabla q_j(\mathbf{p}-\mathbf{p}_0)^T$ in (3.9) takes the form of $\operatorname{trace}\big(\mathbf{Q}_j(\mathbf{A}-\mathbf{A}_0)^T\big)$, where $\mathbf{Q}_i$ is an $m \times m$ matrix corresponding to ∇q_i; $\operatorname{trace}\mathbf{A}$ is a sum of elements standing on the main diagonal of the matrix. The norm $\|\mathbf{p}\|$ transforms to the Euclidean matrix norm $\|\mathbf{A}\| = \big(\operatorname{trace}(\mathbf{A}^T\overline{\mathbf{A}})\big)^{1/2}$.

Let us introduce the $m \times m$ matrices

$$\mathbf{N}_i = \mathbf{v}_i\mathbf{u}_i^T, \quad i = 1,\ldots,d, \quad \mathbf{N} = \sum_{i=1}^{d}\mathbf{N}_i/d. \tag{3.14}$$

These matrices are obtained from the vectors $\mathbf{n}_i$, $\mathbf{n}$ (3.7) after substituting the parameter vector $\mathbf{p}$ by the matrix $\mathbf{A}$ and taking the derivative $\partial\mathbf{A}/\partial p_j$ in the form of the matrix, which has the unit on the (j,l)th position and zeros on the other places. Then Theorems 3.1, 3.3, and Corollary 3.2 can be written in terms of matrices (3.14) for the special case under consideration.

THEOREM 3.4. *Let $\lambda_1,\ldots,\lambda_d$ be simple eigenvalues of the matrix $\mathbf{A}_0$. Then the first order approximation of the stratum λ^d (where the eigenvalues $\lambda_1,\ldots,\lambda_d$ form a nonderogatory eigenvalue λ of multiplicity d) in the space of complex matrices is given by the following system of $d-1$ linear equations*

$$\begin{gathered} q_j^0 + \operatorname{trace}\big(\mathbf{Q}_j(\mathbf{A}-\mathbf{A}_0)^T\big) = 0, \quad j = 1,\ldots,d-1, \\ q_j^0 = \sum_{i=1}^{d} s_{ij}\delta_i^d, \quad \mathbf{Q}_j = \sum_{i=1}^{d}\frac{s_{ij}}{\xi_i}(\mathbf{N}_i - \mathbf{N}). \end{gathered} \tag{3.15}$$

The first order approximations of the multiple eigenvalue λ and the corresponding Jordan chain $\widehat{\mathbf{U}} = [\widehat{\mathbf{u}}_1,\ldots,\widehat{\mathbf{u}}_d]$ on the stratum are given by

$$\lambda = \lambda_0 + \Delta\lambda, \quad \Delta\lambda = \operatorname{trace}\big(\mathbf{N}(\mathbf{A}-\mathbf{A}_0)^T\big), \tag{3.16}$$

$$\begin{gathered} \widehat{\mathbf{U}} = \big(\mathbf{U}(\mathbf{K}+\mathbf{K}') + \mathbf{W}\big)\mathbf{S}, \\ \mathbf{W}^{<i>} = (\mathbf{A}_0 - \lambda_i\mathbf{I} - \overline{\mathbf{v}}_i\mathbf{v}_i^T)^{-1}(\mathbf{UKY} + \Delta\lambda\mathbf{UK} - (\mathbf{A}-\mathbf{A}_0)\mathbf{UK})^{<i>}, \\ \mathbf{K}' = \operatorname{diag}(k_1',\ldots,k_d'), \quad k_i' = -\mathbf{v}_i^T\mathbf{W}\mathbf{S}^{<d>}/\xi_i, \quad i = 1,\ldots,d. \end{gathered} \tag{3.17}$$

COROLLARY 3.5. *Under the conditions of Theorem 3.4 the first order approximation of the matrix $\mathbf{A}_{\min} \in \lambda^d$ nearest to $\mathbf{A}_0$ (with a distance measured by the Euclidean matrix norm) has the form*

$$\mathbf{A}_{\min} = \mathbf{A}_0 + \sum_{j=1}^{d-1}\overline{\mathbf{Q}}_j y_j, \quad \mathbf{y} = \mathbf{P}^{-1}\mathbf{x}, \tag{3.18}$$

where the $(d-1)\times(d-1)$ matrix $\mathbf{P}$ has the elements $p_{ij} = \operatorname{trace}(\mathbf{Q}_i^T\overline{\mathbf{Q}}_j)$; $\mathbf{x}$ and $\mathbf{y}$ are column-vectors of dimension $d-1$ with components $x_j = -q_j^0$ and y_j respectively.

Note that the above approximations use only information on simple eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A}_0$.

3.2. Proof. According to the versal deformation theory [1, 2], a family of matrices $\mathbf{A}(\mathbf{p})$ in the vicinity of the point $\widehat{\mathbf{p}} \in \lambda^d$ can be represented in the form

$$\mathbf{A}(\mathbf{p}) = \mathbf{C}(\mathbf{p})\widetilde{\mathbf{A}}(\widetilde{\mathbf{q}}(\mathbf{p}))\mathbf{C}^{-1}(\mathbf{p}), \tag{3.19}$$

where $\mathbf{C}(\mathbf{p})$ is an $m \times m$ nonsingular matrix smoothly dependent on $\mathbf{p}$; $\widetilde{\mathbf{q}}(\mathbf{p})$ is a vector smoothly dependent on $\mathbf{p}$ such that $\widetilde{\mathbf{q}}(\widehat{\mathbf{p}}) = 0$; $\widetilde{\mathbf{A}}(\widetilde{\mathbf{q}}) = \operatorname{diag}(\mathbf{A}'(\mathbf{q}'), \mathbf{A}''(\mathbf{q}''))$ is a block-diagonal matrix family with

$$\mathbf{A}'(\mathbf{q}') = \widehat{\mathbf{J}} + \begin{pmatrix} q_d & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & q_d & 0 \\ q_1 & q_2 & \cdots & q_d \end{pmatrix}, \quad \mathbf{q}' = (q_1, \ldots, q_d). \tag{3.20}$$

Here $\widehat{\mathbf{J}}$ is the Jordan block of dimension d with the eigenvalue $\widehat{\lambda}$ (3.2). The $d \times d$ block $\mathbf{A}'(\mathbf{q}')$ corresponds to the nonderogatory eigenvalue $\widehat{\lambda}$, while the block $\mathbf{A}''(\mathbf{q}'')$ corresponds to other eigenvalues; $\widetilde{\mathbf{q}} = (\mathbf{q}', \mathbf{q}'')$. According to (3.19), the matrices $\mathbf{A}(\mathbf{p})$ and $\widetilde{\mathbf{A}}(\widetilde{\mathbf{q}}(\mathbf{p}))$ have the same JNF. Note that the block $\mathbf{A}'(\mathbf{q}')$ can be chosen in many different ways, and the special choice (3.20) is made in order to simplify the calculations. The matrix family $\mathbf{C}(\mathbf{p})$ is uniquely determined on any smooth surface $\widetilde{T}$ (in the space of $m \times m$ matrices), which is transversal to the plane

$$\widetilde{P} = \{\mathbf{B} \in \mathrm{C}^{m\times m} : \ \widehat{\mathbf{A}}\mathbf{B} - \mathbf{B}\widetilde{\mathbf{A}}(0) = 0\}, \quad \widehat{\mathbf{A}} = \mathbf{A}(\widehat{\mathbf{p}}), \tag{3.21}$$

at $\mathbf{C}(\widehat{\mathbf{p}}) \in \widetilde{P}$ and $\dim \widetilde{T} + \dim \widetilde{P} = m^2$ [1, 2].

Multiplying (3.19) by $\mathbf{C}(\mathbf{p})$ from right, we obtain $\mathbf{A}(\mathbf{p})\mathbf{C}(\mathbf{p}) = \mathbf{C}(\mathbf{p})\widetilde{\mathbf{A}}(\widetilde{\mathbf{q}}(\mathbf{p}))$. Due to the block-diagonal structure of $\widetilde{\mathbf{A}}$ this equation splits into a set of independent equations for the blocks of $\widetilde{\mathbf{A}}$. The equation corresponding to the first block $\mathbf{A}'(\mathbf{q}')$ takes the form

$$\mathbf{A}(\mathbf{p})\mathbf{C}'(\mathbf{p}) = \mathbf{C}'(\mathbf{p})\mathbf{A}'(\mathbf{q}'(\mathbf{p})), \tag{3.22}$$

where $\mathbf{C}'(\mathbf{p})$ is an $m \times d$ matrix consisting of the first d columns of $\mathbf{C}(\mathbf{p})$. Using the condition $\mathbf{C}(\mathbf{p}) \in \widetilde{T}$ and the block-diagonal structure of $\widetilde{\mathbf{A}}(0)$ in (3.21), we find that the family $\mathbf{C}'(\mathbf{p})$ is uniquely determined on any surface T of dimension $d(m-1)$, which is transversal (in the space of $m \times d$ matrices) to the plane

$$\widehat{P} = \{\mathbf{B}' \in \mathrm{C}^{m\times d} : \ \widehat{\mathbf{A}}\mathbf{B}' - \mathbf{B}'\widehat{\mathbf{J}} = 0\} = \{\widehat{\mathbf{U}}\mathbf{X} : \ \mathbf{X} \in \operatorname{cent} \widehat{\mathbf{J}}\}. \tag{3.23}$$

Here equality (3.2) was used. The *centralizer* $\operatorname{cent} \widehat{\mathbf{J}} = \{\mathbf{X} \in \mathrm{C}^{d\times d} : \ \widehat{\mathbf{J}}\mathbf{X} - \mathbf{X}\widehat{\mathbf{J}} = 0\}$ consists of the matrices

$$\mathbf{X} = \begin{pmatrix} x_1 & x_2 & \ddots & x_d \\ 0 & x_1 & \ddots & \ddots \\ 0 & 0 & \ddots & x_2 \\ 0 & 0 & 0 & x_1 \end{pmatrix}, \tag{3.24}$$

where every upper diagonal is filled by equal numbers; the other places are all zeros [11].

Equation of the stratum λ^d has the form

$$q_1(\mathbf{p}) = \cdots = q_{d-1}(\mathbf{p}) = 0, \tag{3.25}$$

which is equivalent to

$$\mathbf{A}'(\mathbf{q}'(\mathbf{p})) = \mathbf{J}_d = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & \ddots & 0 \\ 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}. \tag{3.26}$$

Here $\lambda = \widehat{\lambda} + q_d(\mathbf{p})$ is an arbitrary constant determining the $d \times d$ Jordan block $\mathbf{J}_d$.

We assume that the point $\mathbf{p}_0$ belongs to the vicinity of $\widehat{\mathbf{p}}$, where (3.22) holds. Then the functions $q_i(\mathbf{p})$ can be represented in the form

$$\begin{gathered} q_i(\mathbf{p}) = q_i^0 + \nabla q_i (\mathbf{p} - \mathbf{p}_0)^T + O(\|\mathbf{p} - \mathbf{p}_0\|^2), \\ q_i^0 = q_i(\mathbf{p}_0), \quad \nabla q_i = \left(\frac{\partial q_i}{\partial p_1}, \ldots, \frac{\partial q_i}{\partial p_n} \right), \quad i = 1, \ldots, d, \end{gathered} \tag{3.27}$$

where derivatives are evaluated at $\mathbf{p}_0$. Using (3.27) in (3.25), we find the first order approximation of the stratum λ^d as follows

$$q_i^0 + \nabla q_i (\mathbf{p} - \mathbf{p}_0)^T = 0, \quad i = 1, \ldots, d-1. \tag{3.28}$$

Equation (3.28) determines a plane σ of codimension $d-1$ in the parameter space; see Figure 2.2.

According to the assumptions of Theorem 3.1, the matrix $\mathbf{A}_0$ has simple eigenvalues $\lambda_1, \ldots, \lambda_d$ which coincide as $\mathbf{p}_0 \to \widehat{\mathbf{p}}$. Hence, $\lambda_1, \ldots, \lambda_d$ are eigenvalues of the matrix $\mathbf{A}_0' = \mathbf{A}'(\mathbf{q}'(\mathbf{p}_0))$. Using (3.20) and comparing the characteristic equation for $\mathbf{A}_0'$

$$\mu^d - q_{d-1}^0 \mu^{d-2} - \cdots - q_2^0 \mu - q_1^0 = 0, \quad \mu = \lambda - \widehat{\lambda} - q_d^0,$$

with $(\lambda - \lambda_1) \cdots (\lambda - \lambda_d) = 0$, we obtain

$$\begin{gathered} \widehat{\lambda} + q_d^0 = \lambda_0, \\ q_j^0 = \sum_{1 \le t_1 < \cdots < t_r \le d} (-1)^{r+1} \delta_{t_1} \cdots \delta_{t_r}, \quad r = d - j + 1, \quad j = 1, \ldots, d-1. \end{gathered} \tag{3.29}$$

Expressions (3.20), (3.29) determine the matrix $\mathbf{A}_0' = \mathbf{A}'(\mathbf{q}'(\mathbf{p}_0))$. Eigenvectors of $\mathbf{A}_0'$ form the $d \times d$ matrix $\mathbf{R}$ satisfying equation $\mathbf{A}_0' \mathbf{R} = \mathbf{R} \mathbf{J}_0$, $\mathbf{J}_0 = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$. Using (3.20), the matrix $\mathbf{R}$ can be found explicitly as follows

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \delta_1 & \delta_2 & \cdots & \delta_d \\ \delta_1^2 & \delta_2^2 & \cdots & \delta_d^2 \\ \vdots & \vdots & \vdots & \vdots \\ \delta_1^{d-1} & \delta_2^{d-1} & \cdots & \delta_d^{d-1} \end{pmatrix}. \tag{3.30}$$

The matrix $\mathbf{A}'_0$ can be expressed in the form

$$\mathbf{A}'_0 = \mathbf{R}\mathbf{J}_0\mathbf{S}, \quad \mathbf{S} = \mathbf{R}^{-1}. \tag{3.31}$$

Elements of the matrix $\mathbf{S}$ are given in (3.8), what can be verified by checking the relation $\mathbf{SR} = \mathbf{I}$. Using (3.20) in (3.31), we can rewrite expressions (3.29) in the form

$$\widehat{\lambda} + q_d^0 = \lambda_0, \quad q_j^0 = \mathbf{R}_{<d>}\mathbf{J}_0\mathbf{S}^{<j>} = \sum_{i=1}^{d} s_{ij}\delta_i^d, \quad j = 1, \dots, d-1. \tag{3.32}$$

Evaluating (3.22) at $\mathbf{p}_0$, multiplying it by $\mathbf{R}$ from the right side, and using expression (3.31), we find

$$\mathbf{A}_0(\mathbf{C}'_0\mathbf{R}) - (\mathbf{C}'_0\mathbf{R})\mathbf{J}_0 = 0, \quad \mathbf{C}'_0 = \mathbf{C}'(\mathbf{p}_0). \tag{3.33}$$

Hence, the ith column of the matrix $\mathbf{C}'_0\mathbf{R}$ is the eigenvector $\mathbf{u}_i$ multiplied by an arbitrary constant k_i, and we obtain

$$\mathbf{C}'_0 = \mathbf{UKS}, \quad \mathbf{K} = \mathrm{diag}(k_1, \dots, k_d). \tag{3.34}$$

We will specify the matrix $\mathbf{K}$ later together with the calculation of the Jordan chain.

The functions $\mathbf{A}'(\mathbf{q}'(\mathbf{p}))$ and $\mathbf{C}'(\mathbf{p})$ can be expressed in the form

$$\begin{aligned} \mathbf{A}'(\mathbf{q}'(\mathbf{p})) &= \mathbf{A}'_0 + \Delta\mathbf{A}' + O(\|\mathbf{p} - \mathbf{p}_0\|^2), \\ \mathbf{C}'(\mathbf{p}) &= \mathbf{C}'_0 + \Delta\mathbf{C}' + O(\|\mathbf{p} - \mathbf{p}_0\|^2), \end{aligned} \tag{3.35}$$

where

$$\Delta\mathbf{A}' = \sum_{i=1}^{n} \frac{\partial \mathbf{A}'}{\partial p_i}(p_i - p_{0i}), \quad \Delta\mathbf{C}' = \sum_{i=1}^{n} \frac{\partial \mathbf{C}'}{\partial p_i}(p_i - p_{0i}); \tag{3.36}$$

derivatives are evaluated at $\mathbf{p}_0$. Substituting (3.35) into (3.22), we obtain the equation for the first order terms

$$\mathbf{A}_0\Delta\mathbf{C}' - \Delta\mathbf{C}'\mathbf{A}'_0 = \mathbf{C}'_0\Delta\mathbf{A}' - \Delta\mathbf{A}\mathbf{C}'_0, \quad \Delta\mathbf{A} = \sum_{i=1}^{n} \frac{\partial \mathbf{A}}{\partial p_i}(p_i - p_{0i}). \tag{3.37}$$

Multiplying (3.37) by $\mathbf{K}^{-1}\mathbf{V}^T$ and $\mathbf{R}$ from left and right respectively, and using expressions (3.6), (3.31), and (3.34), we find

$$\mathbf{J}_0(\mathbf{K}^{-1}\mathbf{V}^T\Delta\mathbf{C}'\mathbf{R}) - (\mathbf{K}^{-1}\mathbf{V}^T\Delta\mathbf{C}'\mathbf{R})\mathbf{J}_0 = \mathbf{S}\Delta\mathbf{A}'\mathbf{R} - \mathbf{K}^{-1}\mathbf{V}^T\Delta\mathbf{A}\mathbf{U}\mathbf{K}. \tag{3.38}$$

Taking the trace of (3.38), the left-hand-side vanishes and we obtain

$$\nabla q_d(\mathbf{p} - \mathbf{p}_0)^T = \mathbf{n}(\mathbf{p} - \mathbf{p}_0)^T = \Delta\lambda, \tag{3.39}$$

where the equality

$$\mathrm{trace}(\mathbf{S}\Delta\mathbf{A}'\mathbf{R}) = \mathrm{trace}(\Delta\mathbf{A}'\mathbf{RS}) = \mathrm{trace}(\Delta\mathbf{A}') = d\nabla q_d(\mathbf{p} - \mathbf{p}_0)^T$$

was used. Expressions (3.29), (3.39) give the first order approximation (3.10) of the multiple eigenvalue $\lambda = \widehat{\lambda} + q_d(\mathbf{p}) = \lambda_0 + \Delta\lambda + O(\|\mathbf{p} - \mathbf{p}_0\|^2)$, $\Delta\lambda = \mathbf{n}(\mathbf{p} - \mathbf{p}_0)^T$, on the stratum λ^d.

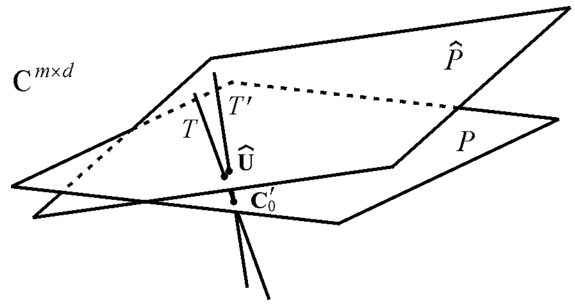

FIG. 3.1. *Geometry of the Jordan chain in the space* $\mathbb{C}^{m\times d}$.

Taking the (i,i)th element of (3.38), the left-hand-side vanishes and we get

$$\mathbf{S}_{<i>}\Delta\mathbf{A}'\mathbf{R}^{<i>} = \mathbf{v}_i^T\Delta\mathbf{A}\mathbf{u}_i = \mathbf{n}_i(\mathbf{p}-\mathbf{p}_0)^T. \tag{3.40}$$

Calculating the value of $\mathbf{S}_{<i>}\Delta\mathbf{A}'\mathbf{R}^{<i>}$ with the use of (3.20) and (3.39), we obtain

$$\sum_{k=1}^{d-1} s_{id}r_{ki}\nabla q_k(\mathbf{p}-\mathbf{p}_0)^T = (\mathbf{n}_i-\mathbf{n})(\mathbf{p}-\mathbf{p}_0)^T, \quad i=1,\ldots,d. \tag{3.41}$$

Multiplying (3.41) by $s_{ij}/s_{id} = s_{ij}/\xi_i$, summing over i from 1 to d, and using relations $\mathbf{RS}=\mathbf{I}$, we find

$$\nabla q_j(\mathbf{p}-\mathbf{p}_0)^T = \sum_{i=1}^{d}\frac{s_{ij}}{\xi_i}(\mathbf{n}_i-\mathbf{n})(\mathbf{p}-\mathbf{p}_0)^T. \tag{3.42}$$

Substituting (3.32) and (3.42) into (3.28), we obtain the first order approximation of the stratum λ^d in the form (3.9).

Now let us find the approximation of the Jordan chain $\widehat{\mathbf{U}} = [\widehat{\mathbf{u}}_1,\ldots,\widehat{\mathbf{u}}_d]$ at points of plane (3.9). Note that the Jordan chain $\widehat{\mathbf{U}}$ is not uniquely determined. All the Jordan chains define the plane $\widehat{P}$ (3.23) of dimension d in the space of $m\times d$ matrices, where any point of the plane $\widehat{P}$ (except the points with $\det\mathbf{X}=0$) represents the Jordan chain. At the same time a set of matrices $\mathbf{C}_0'$ (3.34) determines another plane P of dimension d

$$P = \{\mathbf{UKS}:\ \mathbf{K} = \mathrm{diag}(k_1,\ldots,k_d)\}, \tag{3.43}$$

The planes $\widehat{P}$ and P are sketched on Figure 3.1 (the drawing is very schematic, since the planes P and $\widehat{P}$ usually have only one joint point at the origin).

We are going to find the first order approximation of the Jordan chain $\widehat{\mathbf{U}}$ (or, equivalently, of the plane $\widehat{P}$) using information at the point $\mathbf{C}_0'\in P$. Hence, we need to take the point $\mathbf{C}_0'$, which is close to $\widehat{P}$, in order to get a better approximation. Note that we seek a point $\mathbf{C}_0'\in P$ close to $\widehat{P}$ keeping the norm of $\mathbf{C}_0'$ bounded ($\|\mathbf{C}_0'\|\ge c>0$), because the planes P and $\widehat{P}$ intersect at the origin. Since $\delta_i\to 0$ as $\mathbf{p}_0\to\widehat{\mathbf{p}}$, the limit of the matrix $\mathbf{UK}=\mathbf{C}_0'\mathbf{R}$ as $\mathbf{p}_0\to\widehat{\mathbf{p}}$ is a matrix with equal columns (see the form of the matrix $\mathbf{R}$ in (3.30)). At the same time $\mathbf{C}_0'\to\widehat{\mathbf{U}}\in\widehat{P}$ as $\mathbf{p}_0\to\widehat{\mathbf{p}}$. Hence, if $\mathbf{p}_0$ is close to $\widehat{\mathbf{p}}$ and $\mathbf{C}_0'=\mathbf{UKS}\in P$ is close to $\widehat{P}$, then the matrix $\mathbf{UK}$ has almost equal columns. This condition provides a way for the choice of the appropriate matrix $\mathbf{K}$. Its elements can be taken, for example, in the form (3.8). Expressions (3.8) for the elements k_i of the matrix $\mathbf{K}$ mean that the values of projections of the vectors $k_i\mathbf{u}_i = \mathbf{UK}^{<i>}$ on the direction $\mathbf{u}'$ are equal to 1. The vector $\mathbf{u}'$ can be chosen in different ways. In particular, it can be equal to the direction determined by the eigenvector $\mathbf{u}_1$ (3.8).

The matrix $\mathbf{C}'_0 = \mathbf{UKS}$ represents the zero-order approximation of the Jordan chain $\widehat{\mathbf{U}}$. Analogously, the matrix $\mathbf{VK}^{-1}\mathbf{R}^T$ represents the zero-order approximation of the left Jordan chain $\widehat{\mathbf{V}}$, i.e., $\mathbf{UKS} \to \widehat{\mathbf{U}} \in \widehat{P}$ and $\mathbf{VK}^{-1}\mathbf{R}^T \to \widehat{\mathbf{V}}$ as $\mathbf{p}_0 \to \widehat{\mathbf{p}}$. The matrix $\mathbf{C}'_0$ already represents a good approximation of the Jordan chain; it has the accuracy $O(\|\mathbf{p}-\mathbf{p}_0\|)$, which is enough for many applications. Now let us determine the first-order approximation of the Jordan chain (up to $O(\|\mathbf{p}-\mathbf{p}_0\|^2)$) specified for each point of the stratum λ^d.

The matrix family $\mathbf{C}'(\mathbf{p})$ is uniquely determined on any surface T of dimension $m(d-1)$ (in the space of $m \times d$ matrices) transversal to the plane $\widehat{P}$ at $\widehat{\mathbf{U}} = \mathbf{C}'(\widehat{\mathbf{p}})$; see Figure 3.1 [1, 2]. The surface T can be chosen in the form

$$T = \mathbf{C}'(\widehat{\mathbf{p}}) + \{\mathbf{B} \in \mathrm{C}^{m\times d} :\ \widehat{\mathbf{v}}_j^T \mathbf{B}^{<d>} = 0,\quad j = 1,\dots,d\}. \tag{3.44}$$

It is clear that $\dim T = m(d-1)$. The intersection of T and $\widehat{P}$ is one point $\mathbf{C}'(\widehat{\mathbf{p}})$, since the condition $\widehat{\mathbf{v}}_j^T\mathbf{B}^{<d>} = 0$ fix the numbers x_j standing on the corresponding diagonal of $\mathbf{X}$; see expressions (3.23) and (3.24). Hence, T and $\widehat{P}$ are transversal. Let us consider a plane

$$T' = \mathbf{C}'_0 + \{\mathbf{B} \in \mathrm{C}^{m\times d} :\ \mathbf{R}_{<j>}\mathbf{K}^{-1}\mathbf{V}^T\mathbf{B}^{<d>} = 0,\quad j = 1,\dots,d\} \tag{3.45}$$

obtained from (3.44) if we substitute $\mathbf{C}'(\widehat{\mathbf{p}})$ by $\mathbf{C}'_0$ and $\widehat{\mathbf{v}}_j$ by $(\mathbf{VK}^{-1}\mathbf{R}^T)^{<d+1-j>}$. Since $\mathbf{C}'_0 \to \mathbf{C}'(\widehat{\mathbf{p}})$ and $\mathbf{VK}^{-1}\mathbf{R}^T \to \widehat{\mathbf{V}}$ as $\mathbf{p}_0 \to \widehat{\mathbf{p}}$, the plane T' is a small perturbation of T for $\mathbf{p}_0$ close to $\widehat{\mathbf{p}}$; see Figure 3.1. Hence, T' is transversal to $\widehat{P}$ and we can uniquely choose $\mathbf{C}'(\mathbf{p}) \in T'$. The condition $\mathbf{C}'(\mathbf{p}) \in T'$ yields $\mathbf{R}_{<j>}\mathbf{K}^{-1}\mathbf{V}^T\Delta\mathbf{C}'^{<d>} = 0$ for $j = 1,\dots,d$. These equalities can be written in the equivalent form as follows

$$\mathbf{v}_j^T\Delta\mathbf{C}'^{<d>} = 0,\quad j = 1,\dots,d. \tag{3.46}$$

Multiplying (3.37) by $\mathbf{R}$ from right and using expressions (3.31), (3.34), we obtain

$$\mathbf{A}_0\Delta\mathbf{C}'\mathbf{R} - \Delta\mathbf{C}'\mathbf{R}\mathbf{J}_0 = \mathbf{UKS}\Delta\mathbf{A}'\mathbf{R} - \Delta\mathbf{AUK}. \tag{3.47}$$

Taking the ith column of (3.47), we find

$$(\mathbf{A}_0 - \lambda_i\mathbf{I})(\Delta\mathbf{C}'\mathbf{R})^{<i>} = (\mathbf{UKS}\Delta\mathbf{A}'\mathbf{R} - \Delta\mathbf{AUK})^{<i>}. \tag{3.48}$$

The matrix $\mathbf{A}_0 - \lambda_i\mathbf{I}$ is singular with the simple zero eigenvalue. The solution $(\Delta\mathbf{C}'\mathbf{R})^{<i>}$ of (3.48) exists if and only if [27]

$$\mathbf{v}_i^T(\mathbf{UKS}\Delta\mathbf{A}'\mathbf{R} - \Delta\mathbf{AUK})^{<i>} = 0. \tag{3.49}$$

This condition is satisfied, since

$$\mathbf{v}_i^T(\mathbf{UKS}\Delta\mathbf{A}'\mathbf{R} - \Delta\mathbf{AUK})^{<i>} = k_i(\mathbf{S}_{<i>}\Delta\mathbf{A}'\mathbf{R}^{<i>} - \mathbf{v}_i^T\Delta\mathbf{A}\mathbf{u}_i) = 0,$$

where expression (3.40) was used. Then, the general solution $(\Delta\mathbf{C}'\mathbf{R})^{<i>}$ has the form [27]

$$(\Delta\mathbf{C}'\mathbf{R})^{<i>} = \mathbf{W}^{<i>} + k_i'\mathbf{u}_i,$$
$$\mathbf{W}^{<i>} = (\mathbf{A}_0 - \lambda_i\mathbf{I} - \overline{\mathbf{v}}_i\mathbf{v}_i^T)^{-1}(\mathbf{UKS}\Delta\mathbf{A}'\mathbf{R} - \Delta\mathbf{AUK})^{<i>}, \tag{3.50}$$

where $\mathbf{A}_0 - \lambda_i\mathbf{I} - \overline{\mathbf{v}}_i\mathbf{v}_i^T$ is a nonsingular matrix; k_i' is an arbitrary constant. Hence,

$$\Delta\mathbf{C}' = (\mathbf{W} + \mathbf{UK}')\mathbf{S}, \quad \mathbf{K}' = \mathrm{diag}(k_1', \ldots, k_d'). \tag{3.51}$$

The matrix $\mathbf{S}\Delta\mathbf{A}'\mathbf{R}$ in (3.50), calculated on plane (3.9), with the use of the explicit forms of the matrices $\mathbf{S}$ and $\mathbf{R}$ (3.8), (3.30) is written as follows

$$\mathbf{S}\Delta\mathbf{A}'\mathbf{R} = \mathbf{Y} + \Delta\lambda\mathbf{I}, \tag{3.52}$$

where elements of the matrix $\mathbf{Y}$ are determined in (3.8). Substituting expression (3.51) into conditions (3.46), we find

$$\mathbf{v}_j^T\Delta\mathbf{C}'^{<d>} = \mathbf{v}_j^T\mathbf{WS}^{<d>} + k_j's_{j\,d} = 0,$$
$$k_j' = -\mathbf{v}_j^T\mathbf{WS}^{<d>}/\xi_j.$$

Thus, we obtain the first order approximation of the Jordan chain $[\widehat{\mathbf{u}}_1, \ldots, \widehat{\mathbf{u}}_d] = \mathbf{C}'(\mathbf{p}) = \mathbf{C}_0' + \Delta\mathbf{C}' + O(\|\mathbf{p} - \mathbf{p}_0\|^2)$ in the form (3.11) at any point of plane (3.9). This completes the proof of Theorem 3.1.

The nearest to $\mathbf{p}_0$ point $\mathbf{p}_{\min} \in \lambda^d$ can be found using approximation (3.9) or its matrix form (3.12). In this case $\mathbf{p}_{\min} - \mathbf{p}_0$ is the normal vector to plane (3.9). Hence,

$$\mathbf{p}_{\min} - \mathbf{p}_0 = \sum_{j=1}^{d-1}\overline{\nabla q_j}y_j = \mathbf{y}^T\overline{\mathbf{Q}}, \quad \mathbf{y}^T = (y_1, \ldots, y_{d-1}). \tag{3.53}$$

Substituting (3.53) into (3.12), we get

$$\mathbf{Q}\overline{\mathbf{Q}}^T\mathbf{y} = \mathbf{x}. \tag{3.54}$$

Finding $\mathbf{y}$ from (3.54) and substituting it into (3.53), we obtain expression (3.13) for the nearest point of the stratum.

4. Nonderogatory eigenvalues of real matrices. In this section real matrices $\mathbf{A}(\mathbf{p})$ smoothly dependent on a vector of real parameters $\mathbf{p}$ are considered. In this case there are two different types of nonderogatory eigenvalues: real eigenvalues and complex conjugate pairs of eigenvalues. These eigenvalues form smooth manifolds (strata) in the parameter space. We will denote the stratum corresponding to a real eigenvalue by α^d, and the stratum corresponding to a complex conjugate pair of eigenvalues by $(\alpha \pm i\omega)^d$, where d is the algebraic multiplicity of a nonderogatory eigenvalue. Since versal deformations in the real and complex cases are very similar, the results for real matrices can be obtained after a small change from Theorems 3.1 and 3.4.

4.1. Stratum $(\alpha \pm i\omega)^d$. Let us consider a point $\widehat{\mathbf{p}}$ in the parameter space, where the matrix $\widehat{\mathbf{A}} = \mathbf{A}(\widehat{\mathbf{p}})$ has a nonderogatory complex eigenvalue $\widehat{\lambda}$ with multiplicity d. The corresponding right and left Jordan chains form complex $m \times d$ matrices $\widehat{\mathbf{U}}$

and $\widehat{\mathbf{V}}$ satisfying equations (3.2) and (3.5). The real matrix $\widehat{\mathbf{A}}$ has also a complex conjugate eigenvalue $\overline{\widehat{\lambda}}$ with the corresponding right and left Jordan chains forming matrices $\overline{\widehat{\mathbf{U}}}$ and $\overline{\widehat{\mathbf{V}}}$.

Let us consider a point $\mathbf{p}_0$ in the vicinity of $\widehat{\mathbf{p}}$. We assume that the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ has simple complex eigenvalues $\lambda_1, \ldots, \lambda_d$ such that $\lambda_1, \ldots, \lambda_d \to \widehat{\lambda}$ as $\mathbf{p}_0 \to \widehat{\mathbf{p}}$. If $\mathbf{p}_0$ is close to $\widehat{\mathbf{p}}$, the eigenvalues $\lambda_1, \ldots, \lambda_d$ have the same sign of the imaginary part equal to the sign of $\mathrm{Im}\,\widehat{\lambda}$. The right and left eigenvectors $\mathbf{u}_i$ and $\mathbf{v}_i$ corresponding to λ_i form complex matrices $\mathbf{U}$ and $\mathbf{V}$ satisfying equations (3.6).

A block of the versal deformation, corresponding to a complex nonderogatory eigenvalue $\widehat{\lambda}$, in the case of the real matrix family $\mathbf{A}(\mathbf{p})$ has the form (3.20), where $q_1(\mathbf{p}), \ldots, q_d(\mathbf{p})$ are complex-valued smooth functions of $\mathbf{p}$ ($\mathrm{Re}\, q_i(\mathbf{p})$ and $\mathrm{Im}\, q_i(\mathbf{p})$ are independent smooth functions) [2, 10]. Hence, this case is almost identical to the case of complex matrix families considered in the previous section. The difference is only that the system $q_i(\mathbf{p}) = 0$, $i = 1, \ldots, d-1$, determining the stratum $(\alpha \pm i\omega)^d$ represents $2(d-1)$ real equations (for real and imaginary parts).

THEOREM 4.1. *Let $\lambda_1, \ldots, \lambda_d$ be simple complex eigenvalues of the real matrix $\mathbf{A}_0$, having the same sign of the imaginary part* $\mathrm{sign}(\mathrm{Im}\,\lambda_i)$. *Then the first order approximation of the stratum $(\alpha \pm i\omega)^d$ (where the eigenvalues $\lambda_1, \ldots, \lambda_d$ form a nonderogatory complex eigenvalue λ of multiplicity d) is given by the system of $2(d-1)$ linear real equations (3.9), where every equality represents two equations for real and imaginary parts. The first order approximations of the multiple eigenvalue λ and the corresponding Jordan chain $\widehat{\mathbf{U}}$ on the stratum are given by (3.10) and (3.11).*

COROLLARY 4.2. *Under the conditions of Theorem 4.1 the first order approximation of the point $\mathbf{p}_{\min} \in (\alpha \pm i\omega)^d$ nearest to $\mathbf{p}_0$ has the form*

$$\mathbf{p}_{\min} = \mathbf{p}_0 + \mathbf{x}'^T(\mathbf{Q}'\mathbf{Q}'^T)^{-1}\mathbf{Q}'. \tag{4.1}$$

where $\mathbf{Q}'$ is a $2(d-1) \times n$ real matrix with the rows $\mathrm{Re}\,\nabla q_j$ and $\mathrm{Im}\,\nabla q_j$, $j = 1, \ldots, d-1$; $\mathbf{x}'$ is a real column-vector of dimension $2(d-1)$ with the components $-\mathrm{Re}\, q_j^0$ and $-\mathrm{Im}\, q_j^0$.

If the stratum $(\alpha \pm i\omega)^d$ in the space of real $m \times m$ matrices is considered, we can take all entries of $\mathbf{A}$ as independent parameters (the matrix $\mathbf{A}$ is used instead of the vector of parameters $\mathbf{p}$). In this case the relations of Theorem 4.1 take the form (3.15), (3.16), (3.17), where system (3.15) consists of $2(d-1)$ real equations ($d-1$ equations for the real parts and $d-1$ equations for the imaginary parts). The first order approximation of the matrix $\mathbf{A}_{\min} \in (\alpha \pm i\omega)^d$ nearest to $\mathbf{A}_0$ (with respect to the Euclidean matrix norm) has the form

$$\begin{gathered} \mathbf{A}_{\min} = \mathbf{A}_0 + \sum_{j=1}^{2(d-1)} \mathbf{Q}'_j y'_j, \quad \mathbf{y}' = \mathbf{P}^{-1}\mathbf{x}', \\ x'_{2i-1} = -\mathrm{Re}\, q_i^0,\ x'_{2i} = -\mathrm{Im}\, q_i^0,\ \mathbf{Q}'_{2i-1} = \mathrm{Re}\,\mathbf{Q}_i,\ \mathbf{Q}'_{2i} = \mathrm{Im}\,\mathbf{Q}_i, \end{gathered} \tag{4.2}$$

where the real $2(d-1) \times 2(d-1)$ matrix $\mathbf{P}$ has the elements $p_{ij} = \mathrm{trace}(\mathbf{Q}'^T_i \mathbf{Q}'_j)$; $\mathbf{x}'$ and $\mathbf{y}'$ are real column-vectors of dimension $2(d-1)$ with the components x'_j and y'_j; the $m \times m$ complex matrices $\mathbf{Q}_i$ are determined in (3.14), (3.15).

4.2. Stratum α^d. Let us consider a point $\widehat{\mathbf{p}}$ in the parameter space, where the real matrix $\widehat{\mathbf{A}} = \mathbf{A}(\widehat{\mathbf{p}})$ has a nonderogatory real eigenvalue $\widehat{\lambda}$ with multiplicity d. The corresponding right and left Jordan chains form real $m \times d$ matrices $\widehat{\mathbf{U}}$ and $\widehat{\mathbf{V}}$ satisfying equations (3.2) and (3.5).

Let us consider a point $\mathbf{p}_0$ in the vicinity of $\widehat{\mathbf{p}}$. We assume that the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ has simple eigenvalues $\lambda_1, \ldots, \lambda_d$ such that $\lambda_1, \ldots, \lambda_d \to \widehat{\lambda}$ as $\mathbf{p}_0 \to \widehat{\mathbf{p}}$. The eigenvalues $\lambda_1, \ldots, \lambda_d$ are real or complex conjugate. Let $\mathbf{u}_i$ and $\mathbf{v}_i$ be the right and left eigenvectors corresponding to λ_i and forming the matrices $\mathbf{U}$ and $\mathbf{V}$ that satisfy equations (3.6); eigenvectors corresponding to real eigenvalues are real and eigenvectors corresponding to complex conjugate eigenvalues are complex conjugate.

In the case of real matrix families a block of the versal deformation, corresponding to the real nonderogatory eigenvalue $\widehat{\lambda}$, has the form (3.20), where $q_1(\mathbf{p}), \ldots, q_d(\mathbf{p})$ are real smooth functions of $\mathbf{p}$ [2, 10]. Thus, the case α^d is almost identical to the case λ^d for complex matrix families considered in the previous section. Note that the Jordan chain $\widehat{\mathbf{U}}$ obtained by expressions (3.11) can be complex, if the vector $\mathbf{u}'$ determining the matrix $\mathbf{K}$ is complex. To get the real Jordan chain we have to choose the real vector $\mathbf{u}'$ for determining elements of the matrix $\mathbf{K}$ in (3.8). This can be done as follows

$$\mathbf{K} = \mathrm{diag}(k_1, \ldots, k_d), \quad k_i = 1/(\mathbf{u}_i^T \mathbf{u}'), \quad \mathbf{u}' = \mathrm{Re}\left(\mathbf{u}_1 \Big/ \sqrt{\mathbf{u}_1^T \mathbf{u}_1}\right). \tag{4.3}$$

The matrix $\mathbf{K}$ (4.3) has the required property: if the vectors $\mathbf{u}_i$ are close to $c_i \widehat{\mathbf{u}}_1$ with some constants c_i, then the columns of the matrix $\mathbf{UK}$ will be close to each other. With the use of conditions (4.3) the eigenvectors $(\mathbf{UK})^{<i>} = k_i \mathbf{u}_i$, corresponding to complex conjugate eigenvalues λ_i, are complex conjugate, and the eigenvectors $k_i \mathbf{u}_i$, corresponding to real eigenvalues λ_i, are real. Then it can be shown that though the matrices $\mathbf{U}$, $\mathbf{V}$ and the eigenvalues λ_i can be complex, equations (3.9) and the values of λ and $\widehat{\mathbf{U}}$ found by (3.10) and (3.11) are real.

THEOREM 4.3. *Let $\lambda_1, \ldots, \lambda_d$ be real or complex conjugate simple eigenvalues of the real matrix $\mathbf{A}_0$. Then the first order approximation of the stratum α^d (where the eigenvalues $\lambda_1, \ldots, \lambda_d$ form a nonderogatory real eigenvalue λ with multiplicity d) is given by the system of $d-1$ linear real equations (3.9). The first order approximations of the multiple real eigenvalue λ and the corresponding real Jordan chain $\widehat{\mathbf{U}}$ on the stratum are given by (3.10) and (3.11), where the matrix $\mathbf{K}$ has the form (4.3).*

COROLLARY 4.4. *Under the conditions of Theorem 4.1 the first order approximation of the point $\mathbf{p}_{\min} \in \alpha^d$ nearest to $\mathbf{p}_0$ has the form*

$$\mathbf{p}_{\min} = \mathbf{p}_0 + \mathbf{x}^T (\mathbf{Q}\mathbf{Q}^T)^{-1} \mathbf{Q}, \tag{4.4}$$

where the real matrix $\mathbf{Q}$ and the real vector $\mathbf{x}$ are given by (3.12).

If the stratum α^d in the space of real $m \times m$ matrices is considered, we can take all entries of $\mathbf{A}$ as independent parameters (the matrix $\mathbf{A}$ is used instead of the vector of parameters $\mathbf{p}$). In this case relations of Theorem 4.3 take the form (3.15), (3.16), (3.17) with the matrix $\mathbf{K}$ from (4.3). The first order approximation of the matrix $\mathbf{A}_{\min} \in \alpha^d$ nearest to $\mathbf{A}_0$ has the form (3.18).

Finally, let us consider the case, when the real matrix $\mathbf{A}_0$ has several different nonderogatory multiple eigenvalues.

COROLLARY 4.5. *Let us consider a stratum $\alpha_1^{d_1} \cdots \alpha_t^{d_t} (\alpha_{t+1} \pm i\omega_{t+1})^{d_{t+1}} \cdots (\alpha_s \pm i\omega_s)^{d_s}$, which is determined by matrices having t different nonderogatory real eigenvalues with multiplicities $d_1, \ldots, d_t$ and $s-t$ different pairs of complex conjugate nonderogatory eigenvalues with multiplicities $d_{t+1}, \ldots, d_s$. Then the first order approximation of the stratum is the intersection of planes (3.9) calculated by Theorems 4.1 and 4.3 for every pair of complex conjugate multiple eigenvalues and every multiple real eigenvalue. The multiple eigenvalue and the corresponding Jordan chain are given*

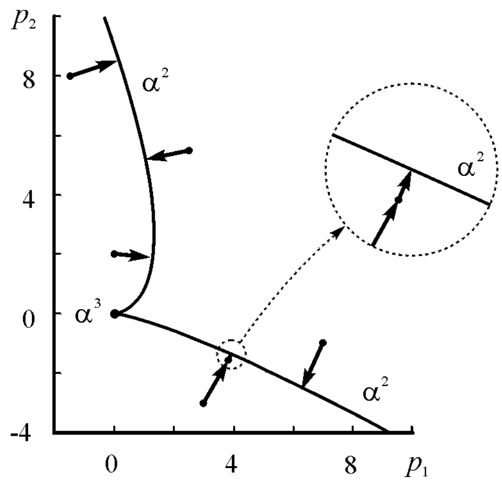

FIG. 4.1. *Approximations of the nearest points of the stratum α^2 from different points $\mathbf{p}_0$.*

by (3.10) and (3.11) for every $\alpha_i^{d_i}$ and $(\alpha_j \pm i\omega_j)^{d_j}$ (the matrix $\mathbf{K}$ for real multiple eigenvalues is taken from (4.3)).

4.3. Example 1. Let us consider a two-parameter matrix family

$$\mathbf{A}(\mathbf{p}) = \begin{pmatrix} 1 & 3 & 0 \\ p_1 & 1 & p_2 \\ 2 & 3 & 1 \end{pmatrix}, \quad \mathbf{p} = (p_1, p_2). \tag{4.5}$$

We will calculate eigenvalues of $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ at different points $\mathbf{p}_0$. Then the nearest to $\mathbf{p}_0$ point of the stratum α^2 is evaluated by Corollary 4.4. In this case ($d = 2$) system (3.9) represents a single linear equation

$$q_1^0 + \nabla q_1 (\mathbf{p} - \mathbf{p}_0)^T = 0, \tag{4.6}$$

where $q_1^0 = \delta^2$, $\nabla q_1 = \delta(\mathbf{n}_2 - \mathbf{n}_1)$, and $\delta = (\lambda_2 - \lambda_1)/2$. Then the nearest to $\mathbf{p}_0$ vector $\mathbf{p}_{\min} \in \alpha^2$ is approximated by

$$\mathbf{p}_{\min} = \mathbf{p}_0 - \frac{q_1^0}{\nabla q_1 \nabla q_1^T} \nabla q_1. \tag{4.7}$$

In calculations we take λ_1 and λ_2 equal to the complex conjugate pair of eigenvalues of $\mathbf{A}_0$. If all eigenvalues of $\mathbf{A}_0$ are real, we consider all three possibilities for λ_1, λ_2 and then take a pair λ_1, λ_2 providing the minimal distance $\|\mathbf{p}_{\min} - \mathbf{p}_0\|$. The result is shown on Figure 4.1. Here the arrows are the vectors $\overrightarrow{\mathbf{p}_0\mathbf{p}_{\min}}$, where $\mathbf{p}_{\min}$ is the nearest point of α^2 found by (4.7) for different $\mathbf{p}_0$. The bifurcation diagram is represented by a solid line (it is found explicitly by calculating the discriminant of the characteristic polynomial of (4.5)). For one point $\mathbf{p}_0$ on Figure 4.1 two iterations were performed, when $\mathbf{p}_{\min}$ is taken as a new starting point $\mathbf{p}_0$. The results show high accuracy of the approximation.

Let us consider a point $\mathbf{p}_0 = (0.3, 9.1)$, where the matrix $\mathbf{A}_0$ has eigenvalues $\lambda_1 = -2.624$, $\lambda_2 = -1.472$, $\lambda_3 = -7.096$. Calculations with the use of (4.7) show that the pair λ_1, λ_2 gives the smallest distance to the stratum α^2. Found by (4.7) and Theorem 4.3 approximate values of the nearest point $\mathbf{p}_{\min} \in \alpha^2$, the double eigenvalue λ, and the corresponding Jordan chain $\widehat{\mathbf{U}} = [\widehat{\mathbf{u}}_1, \widehat{\mathbf{u}}_2]$ (the eigenvector $\widehat{\mathbf{u}}_1$ and the associated vector $\widehat{\mathbf{u}}_2$) at $\mathbf{p}_{\min}$ are as follows

$$\mathbf{p}_{\min} = (-0.0008, 8.9990), \quad \lambda = -2.00006, \quad [\widehat{\mathbf{u}}_1, \widehat{\mathbf{u}}_2] = \begin{pmatrix} 0.7044 & 0.1496 \\ -0.7044 & 0.0852 \\ 0.2346 & -0.1067 \end{pmatrix}.$$

Here the matrices $\mathbf{S}$, $\mathbf{Y}$, and quantities ξ_1, ξ_2 used in (3.11) are

$$\mathbf{S} = \begin{pmatrix} 1/2 & -1/(2\delta) \\ 1/2 & 1/(2\delta) \end{pmatrix}, \; \mathbf{Y} = \frac{\delta}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \; \xi_1 = -1/(2\delta), \; \xi_2 = -\xi_1.$$

The distance from $\mathbf{p}_0$ to α^2 is equal to $\|\mathbf{p}_{\min} - \mathbf{p}_0\| = 0.317$. For comparison, exact values of $\mathbf{p}_{\min}$, λ, and $\widehat{\mathbf{u}}_1$, $\widehat{\mathbf{u}}_2$ (found by the direct analysis of eigenvalues of matrix family (4.5)) are

$$\mathbf{p}_{\min} = (0, 9), \quad \lambda = -2, \quad [\widehat{\mathbf{u}}_1, \widehat{\mathbf{u}}_2] = \begin{pmatrix} 0.7044 & 0.1494 \\ -0.7044 & 0.0854 \\ 0.2348 & -0.1067 \end{pmatrix}.$$

Note that all the approximations were found using only simple eigenvalues and eigenvectors of the matrix $\mathbf{A}_0$ and derivatives of $\mathbf{A}(\mathbf{p})$ with respect to parameters at $\mathbf{p}_0$.

If we consider the stratum α^2 in the space of real matrices (all entries of the matrix $\mathbf{A}$ are parameters), then approximations of the matrix $\mathbf{A}_{\min} \in \alpha^2$ nearest to the matrix $\mathbf{A}_0 = \mathbf{A}(0.3, 9.1)$ and the corresponding Jordan chain are found by (3.18) and (3.17) as follows

$$\mathbf{A}_{\min} = \begin{pmatrix} 0.9774 & 3.0219 & -0.0065 \\ 0.2886 & 1.0119 & 9.0962 \\ 2.0345 & 2.9654 & 1.0107 \end{pmatrix}, \quad [\widehat{\mathbf{u}}_1, \widehat{\mathbf{u}}_2] = \begin{pmatrix} 0.6962 & 0.1480 \\ -0.6961 & 0.0821 \\ 0.2119 & -0.1088 \end{pmatrix}.$$

In this case the distance to the stratum α^2 and the double eigenvalue of $\mathbf{A}_{\min}$ found by (3.16) are equal to $\|\mathbf{A}_{\min} - \mathbf{A}_0\| = 0.0618$ and $\lambda_{\min} = -2.0459$.

Near the origin $\mathbf{p} = 0$, which represents the stratum of higher codimension α^3, approximation (4.6) gives less accurate results. More generally, derived approximations are less accurate, when the distance to the stratum under consideration α^d is of the same magnitude as the distance to some less generic stratum (for example, $\alpha^{d'}$ for $d' > d$). The presence of a less generic stratum means that the point $\mathbf{p}_0$ is near the boundary of α^d. Near the boundary the size of the neighborhood, where transformation to the versal deformation (3.22) holds, decreases. Nevertheless, we always get a good approximation of the least generic stratum lying near $\mathbf{p}_0$, since $\mathbf{p}_0$ is not near its boundary. This property is very useful, because using information on the least generic stratum near $\mathbf{p}_0$, we can determine all the bifurcation diagram in the vicinity [1, 2, 16]. On Figure 4.2 approximations (4.7) of the nearest point of α^2 are shown by solid arrows for different points $\mathbf{p}_0$ near the origin (α^3); for the same points approximations (3.12), (4.4) for the nearest point of the stratum α^3 are shown by dotted arrows. It can be seen from Figure 4.2, that though approximations of the points $\mathbf{p}_{\min} \in \alpha^2$ are less accurate for $\mathbf{p}_0$ near α^3, approximations of the points $\mathbf{p}_{\min} \in \alpha^3$ provide very good results. Having information on the point $\mathbf{p}_{\min} \in \alpha^3$ (triple eigenvalue and the corresponding Jordan chain), we can determine the bifurcation diagram locally, in particular, approximate the cusp singularity of the bifurcation diagram for the example under consideration [16].

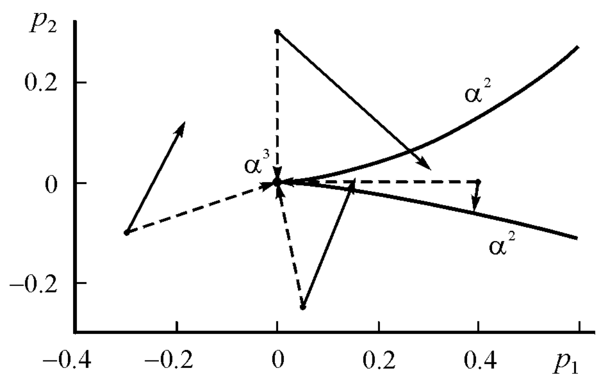

FIG. 4.2. *Behavior of the approximation in presence of a less generic stratum.*

4.4. Example 2. In this example we study accuracy of the approximations of strata and convergence of the method, if we want to find the nearest matrix with the prescribed Jordan structure.

Let us take a 10×10 real matrix $\mathbf{A}_d$ equal to the JNF having a nonderogatory eigenvalue $\lambda = 2$ with multiplicity 4 and simple eigenvalues $\lambda = -4, -3, -2, -1, 0, 4$. We consider perturbations of $\mathbf{A}_d$ in the form

$$\mathbf{A}_0 = \mathbf{A}_d + \mathbf{D},$$

where $\mathbf{D}$ is a 10×10 real normally distributed random matrix having the fixed norm $\|\mathbf{D}\| = 1/2$. Using the nearest to $\lambda = 2$ simple eigenvalues $\lambda_1, \ldots, \lambda_4$ of the matrix $\mathbf{A}_0$ in formulae (3.15), (3.18), we obtain the approximation $\mathbf{A}_{\min}$ of the nearest matrix of the stratum α^4 (in the space of real matrices) and the approximation of the distance from $\mathbf{A}_0$ to α^4 in the form $\operatorname{dist}(\mathbf{A}_0, \alpha^4) = \|\mathbf{A}_{\min} - \mathbf{A}_0\|$. The mean value of this distance, obtained after 100 numerical experiments with different random perturbations $\mathbf{D}$, is equal to 0.087. It is clear that the matrix $\mathbf{A}_{\min}$ should be different from $\mathbf{A}_d$ and the distance from $\mathbf{A}_0$ to $\mathbf{A}_{\min}$ should be much less than 1/2 (the distance to $\mathbf{A}_d$). Since the stratum α^4 has codimension 3, the perturbation $\mathbf{D}$ has, in the average, 3 elements in the normal direction to α^4, while other 97 elements determine perturbations parallel to α^4. Hence, the mean value of the distance from $\mathbf{A}_0$ to α^4 should be $\sqrt{0.03}\|\mathbf{D}\| \approx 0.087$, which confirms correctness of the above results.

To determine accuracy of the approximation $\mathbf{A}_{\min}$, we can find a distance from $\mathbf{A}_{\min}$ to α^4 by taking $\mathbf{A}_0' = \mathbf{A}_{\min}$ as a new starting point. As a result, we obtain the distance $\operatorname{dist}(\mathbf{A}_0', \alpha^4) = \|\mathbf{A}_{\min}' - \mathbf{A}_0'\|$ giving the error of the approximation $\mathbf{A}_{\min}$. Now we can take $\mathbf{A}_0'' = \mathbf{A}_{\min}'$ as a new starting point etc. This sequence of matrices converges to a point of α^4 very fast. The results of such iterations averaged over 100 experiments are shown in the first row of Table 4.1. It turns out that after the third iteration we get a matrix with a nonderogatory real eigenvalue of multiplicity 4 with the accuracy about $8.3 * 10^{-15}$ (the accuracy for the ith iteration is given in the $(i+1)$th column).

Similar calculations were carried out for the stratum $(\alpha \pm i\omega)^4$, where a 10×10 real matrix $\mathbf{A}_d$ with a pair of nonderogatory eigenvalues $\lambda = 1 \pm 2i$ of multiplicity 4 and simple eigenvalues $\lambda = -3, -1$ was considered. The results for 4 iterations averaged over 100 random perturbations $\mathbf{D}$, $\|\mathbf{D}\| = 1/2$, are given in the second row of Table 4.1. Here we calculated matrices $\mathbf{A}_{\min} \in (\alpha \pm i\omega)^4$ having a pair of complex

TABLE 4.1
Iterative calculations of matrices of the strata α^4 and $(\alpha \pm i\omega)^4$. Distances to the strata for 4 iterations averaged over 100 experiments with different perturbations $\mathbf{A}_0 = \mathbf{A}_d + \mathbf{D}$.

	1st iteration	2nd iteration	3rd iteration	4th iteration
$\mathrm{dist}(\mathbf{A}_0, \alpha^4)$	$8.7 * 10^{-2}$	$2.3 * 10^{-3}$	$4.1 * 10^{-7}$	$8.3 * 10^{-15}$
$\mathrm{dist}(\mathbf{A}_0, (\alpha \pm i\omega)^4)$	$1.2 * 10^{-1}$	$2.8 * 10^{-3}$	$3.4 * 10^{-7}$	$3.4 * 10^{-15}$

conjugate nonderogatory eigenvalues of multiplicity 4 using expressions (4.2). One can see that the convergence is very fast and 3 iterations give the matrix of $(\alpha \pm i\omega)^4$ with the accuracy about $3.4 * 10^{-15}$.

5. Approximation of the stratum λ^d from a nongeneric point $\mathbf{p}_0$. In this section the case, when eigenvalues $\lambda_1, \dots, \lambda_d$ of the matrix $\mathbf{A}_0$ are not distinct, is considered. This is the nongeneric case and, hence, it is less important from the practical point of view. Nevertheless, we should study this case to complete the method for approximating matrices with nonderogatory eigenvalues. Results of this section can be useful, when we can not distinguish some of the eigenvalues among $\lambda_1, \dots, \lambda_d$ and, therefore, have to treat some of them as multiple. We consider only complex matrix families (the stratum λ^d in the space of complex parameters C^n). Approximations for real matrix families can be easily derived from the complex case, as it was done in section 4. Since we use the same considerations based on the versal deformation theory, only the main steps of the analysis will be described without going much into details.

Let $\widehat{\lambda}$ be a nonderogatory multiple eigenvalue of the matrix $\widehat{\mathbf{A}} = \mathbf{A}(\widehat{\mathbf{p}})$ with multiplicity d and let $\lambda_1, \dots, \lambda_d$ be eigenvalues of the matrix $\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0)$ such that $\lambda_i \to \widehat{\lambda}$ as $\mathbf{p}_0 \to \widehat{\mathbf{p}}$. According to the singularity theory [1, 2], for points $\mathbf{p}_0$ in the vicinity of $\widehat{\mathbf{p}}$ eigenvalues $\lambda_1, \dots, \lambda_d$ are simple or nonderogatory multiple. Let us denote different eigenvalues among $\lambda_1, \dots, \lambda_d$ by $\lambda'_1, \dots, \lambda'_K$ and their multiplicities by $d_1 \geq \dots \geq d_K$, such that $\lambda_{i_s+1} = \cdots = \lambda_{i_s+d_s} = \lambda'_s$, $s = 1, \dots, K$, where $i_1 = 0$, $i_2 = d_1$, ... , $i_K = d_1 + \dots + d_{K-1}$; $d = d_1 + \cdots + d_K$. The Jordan structure of $\mathbf{A}_0$ corresponding to the eigenvalues $\lambda_1, \dots, \lambda_d$ is described by the $d \times d$ matrix

$$
\mathbf{J}_0 = \begin{pmatrix} \mathbf{J}_0^1 & & \\ & \ddots & \\ & & \mathbf{J}_0^K \end{pmatrix}, \quad \mathbf{J}_0^s = \begin{pmatrix} \lambda'_s & 1 & & \\ & \lambda'_s & \ddots & \\ & & \ddots & 1 \\ & & & \lambda'_s \end{pmatrix}, \tag{5.1}
$$

where $\mathbf{J}_0^s$ is the $d_s \times d_s$ Jordan block corresponding to the eigenvalue λ'_s. The $m \times d$ matrices $\mathbf{U} = [\mathbf{u}_1, \dots, \mathbf{u}_d]$ and $\mathbf{V} = [\mathbf{v}_1, \dots, \mathbf{v}_d]$, satisfying equations (3.6) with the matrix $\mathbf{J}_0$ from (5.1), determine the right and left Jordan chains $\mathbf{u}_{i_s+1}, \dots, \mathbf{u}_{i_s+d_s}$ and $\mathbf{v}_{i_s+d_s}, \dots, \mathbf{v}_{i_s+1}$ corresponding to λ'_s.

The matrix family $\mathbf{A}(\mathbf{p})$ in the vicinity of the point $\widehat{\mathbf{p}} \in \lambda^d$ can be expressed in form (3.22), where $\mathbf{A}'(\mathbf{q}')$ is the block of versal deformation (3.20) corresponding to the nonderogatory multiple eigenvalue $\widehat{\lambda}$ of multiplicity d. We assume that the point $\mathbf{p}_0$ belongs to the vicinity of $\widehat{\mathbf{p}}$, where (3.22) holds. The matrix $\mathbf{A}'_0 = \mathbf{A}'(\mathbf{q}'(\mathbf{p}_0))$ has the eigenvalues $\lambda_1, \dots, \lambda_d$ and, hence, it is determined by expressions (3.20), (3.29). Finding Jordan chains corresponding to the eigenvalues $\lambda'_1, \dots, \lambda'_K$ of $\mathbf{A}'_0$, we determine a $d \times d$ matrix $\mathbf{R}$ with the elements

$$(5.2)\quad r_{ij} = \left\{ \begin{array}{ll} 0, & i < t, \\ C_{i-1}^{t-1}\delta_j^{i-t}, & i \geq t, \end{array} \right. , \quad C_i^t = \frac{i!}{t!(i-t)!}, \quad i_s < j = i_s + t \leq i_s + d_s,$$

where $\delta_j = \lambda_j - \lambda_0$, $\lambda_0 = (\lambda_1 + \cdots + \lambda_d)/d = (d_1\lambda'_1 + \cdots + d_K\lambda'_K)/d$. The matrix $\mathbf{R}$ satisfies the equation $\mathbf{A}'_0\mathbf{R} = \mathbf{R}\mathbf{J}_0$ and the matrix $\mathbf{A}'_0$ is expressed in the form (3.31), where $\mathbf{S} = \mathbf{R}^{-1}$ for the new matrix $\mathbf{R}$ (5.2). Using (3.31), we can rewrite expressions (3.29) for the quantities q_j^0 as follows

$$(5.3)\qquad \widehat{\lambda} + q_d^0 = \lambda_0, \quad q_j^0 = \mathbf{R}_{<d>}\mathbf{J}_0\mathbf{S}^{<j>}, \quad j = 1, \ldots, d-1.$$

The matrix $\mathbf{C}'_0$ can be found from (3.33) as follows

$$(5.4)\qquad \mathbf{C}'_0 = \mathbf{UKS}, \quad \mathbf{K} \in \operatorname{cent} \mathbf{J}_0,$$

where $\mathbf{K} \in \operatorname{cent} \mathbf{J}_0$ is a $d \times d$ matrix commuting with $\mathbf{J}_0$ and having the form [11]

$$(5.5)\quad \mathbf{K} = \begin{pmatrix} \mathbf{K}_1 & & \\ & \ddots & \\ & & \mathbf{K}_K \end{pmatrix}, \quad \mathbf{K}_s = \begin{pmatrix} k_{i_s+1} & k_{i_s+2} & \ddots & k_{i_s+d_s} \\ 0 & k_{i_s+1} & \ddots & \ddots \\ 0 & 0 & \ddots & k_{i_s+2} \\ 0 & 0 & 0 & k_{i_s+1} \end{pmatrix}.$$

The $d_s \times d_s$ matrix $\mathbf{K}_s$ has equal numbers k_{i_s+j} on the jth upper diagonal and zeros below the main diagonal.

For the first order terms $\Delta\mathbf{A}'$ and $\Delta\mathbf{C}'$ (3.36) we have equation (3.38). Taking the sum of (i,j)th elements of (3.38), where $i = j + d_s - t$, $j = i_s + 1, \ldots, i_s + t$, and using (3.20), (5.1), (5.5), (5.10), the left-hand-side vanishes and we obtain the equation

$$(5.6)\qquad \operatorname{trace}(\mathbf{S}\Delta\mathbf{A}'\mathbf{R}\mathbf{T}^T) = \operatorname{trace}(\mathbf{K}^{-1}\mathbf{V}^T\Delta\mathbf{A}\mathbf{U}\mathbf{K}\mathbf{T}^T),$$

where $\mathbf{T}$ is a $d \times d$ matrix having units on the (i,j)th positions for $i = j + d_s - t$, $j = i_s + 1, \ldots, i_s + t$, and zeros on the other places. Since the matrices $\mathbf{T}^T$ and $\mathbf{K}$ commute for any $\mathbf{K} \in \operatorname{cent} \mathbf{J}_0$, we have

$$(5.7)\qquad \operatorname{trace}(\mathbf{K}^{-1}\mathbf{V}^T\Delta\mathbf{A}\mathbf{U}\mathbf{K}\mathbf{T}^T) = \operatorname{trace}(\mathbf{V}^T\Delta\mathbf{A}\mathbf{U}\mathbf{T}^T).$$

With the use of expression (5.7) equation (5.6) takes the form

$$(5.8)\qquad \sum_{l=1}^{d} f_{zl}\nabla q_l(\mathbf{p} - \mathbf{p}_0)^T = \mathbf{m}_z(\mathbf{p} - \mathbf{p}_0)^T,$$

where $z = i_s + t \in \{1, \ldots, d\}$. Elements of the vectors $\mathbf{m}_z = (m_z^1, \ldots, m_z^n)$ and the scalars f_{zl} are as follows

$$m_z^l = \sum_{j=i_s+1}^{z} \mathbf{v}_{j+d_s-t}^T \frac{\partial \mathbf{A}}{\partial p_l}\mathbf{u}_j,$$

$$(5.9)\qquad f_{zl} = \left\{ \begin{array}{ll} \displaystyle\sum_{j=i_s+1}^{z} s_{(j+d_s-t)d}\, r_{lj}, & i_s < z = i_s + t \leq i_s + d_s,\ l \neq d, \\ 0, & i_s < z = i_s + t < i_s + d_s,\ l = d, \\ d_s, & z = i_s + d_s,\ l = d, \end{array} \right.$$

where derivatives are evaluated at $\mathbf{p}_0$.

Taking the sum of equations (5.8) for $z = i_s + d_s$, $s = 1, \ldots, K$, we obtain

$$\Delta\lambda = \nabla q_d(\mathbf{p} - \mathbf{p}_0)^T = \mathbf{m}(\mathbf{p} - \mathbf{p}_0)^T, \tag{5.10}$$

where

$$\mathbf{m} = (m^1, \ldots, m^n), \quad m^l = \frac{1}{d}\sum_{j=1}^{d} \mathbf{v}_j^T \frac{\partial \mathbf{A}}{\partial p_l}\mathbf{u}_j. \tag{5.11}$$

Expression (5.10) determines the approximation of a multiple eigenvalue λ on the stratum λ^d in the form

$$\lambda = \lambda_0 + \Delta\lambda. \tag{5.12}$$

If we substitute the value of $\nabla q_d(\mathbf{p} - \mathbf{p}_0)^T$ from (5.10) into (5.8) and omit the equation for $z = d$, which is now linearly dependent on the others, we find a system of $d-1$ linear equations determining the quantities $\nabla q_j(\mathbf{p} - \mathbf{p}_0)^T$, $j = 1, \ldots, d-1$, as follows

$$\sum_{j=1}^{d-1} f_{ij}\nabla q_j(\mathbf{p} - \mathbf{p}_0)^T = (\mathbf{m}_i - f_{id}\mathbf{m})(\mathbf{p} - \mathbf{p}_0)^T, \quad i = 1, \ldots, d-1. \tag{5.13}$$

Solving (5.13) with respect to $\nabla q_j(\mathbf{p} - \mathbf{p}_0)^T$, we find

$$\nabla q_j(\mathbf{p} - \mathbf{p}_0)^T = \sum_{i=1}^{d-1} h_{ji}(\mathbf{m}_i - f_{id}\mathbf{m})(\mathbf{p} - \mathbf{p}_0)^T, \quad j = 1, \ldots, d-1, \tag{5.14}$$

where $\mathbf{H} = \mathbf{F}^{-1}$ with $\mathbf{H}$ and $\mathbf{F}$ being $(d-1) \times (d-1)$ matrices, whose elements are h_{ij} and f_{ij}, $i, j = 1, \ldots, d-1$.

Expressions (5.3) and (5.14) determine system of $d-1$ linear equations

$$q_j^0 + \nabla q_j(\mathbf{p} - \mathbf{p}_0)^T = 0, \quad j = 1, \ldots, d-1, \tag{5.15}$$

which is the linear approximation of the stratum λ^d.

Finally, we need to find the approximation of the Jordan chain $\widehat{\mathbf{U}}$ on the stratum λ^d. All the Jordan chains $\widehat{\mathbf{U}}$ form the plane $\widehat{P}$ (3.23), and all the matrices $\mathbf{C}'_0$ (5.4) form the plane

$$P = \{\mathbf{UKS} : \ \mathbf{K} \in \operatorname{cent} \mathbf{J}_0\}. \tag{5.16}$$

To get a better approximation of $\widehat{\mathbf{U}} = \mathbf{C}'(\widehat{\mathbf{p}})$ we should take the matrix $\mathbf{C}'_0 \in P$, which is close to $\widehat{P}$. The matrix $\mathbf{C}'_0\mathbf{R} = \mathbf{UK}$ converges to the matrix $\mathbf{C}'(\widehat{\mathbf{p}})\widehat{\mathbf{R}}$ as $\mathbf{p}_0 \to \widehat{\mathbf{p}}$, where nonzero elements of the $d \times d$ matrix $\widehat{\mathbf{R}}$ are $\widehat{r}_{t(i_s+t)} = 1$, $t = 1, \ldots, d_s$, $s = 1, \ldots, K$; see expression (5.2) for elements of the matrix $\mathbf{R}$. Hence, to get the matrix $\mathbf{C}'_0$ close to $\widehat{P}$ we need to choose the matrix $\mathbf{K} \in \operatorname{cent}\mathbf{J}_0$ such that the columns $i_1 + t$, $i_2 + t, \ldots$ (except the cases $i_s + t > i_s + d_s$) of the matrix $\mathbf{UK}$ are close to each other. These columns converge to $\widehat{\mathbf{u}}_t$ as $\mathbf{p}_0 \to \widehat{\mathbf{p}}$. The appropriate matrix $\mathbf{K}$ can be chosen in the form (5.5), where

$$\begin{gathered} k_{i_s+1} = 1\big/(\mathbf{u}_{i_s+1}^T\overline{\mathbf{u}}'), \quad \mathbf{u}' = \mathbf{u}_1\big/\sqrt{\mathbf{u}_1^T\overline{\mathbf{u}}_1}, \\ k_{i_s+t} = -\sum_{z=2}^{t} k_{i_s+t+1-z}\mathbf{u}_{i_s+z}^T\overline{\mathbf{u}}'\Big/(\mathbf{u}_{i_s+1}^T\overline{\mathbf{u}}'), \ t = 2, \ldots, d_s, \ s = 1, \ldots, K. \end{gathered} \tag{5.17}$$

Expressions (5.17) mean that the vectors $k_{i_s+1}\mathbf{u}_{i_s+1} = \mathbf{UK}^{<i_s+1>}$ have the unit projection on the direction $\mathbf{u}'$, while the vectors $\mathbf{UK}^{<i_s+t>}$, $t = 2, \dots, d_s$, have the zero projection on this direction.

The matrix $\Delta\mathbf{C}'$ is determined by equation (3.47), which can be written for each column in the form

$$\begin{gathered}(\mathbf{A}_0 - \lambda'_s\mathbf{I})(\Delta\mathbf{C}'\mathbf{R})^{<i_s+1>} = (\mathbf{C}'_0\Delta\mathbf{A}'\mathbf{R} - \Delta\mathbf{AUK})^{<i_s+1>},\\ (\mathbf{A}_0 - \lambda'_s\mathbf{I})(\Delta\mathbf{C}'\mathbf{R})^{<t>} - (\Delta\mathbf{C}'\mathbf{R})^{<t-1>} = (\mathbf{C}'_0\Delta\mathbf{A}'\mathbf{R} - \Delta\mathbf{AUK})^{<t>},\\ t = i_s + 2, \dots, i_s + d_s, \quad s = 1, \dots, K.\end{gathered} \tag{5.18}$$

A solution of system (5.18) exists for the points $\mathbf{p}$ on the plane (5.15) approximating λ^d. A general form of this solution is as follows [27]

$$\begin{gathered}\Delta\mathbf{C}' = (\mathbf{W} + \mathbf{UK}')\mathbf{S}, \quad \mathbf{K}' \in \mathrm{cent}\mathbf{J}_0,\\ \mathbf{W}^{<i_s+1>} = \mathbf{Z}_s^{-1}(\mathbf{C}'_0\Delta\mathbf{A}'\mathbf{R} - \Delta\mathbf{AUK})^{<i_s+1>},\\ \mathbf{W}^{<t>} = \mathbf{Z}_s^{-1}\big[(\mathbf{C}'_0\Delta\mathbf{A}'\mathbf{R} - \Delta\mathbf{AUK})^{<t>} + \mathbf{W}^{<t-1>}\big],\\ \mathbf{Z}_s = \mathbf{A}_0 - \lambda'_s\mathbf{I} - \overline{\mathbf{v}}_{i_s+d_s}\mathbf{v}^T_{i_s+1},\\ t = i_s + 2, \dots, i_s + d_s, \quad s = 1, \dots, K,\end{gathered} \tag{5.19}$$

where $\mathbf{Z}_s$ is a nonsingular matrix ($\mathbf{v}_{i_s+d_s}$ is the eigenvector corresponding to λ'_s and $\mathbf{v}_{i_s+1}$ is the last vector in the Jordan chain of λ'_s); $\mathbf{K}'$ is an arbitrary matrix of the form (5.5).

In the vicinity of λ^d the matrix family $\mathbf{C}'(\mathbf{p})$ is uniquely determined on the surface T' (3.45) in the space of $m \times d$ matrices. The condition $\mathbf{C}'(\mathbf{p}) \in T'$ leads to equations (3.46). Substituting (5.19) into (3.46), we find the matrix $\mathbf{K}'$ in the form (5.5) with the elements

$$\begin{gathered}k'_{i_s+1} = -\mathbf{v}^T_{i_s+d_s}\mathbf{WS}^{<d>}/s_{(i_s+d_s)d},\\ k'_{i_s+t} = -\Big[\mathbf{v}^T_{i_s+d_s-t+1}\mathbf{WS}^{<d>} + \sum_{z=1}^{t-1} k'_{i_s+z}s_{(i_s+d_s-t+z)d}\Big]/s_{(i_s+d_s)d},\\ t = 2, \dots, d_s,\end{gathered} \tag{5.20}$$

where s_{ij} are elements of the matrix $\mathbf{S}$. Expressions (5.4), (5.17), (5.19), and (5.20) give the first order approximation of the Jordan chain $\widehat{\mathbf{U}} = \mathbf{C}'_0 + \Delta\mathbf{C}'$ on the stratum λ^d.

The results of this section can be represented in the form of a theorem.

THEOREM 5.1. *Let $\lambda'_1, \dots, \lambda'_K$ be simple or nonderogatory multiple eigenvalues of the matrix $\mathbf{A}_0$ with multiplicities $d_1, \dots, d_K$ respectively; $d = d_1 + \dots + d_K$. Then the first order approximation of the stratum λ^d (where the eigenvalues $\lambda'_1, \dots, \lambda'_K$ form a nonderogatory eigenvalue λ of multiplicity d) is given by the system of $d-1$ linear equations (5.15) with q^0_j and ∇q_j determined in (5.1)–(5.3), (5.11), (5.9), and (5.14). The first order approximations of the multiple eigenvalue λ and the corresponding Jordan chain $\widehat{\mathbf{U}} = \mathbf{C}'_0 + \Delta\mathbf{C}'$ on the stratum λ^d are given by (5.10)–(5.12) and (5.4), (5.5), (5.17), (5.19), (5.20).*

COROLLARY 5.2. *Under the conditions of Theorem 5.1 the first order approximation of the point $\mathbf{p}_{\min} \in \lambda^d$ nearest to $\mathbf{p}_0$ (a distance is measured by the Euclidean norm) has the form*

$$\mathbf{p}_{\min} = \mathbf{p}_0 + \mathbf{x}^T(\overline{\mathbf{Q}}\mathbf{Q}^T)^{-1}\overline{\mathbf{Q}}, \tag{5.21}$$

where the matrix $\mathbf{Q}$ *and the vector* $\mathbf{x}$ *are determined in (3.12) with the quantities* q_j^0 *and the vectors* ∇q_j *determined by Theorem 5.1.*

Note that Theorem 5.1 gives local approximations of λ^d, multiple eigenvalue λ, and the Jordan chain $\widehat{\mathbf{U}}$ even if $\mathbf{p}_0$ belongs to λ^d. Theorem 3.1 is a special case of Theorem 5.1, when all the eigenvalues $\lambda'_1, \dots, \lambda'_K$ are simple. Similar results for real matrix families can be easily obtained from Theorem 5.1 in the same way, as it was done in section 4.

5.1. Example. Let us consider matrix family (4.5). At the point $\mathbf{p}_0 = (2/3, 1/3)$ the matrix $\mathbf{A}_0$ has a double nonderogatory eigenvalue $\lambda_1 = \lambda_2 = \lambda'_1 = 0$ and a simple eigenvalue $\lambda_3 = \lambda'_2 = 3$ (multiplicities are $d_1 = 2$ and $d_2 = 1$). Hence, $\mathbf{p}_0$ is a nongeneric point of the parameter space. Let us find the approximation of the stratum λ^3 using information at $\mathbf{p}_0$. The matrices $\mathbf{J}_0$ and $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$, $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ in this case have the form

$$\mathbf{J}_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \ \mathbf{U} = \begin{pmatrix} -1 & -1 & 1 \\ 1/3 & 0 & 2/3 \\ 1 & 3 & 2 \end{pmatrix}, \ \mathbf{V} = \begin{pmatrix} -2/3 & 0 & 1/3 \\ 5/3 & -1 & 2/3 \\ -2/9 & 1/3 & 1/9 \end{pmatrix},$$

where $\mathbf{u}_1$, $\mathbf{u}_3$ and $\mathbf{v}_2$, $\mathbf{v}_3$ are right and left eigenvectors of λ'_1, λ'_2; $\mathbf{u}_2$ and $\mathbf{v}_1$ are right and left associated vectors corresponding to λ'_1. The matrices $\mathbf{R}$, $\mathbf{S}$, and $\mathbf{K}$ have the form

$$\mathbf{R} = \begin{pmatrix} 1 & 0 & 1 \\ \delta_1 & 1 & \delta_3 \\ \delta_1^2 & 2\delta_2 & \delta_3^2 \end{pmatrix}, \ \mathbf{S} = \mathbf{R}^{-1}, \ \mathbf{K} = \begin{pmatrix} k_1 & k_2 & 0 \\ 0 & k_1 & 0 \\ 0 & 0 & k_3 \end{pmatrix},$$

where $\delta_1 = \delta_2 = \lambda'_1 - \lambda_0 = -1$, $\delta_3 = \lambda'_2 - \lambda_0 = 2$, $\lambda_0 = (2\lambda'_1 + \lambda'_2)/3 = 1$. Using expressions (5.3), (5.11), (5.9), and (5.14), we find two equations (5.15) for the approximation of the stratum λ^3 as follows

$$\begin{aligned} 2 + 6(p_2 - 1/3) &= 0, \\ 3 + 3(p_1 - 2/3) + 3(p_2 - 1/3) &= 0. \end{aligned} \tag{5.22}$$

System (5.22) have a solution $\widehat{\mathbf{p}} = (0, 0)$, which is the approximation of λ^3. The triple nonderogatory eigenvalue λ and the Jordan chain $\widehat{\mathbf{U}} = [\widehat{\mathbf{u}}_1, \widehat{\mathbf{u}}_2, \widehat{\mathbf{u}}_3]$ at $\widehat{\mathbf{p}}$ calculated by formulae (5.10)–(5.12) and (5.4), (5.17), (5.19), (5.20) have the form

$$\lambda = 1, \quad \widehat{\mathbf{U}} = \frac{1}{209}\begin{pmatrix} 0 & 189 & 1 \\ 0 & 0 & 63 \\ 378 & 191 & -20 \end{pmatrix}. \tag{5.23}$$

Direct calculations show that the point $\widehat{\mathbf{p}} = (0, 0)$, quantity $\lambda = 1$, and the matrix $\widehat{\mathbf{U}}$ (5.23) represent the exact stratum λ^3, multiple eigenvalue, and corresponding Jordan chain. The reason for such accuracy is that $\mathbf{q}'(\mathbf{p})$ and $\mathbf{C}'(\mathbf{p})$ are linear functions determined in the whole parameter space

$$\mathbf{A}(\mathbf{p})\mathbf{C}'(\mathbf{p}) = \mathbf{C}'(\mathbf{p})\mathbf{A}'(\mathbf{q}'(\mathbf{p})), \quad \mathbf{q}'(\mathbf{p}) = (q_1, q_2, q_3) = (6p_2, 3(p_1 + p_2), 0),$$

$$\mathbf{C}'(\mathbf{p}) = \begin{pmatrix} 6p_2 & 1 + 2(p_1 + p_2) & 1 \\ 2p_2 & p1 + 3p_2 & (1 + 2(p_1 + p_2))/3 \\ 2(1 - p_1 + 2p_2) & 3 + 2(p_1 + p_2) & 3 \end{pmatrix},$$

and Newton's method, whose idea was used for finding the approximations, gives exact solutions for linear functions.

6. Conclusion. In this paper a practical numerical method of finding nonderogatory multiple eigenvalues and corresponding Jordan chains of complex and real matrices dependent on several parameters is developed. Similar problem in the space of all matrices is considered as a special case. The advantage of the suggested approach is that given a point $\mathbf{p}_0$ in the parameter space we can approximate the nearest point $\mathbf{p}_{\min} \in \lambda^d$, where the matrix $\mathbf{A}(\mathbf{p}_{\min})$ has a nonderogatory eigenvalue with multiplicity d, using only information at $\mathbf{p}_0$. Moreover, we can determine the geometry of the stratum λ^d in the parameter space and find approximations of the multiple eigenvalue and the corresponding Jordan chain at points of this stratum. The approach is based on the versal deformation theory. As a result, it gives quantitative information about the stratified structure of the parameter space near the point $\mathbf{p}_0$, which can not be achieved by other methods. This information allows studying multiple eigenvalues of special form, like eigenvalues with a given real part or absolute value, which is important in stability problems.

Multiple eigenvalues play significant role in problems of stability and dynamics, since they are associated with the singular behavior of a system. Therefore, it is important to know whether the system matrix has multiple eigenvalues. Assuming that the vector of parameters $\mathbf{p}_0$ is given with the accuracy ε, we need to find a point $\mathbf{p}$ in the ε-vicinity of $\mathbf{p}_0$ such that the matrix $\mathbf{A}(\mathbf{p})$ has an eigenvalue of highest multiplicity (more correctly, the matrix with the least generic Jordan structure). This represents a very important problem of numerical calculation of multiple eigenvalues, since the least generic Jordan structure determines local characteristics of the spectrum [1, 2]. The results of this paper solve this problem in the part concerning nonderogatory eigenvalues, i.e., we can approximate a point $\mathbf{p}$, $\|\mathbf{p}-\mathbf{p}_0\| < \varepsilon$, where the matrix $\mathbf{A}(\mathbf{p})$ has a nonderogatory eigenvalue of the highest multiplicity.

Analysis of derogatory eigenvalues of matrices $\mathbf{A}(\mathbf{p})$ lead to additional difficulties, which require modification of the method. These difficulties are associated with the fact that a versal deformation is not *universal* in the derogatory case [1, 2]. This means that the function $\mathbf{q}'(\mathbf{p})$ is not uniquely determined by the matrix family $\mathbf{A}(\mathbf{p})$. As a result, the matrices $\mathbf{A}'_0$ and $\Delta\mathbf{A}'$ can not be found only from the analysis of the equation $\mathbf{A}(\mathbf{p})\mathbf{C}'(\mathbf{p}) = \mathbf{C}'(\mathbf{p})\mathbf{A}'(\mathbf{q}'(\mathbf{p}))$. Hence, it is necessary to use some additional considerations on the geometry of the stratification near points $\widehat{\mathbf{p}}$, where derogatory eigenvalues exist. This will be done in the next paper.

Application of the versal deformation theory to calculation of nongeneric normal forms can be carried out in the same way for special types of matrices such as Hamiltonian or reversible matrices. Using versal deformations of these matrices in the corresponding matrix spaces, one can derive approximations of strata, multiple nonderogatory eigenvalues, and corresponding Jordan chains specified for different types of matrices. These approximations can be useful for the analysis of stability and dynamics of special types of mechanical and physical systems. Similarly, versal deformations can be used for calculation of the nongeneric Kronecker canonical form of a matrix pencil.

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