Unique Continuation Principles for the Benjamin-Ono Equation.

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Abstract

Let σ denote the Hilbert transform and $\mu \geq 0$. We prove that if $u \in C\left([0,T], H^2(\mathbb{R}) \cap L^2_2(\mathbb{R})\right)$ is a solution of

$$\partial_t u + 2\sigma\partial_x^2 u + u\partial_x u = \mu\partial_x^2 u$$

such that there are $t_0 < t_1 < t_2$ satisfying $u(t_j) \in H^4(\mathbb{R}) \cap L^2_4(\mathbb{R})$, $j = 0,1,2$, then $u(t) = 0$ for all $t \in [0,T]$.

1 Introduction.

Let $H^s(\mathbb{R})$, $s \in \mathbb{R}$, denote the L^2 type Sobolev spaces and let $L^2_r(\mathbb{R}) = (H^r(\mathbb{R}))^{\wedge}$, $r \in \mathbb{R}$, denote the usual weighted L^2 spaces, where $\widehat{f} = Ff$ is the Fourier transform of $f \in \mathcal{S}'(\mathbb{Z})$. Define weighted Sobolev spaces by the formula,

$$\mathcal{F}_{s,r} = \mathcal{F}_{s,r}(\mathbb{R}) = H^s(\mathbb{R}) \cap L^2_r(\mathbb{R}). \tag{1}$$

In [4] (see also [6]), we proved the following results.

Theorem 1 *Let $r \in \mathbb{N}$ and $\phi \in \mathcal{F}_{r,r}$. Then $x^j\partial_x^k\phi \in L^2(\mathbb{R})$ for all $j,k \in \mathbb{N}$ such that $0 \leq j+k \leq r$.*

Theorem 2 *Let $\phi \in \mathcal{F}_{2,2}$ and $\mu \geq 0$. Then*
(a) there exists a unique solution $u \in C([0,\infty), \mathcal{F}_{2,2})$ to the problem

$$\begin{aligned} \partial_t u(t) + 2\sigma\partial_x^2 u(t) + u(t)\,\partial_x u(t) &= \mu\partial_x^2 u(t) \\ u(0) &= \phi, \end{aligned} \tag{2}$$

with the time derivative computed in the norm of $L^2(\mathbb{R})$.
(b) if $u \in C([0,T], \mathcal{F}_{4,4})$, for some $T > 0$, u must be identically zero.

The purpose of this article is to show that the unique continuation principle in the previous statements (part (b)) can be improved as described in the next theorem.

Theorem 3 *Let $u \in C([0,T], \mathcal{F}_{2,2})$ be a solution of (2) and assume that there exists $t_0 < t_1 < t_2$ such that $u(t_j) \in \mathcal{F}_{4,4}$, $j = 0,1,2$. Then $u \equiv 0$.*

There are unique continuation theorems for other evolution equations Saut and Scheurer ([8]) proved that if u is a sufficiently smooth solution of the Korteweg-de Vries equation (KdV)

$$\partial_t u(t) + \partial_x^3 u(t) + u(t)\,\partial_x u(t) = 0 \tag{3}$$

such that $suppu(t) \subset (a,b)$ for all values of t in some interval (τ_1, τ_2) then $u \equiv 0$. Their arguments employ a Carleman type estimate. Bourgain ([1]) reobtained this unique continuation principle using complex variable techniques. Stronger results were obtained in Zhang ([9]) and Kenig Ponce and Vega in ([7]). Both works assume there are $t_1 < t_2$ such that $u(t_j)$, $j = 1,2$, are supported in a half line. It deserves remark that [9] deals with KdV using the inverse scattering transform while in [7] the authors consider the generalized KdV equation, make use of Carlemann type estimates to obtain their results. It should also be noted that ([1]) and ([7]) deal with equations which are much more general than KdV.

This article is organized as follows. In Section 2 we collect some definitions, results and remarks that will be useful in the sequel. Section 3 deals with the $\mathcal{F}_{3,3}$ theory, while in Section 4 we prove Theorem 3.

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2 Preliminaries.

We begin with the Cauchy problem associated to the linear BO-Burgers equation, that is

$$\begin{aligned} \partial_t v(t) + 2\sigma\partial_x^2 v(t) &= \mu\partial_x^2 v(t), \\ v(0) &= \phi. \end{aligned} \tag{4}$$

Before preceding, it is convenient to introduce some definitions and recall a few well known facts. If Y and X are Banach spaces, we will denote the collection of all bounded operators from Y into X by $\mathcal{B}(Y,X)$. If $Y = X$ we write simply $\mathcal{B}(Y)$. The Hilbert transform σ is defined by

$$(\sigma f)(x) = pv\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{y-x}dx. \tag{5}$$

It is possible to show (see [2] or [3] for example) that $\sigma \in \mathcal{B}(L^p(\mathbb{R}))$, $p \in (1,\infty)$ and that

$$(\sigma f)^{\wedge}(\xi) = ih(\xi)\,\widehat{f}(\xi), \ \forall f \in \mathcal{S}(\mathbb{R}), \tag{6}$$

where

$$h(\xi) = \begin{cases} -1, & \xi < 0, \\ 1, & \xi > 0. \end{cases} \tag{7}$$

In view of this formula, we can define

$$\sigma f = \left(ih\widehat{f}\right)^{\vee}, \ \forall f \in H^s(\mathbb{R}). \tag{8}$$

Note that $\sigma : H^s(\mathbb{R}) \longrightarrow H^s(\mathbb{R})$ is unitary for all $s \in \mathbb{R}$ and $\sigma^2 = -1$. With these remarks in mind, the solution of (4) is given by

$$v(t) = E_\mu(t)\phi = \exp(-Q_\mu t)\phi = \left[F_\mu(t,\cdot)\,\widehat{\phi}(\cdot)\right]^{\vee}, \tag{9}$$

where

$$-Q_\mu = \mu\partial_x^2 - 2\sigma\partial_x^2, \tag{10}$$

and

$$F_\mu(t,\xi) = \exp\left[(-\mu + 2ih(\xi))\,\xi^2 t\right]. \tag{11}$$

The following result describes the behavior of $v(t)$ in the spaces $F_{r,r}$.

Theorem 4 *Let $r \in \mathbb{N}$ and $\phi \in \mathcal{F}_{r,r}(\mathbb{R})$. Then for each fixed $\mu \geq 0$ we have*
(a) $v_\mu \in C([0,\infty);\mathcal{F}_{r,r})$ if $r = 0, 1, 2$ and satisfies

$$\|v_\mu(t)\|_{r,r} \leq \Theta_\mu(t)\,\|\phi\|_{r,r}, \tag{12}$$

where $\Theta_\mu(t)$ is a polynomial of degree r with positive coefficients depending only on r and μ.
(b) if $r \geq 3$, the following statements are equivalent

$$v_\mu \in C([0,\infty);\mathcal{F}_{r,r}), \tag{13}$$

$$\exists\, t_1 < t_2 \text{ such that } v_\mu(t_j) \in \mathcal{F}_{r,r},\ j = 1, 2, \tag{14}$$

$$\partial_\xi^j \widehat{\phi}(0) = 0,\ j = 0, 1, ..., r-3, \tag{15}$$

$$\partial_\xi^j \widehat{v}_\mu(t,0) = 0,\ j = 0, 1, ..., r-3,\ \forall t \in [0,\infty). \tag{16}$$

In this case an estimate of the form (12) also holds.

Proof. The proof of this result is contained in ([4]) and ([6]). It consists in combining Leibniz's rule with Theorem 1 and the formulas

$$\partial_{\xi} F_{\mu}(t,\xi) = (-2t\xi)(\mu - 2ih(\xi)) F_{\mu}(t,\xi) \tag{17}$$

$$\partial_{\xi}^{2} F_{\mu}(t,\xi) = -2t(\mu - 2ih(\xi)) F_{\mu}(t,\xi) + (-2t\xi)^{2}(\mu - 2ih(\xi))^{2} F_{\mu}(t,\xi). \tag{18}$$

$$\begin{aligned} \partial_{\xi}^{3} F_{\mu}(t,\xi) &= 8it\delta + 3(-2t)^{2}\xi^{2}(\mu - 2ih(\xi))^{2} F_{\mu}(t,\xi) \\ &+ (-2t)^{3}\xi^{3}(\mu - 2ih(\xi))^{3} F_{\mu}(t,\xi) \end{aligned} \tag{19}$$

$$\partial_{\xi}^{j} F_{\mu}(t,\xi) = 8it\delta^{(j)} + \sum_{k=1}^{j} a_k t^k \xi^{m(k)} (\mu - 2ih(\xi))^{k} F_{\mu}(t,\xi)\ ,\ j \geq 4, \tag{20}$$

where δ denotes Dirac's delta function, $a_k \in \mathbb{R}$ and $m(k) \in \{1,2,...,k\}$, to show that for any fixed $t \in (0,\infty)$, and $r \geq 3$ we have

$$(x^{r} v_{\mu}(t))^{\wedge}(\xi) = 8it \sum_{j=0}^{r-3} c_j \delta^{(j)} \phi^{(j)} + (R(t))^{\wedge}(\xi), \tag{21}$$

with $c_j \in \mathbb{R}$ for all values of j and $R \in C\left([0,\infty), L^{2}(\mathbb{R})\right)$. ■

It follows that if

$$\phi,\ u(t_0) \in \mathcal{F}_{\infty,\infty} = \cap_r \mathcal{F}_{r,r} \tag{22}$$

for some $t_0 > 0$, then $\widehat{\phi}$ has a zero of infinite order at $\xi = 0$. Thus, if ϕ has compact support we conclude that $\phi = 0$ because

$$\widehat{\phi}(z) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-a}^{a} \exp(-izx)\,\phi(x)\,dx \tag{23}$$

is an entire function for all a sufficiently large. Hence $u(t) = 0$ for all t.

Note that the results in Theorems 2 and 3 are far stronger than the ones just described. In the case of Theorem 2, the proof is roughly as follows. First we prove that if u is as in the statement of the theorem, then we must have $\widehat{u}(t,0) = \partial_{\xi}\widehat{u}(t,0) = 0$ for all $t \in [0,T]$. Then we observe that the time evolution of the quantity

$$(2\pi)^{\frac{1}{2}} \partial_{\xi}\widehat{u}(t,0) = \int_{\mathbb{R}} xu(t,x)\,dx \tag{24}$$

can be computed explicitly. Indeed, differentiating with respect to time, using the differential equation and integrating by parts we obtain

$$\partial_t \int_{\mathbb{R}} xu(t,x)\,dx = \frac{1}{2}\|u(t)\|_{0}^{2}. \tag{25}$$

Therefore

$$\int_{\mathbb{R}} xu(t,x)\,dx = \int_{\mathbb{R}} x\phi(x)\,dx + \frac{1}{2}\int_0^t \|u(t')\|_0^2\,dt'. \tag{26}$$

Since $\widehat{u}(t,0) = \partial_\xi \widehat{u}(t,0) = 0$ for all $t \in [0,T]$, it follows that $\|u(t')\|_0^2 = 0$ for all $t' \in [0,T]$. As we shall see, the proof of Theorem 3 is a refinement of this argument. Note that (26) suggests that all we need to show is $\partial_\xi \widehat{u}(\tau_j, 0) = 0$, $j = 1,2$ with $\tau_1 < \tau_2$.

3 The $\mathcal{F}_{3,3}$ Theory.

Let $E_\mu(t)$ be as in (9). Then the Cauchy problem

$$\begin{aligned} \partial_t u(t) + 2\sigma\partial_x^2 u(t) + u(t)\,\partial_x u(t) &= \mu\partial_x^2 u(t) \\ u(0) &= \phi \in \mathcal{F}_{2,2}, \end{aligned} \tag{27}$$

is equivalent to the integral equation

$$u(t) = E_\mu(t)\,\phi - \int_0^t E_\mu(t-t')\,u(t')\,\partial_x u(t')\,dt'. \tag{28}$$

Remark 5 *This equivalence is strictly true if $\mu > 0$ due to the regularizing properties of $E_\mu(t)$. If $\mu = 0$, we regard (28) as an identity in $H^1(\mathbb{R})$ because of the loss of derivatives in the integral term. This will not hamper the arguments that follow.*

Our strategy in this and the next section consists in multiplying the integral equation by x^k, $k = 3,4$, using Leibniz's rule together with formulas (17)-(20) to separate the singular and regular parts of $\left(x^k u(t)\right)^\wedge$ to prove that certain moments must be zero. In this connection, it is convenient to introduce the following (non-standard) definition.

Definition 6 *Let*

$$\mathbb{R}_+^n = \{\theta = (\theta_1, \theta_2, ..., \theta_n)\,|\theta_\ell \geq 0, \ell = 1, 2..., n\}. \tag{29}$$

If $n = 1$ we write $\mathbb{R}_+^1 = \mathbb{R}_+$. A function $R : \mathbb{R}_+^n \longrightarrow \mathcal{S}'(\mathbb{R})$ is called a regular map if there exists a finite number of (Banach) subspaces $X_j \subset \mathcal{S}'(\mathbb{R})$, $j = 1,2,...,N$ and maps $R_j \in C\left(\mathbb{R}_+^n, X_j\right)$ such that

$$R = \sum_{j=1}^N R_j \tag{30}$$

and $\widehat{R}_j(t) = (R_j(t))^\wedge$ is a measurable function for all $j = 1,2,...,N$. In particular, $\widehat{R}(t) = (R(t))^\wedge$ is a measurable function. We will write $R \in reg(N, X_1, X_2, ..., X_N.)$

Remark 7 *In the sequel, the letter R will be reserved to denote regular maps, whose precise form is not relevant to our arguments. Thus, in different formulas, R may represent different regular maps, but in all instances R has the properties described in Definition 6.*

We are now in position to prove the main result of this section.

Theorem 8 *Let $u \in C([0,\infty), \mathcal{F}_{2,2})$ be the solution described in Theorem 2, and assume that there exist $0 \leq t_0 < t_1$ such that $u(t_j) \in \mathcal{F}_{3,3}$, $j = 0, 1$. Then $\widehat{u}(t,0) = 0$ for all $t \in [0,\infty)$.*

Proof. Assume, without loss of generality that $t_0 = 0$. Multiplying (28) by x^3 we obtain

$$x^3 u(t) = x^3 E_\mu(t)\,\phi - \int_0^t x^3 E_\mu(t-t')\,u(t')\,\partial_x u(t')\,dt' \forall\, t \in [0,\infty). \tag{31}$$

Combining the assumption that $\phi \in \mathcal{F}_{3,3}$ with

$$\left(x^3 E_\mu(t)\,\phi\right)^\wedge(\xi) = -i\partial_\xi^3\left(F_\mu(t,\xi)\,\widehat{\phi}(\xi)\right) = -i\sum_{j=0}^{3} c_j \partial_\xi^j F_\mu(t,\xi)\,\partial_\xi^{3-j}\widehat{\phi}(\xi) \tag{32}$$

and with formulas (17)-(19) we conclude that there exists a constant $C > 0$ and a regular map $R \in reg\left(1, L^2(\mathbb{R})\right)$ such that

$$x^3 E_\mu(t)\,\phi = Ct\widehat{\phi}(0) + R(t),\ \forall t \in [0,\infty). \tag{33}$$

The nonlinear term is more difficult to handle because $u(t)\,\partial_x u(t) \notin \mathcal{F}_{3,3}$ in general. In fact all we can say now is $u(t)\,\partial_x u(t) \in \mathcal{F}_{1,2}$. However, combining this property with Theorems 1, 2 (part (a)), Leibniz's rule, (17)-(19) and the fact that the product of $\varphi \in H^s(\mathbb{R})$ with $f \in H^{-s}(\mathbb{R})$, $s > \frac{1}{2}$ is well defined and belongs to H^{-s}, one is able to show that

$$\begin{aligned} &x^3 E_\mu(t-t')\,u(t')\,\partial_x u(t') \\ &= C(t-t')\left(u(t')\,\partial_x u(t')\right)^\wedge(0) + R(t,t'),\forall t,t' \in [0,\infty), \end{aligned} \tag{34}$$

where $R \in reg\left(2, H^{-2}(\mathbb{R}), L^1(\mathbb{R})\right)$. But

$$\left(u(t')\,\partial_x u(t')\right)^\wedge(0) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} u(t',x)\,\partial_x u(t',x)\,dx = 0,\ \forall t' \in [0,\infty). \tag{35}$$

Therefore (27) together with (31)-(33) imply that

$$x^3 u(t) = Ct\widehat{\phi}(0) + R(t),\ \forall t \in [0,\infty). \tag{36}$$

By assumption $x^3 u(t_2) \in L^2(\mathbb{R})$ so that

$$\left(x^3 u(t_2) - R(t)\right)^\wedge = Ct\widehat{\phi}(0)\,\delta, \tag{37}$$

must be measurable. But this is true if and only if $\widehat{\phi}(0) = 0$. Since

$$\widehat{u}(t,0) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} u(t,x)\, dx, \tag{38}$$

is a conserved quantity, the proof is complete. ■

In view of the preceding theorem it is natural to look for solutions of (2) in

$$\widetilde{\mathcal{F}}_{3,3} = \left\{ \psi \in \mathcal{F}_{3,3} \,\middle|\, \widehat{\psi}(0) = 0 \right\}. \tag{39}$$

As already proved in [4], we have the following result.

Theorem 9 *Assume that $\phi \in \widetilde{\mathcal{F}}_{3,3}$ and let $\mu \geq 0$ be fixed. Then there exists a unique $u \in C\left([0,\infty), \widetilde{\mathcal{F}}_{3,3}\right)$ satisfying (2) with the time derivative computed in the norm of $H^1(\mathbb{R})$.*

4 The Unique Continuation Principle.

We will now prove Theorem 3. In view of 26 we must show that there exist $\tau_1 < \tau_2$ such that

$$\int_{\mathbb{R}} xu(\tau_j, x)\, dx = 0,\ j = 1, 2.$$

Once again, it is enough to assume that $t_0 = 0$. Since ϕ and $u(t_1)$ both belong $\mathcal{F}_{4,4} \hookrightarrow \mathcal{F}_{3,3}$, Theorem (8) implies that

$$\widehat{u}(t,0) = \widehat{\phi}(0) = 0 \forall t \in [0,\infty), \tag{40}$$

Thus $u \in C\left([0,\infty), \widetilde{\mathcal{F}}_{3,3}\right)$ in view of Theorem 9. Multiply (28) by x^4 and consider first the linear term. Since $\phi \in \mathcal{F}_{4,4}$ we get

$$\begin{aligned} \left(x^4 E_\mu(t)\,\phi\right)^{\wedge}(\xi) &= \partial_\xi^4 \left(F_\mu(t,\xi)\,\widehat{\phi}(\xi)\right) \\ &= \sum_{j=0}^{4} c_j \partial_\xi^j F_\mu(t,\xi)\, \partial_\xi^{4-j} \widehat{\phi}(\xi) \\ &= 8it\delta' \widehat{\phi} + c_3 8it\delta \widehat{\phi}' + \widehat{R}(t), \end{aligned} \tag{41}$$

and $R \in reg\left(1, L^2(\mathbb{R})\right)$. Let $\psi \in \mathcal{F}_{4,4}$. Then for all appropriate test functions φ we have

$$\begin{aligned} \langle \psi\delta', \varphi \rangle &= \langle \delta', \psi\varphi \rangle = -\left\langle \delta, (\psi\varphi)' \right\rangle \\ &= -\left(\psi'\varphi + \psi\varphi'\right)(0) \\ &= -\psi'(0)\,\varphi(0) - \psi(0)\,\varphi'(0) \\ &= -\langle \psi'(0)\,\delta, \varphi \rangle - \langle \psi(0)\,\delta, \varphi' \rangle = \\ &= -\langle \psi'(0)\,\delta, \varphi \rangle + \langle \psi(0)\,\delta', \varphi \rangle. \end{aligned} \tag{42}$$

Therefore,

$$\psi\delta' = -\psi'(0)\,\delta + \psi(0)\,\delta'. \tag{43}$$

Taking $\psi = \widehat{\phi}$ it follows that $\widehat{\phi}\delta' = -\widehat{\phi}'(0)\,\delta$, because of (40). Hence (41) becomes

$$\left(x^4 E_\mu(t)\,\phi\right)^\wedge(\xi) = -8it\,(1+c_3)\,\widehat{\phi}'(0)\,\delta + \widehat{R}(t),\ \forall t\in[0,\infty), \tag{44}$$

with $R \in reg\left(1, L^2(\mathbb{R})\right)$. Now we turn to the nonlinear term. Using the same ingredients as in the proof of (34) we obtain

$$\begin{aligned}
&\left(x^4 E_\mu(t-t')\,u(t')\,\partial_x u(t')\right)^\wedge(\xi)\\
&= \underbrace{-8i\,(t-t')\,\delta'\left(u(t')\,\partial_x u(t')\right)^\wedge(\xi) + \widehat{R}}_{j=4}\\
&\underbrace{-8i\,(t-t')\,\delta c_3\partial_\xi\left(u(t')\,\partial_x u(t')\right)^\wedge(\xi) + \widehat{R}}_{j=3} + \underbrace{\widehat{R}}_{j\le 2}.
\end{aligned} \tag{45}$$

where $R \in reg\left(2, H^{-4}(\mathbb{R}), L^1(\mathbb{R})\right)$. Since

$$\left(u(t')\,\partial_x u(t')\right)^\wedge(0) = 0 \tag{46}$$

(43) and (45) imply that

$$\begin{aligned}
&\left(x^4 E_\mu(t-t')\,u(t')\,\partial_x u(t')\right)^\wedge(\xi)\\
&= -8i\,(t-t')\,\delta\,(1+c_3)\,\partial_\xi\left(u(t')\,\partial_x u(t')\right)^\wedge(0) + \widehat{R}.
\end{aligned} \tag{47}$$

Multiplying (27) by x^4 and using (44) and (47) we get

$$\begin{aligned}
&\left(x^4 u(t)\right)^\wedge(\xi)\\
&= -8it\delta\,(1+c_3)\,\widehat{\phi}'(0) + \int_0^t 8i\,(t-t')\,\delta\,(1+c_3)\,\partial_\xi\left(u(t')\,\partial_x u(t')\right)^\wedge(0)\,dt' + \widehat{R}
\end{aligned} \tag{48}$$

$$= -8i\,(1+c_3)\,\delta\left[t\widehat{\phi}'(0) - \int_0^t (t-t')\,\partial_\xi\left(u(t')\,\partial_x u(t')\right)^\wedge(0)\,dt'\right] + \widehat{R},$$

with $R \in reg\left(2, H^{-4}(\mathbb{R}), L^1(\mathbb{R})\right)$. Now note that the following relations hold,

$$\partial_\xi\widehat{\phi}(0) = \frac{-i}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} x\phi(x)\,dx, \tag{49}$$

$$\begin{aligned}
\partial_\xi\left(u(t)\,\partial_x u(t)\right)^\wedge(0) &= \frac{-i}{(2\pi)^{\frac{1}{2}}}\int_{\mathbb{R}} x\partial_x\frac{u^2(t)}{2}dx\\
&= \frac{i}{(2\pi)^{\frac{1}{2}}}\frac{\|u(t)\|_0^2}{2},\forall t\in[0,\infty)
\end{aligned} \tag{50}$$

$$\partial_t \int_{\mathbb{R}} xu(t)\,dx = \frac{\|u(t)\|_0^2}{2}, \forall t \in [0,\infty). \tag{51}$$

Combining (48)-(51) we obtain

$$\begin{aligned} &\left(x^4 u(t)\right)^{\wedge}(\xi) \\ &= K\delta\left[t\int_{\mathbb{R}} x\partial_x\phi(x)\,dx + \int_0^t (t-t')\frac{\|u(t')\|_0^2}{2}dt'\right] + \widehat{R}, \end{aligned} \tag{52}$$

where

$$K = \left(\frac{i(1+c_3)}{(2\pi)^{\frac{1}{2}}}\right)8i. \tag{53}$$

Integrating by parts in (52) with respect to t' we get

$$\begin{aligned} &\left(x^4 u(t)\right)^{\wedge}(\xi) = K\delta\left(t\int_{\mathbb{R}} x\phi(x)\,dx\right) \\ &+ K\delta\left(\left[(t-t')\int_{\mathbb{R}} xu(t')\,dx\right]_0^t - \int_0^t (\partial_{t'}(t-t'))\int_{\mathbb{R}} xu(t')\,dxdt'\right) + \widehat{R} \\ &= K\delta\left[t\int_{\mathbb{R}} x\phi(x)\,dx - t\int_{\mathbb{R}} x\phi(x)\,dx + \int_0^t\left(\int_{\mathbb{R}} xu(t')\,dx\right)dt'\right] + \widehat{R} \\ &= K\delta\int_0^t\left(\int_{\mathbb{R}} xu(t')\,dx\right)dt' + \widehat{R}. \end{aligned} \tag{54}$$

Since $x^4 u(t_1) \in L^2(\mathbb{R})$, (54) implies that

$$\int_0^{t_1}\left(\int_{\mathbb{R}} xu(t')\,dx\right)dt' = 0 \tag{55}$$

Therefore,

$$F(t) = \int_0^t\left(\int_{\mathbb{R}} xu(t')\,dx\right)dt', \tag{56}$$

satisfies $F(0) = F(t_1) = 0$. It follows that there exists a $\tau_1 \in (0,t_1)$ such that

$$\int_{\mathbb{R}} xu(\tau_1)\,dx = 0. \tag{57}$$

Now repeat the preceeding argument with ϕ replaced by $u(t_1)$ and $u(t_1)$ replaced by $u(t_2)$ to obtain τ_2 such that

$$\int_{\mathbb{R}} xu(\tau_2)\,dx = 0$$

This concludes the proof.

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