Bundle Bispectrality for Matrix Differential Equations

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Abstract

We consider the fundamental solutions of a wide class of first order systems with polynomial dependence on the spectral parameter and rational matrix potentials. Such matrix potentials are rational solutions of a large class of integrable nonlinear equations, which play an important role in different mathematical physics problems. The concept of bispectrality, which was originally introduced by Grünbaum, is extended in a natural way for the systems under consideration and their bispectrality is derived via the representation of the fundamental solutions. This bispectrality is preserved under the flows of the corresponding integrable nonlinear equations. For the case of Dirac type (canonical) systems the complete characterization of the bispectral potentials under consideration is obtained in terms of the system's spectral function.

1 Introduction

The striking characteristic of certain aspects of operator theory is its deep connections with other apparently distant mathematical fields. An example of such connections is provided by time and band limiting operators [BGK97], [GKW93], [Sle83]. The present work has one of its roots in the works of D. Slepian and collaborators motivated by applications to communication theory [SP61], [Sle64], [Sle78]. The simplest instance of the problem that was investigated by D. Slepian concerns the problem of determining in an effective way the eigenvalues and eigenfunctions of the time and band limiting operator:

$$f(x) \longmapsto \int_{-T}^{T} \frac{\sin(\Omega(x-s))}{x-s} f(s)\, ds \ .$$

Here, $2T$ is the size of the time interval and 2Ω is the size of the frequency interval. The question resurfaced in extensions developed by A. Grünbaum stimulated by problems in computerized tomography. For a historical description of this problem the reader is referred to [Sle83], [Grü82], [Zub97] and references therein. The time and band limiting problem turned out to have fundamental connections with the main object of the present article, which is the so-called *bispectral problem* [Grü84], [Grü92], [Grü94]. The latter goes as follows:

When does an ordinary differential operator $L(x, \partial_x)$ admit a family of eigenfunctions $\varphi(x, \lambda)$ that also satisfies a differential equation in the spectral parameter λ of the form

$$B(\lambda, \partial_\lambda)\varphi(x, \lambda) = \Theta(x)\varphi(x, \lambda) \ , \tag{1.1}$$

where $\Theta(x)$ is independent of λ and B is a positive order ordinary differential operator independent of x. In other words, φ satisfies both (1.1) and

$$L(x, \partial_x)\varphi(x, \lambda) = \lambda\varphi(x, \lambda) \ . \tag{1.2}$$

The presence of two spectral problems, one with spectral variable λ and the other with (bi)spectral variable $\Theta(x)$, leads to the terminology *bispectral problem.*

The bispectral problem for the scalar Schrödinger operator case $L = -\partial_x^2 + u$ was completely characterized by Duistermaat and Grünbaum [DG86]. One of the directions towards which the bispectral property was explored was that of systems, such as those studied by Zakharov-Shabat [ZS71] and Ablowitz-Kaup-Newel-Segur [AKNS74] as well as more general matrix differential operators. This was pursued, for example, in [Zub90], [Zub92b], [BHY96].

The bispectral problem presented very surprising connections with the field of integrable systems and soliton theory. Besides some elementary examples, it turned out that the class of bispectral Schrödinger potentials included the *rational solutions* of the Korteweg-de Vries (KdV) hierarchy [AS81], [AMM77], [AM78], [CC77], which play an important role in soliton theory. Such rational functions are limiting cases of N-soliton potentials, which in turn are limiting cases of algebraic-geometric solutions of the Kadomtsev-Petviashvili hierarchy [NMPZ84]. The connections between soliton theory and the bispectral problem did not stop here. Besides the rational solutions of the KdV, the remaining bispectral Schrödinger potentials showed an even deeper connection with infinite dimensional integrable systems. In an nutshell, they turned out to be rational solutions of the master symmetries of the KdV hierarchy. See [ZM91], [ZV00].

Although the scalar bispectral differential operators have been thoroughly investigated, the picture for matrix rational coefficients seems to be much less understood. See [HK98] and references therein for a recent account.

In this paper we shall consider systems with polynomial dependence on the spectral parameter λ:

$$\partial_x w(x,\lambda) = G(x,\lambda)w(x,\lambda) , \tag{1.3}$$

$$G(x,\lambda) = -\sum_{k=0}^{r} \lambda^k q_k(x) . \tag{1.4}$$

The function w in (1.3) is an absolutely continuous $m \times p$ matrix valued function. It can be either a fundamental solution or a vector function, in particular. Many systems auxiliary to important integrable nonlinear equations have this form. For systems of the form (1.3), we shall present large classes of rational coefficients where different forms of bispectrality manifest itself and construct the corresponding fundamental solutions of the form $\tilde{w}(x,\lambda) = w_A(x,\lambda)e^{ix\lambda^r d}$, where d is a diagonal matrix and w_A is rational both in x and λ. After taking into account the additional time variable t, we show that such rational potentials satisfy different integrable nonlinear equations.

The main objective of this paper is to explore the bispectral property of system (1.3) in case $r = 1$ and to generalize it in a natural way for $r > 1$. Another goal is to get a better understanding of the close connection between the spectral and bispectral problems. In the case of the canonical system (self-adjoint ZS-AKNS operator) our rational potentials belong to the so called pseudo-exponential class studied in [GKS98a] and we use the results of [GKS98a] to characterize the spectral functions associated to the canonical systems that correspond to such potentials.

Our fundamental tool is one of the generalizations of the classical Darboux transformation. The well-known Darboux transformations are widely used to construct the solutions of nonlinear integrable equations. See [Miu76], [AS81], [Mar88] [MS91] and references therein. One of the crucial ingredients in proving that a certain operator is bispectral was the possibility of writing its eigenfunctions in a fairly explicit way making use of the Darboux transformations [DG86], [Zub90]. The theory of the *generalized Bäcklund-Darboux transformation* (GBDT) that we apply here was worked out in the papers [Sak93]-[Sak98]. It is based on the representation of the gauge transformation in the form of a transfer matrix function, which is a familiar tool in control theory and dynamical system theory. It uses also the "generalized" eigenfunctions, where arbitrary parameter matrices substitute the traditionally used eigenvalues.

The plan of this paper is the following:

Section 2 reviews the necessary results on GBDT. Section 3 specializes the construction of GBDT to the situation where a certain parameter matrix is nilpotent. This leads to rational coefficients in the systems (1.3). In Section 4 we show that they are bispectral. Section 5 is concerned with ZS-AKNS examples of bispectral systems and the spectral characterization of the corresponding subclass of canonical systems. Section 6 is concerned with examples of bispectral systems auxiliary to the N-wave problem. Finally, in Section 7 we close with a few suggestions for further research.

2 Preliminaries

Consider a system of the form (1.3) with polynomial dependence on the spectral parameter λ and $x \in \Gamma$, where the coefficients $q_k(x)$ are $m \times m$ matrix functions locally summable in Γ, where Γ denotes either $[0,\, c)$ or $(-c,\, c)$ or $(-c, 0]$ $(c \leq \infty)$. GBDT of the system (1.3) will be determined by three $n \times n$ matrices A_1, A_2, $S(0)$ ($\det S(0) \neq 0$), and two $n \times m$ matrices $\Pi_1(0)$ and $\Pi_2(0)$, satisfying the operator identity

$$A_1 S(0) - S(0) A_2 = \Pi_1(0)\Pi_2(0)^*. \tag{2.1}$$

Suppose such parameter matrices are fixed. Then, we can introduce the matrix functions $\Pi_1(x)$, $\Pi_2(x)$, and $S(x)$, with the values $\Pi_1(0)$, $\Pi_2(0)$ and $S(0)$ at $x = 0$ as the solutions of the linear differential equations

$$\Pi_1'(x) = \sum_{p=0}^{r} A_1^p \Pi_1(x) q_p(x), \quad \Pi_2'(x) = -\sum_{p=0}^{r} (A_2^*)^p \Pi_2(x) q_p(x)^*, \tag{2.2}$$

$$S'(x) = \sum_{p=1}^{r} \sum_{j=1}^{p} A_1^{p-j} \Pi_1(x) q_p(x) \Pi_2(x)^* A_2^{j-1}. \tag{2.3}$$

Notice that equations (2.2) and (2.3) are chosen in such a way that the identity

$$A_1 S(x) - S(x) A_2 = \Pi_1(x)\Pi_2(x)^* \tag{2.4}$$

follows from (2.1)-(2.3) for all $x \in \Gamma$; as could be verified by direct differentiation of both sides of (2.4).

Since $\det S(x) \not\equiv 0$, we can define the matrix function

$$w_A(x, \lambda) = I_m - \Pi_2(x)^* S(x)^{-1} (A_1 - \lambda I_n)^{-1} \Pi_1(x), \tag{2.5}$$

where I_m is the $m \times m$ identity matrix. The points of invertibility of the matrix S will be called *regular points* of S.

The matrix w_A is the so-called transfer matrix function, which is a well known tool in control theory. Matrices of the form (2.5) with the additional property (2.4) were introduced by L.Sakhnovich in the context of his method of operator identities [Sak76]-[Sak99]. From [Sak96] it follows:

Theorem 2.1 *Suppose that the matrix functions w, Π_1, Π_2, and S satisfy equations (1.3) and (2.1) - (2.3). Then, in the regular points of S we have that*

$$\tilde{w}(x,\lambda) = w_A(x,\lambda)w(x,\lambda) \tag{2.6}$$

satisfies the system

$$\tilde{w}'(x,\lambda) = \tilde{G}(x,\lambda)\tilde{w}(x,\lambda), \quad \tilde{G}(x,\lambda) = -\textstyle\sum_{k=0}^{r} \lambda^k \tilde{q}_k(x) \tag{2.7}$$

with coefficients

$$\tilde{q}_k(x) = q_k(x) - \sum_{p=k+1}^{r} (q_p(x)Y_{p-k-1}(x) - X_{p-k-1}(x)q_p(x) + \sum_{j=k+2}^{p} X_{p-j}(x)q_p(x)Y_{j-k-2}(x)), \tag{2.8}$$

where

$$X_k(x) = \Pi_2(x)^* S(x)^{-1} A_1^k \Pi_1(x), \qquad Y_k(x) = \Pi_2(x)^* A_2^k S(x)^{-1} \Pi_1(x). \tag{2.9}$$

The transformations (2.6) and (2.8) are called GBDT of the initial system (1.3). Notice that this construction makes use of "generalized eigenfunctions" Π_1 and Π_2^* with matrices A_1 and A_2 instead of eigenvalues a_1 and a_2.

Let us now add the time variable t in our considerations. Take Γ_1 and Γ_2 two subintervals of the form $[0,\, c_i)$ or $(-c_i,\, c_i)$ or $(-c_i, 0]$ $(c_i \leq \infty)$ with $i = 1, 2$. Suppose that we are given two $m \times m$ matrix polynomials in λ depending also on the variables $x \in \Gamma_1$ and $t \in \Gamma_2$

$$G(x,t,\lambda) = -\sum_{k=0}^{r} \lambda^k q_k(x,t), \quad F(x,t,\lambda) = -\sum_{k=0}^{R} \lambda^k Q_k(x,t), \tag{2.10}$$

and that $w(x,t,\lambda)$ is a nondegenerate $m \times m$ solution of the matrix equations

$$w_x(x,t,\lambda) = G(x,t,\lambda)w(x,t,\lambda), \quad w_t(x,t,\lambda) = F(x,t,\lambda)w(x,t,\lambda) \quad (w_x = \frac{\partial}{\partial x} w). \tag{2.11}$$

The compatibility condition for equations (2.11) has the well-known "zero-curvature" form

$$G_t(x,t,\lambda) - F_x(x,t,\lambda) + [G(x,t,\lambda), F(x,t,\lambda)] = 0 \quad ([G,F] = GF - FG), \tag{2.12}$$

which for the integrable nonlinear equations is equivalent to the famous Lax equality. See [AKNS74], [WE75], [FT87] and references therein. The five matrices of parameters are now A_1, A_2, $\Pi_1(0,0)$, $\Pi_2(0,0)$ and $S(0,0)$ ($\det S(0,0) \neq 0$), satisfying the operator identity

$$A_1 S(0,0) - S(0,0) A_2 = \Pi_1(0,0)\Pi_2(0,0)^*. \tag{2.13}$$

The matrix functions $\Pi_1(x,t)$, $\Pi_2(x,t)$ and $S(x,t)$ are determined by their initial values $\Pi_1(0,0)$, $\Pi_2(0,0)$ and $S(0,0)$, the differential equations

$$\frac{\partial}{\partial x}\Pi_1(x,t) = \sum_{p=0}^{r} A_1^p \Pi_1(x,t) q_p(x,t), \quad \frac{\partial}{\partial x}\Pi_2(x,t) = -\sum_{p=0}^{r} (A_2^*)^p \Pi_2(x,t) q_p(x,t)^*, \tag{2.14}$$

$$\frac{\partial}{\partial x} S(x,t) = \sum_{p=1}^{r}\sum_{j=1}^{p} A_1^{p-j}\Pi_1(x,t) q_p(x,t)\Pi_2(x,t)^* A_2^{j-1} \tag{2.15}$$

with respect to x, and differential equations

$$\frac{\partial}{\partial t}\Pi_1(x,t) = \sum_{p=0}^{R} A_1^p \Pi_1(x,t) Q_p(x,t), \quad \frac{\partial}{\partial t}\Pi_2(x,t) = -\sum_{p=0}^{R} (A_2^*)^p \Pi_2(x,t) Q_p(x,t)^*, \tag{2.16}$$

$$\frac{\partial}{\partial t} S(x,t) = \sum_{p=1}^{R}\sum_{j=1}^{p} A_1^{p-j}\Pi_1(x,t) Q_p(x,t)\Pi_2(x,t)^* A_2^{j-1} \tag{2.17}$$

with respect to t. Applying Theorem 2.1 twice: with respect to variable x and to variable t, one can easily obtain [Sak98]

Theorem 2.2 *Suppose that the matrix functions G and F, of the form (2.10), satisfy the zero-curvature equation (2.12) and that the coefficients q_k and Q_k are continuously differentiable for $x \in \Gamma_1$ and $t \in \Gamma_2$. Let also parameter matrices satisfy (2.13),* $\det S(0,0) \neq 0$. *Then, the equations (2.11) are compatible, systems (2.14) and (2.16) are compatible, and systems (2.15) and (2.17) are compatible also. Moreover, in the regular points of the matrix function $S(x,t)$ we have that the matrix function*

$$\tilde{w}(x,t,\lambda) = w_A(x,t,\lambda) w(x,t,\lambda) \ , \tag{2.18}$$

where w satisfies (2.11) and w_A is given by

$$w_A(x,t,\lambda) = I_m - \Pi_2(x,t)^* S(x,t)^{-1}(A_1 - \lambda I_n)^{-1}\Pi_1(x,t), \tag{2.19}$$

satisfies

$$\tilde{w}_x(x,t,\lambda) = \tilde{G}(x,t,\lambda)\tilde{w}(x,t,\lambda), \quad \tilde{w}_t(x,t,\lambda) = \tilde{F}(x,t,\lambda)\tilde{w}(x,t,\lambda), \tag{2.20}$$

where

$$\tilde{G}(x,t,\lambda) = -\sum_{k=0}^{r}\lambda^k \tilde{q}_k(x,t); \quad \tilde{F}(x,t,\lambda) = -\sum_{k=0}^{R}\lambda^k \tilde{Q}_k(x,t), \tag{2.21}$$

$$\tilde{q}_k(x,t) = q_k(x,t) - \sum_{p=k+1}^{r}\Big(q_p(x,t)Y_{p-k-1}(x,t) - X_{p-k-1}(x,t)q_p(x,t) + $$
$$+ \sum_{j=k+2}^{p} X_{p-j}(x,t)q_p(x,t)Y_{j-k-2}(x,t)\Big), \tag{2.22}$$

$$\tilde{Q}_k(x,t) = Q_k(x,t) - \sum_{p=k+1}^{R}\Big(Q_p(x,t)Y_{p-k-1}(x,t) - X_{p-k-1}(x,t)Q_p(x,t) + $$
$$+ \sum_{j=k+2}^{p} X_{p-j}(x,t)Q_p(x,t)Y_{j-k-2}(x,t)\Big), \tag{2.23}$$

$$X_k(x,t) = \Pi_2(x,t)^* S(x,t)^{-1}A_1^k\Pi_1(x,t), \qquad Y_k(x,t) = \Pi_2(x,t)^*A_2^k S(x,t)^{-1}\Pi_1(x,t). \tag{2.24}$$

Furthermore, the matrix functions $\tilde{G}$ and $\tilde{F}$ defined in this way satisfy the zero-curvature equation

$$\tilde{G}_t(x,t,\lambda) - \tilde{F}_x(x,t,\lambda) + [\tilde{G}(x,t,\lambda), \tilde{F}(x,t,\lambda)] = 0. \tag{2.25}$$

3 Rational Coefficients

In this section GBDT will be used to get a wide class of systems (2.7) with rational coefficients and the nondegenerate $m \times m$ solutions $\tilde{w}$ of the form

$$\tilde{w}(x,\lambda) = w_A(x,\lambda)e^{x\lambda^r d}, \tag{3.1}$$

where w_A is rational both in x and λ. The constant diagonal matrix d is given by

$$d = \operatorname{diag}\{d_1,\, d_2,\, \ldots,\, d_m\}. \tag{3.2}$$

For this purpose, we put in (1.3):

$$q_r(x) \equiv -d, \quad q_k(x) \equiv 0 \quad (0 \le k < r). \tag{3.3}$$

Then, from (1.3) and (2.6) one has the representation (3.1) for some transfer matrix w_A. If (3.3) holds, then formulas (2.2), (2.3) and (3.2) imply

$$\begin{aligned} \Pi_1(x) &= \left[e^{-d_1xA_1^r}\Pi_{11} \quad e^{-d_2xA_1^r}\Pi_{12} \; \ldots \; e^{-d_mxA_1^r}\Pi_{1m}\right], \\ \Pi_2(x) &= \left[e^{\bar{d}_1x(A_2^*)^r}\Pi_{21} \quad e^{\bar{d}_2x(A_2^*)^r}\Pi_{22} \; \ldots \; e^{\bar{d}_mx(A_2^*)^r}\Pi_{2m}\right], \end{aligned} \tag{3.4}$$

and

$$S(x) = S(0) - \sum_{j=1}^{r} A_1^{r-j} \int_0^x \left(\sum_{l=1}^{m} d_l e^{-d_l u A_1^r} \Pi_{1l} \Pi_{2l}^* e^{d_l u A_2^r} \right) du A_2^{j-1}, \qquad (3.5)$$

where Π_{kl} denotes the l-th column of the matrix $\Pi_k(0)$. Thus, from Theorem 2.1 it follows that:

Theorem 3.1 *Let's assume that the matrix functions Π_1, Π_2 and S are given by (3.4) and (3.5), where $\Pi_1(0)$, $\Pi_2(0)$ and $S(0)$ satisfy the operator identity (2.1). Then, in the regular points of S, the matrix function $\tilde{w}$ of the form (3.1), with w_A defined in (2.5), satisfies the system (2.7), where*

$$\begin{cases} \tilde{q}_r \equiv -d \\ \tilde{q}_k(x) = dY_{r-k-1}(x) - X_{r-k-1}(x)d + \sum_{j=k+2}^{r} X_{r-j}(x) d Y_{j-k-2}(x) \quad (0 \leq k < r) \end{cases}, \qquad (3.6)$$

with X_k and Y_k given by (2.9).

Notice that now w_A is always rational in λ. Theorem 3.1 yields the next result.

Theorem 3.2 *Let the conditions of Theorem 3.1 be fulfilled and suppose additionally that the matrices A_1 and A_2 are nilpotent, Then, w_A as well as the coefficients $\tilde{q}_k$ of the corresponding system (2.7) are rational in x.*

Proof. Since A_1 and A_2 are nilpotent it follows from (3.4) and (3.5) that the matrix functions Π_1, Π_2 and S are matrix polynomials. Hence, in view of (2.5) and (2.9) we see that w_A, X_k and Y_k are rational in x. Finally, from (3.6) it follows the rationality of the coefficients $\tilde{q}_k$. ■

The particular case $r = 1$, $d = J := \operatorname{diag}\{I_s, -I_s\}$ was studied earlier in [Zub92b], [Zub92a], [GKS98a] and [GKS98b].

To consider the flows of the coefficients put

$$q_r(x,t) \equiv -d, \quad q_k(x,t) \equiv 0 \quad (0 \leq k < r); \qquad (3.7)$$

$$Q_R(x,t) \equiv -D, \quad Q_k(x,t) \equiv 0 \quad (0 \leq k < R), \quad D = \operatorname{diag}\{D_1, D_2, \ldots, D_m\}. \qquad (3.8)$$

It's immediate that G and F with coefficients as in (3.7) and (3.8) satisfy the conditions of Theorem 2.2. Moreover, we can put

$$w(x,t,\lambda) = e^{(x\lambda^r d + t\lambda^R D)}. \qquad (3.9)$$

From (2.14) and (2.16) we derive also

$$\begin{aligned} \Pi_1(x,t) &= \left[e^{-(d_1 x A_1^r + D_1 t A_1^R)}\Pi_{11} \quad e^{-(d_2 x A_1^r + D_2 t A_1^R)}\Pi_{12} \ \ldots \ e^{-(d_m x A_1^r + D_m t A_1^R)}\Pi_{1m} \right], \\ \Pi_2(x,t) &= \left[e^{(\bar{d}_1 x (A_2^*)^r + \bar{D}_1 t (A_2^*)^R)}\Pi_{21} \quad e^{(\bar{d}_2 x (A_2^*)^r + \bar{D}_2 t (A_2^*)^R)}\Pi_{22} \ \ldots \ e^{(\bar{d}_m x (A_2^*)^r + \bar{D}_m t (A_2^*)^R)}\Pi_{2m} \right], \end{aligned} \tag{3.10}$$

where Π_{kl} denotes the l-th column of the matrix $\Pi_k(0,0)$. Finally formulas (2.15), (2.17) and (3.5) yield

$$\begin{aligned} S(x,t) &= S(x,0) + \int_0^t \frac{\partial}{\partial u} S(x,u)du \\ &= S(0) - \sum_{j=1}^r A_1^{r-j} \int_0^x \left(\sum_{l=1}^m d_l e^{-d_l u A_1^r} \Pi_{1l} \Pi_{2l}^* e^{d_l u A_2^r} \right) du A_2^{j-1} - \\ &- \int_0^t \sum_{j=1}^R A_1^{R-j} \Pi_1(x,u) D \Pi_2(x,u)^* A_2^{j-1} du. \end{aligned} \tag{3.11}$$

From Theorem 2.2 now follows:

Theorem 3.3 *Let the matrix functions Π_1, Π_2 and S be given by (3.10) and (3.11), where $\Pi_1(0,0)$, $\Pi_2(0,0)$ and $S(0,0)$ satisfy the operator identity (2.13). Then, in the regular points of S, the matrix function $\tilde{w}$ of the form*

$$\tilde{w}(x,t,\lambda) = w_A(x,t,\lambda) e^{(x\lambda^r d + t\lambda^R D)}, \tag{3.12}$$

with w_A defined in (2.19), satisfies the system (2.20), where

$$\tilde{q}_r \equiv -d \ , \tag{3.13}$$

$$\tilde{q}_k = dY_{r-k-1} - X_{r-k-1}d + \sum_{j=k+2}^r X_{r-j} d Y_{j-k-2} \quad (k < r), \tag{3.14}$$

$$\tilde{Q}_R \equiv -D \ , \tag{3.15}$$

$$\tilde{Q}_k = DY_{R-k-1} - X_{R-k-1}D + \sum_{j=k+2}^R X_{R-j} D Y_{j-k-2} \quad (k < R), \tag{3.16}$$

where $X_k = X_k(x,t)$ and $Y_k = Y_k(x,t)$ are given in (2.24).

Analogously to Theorem 3.1 yields Theorem 3.2, we immediately derive from Theorem 3.3 the following:

Theorem 3.4 *Let the conditions of Theorem 3.3 be fulfilled. Suppose additionally that* $\det S(0,0) \neq 0$ *and the matrices A_1 and A_2 are nilpotent. Then, w_A as well as the coefficients $\tilde{q}_k$ and $\tilde{Q}_k$ of the corresponding systems (2.11) are rational in x and t.*

Remark 3.5 *If the conditions of Theorem 3.2 are met, then one can factor out from w_A a scalar function $\eta(x)^{-1} = 1/\det S(x)$, which is independent of λ, in such a way that $\eta(x)w_A(x,\lambda)$ is a polynomial in x. Analogously, if the conditions of Theorem 3.4 are met, then one can factor out from w_A a scalar function $\eta(x,t)^{-1}$, which is again independent of λ, in such a way that $\eta(x,t)w_A(x,t,\lambda)$ is a polynomial in x.*

4 Bispectralities

In this section we shall present some results connected to the notion of bispectrality. The key point behind this concept, which is present in a number of situations associated to integrable systems and special functions, is the idea that certain privileged families of eigenfunctions also satisfy differential equations in the spectral parameter.

As explained in the Introduction, the first notion of bispectrality was introduced by F.A. Grünbaum in relation to some problems in computerized tomography. It was then studied in detail by J.J. Duistermaat and F.A. Grünbaum [DG86] for the Schrödinger operator case, by M.Reach [Rea88] for the continuous-discrete context, and in [Zub90]–[Zub92a] for the matrix case. In all these situations, for a given class of differential operators $L(x,\partial_x)$ one seeks a family of eigenfunctions φ: $L(x,\partial_x)\varphi(x,\lambda) = \lambda\varphi(x,\lambda)$ and a differential (or difference) operator $B(\lambda,\partial_\lambda)$ independent of x, such that

$$B\varphi(x,\lambda) = \Theta(x)\varphi(x,\lambda) \ , \tag{4.1}$$

where $\Theta(x)$ does not depend on λ. Notice that both operators L and B are supposed to be of positive orders and the bispectral problem, as posed above, is local, i.e., we are not imposing any boundary conditions. The issue of boundary conditions is an interesting one and to our knowledge still unexplored.

In the transition from the scalar bispectral problem to the case of a system, we have to consider the issue of matrix coefficients in the differential operator $B(\lambda,\partial_\lambda)$. In this case, it is still unclear what kind of conditions should be placed on the matrix coefficients to avoid pathological situations. Therefore we call a positive-order differential operator B *weakly nondegenerate* if the rows of $B(\lambda,\partial_\lambda)-C$ are linearly independent almost everywhere for any constant matrix C. We shall say that $B(\lambda,\partial_\lambda)$ is *nondegenerate* or *strongly nondegenerate* if as in [Zub90] the rows of the leading coefficient in B are linearly independent almost everywhere (in other words the leading coefficient is *nonsingular*).

We have already mentioned above that the notion of bispectrality is closely associated with the nonlinear integrable systems. As the corresponding auxiliary linear systems often depend polynomially on the spectral parameter it seems important to generalize the notion of bispectrality for the case of the *bundles* of operators and for the corresponding systems. In this article we are considering the first-order differential equations of the form (1.3) with polynomial dependence on the spectral parameter λ. Generalizing in a natural way the notions of bispectrality from [DG86] and in [Zub90] we say that a system of the form (1.3) posesses the *bispectral* (sometimes we call it *bundle bispectral*) property if there exist a family of solutions $\varphi(x, \lambda)$, a differential operator $B(\lambda, \partial_\lambda)$ and a function $\Theta(x)$, such that (4.1) holds. A *rank of bispectrality* was introduced in [DG86] as the number of families of linearly independent bispectral vector eigenfunctions $\varphi_k(x, \lambda)$. We shall say that (1.3) is *ρ-rank bispectral* if there exists an operator $B(\lambda, \partial_\lambda)$ such that ρ linearly independent families φ_k and corresponding functions Θ_k satisfy (1.3) and (4.1) simultaneously. Denote by e_k the k-th column vector of the canonical basis of $\mathbb{C}^m$. Taking into account Remark 3.5 and following the scheme of the proof of bispectrality from [Zub92b], we obtain the next result.

Theorem 4.1 *Assume that the conditions of Theorem 3.2 are satisfied and put*

$$\varphi_k(x, \lambda) := \tilde{w}(x, \lambda) e_k. \tag{4.2}$$

Then, for any fixed set Ω of $m-1$ different values of the index $k \in \{1, \cdots, m\}$ and for an arbitrary constant scalar c there exists a weakly nondegenerate differential operator $B(\lambda, \partial_\lambda)$ such that $B(\lambda, \partial_\lambda)\varphi_k(x, \lambda) = c\varphi_k(x, \lambda)$. So, system (2.7) is $(m-1)$-rank (bundle) bispectral. Assume now that the conditions of Theorem 3.4 are satisfied and put

$$\varphi_k(x, t, \lambda) := \tilde{w}(x, t, \lambda) e_k \ .$$

Then, for any fixed set Ω of $m-1$ values of the index k and for an arbitrary scalar function $c(t)$ there exists a weakly nondegenerate differential operator $B(\lambda, \partial_\lambda, t)$ such that $B(\lambda, \partial_\lambda, t)\varphi_k(x, t, \lambda) = c(t)\varphi_k(x, t, \lambda)$. So, the flow of the first system in (2.20) is $(m-1)$-rank bispectral. The same goes with the second system in (2.20) after we interchange the variables x and t there.

Proof. Let's introduce the auxiliary variable $\zeta = \lambda^r$, and notice that the claim about the existence of a differential operator in the spectral parameter λ is equivalent to the existence of an operator in ζ. Moreover, the weak or strong non-degeneracy in ζ yields the corresponding nondegeneracy in λ. To prove the first statement of the theorem it suffices therefore to construct a weakly nondegenerate operator

$$B(\zeta, \partial_\zeta) = \sum_{l=0}^{N} c_l(\zeta) \frac{\partial^l}{\partial \zeta^l} \tag{4.3}$$

such that

$$B(\zeta,\partial_\zeta)\eta(x)\varphi_k(x,\zeta^{1/r}) = c\eta(x)\varphi_k(x,\zeta^{1/r}) \tag{4.4}$$

for some $\eta(x)\neq 0$ and for all $k\in\Omega$. We shall prove (4.4) for $\eta(x)$ mentioned in Remark 3.5. Put in (4.3) $c_0 = cI_m$ and recall (3.1) and (4.2) to rewrite (4.4) as

$$\sum_{l=1}^{N} c_l(\zeta)\frac{\partial^l}{\partial\zeta^l}\eta(x)w_A(x,\zeta^{1/r})e^{x\zeta d}e_k = 0.$$

The last formula is equivalent to the relation

$$\sum_{l=1}^{N} c_l(\zeta)(\frac{\partial}{\partial\zeta}+d_k x)^l\omega_k(x,\zeta^{1/r}) = 0, \qquad \omega_k(x,\lambda)=\eta(x)w_A(x,\lambda)e_k. \tag{4.5}$$

Hence, it suffices to show the existence of a family $c_l(\zeta)$, where $l=1,\cdots,N$, such that (4.5) holds. By Remark 3.5 $\eta(x)w_A(x,\lambda)$ is a matrix polynomial in x. Denote its degree by $\varkappa$. Then, the vector functions ω_k admit expansions in x: $\omega_k(x,\zeta^{1/r}) = \sum_{s=0}^{\varkappa} x^s f^k_{s0}(\zeta)$. Using this expansion we can transform (4.5) into the linear algebraic equation

$$\sum_{l=1}^{N} c_l(\zeta)\sum_{s=0}^{\varkappa+l} x^s f^k_{sl}(\zeta) = 0\ , \tag{4.6}$$

where f^k_{sl} depends only on the variable ζ. Rewrite (4.6) in a matrix form

$$\sum_{s=0}^{\varkappa+N} x^s\mathcal{C}\mathcal{V}^k_s = 0, \qquad \mathcal{C} := [c_1,\cdots,c_N], \qquad \mathcal{V}^k_s := \begin{bmatrix} f^k_{s1} \\ \vdots \\ f^k_{sN}\end{bmatrix}. \tag{4.7}$$

We claim that the $m\times Nm$ matrix $\mathcal{C}$ can be chosen of rank m so that $\mathcal{C}\mathcal{V}^k_s = 0$ for $0\le s\le \varkappa+N$ and for all $k\in\Omega$. Indeed, just look at the vectors $\mathcal{V}^k_s$ as elements of $\mathbb{C}^{mN}$. The problem then boils down to finding m linearly independent vectors in $(\mathbb{C}^{mN})^*\equiv\mathbb{C}^{mN}$ which annihilate $\mathcal{V}^k_s$. Such a set of vectors always exists provided that N is chosen sufficiently large so that $mN-(m-1)(\varkappa+N+1)\ge m$. The equalities $\mathcal{C}\mathcal{V}^k_s = 0$ imply (4.7), i.e. the operator B defined by (4.3) satisfies (4.4). Finally the maximal rank of $\mathcal{C}$ yields the weak non-degeneracy of B and the first statement of the theorem is proved. Two other statements of the theorem are proved quite analogously. ∎

Remark 4.2 *Obviously, given that the dependence of f_{sl} is polynomial in $\zeta^{-1/r}$ it follows that the dependence of c_l can be taken rational in $\zeta^{-1/r}$.*

It could be suggested that "bona fide" bispectral parameter $\Theta(x)$ should be non-constant. We are presently working in this direction.

Let's return to the representation (3.1) and rewrite it in the form

$$\tilde{w}(x,\lambda) = \frac{1}{\det S(x)}\Upsilon(x,\lambda)e^{x\lambda^r d}, \tag{4.8}$$

$$\Upsilon(x,\lambda) = \det S(x)\,(I_m - \Pi_2(x)^* S(x)^{-1}(A_1-\lambda I_n)^{-1}\Pi_1(x)). \tag{4.9}$$

It follows from the given above arguments that a possibility to factorize the fundamental (i.e. $m\times m$ and nondegenerate at $x=0$) solution $\tilde{w}$ of (2.7) in the form (4.8) with Υ a polynomial in x is sufficient for the corresponding system to be bispectral. Equation (4.9) enables us also to factor out from $\Upsilon(x,\lambda)-\det S(x)\,I_m$ a matrix function that depends on λ: $\Upsilon(x,\lambda)-\det S(x)\,I_m = \Upsilon_1(x)(A_1-\lambda I_n)^{-1}\Upsilon_2(x)$. The spectrum of a matrix A will be denoted by $\sigma(A)$. When $\sigma(A_1)=\{c\}$ for some $c\in\mathbb{C}$, we have $(A_1-cI_n)^i=0$ for some $i\geq 0$ and therefore the resolvent $(A_1-\lambda I_n)^{-1}$ is a polynomial of $(\lambda-c)^{-1}$:

$$(A_1-\lambda I_n)^{-1} = -\sum_{p=1}^{i}\frac{(A_1-cI_n)^{p-1}}{(\lambda-c)^p}. \tag{4.10}$$

This interesting truncation is reflected in the notion of the *modified bispectrality* that was introduced in [GKS98a] and [GKS98b] for the canonical and pseudo-canonical systems. The notion, the results and the proofs are easily transfered to our more general setting. We say that system (2.7) has the *modified bispectral property* if there is a set of complex numbers $\{c_l\}$, $0\leq l\leq N$, such that a fundamental solution $\tilde{w}(x,\lambda)$ of system (2.7) satisfies for each x a differential equation in λ:

$$\sum_{l=1}^{N} c_l(\lambda-c_0)^l\frac{\partial^l}{\partial\lambda^l}(\tilde{w}(x,\lambda)e^{-x\lambda^r d}) = 0 \quad (\sum_{l=1}^{N}|c_l|\neq 0).$$

Theorem 4.3 *Assume that the conditions of Theorem 3.1 are satisfied and $\sigma(A_1)=\{c\}$. Then, system (2.7) has the modified bispectral property.*

Proof. Put $c_0=c$ and take into account (4.8)–(4.10). The proof of the theorem then boils down to proving that there exists a set $\{c_l\}$, $1\leq l\leq N$ such that

$$\sum_{l=1}^{N} c_l(\lambda-c)^l\frac{\partial^l}{\partial\lambda^l}(\lambda-c)^{-p}=0,\quad p=1,\ldots,i;\qquad \sum_{l=1}^{N}|c_l|\neq 0,$$

or, equivalently,

$$\sum_{l=1}^{N} c_l\nu_{lp}=0,\quad p=1,\ldots,i;\qquad \sum_{l=1}^{N}|c_l|\neq 0,\qquad \nu_{lp}:=(\lambda-c)^p\frac{\partial^l}{\partial\lambda^l}(\lambda-c)^{-p}. \tag{4.11}$$

A nontrivial solution $[c_1, \ldots, c_N]$ of the linear algebraic system (4.11), evidently, exists for $N > i$. ■

Remark 4.4 *It's evident that if $\sigma(A_1) = \{c\}$, the modified bispectality is preserved by the flow described in Theorem 3.3.*

5 ZS-AKNS Systems

One of the instrumental developments in the inverse scattering transform (IST) method was due to the works of Zakharov & Shabat [ZS71] and Ablowitz, Kaup, Newel, & Segur [AKNS74]. They have shown that the auxiliary role played by the scalar Schrödinger equation in the IST integration of the KdV could be applied to certain first order matrix systems. This lead to the integration of other significant equations in mathematical physics, such as the nonlinear Schrödinger equation, the modified KdV, and the sine-Gordon. The auxiliary system studied in [AKNS74] took the form

$$\partial_x y(x,\lambda) = i(\lambda J + JV(x))y(x,\lambda), \qquad V(x) := \begin{bmatrix} 0 & v_1(x) \\ v_2(x) & 0 \end{bmatrix}, \tag{5.1}$$

where v_1 and v_2 are scalar fields and $J = \operatorname{diag}\{1, -1\}$. The bispectral problem for such systems was investigated in [Zub92b] and [Zub92a]. It was shown therein that certain (generic) families of rational solutions to the AKNS hierarchy are bispectral.

In this section we start by proving the bispectrality of (5.1) for a class of ZS-AKNS systems with *rational matrix* fields v_1 and v_2. Then, we characterize the spectral density function for the self-adjoint class of bispectral ZS-AKNS systems under consideration. Finally, we show that the bispectral systems remain rational by the flow of the nonlinear Schrödinger equation. We set $J := \operatorname{diag}\{I_s, -I_s\}$ and consider ZS-AKNS type systems of the form (5.1) where $m = 2s$ and v_1 and v_2 are $s \times s$ matrices. System (5.1) corresponds to the particular case of system (1.3), where

$$r = 1, \qquad d = iJ. \tag{5.2}$$

After substitution of (5.2) into (3.3) formulas (3.6) can be rewritten as

$$\tilde{q}_1 \equiv -iJ, \quad \tilde{q}_0(x) = i(JY_0(x) - X_0(x)J). \tag{5.3}$$

According to (5.3) system (2.7) now takes the form (5.1), where

$$V(x) = iJ\tilde{q}_0(x) = JX_0(x)J - Y_0(x). \tag{5.4}$$

Denote the block of the first s columns of $\Pi_k(0)$ by Φ_{k1} and the block of the second s columns by Φ_{k2} ($k = 1, 2$), i.e. put $\Pi_k(0) =: [\Phi_{k1} \quad \Phi_{k2}]$. Then, equalities (3.4) and (3.5) can be transformed into

$$\Pi_1(x) = \left[e^{-ixA_1}\Phi_{11} \quad e^{ixA_1}\Phi_{12}\right], \qquad \Pi_2(x) = \left[e^{-ixA_2^*}\Phi_{21} \quad e^{ixA_2^*}\Phi_{22}\right], \tag{5.5}$$

$$S(x) = S(0) - i\int_0^x \left(e^{-iuA_1}\Phi_{11}\Phi_{21}^* e^{iuA_2} - e^{iuA_1}\Phi_{12}\Phi_{22}^* e^{-iuA_2}\right) du. \tag{5.6}$$

Recall now that by (2.9) we have

$$X_0(x) = Y_0(x) = \Pi_2(x)^* S(x)^{-1}\Pi_1(x). \tag{5.7}$$

From (5.4), (5.5) and (5.7) we derive

$$v_1(x) = -2\Phi_{21}^* e^{ixA_2} S(x)^{-1} e^{ixA_1}\Phi_{12}, \qquad v_2(x) = -2\Phi_{22}^* e^{-ixA_2} S(x)^{-1} e^{-ixA_1}\Phi_{11}. \tag{5.8}$$

By Theorem 4.1 we get

Corollary 5.1 *Assume that v_1 and v_2 of the ZS-AKNS system (5.1) are defined by (5.8), where the matrices A_1, A_2, $S(0)$ and $\Pi_k(0) = [\Phi_{k1} \quad \Phi_{k2}]$ satisfy the operator identity (2.1),* $\det S(0) \neq 0$, *and the matrices A_1 and A_2 are nilpotent. Then, the system (5.1) is $(m-1)$-rank bispectral.*

The cases where the system (5.1) satisfies $v_2 = \pm v_1^*$ are of special interest. Indeed, these are associated to the study of the nonlinear Schrödinger equation. From (5.5) and (5.6) one can easily see that if

$$A_2 = A_1^*, \qquad S(0) = S(0)^*, \qquad \Pi_2(0) = -i\Pi_1(0)J^p \qquad (p = 0, 1), \tag{5.9}$$

we have

$$S(x) = S(x)^*, \qquad \Pi_2(x) = -i\Pi_1(x)J^p. \tag{5.10}$$

Formulas (5.8)-(5.10) imply

$$v_2(x) = v_1(x)^* \quad (\text{for} \quad \mathrm{p} = 1), \qquad v_2(x) = -v_1(x)^* \quad (\text{for} \quad \mathrm{p} = 0). \tag{5.11}$$

Thus, when $p = 1$ we get the Dirac type (or canonical) subclass of ZS-AKNS systems, whereas when $p = 0$ we have the pseudocanonical subclass of ZS-AKNS systems:

$$y_x(x,\lambda) = i\left(\lambda J + J\begin{bmatrix} 0 & v_1(x) \\ (-1)^{p+1}v_1(x)^* & 0 \end{bmatrix}\right) y(x,\lambda). \tag{5.12}$$

In view of (5.11), Corollary 5.1 implies the following:

Corollary 5.2 *Let's assume that the canonical system (5.12) (the case $p = 1$) and the pseudocanonical system (5.12) (the case $p = 0$) are given and that v_1 is defined by (5.8), where the matrices A_1, $S(0)$, Φ_{21} and Φ_{12} satisfy the operator identity*

$$A_1 S(0) - S(0) A_1^* = i(\Phi_{21}\Phi_{21}^* + (-1)^p \Phi_{12}\Phi_{12}^*),$$

$\det S(0) \neq 0$, $A_2 = A_1^*$ *and the matrix* A_1 *is nilpotent. Then, (5.12) is* $(m-1)$*-rank bispectral.*

Remark 5.3 *If $p = 1$ and $S(0) > 0$, then, by (5.6) the inequality $S(x) > 0$ is valid for all $x > 0$ and so system (5.1) is well-defined on the half-line $0 \leq x < \infty$. Moreover, without loss of generality, we can introduce the condition $S(0) = I_n$ instead of $S(0) > 0$.*

The transfer to the parameter matrices with $S(0) = I_n$ is conducted by the transformation of the initial matrices A_1, A_2, $S(0)$, $\Pi_1(0)$ and $\Pi_2(0)$ into the matrices $S(0)^{-\frac{1}{2}} A_1 S(0)^{\frac{1}{2}}$, $S(0)^{\frac{1}{2}} A_2 S(0)^{-\frac{1}{2}}$, I_n, $S(0)^{-\frac{1}{2}}\Pi_1(0)$ and $S(0)^{-\frac{1}{2}}\Pi_2(0)$. This transformation evidently does not change the values of v_1 and v_2 in (5.8). Let us simplify accordingly our formulas and notation. Put

$$A := A_1 = A_2^*, \qquad \Pi(x) := \Pi_1(x) = i\Pi_2(x)J, \qquad S(0) = I_n, \qquad \Pi(0) =: [\Phi_1 \quad \Phi_2], \tag{5.13}$$

and write down the canonical system as

$$y_x(x,\lambda) = i(\lambda J + JV(x))y(x,\lambda), \qquad V(x) := \begin{bmatrix} 0 & v(x) \\ v(x)^* & 0 \end{bmatrix} \qquad (0 \leq x < \infty), \tag{5.14}$$

where

$$v(x) = v_1(x) = v_2(x)^* = -2i\Phi_1^* e^{ixA^*} S(x)^{-1} e^{ixA}\Phi_2. \tag{5.15}$$

Here, we suppose that

$$S(x) = I_n + \int_0^x \left(e^{-iuA}\Phi_1\Phi_1^* e^{iuA^*} + e^{iuA}\Phi_2\Phi_2^* e^{-iuA^*}\right) du, \tag{5.16}$$

and A and Π satisfy the operator identity

$$A - A^* = i(\Phi_1\Phi_1^* - \Phi_2\Phi_2^*). \tag{5.17}$$

The spectral theory of the canonical systems with the potentials v as given above (i.e. v determined via (5.15) and (5.16) by the parameter matrices A, Φ_1 and Φ_2 satisfying (5.17)) was studied in [GKS98a]. The potentials v were said to be *determined by the triple* A, Φ_1 and Φ_2 (satisfying (5.17)) and the class of such potentials was denoted by the acronym PE (pseudo-exponential).

In view of the close connection between the notion of bispectrality and the spectral properties of the system it would be interesting to characterize the constructed class of bispectral potentials in terms of the spectral theory. For this purpose we shall shortly describe some definitions and results.

As usual, we shall denote by $L^2_m(0,\infty)$ the square summable functions defined almost everywhere (a.e) on $(0,\infty)$ w.r.t. the Lebesgue measure and taking values in $\mathbb{C}^m$. Similarly, for a given measure $d\tau$ on $\mathbb{R}$, $L^2_s(\mathbb{R}, d\tau)$ denotes the square summable functions defined $d\tau$-a.e. on $\mathbb{R}$ and taking values in $\mathbb{C}^s$. Assume that u is the $m\times m$ solution of (5.14) normalized by the condition $u(0,\lambda) = I_m$.

Recall that a nondecreasing $s \times s$ matrix function τ is called a *spectral function of system* (5.14) with boundary condition

$$[\, I_s \quad -I_s \,]\, f(0) = 0, \tag{5.18}$$

if the operator U, given by the formula

$$Ug(z) := \frac{1}{\sqrt{2\pi}} \int_0^\infty [I_s \quad I_s] u(x,z)^* \, g(x)\, dx, \quad z \in \mathbb{R},$$

is an isometric mapping from $L^2_m(0,\infty)$ into $L^2_s(\mathbb{R}, d\tau)$.

Theorem 5.4 *[GKS1] Assume that the potential v of the half-line canonical system (5.14) is determined by the triple A, Φ_1 and Φ_2 (satisfying (5.17)) and put*

$$B = A + \Phi_1\Phi^*, \qquad \Phi = i(\Phi_1 + \Phi_2). \tag{5.19}$$

Let z_j be the real points of $\sigma(B)$, i.e.,

$$z_j \in (\sigma(B) \cap \mathbb{R}), \quad 1 \le j \le k, \tag{5.20}$$

and let the matrices ν_j and the matrix function W be given by the equalities

$$\nu_j = 2\pi \mathrm{res}_{z=z_j} \Phi_1^*(zI_n - B)^{-1}\Phi_1 \quad (1 \le j \le k), \tag{5.21}$$

$$W(z) = I_s + \hat{C}(zI_{2n} - \hat{A})^{-1}\hat{B}, \tag{5.22}$$

where the matrices $\hat{A}$, $\hat{B}$ and $\hat{C}$ are defined by

$$\hat{C} = \begin{bmatrix} \Phi_1^* & \Phi^* \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \Phi \\ \Phi_1 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} B^* & \Phi\Phi^* \\ 0 & B \end{bmatrix}. \tag{5.23}$$

Then, the matrices ν_j are nonnegative, the matrix function W is nonnegative without poles on $\mathbb{R}$ and the matrix function

$$\tau(z) = \int_0^z W(\mu)d\mu + \sum_{z_j<z} \nu_j \tag{5.24}$$

is the nondecreasing piecewise absolutely continuous spectral function of the canonical system (5.14).

The inverse problem consists of recovering the potential v from the spectral function of the form (5.24) with a finite number of jumps and a rational spectral density W that goes to I_s at infinity. This problem was also solved explicitly in [GKS98a]. It follows from the general theory (see [Sak99] and references therein) that there is no more than one solution of the inverse spectral problem for system (5.14) with the boundary condition (5.18) in the class of locally bounded potentials. Using these results we derive a spectral characterization of the subclass of bispectral potentials that we obtained above.

Theorem 5.5 *A locally bounded potential v is determined by a triple with a nilpotent matrix A if, and only if, the spectral function τ of the corresponding canonical system (5.14) with the boundary condition (5.18) has the form*

$$\tau(z) = \int_0^z W(\mu)d\mu + \begin{cases} 0 & \text{for} \quad z<0, \\ \nu & \text{for} \quad z>0; \end{cases} \qquad \left(W(\mu) \geq 0, \quad \lim_{\mu\to\infty} W(\mu) = I_s, \quad \nu \geq 0\right), \tag{5.25}$$

where $W(z)^{-1}$ is a matrix polynomial of the variable z^{-1}.

Proof. Assuming v is determined by a triple and A is nilpotent we shall prove at first the necessary part of the theorem. By Theorem 5.4, τ admits representation (5.24). To show that τ has at most one jump, and it may occur only at $z=0$, we can use some arguments from [GKS98a] on the connection between $\sigma(B)$ and $\sigma(A)$. Indeed, according to (5.17) and (5.19) we have

$$B - B^* = i(\Phi_1\Phi_1^* - \Phi_2\Phi_2^*) - i\Phi_1(\Phi_1+\Phi_2)^* - i(\Phi_1+\Phi_2)\Phi_1^* = -i\Phi\Phi^*. \tag{5.26}$$

Let now f be an eigenvector of B: $Bf = zf$, $z \in \mathbb{R}$, $f \neq 0$. Then we get $f^*(B-B^*)f = 0$ and (5.26) yields $f^*(\Phi\Phi^*)f = 0$, i.e. $\Phi^* f = 0$. So by (5.19) $Af = Bf = zf$ and hence

$z = 0$. Therefore $(\sigma(B) \cap \mathbb{R}) \setminus \{0\} = \emptyset$ and using formula (5.20) we immediately derive representation (5.25) from (5.24). It remains to show that W^{-1} is a polynomial of z^{-1}. By the well-known fact from system theory (see [KFA69]) "realization" (5.22) implies

$$W(z)^{-1} = I_s - \hat{C}(zI_{2n} - \hat{A}^{\times})^{-1}\hat{B}, \qquad \hat{A}^{\times} = \hat{A}^{-}\hat{B}\hat{C}. \tag{5.27}$$

In view of (5.19) and (5.23) we see that

$$\hat{A}^{\times} = \begin{bmatrix} A^* & 0 \\ -\Phi_1\Phi_1^* & A \end{bmatrix}. \tag{5.28}$$

According to (5.28), $\hat{A}^{\times}$ is nilpotent. Therefore, (5.27) means that W^{-1} is indeed a polynomial.

To prove the sufficiency part, we shall introduce some more elements from the system theory (see [KFA69] and [BGK79]). A rational matrix function $K(z)$, such that $\lim_{z\to\infty} K(z) = I_s$, always admits a representation

$$K(z) = I_s + C_K(zI_l - A_K)^{-1}B_K \qquad (A_K,\, B_K,\, C_K, -\text{ matrices}) \tag{5.29}$$

that is called a *realization* of K; the realization is minimal when the order l of A_K is as small as possible. If the realization is minimal, then the set of the poles of K coincides precisely with $\sigma(A_K)$. The dimension of a minimal realization is called the McMillan degree of K. A factorization $K = K_1K_2$ of K into two proper $\mathbb{C}^{s\times s}$-valued rational functions is called *minimal* if the McMillan degrees of K_1 and K_2 add up to the McMillan degree of K. If our K is nonnegative on $\mathbb{R}$ it admits [Ran82] a minimal factorization:

$$K(z) = K_1(z)K_1(\bar{z})^* \qquad (\lim_{z\to\infty} K_1(z) = I_s). \tag{5.30}$$

If in addition K^{-1} has no poles on the real line, we can choose K_1 with all the poles of K_1^{-1} in the lower half-plane $\mathbb{C}_-$. In terms of a minimal realization

$$K_1(z) = I_s - \gamma_2^*(zI_{n_0} - A_0)^{-1}\gamma_1, \tag{5.31}$$

it means that

$$\sigma(B_0) \subset \mathbb{C}_-, \qquad B_0 = A_0 + \gamma_1\gamma_2^*. \tag{5.32}$$

Moreover we can choose a minimal realization of such K_1 with matrices A_0, γ_1 and γ_2 also satisfying [GKS98a] the operator identity

$$A_0 - A_0^* = i\gamma_1\gamma_1^* - i(\gamma_1 + i\gamma_2)(\gamma_1 + i\gamma_2)^*. \tag{5.33}$$

Suppose now that we are given a matrix function τ of the form (5.25) such that W^{-1} is a matrix polynomial of z^{-1}. Then, $K = W^{-1}$ admits a minimal factorization (5.30) with K_1 chosen as mentioned above. That is K_1 admits realization (5.31), where n_0 denotes the McMillan degree of K_1 and A_0, γ_1 and γ_2 satisfy (5.33). Put

$$C_K = -\begin{bmatrix} \gamma_1^* & \gamma_2^* \end{bmatrix}, \quad B_K = \begin{bmatrix} \gamma_2 \\ \gamma_1 \end{bmatrix}, \quad A_K = \begin{bmatrix} A_0^* & 0 \\ -\gamma_1\gamma_1^* & A_0 \end{bmatrix}. \tag{5.34}$$

It can be directly derived from (5.30) and (5.31) that $K = W^{-1}$ admits representation (5.29) with A_K, B_K and C_K given in (5.34) and $l = 2n_0$. As the factorization (5.30) and the realization (5.31) are both minimal, so our A_K, B_K and C_K provide a minimal realization of W^{-1}. Recall now that W^{-1} is a polynomial of z^{-1}, i.e. the only pole of W^{-1} is at $z = 0$, to obtain $\sigma(A_K) = \{0\}$. As A_K is nilpotent, so by (5.34) A_0 is also nilpotent. Denote finally the rank of ν by n_1 and introduce a triple

$$A = \begin{bmatrix} A_0 & 0 \\ -\chi\gamma_2^* & 0 \end{bmatrix}, \qquad \Phi_1 = \begin{bmatrix} \gamma_1 \\ \chi \end{bmatrix}, \qquad \Phi_2 = -\begin{bmatrix} \gamma_1 + i\gamma_2 \\ \chi \end{bmatrix}, \tag{5.35}$$

where χ is a $n_1 \times s$ matrix such that $\nu = 2\pi\chi^*\chi$. In view of (5.33) it is now immediate that (5.17) is satisfied. Notice also that A is nilpotent. Thus the potential determined by A, Φ_1 and Φ_2 belongs to our class. As according to the general uniqueness results [Sak99] there is no other potential with the same spectral function, it remains to prove that given τ is the spectral function of system (5.12) with the potential v determined by A, Φ_1 and Φ_2. From (5.19),(5.32) and (5.35) we get

$$\Phi = \begin{bmatrix} \gamma_2 \\ 0 \end{bmatrix}, \qquad B = \begin{bmatrix} B_0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.36}$$

By Theorem 5.4 and taking into account (5.32), (5.35) and (5.36) the jump of the spectral function of our system equals

$$2\pi \mathrm{res}_{z=0} \Phi_1^*(zI_n - B)^{-1}\Phi_1 = 2\pi\chi^*\chi, \qquad n = n_0 + n_1,$$

which by the definition of χ is just ν. Using Theorem 5.4 again and taking into account (5.35) and (5.36) we get that the spectral density of our system equals

$$I_s + \begin{bmatrix} \Phi_1^* & \Phi^* \end{bmatrix} \left(zI_{2n} - \begin{bmatrix} B^* & \Phi\Phi^* \\ 0 & B \end{bmatrix} \right)^{-1} \begin{bmatrix} \Phi \\ \Phi_1 \end{bmatrix} =$$

$$= I_s + \left[\begin{array}{cc} \gamma_1^* & \gamma_2^* \end{array}\right] \left(zI_{2n_0} - \left[\begin{array}{cc} B_0^* & \gamma_2\gamma_2^* \\ 0 & B_0 \end{array}\right] \right)^{-1} \left[\begin{array}{c} \gamma_2 \\ \gamma_1 \end{array}\right]. \quad (5.37)$$

Recall that W^{-1} admits representation (5.29), (5.34), i.e.

$$W(z) = I_s - C_K(zI_l - (A_K - B_KC_K))^{-1}B_K =$$

$$= I_s + \left[\begin{array}{cc} \gamma_1^* & \gamma_2^* \end{array}\right] \left(zI_{2n_0} - \left[\begin{array}{cc} (A_0 + \gamma_1\gamma_2^*)^* & \gamma_2\gamma_2^* \\ 0 & A_0 + \gamma_1\gamma_2^* \end{array}\right] \right)^{-1} \left[\begin{array}{c} \gamma_2 \\ \gamma_1 \end{array}\right]. \quad (5.38)$$

In view of (5.32) the right-hand sides of (5.37) and (5.38) coincide. Hence, W is the spectral density of our system. Therefore, τ is the spectral function, which finishes the proof. ∎

To study the flows of (canonical and pseudocanonical) systems (5.12), following Section 2, let's add the time variable t and put in (3.7) and (3.8)

$$r = 1, \qquad R = 2, \qquad d = D = iJ. \quad (5.39)$$

In this context, the reduction (5.9) is substituted by the reduction

$$A := A_1 = A_2^*, \qquad S(0,0) = S(0,0)^*, \qquad \Pi(0,0) := \Pi_1(0,0) = i\Pi_2(0,0)J^p \quad (5.40)$$

$$(p = 0,\, 1, \qquad \Pi(0,0) =: [\Phi_1 \quad \Phi_2]),$$

and rewrite therefore (2.1) as

$$AS(0,0) - S(0,0)A^* = i\Pi(0,0)J^p\Pi(0,0)^*. \quad (5.41)$$

Analogously to (5.10) and (5.13) formulas (2.14)-(2.17) and (5.40) yield

$$\begin{cases} \Pi(x,t) := \Pi_1(x,t) = i\Pi_2(x,t)J^p = [e^{-i(xA+tA^2)}\Phi_1 \quad e^{i(xA+tA^2)}\Phi_2]\ , \\ S(x,t) = S(x,t)^*\ . \end{cases} \quad (5.42)$$

We apply now the transformations (3.13) and (3.14). Analogously to (5.3), (5.4), 5.11), and in view of (5.42), we obtain

$$\tilde{q}_1 \equiv -iJ, \quad \tilde{q}_0(x,t) = i(JY_0(x,t) - X_0(x,t)J) = -iJV(x,t), \quad (5.43)$$

where $V(x,t) = \left[\begin{array}{cc} 0 & v(x,t) \\ (-1)^{p+1}v(x,t)^* & 0 \end{array}\right]$,

$$v(x,t) = -2i\Phi_1^* e^{i(xA^*+t(A^*)^2)}S(x,t)^{-1}e^{i(xA+tA^2)}\Phi_2. \quad (5.44)$$

By (2.21) and (5.43) we have

$$\tilde{G}(x,t,\lambda) = i\lambda J + iJV(x,t). \tag{5.45}$$

Hence the first of the corresponding systems (2.20) can be presented in the form

$$y_x(x,t,\lambda) = i(\lambda J + JV(x,t))y(x,t,\lambda), \qquad V(x,t) = \begin{bmatrix} 0 & v(x,t) \\ (-1)^{p+1}v(x,t)^* & 0 \end{bmatrix}. \tag{5.46}$$

Therefore the system preserves its canonical or pseudocanonical form during the evolution.

According to [Sak94] (see [Sak98] for the details) the transformation (3.15), (3.16) yields

$$\tilde{Q}_2 \equiv -iJ, \quad \tilde{Q}_1(x,t) = -iJ^pV(x,t), \quad \tilde{Q}_0(x,t) = \frac{i}{2}((-1)^pJV(x,t)^2 + iJ^{p+1}V_x(x,t)). \tag{5.47}$$

By (2.21) and (5.47) we have

$$\tilde{F}(x,t,\lambda) = i\lambda^2 J + i\lambda J^pV(x,t) - \frac{i}{2}((-1)^pJV(x,t)^2 + iJ^{p+1}V_x(x,t)). \tag{5.48}$$

By Theorem 3.3, systems (2.20) are compatible, i.e. (2.25) holds. The zero curvature equation (2.25) with $\tilde{G}$ and $\tilde{F}$ given by (5.45) and (5.48), where V is of the form (5.46), is equivalent [ZS73] to the famous nonlinear Schrödinger equation:

$$2v_t(x,t) + i(v_{xx}(x,t) + 2(-1)^pv(x,t)v(x,t)^*v(x,t)) = 0. \tag{5.49}$$

Thus, from Theorem 4.1 we derive

Corollary 5.6 *Let's assume that the canonical system (5.46) (the case $p = 1$) and the pseudocanonical system (5.46) (the case $p = 0$) are given and that v is defined by (5.44), where the matrices A, $S(0,0)$ and $\Pi(0,0) = [\Phi_1 \quad \Phi_2]$ satisfy the operator identity (5.41),* $\det S(0,0) \neq 0$ *and the matrix A is nilpotent. Then, these systems evolve by the nonlinear Schrödinger equation (5.49) flow and they remain $(m-1)$-rank bispectral by this flow.*

Remark 5.7 *One should notice that the second system in (2.20) satisfies the conditions of Theorem 4.1 also, i.e. we have obtained simultaneously an example of bundle bispectrality: both auxiliary systems (2.20) in the case of nilpotent matrix A are bispectral with respect to our definition of the bundle bispectrality.*

Corollary 5.8 *Let's consider for $p = 0$ or $p = 1$ the system*

$$y_t(x,t,\lambda) = (i\lambda^2 J + i\lambda J^p V(x,t) - \frac{i}{2}((-1)^p JV(x,t)^2 + iJ^{p+1}V_x(x,t)))y(x,t,\lambda) \ . \quad (5.50)$$

Suppose that v is defined by (5.44), where the matrices A, $S(0,0)$ and $\Pi(0,0) = [\Phi_1 \quad \Phi_2]$ satisfy the operator identity (5.41), $\det S(0,0) \neq 0$ and the matrix A is nilpotent. Then, the system (5.50) is $(m-1)$-rank bundle bispectral for each fixed x.

6 The N-Wave Problem and Other Examples

In this section we shall consider the N-wave model. It was studied by V.E. Zaharov and S.V. Manakov [ZM75] by means of a zero curvature representation. It describes various interactions of the wave packets in a medium with quadratic nonlinearity: merger, breakdown, explosive instability, and so on. In the stationary case the N-wave equation reduces to Euler's equation [Man76]. Following [ZM75] and [Man76] an intensive research was conducted, see [FT87] for some references. The linear auxiliary systems for the N-wave equation are of the form $y_x = (\lambda d + V)y$ more general than ZS-AKNS systems.

Suppose now that in the equalities (3.7) and (3.8) we put

$$R = r = 1. \quad (6.1)$$

Formulas (3.14) and (3.16) in this case are essentially simplified:

$$\tilde{q}_1 \equiv -d, \quad \tilde{q}_0(x,t) = dY_0(x,t) - X_0(x,t)d; \quad \tilde{Q}_1 \equiv -D, \quad \tilde{Q}_0(x,t) = DY_0(x,t) - X_0(x,t)D. \quad (6.2)$$

Set

$$X(x,t) := \Pi_2(x,t)^* S(x,t)^{-1} \Pi_1(x,t) \quad (6.3)$$

and recall that $[d, X]$ denotes the commutator $dX - Xd$. According to (2.24) and (6.2) we have

$$X(x,t) = X_0(x,t) = Y_0(x,t); \quad \tilde{q}_0(x,t) = [d, X(x,t)]; \quad \tilde{Q}_0(x,t) = [D, X(x,t)]. \quad (6.4)$$

As the conditions of Theorem 4.1 are satisfied we get

Corollary 6.1 *Let X be defined by (6.3), where Π_1, Π_2 and S are given by (3.10) and (3.11) with $R = r = 1$; $\Pi_1(0,0)$, $\Pi_2(0,0)$ and $S(0,0)$ satisfy operator identity (2.13);*

$\det S(0,0) \neq 0$ *and the matrices* A_1 *and* A_2 *are nilpotent. Then, a one-parameter family of systems*

$$y_x(x,t,\lambda) = (\lambda d - [d, X(x,t)])y(x,t,\lambda) \quad (y \in \mathbb{C}^m) \tag{6.5}$$

is $(m-1)$*-rank bispectral for each* t.

Notice that $G = \lambda d$ and $F = \lambda D$ satisfy the conditions of Theorem 2.2 too. Thus, by Theorem 2.2, the matrix functions

$$\tilde{G}(x,t,\lambda) := \lambda d - [d, X(x,t)], \quad \tilde{F}(x,t,\lambda) := \lambda D - [D, X(x,t)] \tag{6.6}$$

satisfy the compatibility equality (2.25). It's easy to see that in the case (6.6) condition (2.25) takes the form

$$[d, X_t(x,t)] - [D, X_x(x,t)] - [[d, X(x,t)], [D, X(x,t)]] = 0, \tag{6.7}$$

i.e., X satisfies (6.7). Equation (6.7) with $d = d^*$, $D = D^*$ and an additional reduction demand

$$X(x,t) = BX(x,t)^*B \qquad (B = \mathrm{diag}\,\{b_1, \ldots, b_m\}, \quad b_p = \pm 1\,(1 \leq p \leq m)) \tag{6.8}$$

is the physically important N-wave (or nonlinear optics) equation. It's easy to see [Sak93], [Sak94], that if

$$d = d^*, \quad D = D^*, \quad A_2 = -A_1^*, \quad S(0,0) = S(0,0)^*, \quad \Pi_2(0,0) = \Pi_1(0,0)B, \tag{6.9}$$

then we have

$$S(x,t) = S(x,t)^*, \quad \Pi_2(x,t) = \Pi_1(x,t)B, \tag{6.10}$$

and thus, in view of (6.3), the reduction conditions (6.8) are fulfilled.

Corollary 6.2 *Let's assume that the conditions of the Corollary 6.1 are satisfied and a* t*-parameter family of systems (6.5) is given. Assume additionally that conditions (6.9) for* d*,* D *and the set of parameter matrices hold. Then, the family of systems (6.5) evolves by the* N*-wave ((6.7) with reduction (6.8)) flow and it remains* $(m-1)$*-rank bispectral by the flow.*

Let's construct now the examples of the m-rank bispectral canonical and pseudo-canonical systems (5.46) with the irrational in λ families of the bispectral eigenfunctions.

As before we shall consider the canonical and pseudocanonical cases together because the formulas in both cases are rather similar. For this purpose put $m = 2$ and insert

$$v(x,t) = i\exp\{(-1)^{p+1}it\} \qquad p = 0,\, 1. \tag{6.11}$$

One can easily check that this v satisfies the nonlinear Schrödinger equation (5.49). In the case $p = 0$ the "dressing" of our v was used in [TCL84] and [AK86] to construct and study the so called n-modulation solutions of (5.49). See also [Sak94], where the Weyl functions for such solutions were constructed. We shall use three 2×2 matrix functions:

$$C_0(\lambda) := \begin{bmatrix} 1 & 1 \\ i(\lambda + i^p\sqrt{1+(-1)^p\lambda^2}) & i(\lambda - i^p\sqrt{1+(-1)^p\lambda^2}) \end{bmatrix},$$

$$C_1(\lambda) := -i^{p+1}\sqrt{1+(-1)^p\lambda^2}\,J, \quad C_2(t) := \exp\{\frac{1}{2}(-1)^{p+1}itJ\}.$$

These matrices are chosen in such a way, that

$$C_2(t)C_0(\lambda)C_1(\lambda)C_0(\lambda)^{-1}C_2(t)^{-1} = i(\lambda J + JV(x,t)),$$

where v is given in (6.11). Therefore, one can see that

$$w(x,t,\lambda) := C_2(t)C_0(\lambda)\exp\{xC_1(\lambda)\} \tag{6.12}$$

is an 2×2 solution of (5.46). We see also that

$$\frac{\partial w}{\partial \lambda}(x,t,\lambda) = (C_2(t)\frac{\partial C_0}{\partial \lambda}(\lambda) + xC_2(t)C_0(\lambda)\frac{\partial C_1}{\partial \lambda}(\lambda))\exp\{xC_1(\lambda)\}. \tag{6.13}$$

Put now $B(\lambda,t) := B_1(\lambda,t)\frac{\partial}{\partial\lambda} + B_0(\lambda,t)$, $\qquad B_1 := C_2(t)C_0(\lambda)\frac{\partial C_1}{\partial \lambda}(\lambda)^{-1}C_0(\lambda)^{-1}C_2(t)^{-1}$,

$$B_2 := -C_2(t)C_0(\lambda)\frac{\partial C_1}{\partial \lambda}(\lambda)^{-1}C_0(\lambda)^{-1}\frac{\partial C_0}{\partial \lambda}(\lambda)C_0(\lambda)^{-1}C_2(t)^{-1}.$$

Applying now operator B to w and taking into account (6.12) and (6.13) we get the result:

$$B(\lambda,t)w(x,t,\lambda) = xw(x,t,\lambda),$$

i.e. the columns of w are bispectral eigenfunctions of (5.46). Thus systems (5.46) with $v(x,t) = i\exp\{(-1)^{p+1}it\}$ remain 2-rank bispectral by the nonlinear Schrödinger equation flow.

7 Conclusions and Suggestions for Further Research

In this article we have extended the notion of bispectrality for systems with polynomial dependence on the spectral parameter. Such systems are auxiliary to many important integrable nonlinear evolution equations obtained as zero curvature conditions. It turned out that by making use of generalized Darboux transformations we were able to construct families of rational potentials so that the corresponding systems are bispectral. As in earlier works, such as [DG86], [Zub90], the bispectral property of the underlying system was preserved by the flows of the corresponding nonlinear integrable equations. We provided a number of interesting examples where the results apply, such as ZS-AKNS type systems and N-wave equations.

It should be mentioned that the description of the classes of non-constant Θ that work as bispectral parameters with the potentials presented herein is an interesting theme that was not discussed here. This would be a natural follow up of the present work.

We also did not discuss systems related to matrix KdV equations. The methods employed here can be transfered also to the matrix KdV equations, extending in this way the previous results of [DG86] for the scalar case and of [Zub90] for AKNS systems. We shall pursue this in a future publication.

For an important sub-class of ZS-AKNS systems, we provided a spectral characterization of the bispectral potentials obtained by the GBDT method.

Finally there seems to exist an intriguing connection with the results by G. Wilson [Wil98] where operator identities and nilpotent matrices are both used, though in quite different way, to obtain bispectral eigenfunctions associated to the KP hierarchy.

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