Chapter 1

The intermittency route to chaotic dynamics

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To Floris, whose work has been a continuous source of inspiration

The expression *intermittency* describes a mechanism of transition from simple behaviour to turbulence in dissipative convective fluids, and many other dissipative dynamical systems. The pioneer work of Pomeau, Manneville [26] analyzed intermittency in the Lorenz model, as well as in families of systems unfolding a saddle-node, a flip, or a Hopf bifurcation. Their article presented numerical evidence indicating that in these bifurcations the Lyapunov exponent grows continuously from zero beyond the bifurcation threshold.

A conceptual formulation of intermittency in a broad setting was proposed by Floris Takens in [30]: An arc (1-parameter family) of diffeomorphisms $(\phi_\mu)_\mu$ on a manifold has an *intermittency bifurcation for* $\mu = \mu_0$ *at a compact invariant set* K if

- for every $\mu < \mu_0$ the diffeomorphism ϕ_μ has an attracting compact set K_μ (not necessarily transitive), converging to K in the Hausdorff sense when μ tends to μ_0 from below;
- for $\mu > \mu_0$ close to μ_0 there are no ϕ_μ-attracting sets near K, yet the ϕ_μ-orbit of Lebesgue almost every point in a neighbourhood of K returns close to K infinitely often.

Such bifurcations are accompanied by profound changes of the dynamics, both at the local level (in a neighbourhood of the compact set K) and

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at the global level. As we shall see, these global changes are mainly influenced by the way points return to the vicinity of K, for the bifurcation parameter.

The best studied situations correspond to the case where the set K consists of a unique fixed (or periodic) orbit, of *saddle-node* type: one multiplier is equal to 1, all the others are less than 1 in norm. This is also the setting we have in mind in this review, specially when the global recurrence stems from the presence of a *cycle*, that is periodic points with cyclic intersections of their stable and unstable manifolds.

Other interesting cases include, for example, transitions from Anosov to derived from Anosov diffeomorphisms [29, 32], as well as certain bifurcations of partially hyperbolic sets in dimension 3 or bigger.

1.1 Saddle-nodes of diffeomorphisms

1.1.1 Definitions and basic facts

A *saddle-node* of a C^r diffeomorphism $\phi : M \to M$ is a fixed (or periodic) point P of ϕ, such that $D\phi(P)$ has a multiplier equal to 1 and all the others less than 1 in norm. The tangent space $T_P M$ splits into two $D\phi$-invariant spaces, the one-dimensional *central* space E^c, which is the eigenspace associated to the multiplier 1, and the *stable* space E^{ss}, corresponding to the remaining multipliers.

By normal hyperbolicity theory [11, 19], there are locally invariant immersed *central manifold* W^c and *strong stable manifold* W^{ss}, tangent at P to E^c and to E^{ss}, respectively. The stable manifold is unique and of class C^r. In general, there are several central manifolds, and they may be less smooth than the diffeomorphism ϕ.

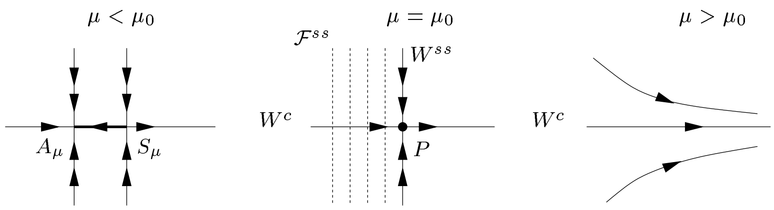

Figure 1.1. Local dynamics at a saddle-node bifurcation

It is part of the definition of saddle-node that, for some choice of W^c, the restriction of ϕ to the central manifold has a non-vanishing 2-jet at P: there is a coordinate x on W^c (with P corresponding to $x = 0$) such that

$$\phi(x) = x + \alpha\, x^2 + \mathcal{O}(|x|^3) \quad \text{with } \alpha \neq 0.$$

Then P is a semi-attractor restricted to W^c, as depicted at the center of Figure 1.1, and it also follows that the central manifold is of class C^r. The unstable manifold W^u of P is an immersed half-line contained in W^c. The stable manifold W^s is a closed half-space with W^{ss} as its boundary.

Moreover, there is a unique ϕ-invariant foliation of the stable manifold of P by co-dimension 1 sub-manifolds having W^{ss} as a leaf. It is called the *strong stable foliation* $\mathcal{F}^{ss}$ of the saddle-node.

1.1.2 Unfolding saddle-nodes

Saddle-nodes are obtained by collapsing a saddle S_μ and a periodic attractor (node) A_μ into a single point, as described in Figure 1.1. Afterwards, the periodic points disappear and there is no attracting set in the region where the saddle-node P was formed. The first part of Takens' definition of intermittency is fulfilled taking $K = P$ and K_μ to be the closure of the separatrix connecting S_μ to A_μ. To have the second one, we shall assume later that the saddle-node is part of a cycle.

An arc of diffeomorphisms $(\phi_\mu)_\mu$ *unfolds generically* a saddle-node P of a diffeomorphism $\phi = \phi_{\mu_0}$ if it cuts the hyper-surface of diffeomorphisms with a saddle-node point transversely at ϕ. Here is an alternative formulation, in terms of local expressions.

One considers a continuation W^c_μ of the central manifold W^c, for nearby parameter values (this exists because the invariant manifold W^c is normally hyperbolic [11] for the diffeomorphism $\phi = \phi_{\mu_0}$). Generic unfolding means that, up to a convenient choice of coordinates x in W^c_μ, and a re-parameterization of the family, the restriction of ϕ_μ to W^c_μ has the form

$$\phi_\mu(x) = x + \mu + \alpha\, x^2 + \beta\, x\, \mu + \gamma\, \mu^2 + \mathcal{O}(|\mu|^3 + |x|^3).$$

After re-parameterization, the bifurcation parameter has become $\mu = 0$. From now on we shall always consider $\mu_0 = 0$.

The notion of saddle-node may be extended to include other non-hyperbolic periodic points obtained by collapsing two saddle-points with different stable dimensions: they have a unique multiplier equal to 1, and all the others are different from 1 in norm. See [7] for results in this setting.

1.1.3 Saddle-node cycles

A diffeomorphism ϕ has a *saddle-node k-cycle*, $k \in \mathbb{N}$, if there are a saddle-node p_0 and hyperbolic periodic saddles $p_1, \ldots, p_{k-1}$, such that $W^u(p_{j-1})$ intersects transversely $W^s(p_j)$ for every j and $W^u(p_{k-1})$ meets $W^s(p_0)$. The cycle is *critical* if $W^u(p_{k-1})$ is non-transverse to the strong stable foliation of the saddle-node. Otherwise, it is called *non-critical*. Figure 1.2 exhibits three different types of saddle-node cycles: from left to right we have a critical 1-cycle, a *critical saddle-node horseshoe*, and a non-critical 2-cycle (non-critical saddle-node horseshoe).

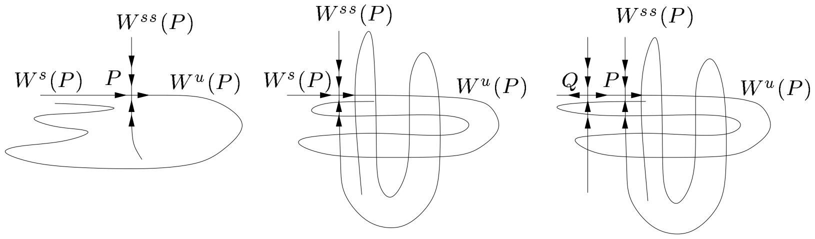

Figure 1.2. Saddle-node cycles

An arc of diffeomorphisms $(\phi_\mu)_\mu$ *unfolds generically a saddle-node cycle* of $\phi = \phi_0$ if it unfolds generically the saddle-node p_0 involved in that cycle. This is a remarkably rich mechanism of bifurcation. For instance,

Theorem 1. (Newhouse, Palis, Takens [19]) If an arc $(\phi_\mu)_\mu$ of surface diffeomorphisms unfolds generically a critical saddle-node cycle of ϕ_0, then there is a sequence of parameters $\nu_n \to 0$ such that, for every ν_n, the diffeomorphism ϕ_{ν_n} has a homoclinic tangency which is unfolded generically by the family $(\phi_\mu)_\mu$.

This result extends to arbitrary dimension, see [8]. Moreover, the converse is also true (L. Mora): the generic unfolding of a homoclinic tangency by a family of surface diffeomorphisms always includes the formation and generic unfolding of critical saddle-node cycles.

From Theorem 1 one deduces that any phenomena occurring during a homoclinic bifurcation (e.g. the creation of attractors) are also present when a critical saddle-node cycle is unfolded. However, saddle-node bifurcations have a very distinctive feature, that we state as the following informal principle: *persistent phenomena (positive Lebesgue measure of values of μ) are, actually, prevalent (positive Lebesgue density at $\mu = 0$).* More precise statements and an explanation of the mechanism behind this property are provided in the next sections.

1.1.4 Persistence and prevalence

Let $(\phi_\mu)_\mu$ be an arc of diffeomorphisms on a manifold M, going through some bifurcation at $\mu = 0$. Let $\mathcal{P}$ be some dynamical property, like hyperbolicity, co-existence of infinitely many sinks, or presence of non-hyperbolic strange attractors.

The property $\mathcal{P}$ is *persistent* after the bifurcation if for every $\varepsilon > 0$ the subset $E_\varepsilon \subset [0, \varepsilon]$ of parameter values for which ϕ_μ verifies $\mathcal{P}$ has positive

Lebesgue measure. $\mathcal{P}$ is called *prevalent* at the bifurcation if

$$\liminf_{\varepsilon \to 0} \frac{|E_\varepsilon|}{\varepsilon} > 0,$$

where $|E_\varepsilon|$ denotes the Lebesgue measure of E_ε. Finally, $\mathcal{P}$ is *fully prevalent* if the limit is 1.

For instance, Newhouse, Palis, Takens [18, 20, 21] prove that hyperbolicity is fully prevalent in arcs of surface diffeomorphisms unfolding homoclinic tangencies associated to hyperbolic sets with Hausdorff dimension less than 1. This is not true if the Hausdorff dimension is bigger than 1, by Palis, Yoccoz [23], but the union of hyperbolicity and persistent tangencies (Newhouse's phenomenon [17]) is always fully prevalent at homoclinic bifurcations in dimension 2, by Moreira, Yoccoz [15].

In the same setting, Mora, Viana [13] proved that existence of non-hyperbolic strange attractors is a persistent phenomenon. By a recent result of Palis, Yoccoz [25], it can not be prevalent. On the other hand, as we shall see in a while, non-hyperbolic strange attractors are always a prevalent phenomenon in the unfolding of critical saddle-node cycles. This is a striking realization of the informal principle we stated before: In saddle-node bifurcations, persistent properties tend to be prevalent.

This remarkable feature results from the existence of a *repetition pattern in parameter space* that is characteristic of intermittency bifurcations: One can find sequences μ_n converging to the bifurcation value 0 such that the arcs obtained by restricting the parameter to each interval $[\mu_{n+1}, \mu_n]$ have roughly the same dynamics for all large n, up to convenient parameterization.

This is properly explained by means of the following construction of Newhouse, Palis, Takens [19], that plays a crucial role in the sequel. For clarity, we shall restrict ourselves to the case of surface diffeomorphisms. However, this construction extends to any dimension [8].

1.2 Transition maps

Let $(\phi_\mu)_\mu$ be an arc of diffeomorphisms unfolding generically a saddle-node of $\phi = \phi_0$. Fix, once and for all, a continuation W^c_μ of a central manifold, and a coordinate system x in each W^c_μ so that

$$\phi_\mu(x) = x + \mu + \alpha\, x^2 + \beta\, x\, \mu + \gamma\, \mu^2 + \mathcal{O}(|\mu|^3 + |x|^3).$$

It is no restriction to assume $\alpha > 0$. Then, for $\mu = 0$, the subsets $\{x < 0\}$ and $\{x > 0\}$ of the central manifold of the saddle-node are contained in its stable and unstable manifolds, respectively. See Figure 1.1.

1.2.1 Finite-time transition maps

For $\mu = 0$, the presence of the fixed point prevents the transition of orbits from the left $\{x < 0\}$ to the right hand side $\{x > 0\}$. However, this obstruction disappears when the parameter μ becomes positive. We can then define transition maps, in the following way.

Fix compact fundamental domains $D^- \subset \{x < 0\}$ and $D^+ \subset \{x > 0\}$ of ϕ_μ restricted to W^c_μ. Their dependence on μ is not relevant here, so we omit it in our notations. For each $\mu > 0$ let $k = k(\mu)$ be the smallest integer such that $\phi^k_\mu(D^-)$ intersects D^+. As μ decreases to zero, more and more iterates are needed for D^- to reach D^+, which means that $k(\mu) \to \infty$ as μ tends to zero from above. There is a decreasing sequence of parameters $\mu_n \to 0$ such that $k(\mu) = n$ for all $\mu \in [\mu_{n+1}, \mu_n)$ and $\phi^n_{\mu_n}(D^-) = D^+$. See Figure 1.3.

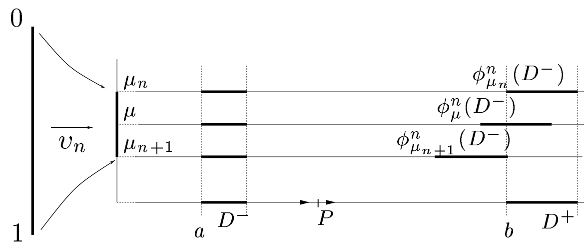

Figure 1.3. Dynamical normalizations of parameter space

It is useful to identify points in $\{x < 0\}$ if they are in the same orbit of ϕ_μ, and similarly in $\{x > 0\}$, and we shall often do it in the sequel. This identification turns D^- and D^+ into smooth circles. For each large n and $\mu \in [\mu_{n+1}, \mu_n)$, we consider the circle map

$$\tilde{T}_n(\mu, \cdot) : D^- \to D^+$$

induced by the nth iterate ϕ^n_μ, and call it the *time-n transition map* of the saddle-node arc $(\phi_\mu)_\mu$.

The repetition pattern we announced before comes from the fact that these arcs of finite-time transitions behave roughly the same when n is large: up to dynamically defined normalizations of the domain in parameter space, the arcs $\tilde{T}_n$ converge to some limit T_∞ when n tends to infinity.

1.2.2 Parameter normalization and infinite-time transition

A 1-parameter family of vector fields $(X_\mu)_\mu$ is a *saddle node arc* if (in local coordinates around the origin) the vector fields are of the form

$$X_\mu(x) = \mu + \alpha\, x^2 + \beta\, x\, \mu + \gamma\, \mu^2 + \mathcal{O}(|\mu|^3 + |x|^3),$$

for some constants α, β, γ with $\alpha > 0$.

The arc $(X_\mu)_\mu$ is *adapted* to $(\phi_\mu)_\mu$ if $\phi_\mu(x)$ coincides with $X_\mu^1(x)$ for all $\mu \geq 0$ and x close to zero, where X_μ^1 is the time-1 flow map of the vector field X_μ ([19] use a weaker condition, the present definition is from [8]). For the existence of adapted arcs of vector fields see [12, 33].

Let us write $D^- = [a, \phi_\mu(a)]$ and $D^+ = [b, \phi_\mu(b)]$ with $a < 0 < b$. By the definition of the μ_n, the point $\phi_{\mu_n}^{n+1}(a)$ coincides with the right end-point $\phi_\mu(b)$ of D^+, whereas $\phi_{\mu_{n+1}}^{n+1}(a)$ coincides with the left end-point b of D^+. Moreover,

$$[\mu_{n+1}, \mu_n] \ni \mu \mapsto \phi_\mu^{n+1}(a) \in D^+$$

is increasing (if n is large).

For each $\mu \in [\mu_{n+1}, \mu_n]$ we denote $\xi_n(\mu)$ the time the flow of the adapted arc of vector fields X_μ takes to go from $\phi_\mu^{n+1}(a)$ to $\phi_\mu(b)$. That is,

$$X_\mu^{\xi_n(\mu)}\left(\phi_\mu^{n+1}(a)\right) = \phi_\mu(b) \quad \Leftrightarrow \quad X_\mu^{n+\xi_n(\mu)}(a) = b$$

ξ_n maps $[\mu_{n+1}, \mu_n]$ onto $[0, 1]$ in a decreasing fashion. We define the nth parameter space normalization $\upsilon_n: [0, 1] \to [\mu_{n+1}, \mu_n]$ to be the inverse of this map ξ_n.

The adapted arc $(X_\mu)_\mu$ also allows us to exhibit *infinite-time transition maps* $T_\infty: [0, 1] \times D^- \to D^+$, given by

$$T_\infty(\sigma, x) = X_0^{t(x)-\sigma}(b),$$

where $t(x)$ is the time the flow spends from a to x, that is, $X_0^{t(x)}(a) = x$. Keep in mind that we think of D^- and D^+ as circles, under identifications of points in the same orbit.

Note that, if one takes $t(x) \bmod 1$ as a new coordinate in D^- and, similarly, considers the time the flow of X_0 takes to go from b to any point in D^- as a new coordinate in D^-, these $T_\infty(\sigma, \cdot)$ become circle isometries. In fact, each $T_\infty(\sigma, \cdot)$ is obtained composing $T_\infty(0, \cdot)$ with the rigid rotation of angle $-\sigma$.

1.2.3 Convergence and distortion properties

Let T_n be the arcs of transformations from D^- to D^+ obtained by re-parameterizing the finite-time transitions $\tilde{T}_n$ according to υ_n:

$$T_n: [0, 1] \times D^- \to D^+, \quad T_n(\sigma, x) = \tilde{T}(\upsilon_n(\sigma), x).$$

That is, $T_n(\sigma, \cdot)$ is the map induced by the restriction of $\phi_{\upsilon_n(\sigma)}^n(x)$ to the central manifold, in the quotient spaces obtained by identifying points in the same orbit, on $\{x < 0\}$ and on $\{x > 0\}$. Here is the convergence statement we had announced:

Theorem 2. (Newhouse, Palis, Takens [19], Díaz, Rocha, Viana [8]) The sequence of maps

$$T_n: [0,1] \times D^- \to D^+$$

converges to $T_\infty: [0,1] \times D^- \to D^+$ in the C^r-topology when $n \to \infty$.

Most important for the kind of problems we want to deal with, the re-parameterizations υ_n have uniformly bounded distortion:

Proposition 3. ([8, Proposition 2.2]) For every $\varepsilon > 0$ there is n_0 such that

$$(1-\varepsilon)|A| < \frac{|\upsilon_n(A)|}{\mu_n - \mu_{n+1}} < (1+\varepsilon)|A|$$

for every measurable subset A of $[0,1]$ and every $n \geq n_0$.

We have been concerned only with the dynamics restricted to the central manifold. The reason is that the dynamics of the transition maps transverse to W^c_μ vanishes when μ approaches zero: all that is left is the dynamics along the central manifold, described by T_∞. Here is a more precise explanation.

Consider neighbourhoods C^- and C^+ of D^- and D^+. If C^- and C^+ are conveniently chosen, their quotients after identification of points in the same orbit (that we continue denoting in the same way) are diffeomorphic to cylinders $D^\pm \times [-1,1]$. Define $\hat{T}_n(\sigma,\cdot)$ to be the map from C^- to C^+ induced by the diffeomorphism $\phi^n_{\upsilon_n(\sigma)}(x)$ (now we do not restrict to the central manifold). Since our diffeomorphisms are contracting transversely to the central manifold, the image of $\hat{T}_n(\sigma,\cdot)$ gets closer and closer to the equator $D^+ \times \{0\}$ of C^+ when n increases. Indeed, we have the following higher dimensional version of Theorem 2:

Theorem 4. ([8, Theorem 2.6]) The sequence $\hat{T}_n: [0,1] \times C^- \to \times C^+$ converges to the arc

$$\hat{T}_\infty: [0,1] \times C^- \to C^+, \quad \hat{T}_\infty(\sigma,x,y) = (T_\infty(\sigma,x),0)$$

in the C^r topology, when $n \to \infty$.

1.3 Global aspects: ghost dynamics

Now we analyze the unfolding a saddle-node cycles, from the global point of view. The situation when the saddle-node is the unique periodic point involved in the cycle deserves a separate treatment.

1.3.1 A return map for 1-cycles

Let $(\phi_\mu)_\mu$ be an arc of diffeomorphisms generically unfolding a critical 1-cycle. Fix fundamental domains D^- and D^+, as in the previous section. We assume that the unstable manifold of the saddle-node P is contained in its stable manifold. Then there exists $l \geq 1$ such that $\phi_0^l(D^+)$ is contained in the region $\{x < 0\}$, inside the local stable manifold of P. See Figure 1.4.

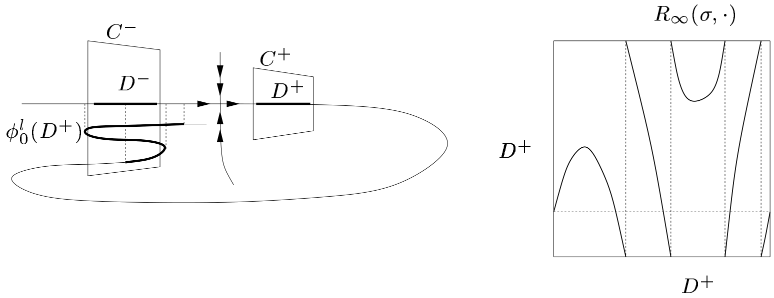

Figure 1.4. Ghost circle maps

Fix fundamental regions $C^- \supset D^-$ and $C^+ \supset D^+$, as before, such that $\phi_\mu^l(C^+)$ is contained in $\{x < 0\}$ for every μ close to zero, and the orbit of any point of $\phi_\mu^l(C^+)$ has a representative in C^-: it suffices that that C^+ be sufficiently short, and C^- be long enough along the vertical (strong-stable) direction. Then, identifying points in the same ϕ_μ orbit as we have been doing, there is a well defined arc of smooth maps

$$\Psi_\mu : C^+ \to C^-$$

from the cylinder C^+ to the cylinder C^-, induced by ϕ_μ^l.

Moreover, if π denotes the projection from the stable manifold onto W^c along the leaves of the strong-stable foliation, we can define a smooth circle map

$$\psi_0 : D^+ \to D^-$$

from the circle D^+ to the circle D^-, induced by $\pi \circ \phi_0^l$. Observe that if the cycle is critical then this circle map exhibits (at least two) critical points. This is the case Figure 1.4 refers to, and the one we are most interested in for the time being.

Composing the Ψ_μ with the transition maps that were introduced before, we obtain arcs of *global return maps*

$$R_n : [0,1] \times C^+ \to C^+, \quad R_n(\sigma, \cdot) = \hat{T}_n(\sigma, \cdot) \circ \Psi_{v_n(\sigma)}(\cdot).$$

These maps encode the whole dynamics of the diffeomorphisms ϕ_μ close to the cycle. Moreover, by Theorems 2 and 4, the sequence R_n converges, in the C^r topology, to the arc of *ghost maps*

$$R_\infty\colon [0,1]\times C^+ \to C^+,\ \ R_\infty(\sigma,x,y) = \hat{T}_\infty(\sigma,\psi_0(x),0) = (T_\infty(\sigma,\psi_0(x)),0).$$

It is important to observe that, since the last variable y plays no role in R_∞, we may also think of it as an arc of circle maps:

$$R_\infty\colon [0,1]\times D^+ \to D^+, \quad R_\infty(\sigma,x) = T_\infty(\sigma,\psi_0(x)).$$

Thus, the unfolding of the saddle-node cycle may, to some extent, be reduced to a 1-dimensional problem: From understanding the dynamics of these circle maps $R_\infty(\sigma,\cdot)$, one may draw conclusions about the behaviour of ϕ_μ for small $\mu > 0$. Next comes an important application of this idea.

1.3.2 Prevalence of hyperbolicity

Suppose $\mathcal{P}$ is a robust property, that is, the set of dynamical systems that satisfy $\mathcal{P}$ is open. Suppose, in addition, that $\mathcal{P}$ holds for some ghost circle map $R_\infty(\sigma,\cdot) : D^+ \to D^+$. Then, by robustness, $\mathcal{P}$ is satisfied by $R_n(\sigma,\cdot)$ for every large n and every σ in some interval $J \subset [0,1]$. Since each $R_n(\sigma,\cdot)$ is a quotient map of an iterate of $\phi_{\upsilon_n(\sigma)}$ (identification of points in the same orbit), we conclude that, up to convenient translation, property $\mathcal{P}$ is satisfied by ϕ_μ for all parameters μ in the set $E = \bigcup_n \upsilon_n(J)$.

On the other hand, by the bounded distortion property in Proposition 3,

$$\frac{|E\cap[\mu_{n+1},\mu_n]|}{|[\mu_{n+1},\mu_n]|} \geq (1-\varepsilon)|J| \geq \frac{1}{2}|J|$$

for every large n. So, E has positive density at $\mu = 0$. In other words, *the property $\mathcal{P}$ is prevalent at the bifurcation for the arc $(\phi_\mu)_\mu$.*

For instance, take $\mathcal{P}$ to be hyperbolicity (Axiom A plus strong transversality [29]). It is not difficult to ensure, for a critical saddle-node arc $(\phi_\mu)_\mu$, that some ghost circle map $R_\infty(\sigma,\cdot)$ is hyperbolic. For instance, one may choose $R_\infty(1/2,\cdot)$ such that it has exactly two critical points, both contained in the basin of attraction of a fixed point s_0, and the norm of the derivative is larger than 1 outside neighbourhoods of the critical points contained in the basin of s_0. Then the non-wandering set of $R_\infty(1/2,\cdot)$ is hyperbolic (implying the Axiom A) and the map satisfies the strong transversality condition. It follows, by robustness of hyperbolicity, that ϕ_μ is hyperbolic for a sizable subset of parameters μ. Along these lines one gets

Theorem 5. (Díaz, Rocha, Viana [8]) There exists an open set of arcs of diffeomorphisms unfolding a critical saddle-node 1-cycle for which hyperbolicity is a prevalent property at the bifurcation.

This result extends to critical saddle-node l-cycles, any $l \geq 1$ [8].

Question 1. Is prevalence of hyperbolicity a generic property (open and dense) among arcs of diffeomorphisms unfolding critical saddle-node cycles with finitely many criticalities (for the ghost circle maps) ?

One way to prove this would be to show that given a generic multimodal map R of the circle (finitely many critical points), there exists σ such that $R - \sigma$ (composition with the rotation by $-\sigma$) is hyperbolic.

1.3.3 Saddle-node horseshoes

The kind of systems described in the central part of Figure 1.2 was first treated by Zeeman [34], and was pointed out by Takens [30] as an important model of intermittency.

One considers a 2-dimensional disk D and an embedding $\phi: D \to D$ whose limit set in D consists of a horseshoe Λ and a periodic attractor. Then one lets the attractor and the accessible fixed point of the horseshoe collapse into a saddle-node. At the bifurcation, the limit set Λ_0 is topologically conjugate to the initial horseshoe, but it is no longer hyperbolic, as it contains the saddle-node. Since Λ_0 has a dense subset of periodic points, the diffeomorphism exhibits saddle-node l-cycles for any $l \geq 2$.

A key difference with respect to the case of 1-cycles we discussed above is that now the unstable manifold of the saddle-node P is not completely contained in its stable manifold: for instance, $W^u(P)$ intersects the stable manifolds of all the other periodic points in the non-hyperbolic horseshoe Λ_0. This means that there is no family of global returns maps, as we were able to construct in the previous case.

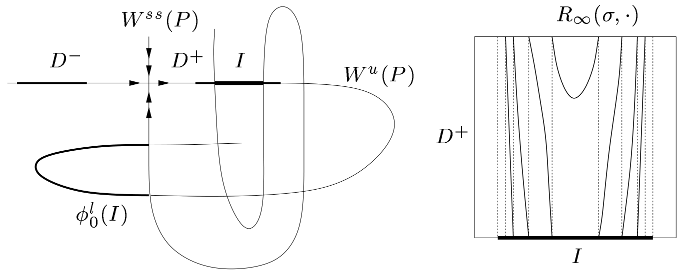

Figure 1.5. Saddle-node horseshoes: partially defined ghost maps

However, it is possible to construct partially defined return maps, as follows. One fixes fundamental domains D^- and D^+ as before, and considers a maximal open subinterval I of D^+ contained in $W^u(P)$ and whose extremes are points of the strong stable manifold $W^{ss}(P)$. Then one defines, in much the same way as before, an arc of ghost return maps $R_\infty(\sigma, \cdot)$

from I to D^+. In the example described in Figure 1.5, the return maps have a unique critical point. Note that the norm of the derivative goes to infinity at the boundary of I. The convergence Theorems 2, 4 remain valid on compact subsets of I.

Partially defined ghost maps are used by Costa [4] in her proof that global strange attractors are a prevalent phenomena in the unfolding of saddle-node horseshoes, in a robust (open) class of cases. Prevalence of hyperbolicity had been proven in [8], for another robust class. A detailed study of these return maps $R_\infty(\sigma, \cdot)$ is carried out by Díaz, Rios [6], who provide a geometric model for the unfolding of saddle-node horseshoes. Another use of partially defined return maps, by Díaz, Ures [9], will be discussed in a forthcoming section.

In a related setting, Crovisier [5] shows, in great generality, that saddle-node horseshoes give rise to true (hyperbolic) horseshoes when the saddle-node is unfolded in the direction of negative parameters. Cao, Kiriki [3] study the unfolding of non-critical horseshoes, as on the right hand side of Figure 1.2.

1.4 Prevalence of local and global strange attractors

An *attractor* of a diffeomorphism $\phi : M \to M$ is a compact invariant subset Λ of M that is transitive (dense orbits) and whose *basin* (or stable set)

$$W^s(\Lambda) = \{x \in M : \phi^n(x) \to \Lambda \text{ as } n \to +\infty\}$$

has positive Lebesgue measure. A *repeller* of f is just an attractor of the inverse map f^{-1}. One calls the attractor *strange* if orbits in the basin are sensitive with respect to initial conditions: almost every pair of orbits starting in nearby points diverge from each other as time increases.

In this section we discuss saddle-node cycles as a privileged mechanism for creating strange attractors, specially non-hyperbolic ones.

1.4.1 A general prevalence result

According to Theorem 1, the generic unfolding of a critical saddle-node cycle always involves the formation and generic unfolding of homoclinic tangencies. On the other hand, Mora, Viana [13] prove, based on the work of Benedicks, Carleson [1], that the presence of non-hyperbolic strange attractors is a persistent phenomenon in generic arcs of surface diffeomorphisms unfolding a homoclinic tangency. See also [28, 31] for the extension to arbitrary dimension. It follows that strange attractors are persistent also in the unfolding of saddle-node critical cycles.

In view of the ideas discussed in Section 1.3.2, one may expect the presence of strange attractors to be a *prevalent* phenomenon in this setting of saddle-node cycles. However, one should stress that the situation is much more subtle than in the case of hyperbolicity, that we settled in Section 1.3.2, because in the present context one lacks robustness: the sets of systems constructed in [1, 13, 31], for which strange attractors are known to exist, have empty interior. Thus, a delicate analysis of the bifurcation mechanisms is needed to justify that expectation:

Theorem 6. (Díaz, Rocha, Viana [8]) Existence of non-hyperbolic strange attractors is a prevalent property at the bifurcation for every arc of diffeomorphisms $(\phi_\mu)_\mu$ unfolding generically a critical saddle-node cycle.

1.4.2 Global strange attractors

The strange attractors obtained by the previous construction have a local nature: they are periodic, with high periods, and their basins have a large number of connected components, with small total Lebesgue measure. This is entirely in the nature of things: without further assumptions about the geometry at the bifurcation, the set of points whose forward orbits remain forever close to the cycle may have small volume, for all positive values of the parameter of μ.

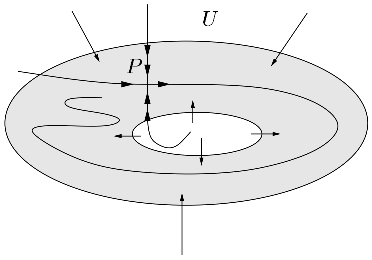

Figure 1.6. Global invariant region for 1-cycles

On the other hand, in some relevant cases one can identify a global region around the cycle that remains forward invariant for all parameters close to zero. An important example, corresponding to a saddle-node 1-cycle, is described in Figure 1.6, where the invariant region is an annulus. In such cases, it is natural to ask whether a unique attractor can be found, in a persistent or even prevalent way, that accounts for the whole dynamical behaviour, in the sense that its basin contains the entire invariant region. The first construction of non-hyperbolic strange attractors with such a global character was given by the following

Theorem 7. (Díaz, Rocha, Viana [8]) Presence of a global non-hyperbolic strange attractor is prevalent at the bifurcation for an open class of arcs of diffeomorphisms unfolding a critical saddle-node 1-cycle.

Other constructions appeared subsequently, including [4] in the setting of saddle-node horseshoes, where one may take a disk as the forward invariant region.

1.5 Persistence of tangencies

In this section we discuss fractal dimensions and the phenomenon of persistent tangencies in the context of saddle-node bifurcations.

1.5.1 Fractal dimensions in homoclinic bifurcations

Starting in the early seventies, works of Newhouse, Palis, Takens [18, 21, 20] and, later, also Yoccoz, Moreira [23, 15], have unveiled a deep connection between fractal dimensions (such as the Hausdorff dimension) of invariant sets, and the frequency of hyperbolicity in the unfolding of homoclinic tangencies of surface diffeomorphisms. Let us outline this connection.

One considers a homoclinic tangency associated to a periodic point P contained in a horseshoe Λ. See Figure 1.7. The existence of a homoclinic tangency implies that the invariant (stable and unstable) foliations of Λ are tangent along a differentiable curve γ containing the homoclinic point in its interior and transverse to both foliations. The intersection of γ with the leaves of the foliations corresponding to points of the hyperbolic set Λ defines two Cantor sets Λ^s and Λ^u.

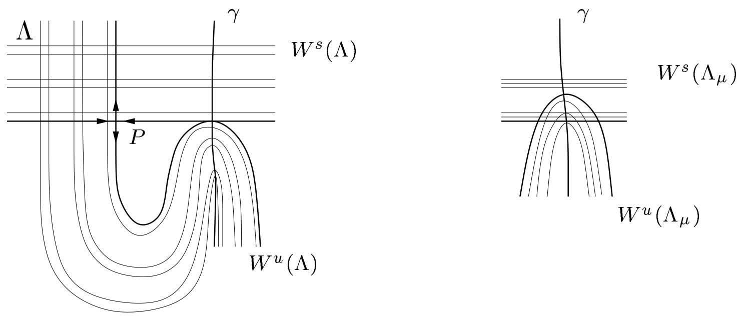

Figure 1.7. Persistent tangencies between invariant foliations

Given an arc $(\phi_\mu)_\mu$ of diffeomorphisms unfolding the tangency, one considers the corresponding intersections Λ^s_μ and Λ^u_μ of γ with the stable and unstable leaves through the points of the hyperbolic continuation Λ_μ of Λ. Clearly, if the sets Λ^s_μ and Λ^u_μ have non-empty intersection there is a homoclinic tangency associated to Λ_μ. Identifying γ with an interval of $\mathbb{R}$ one can think of Λ^s_μ and Λ^u_μ as μ-translations of the cantor sets Λ^s and Λ^u.

Newhouse [16] introduced a notion of *thickness*, that allowed him to give a sufficient criterion for two Cantor sets to intersect. It is defined as follows. Consider the process of construction of the Cantor set by, successively removing the corresponding gaps, in a non-increasing order of their lengths. Each time a gap is removed, compute the ratio between the lengths of the two remaining nearby intervals and the length of the gap itself. The thickness is the infimum of all these ratii.

Newhouse's gap lemma [16] states that two Cantor sets such that the product of their thicknesses is larger than 1 must intersect, unless one of them is contained in a gap of the other. Building on this, he was able to construct examples of arcs of diffeomorphisms $(\phi_\mu)_\mu$ generically unfolding a homoclinic tangency of $\phi = \phi_0$ such that for a dense subset of a whole interval $[0, \varepsilon]$ of values of μ the diffeomorphism ϕ_μ has another homoclinic tangency. One speaks of interval of *persistent tangencies*. Later, in [17], he proved that persistent tangencies occur in *any* generic unfolding of any homoclinic tangency by an arc of surface diffeomorphisms.

Then, the series of papers by Newhouse, Palis, Takens, Yoccoz, Moreira mentioned above identified the Hausdorff dimension as a key fractal invariant determining the frequency of hyperbolicity in the unfolding of homoclinic tangencies on surfaces. In general terms, *hyperbolicity is prevalent at the bifurcation if and only if the Hausdorff dimension of the horseshoe* Λ *is less than* 1.

More recent results of Moreira, Palis, Viana [14, 24] and Romero [28] have shown that this principle remains valid on manifolds with arbitrary dimension. In dimension larger than 2 there are other mechanisms (not involving fractal dimensions explicitly) yielding persistence of tangencies in the C^1 topology, see Bonatti, Díaz [2]. Moreover, Rios [27] extended many of the previous results to the unfolding of homoclinic tangencies accumulated by periodic points (the homoclinic orbit is contained in the limit set of the diffeomorphism).

1.5.2 Thick horseshoes in saddle-node cycles

Saddle-node cycles exhibit some original features, from the point of view of the discussion in the previous section. One of the most striking is the possibility of thick horseshoes to be created, "out of nowhere", immediately after the bifurcation. In fact, such horseshoes may be seen as a kind of continuation of thick invariant sets of the ghost return maps. Let us explain this in the case of critical 1-cycles.

We may construct examples of critical saddle-node 1-cycles such that the ghost circle map $R_\infty(\sigma, \cdot)$ has a hyperbolic Cantor set with large thickness for some subset of parameters $\sigma \in [0, 1]$. For instance, one may take for $R_\infty(\sigma, \cdot)$ a circle map such that the derivative is larger than 1 in norm outside two intervals Δ_1 and Δ_2 (around the critical points) with length

δ bounded by some small $\delta > 0$. Then the maximal invariant set Λ_σ of $R_\infty(\sigma,\cdot)$ in the complement of $\Delta_1 \cup \Delta_2$ is hyperbolic and its thickness is of order of $1/\delta$.

Then, using the convergence Theorems 2 and 4, and the continuous dependence of the thickness on the diffeomorphism [17], one gets that the diffeomorphism ϕ_μ, $\mu = v_k(\sigma)$ has a hyperbolic set with stable thickness (transverse thickness of the stable foliation) of order $1/\delta$, for every large k.

This observation is at the origin of a result of Díaz, Ures [9] we are going to state next, saying that the unfolding of certain saddle-node cycles leads to an interval of persistence of tangencies immediately after the bifurcation (the interval is of the form $[0, \varepsilon_0]$ for some $\varepsilon_0 > 0$), even if the Hausdorff dimension of the limit set at the bifurcation is smaller than 1.

However, the previous construction is not sufficient to prove such a result. One problem is that it proves the existence of thick horseshoes only for certain subintervals in the space of parameters μ. Another, more serious, difficulty is that the hyperbolic sets one gets in this way might have very small *unstable* thickness, and so the gap lemma might not apply to them.

1.5.3 Thick horseshoes from saddle-node horseshoes

These difficulties can be bypassed for certain robust classes of arcs of diffeomorphisms unfolding a saddle-node horseshoe: one obtains hyperbolic sets with large product of stable and unstable thicknesses for all small values of the parameter μ, even if the saddle-node horseshoes itself is thin.

As we have seen in Section 1.3.3, in this situation ghost return maps $R_\infty(\sigma,\cdot)$ may be defined on convenient subintervals I of the fundamental domain D^+. The end-points of I correspond to points of the strong stable manifold of the saddle-node and that the norm of the derivative of $R_\infty(\sigma,\cdot)$ goes to infinity at the end-points. See Figures 1.5 and 1.8.

One proves that, in an open class of cases, the map $R_\infty(\sigma,\cdot)$ has a hyperbolic Cantor set Λ_σ with large stable thickness, for every parameter σ. In fact, the stable thickness admits a lower bound M that is of the order of $1/|B|$ where B is the smallest of the following intervals: the connected components of $(D^+ \setminus I)$ and an interval around the critical point outside of which the derivative is larger than 1. Assuming the gap of the initial horseshoe is big enough, we can take I proportionally big in D^+, and then we can make M as large as we like.

Next, one has to ensure that the unstable thickness remains bounded from zero, by some small constant that may be fixed independently of M. For this one argues that almost all (a subset with nearly the same thickness) of the initial saddle-node horseshoe persists, as a hyperbolic horseshoe, after the unfolding of the saddle-node. This uses also the continuity of the thickness with the dynamics. Since the unstable thickness of the saddle-

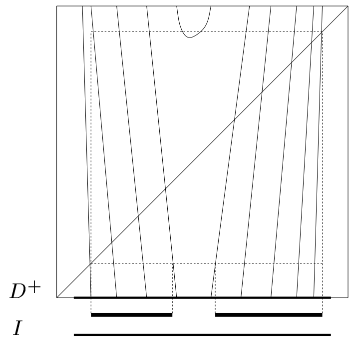

Figure 1.8. Thick invariant Cantor sets for the maps $R_\infty(\sigma, \cdot)$

node horseshoe is positive, we conclude that the unstable thickness of these hyperbolic sub-horseshoe are bounded from zero by some $m > 0$.

Since M and m depend on the geometry of the saddle-node horseshoe in different directions (respectively stable and unstable), we may indeed increase M without reducing m, so that their product is larger than 1. This is a main ingredient in the proof of

Theorem 8. (Díaz, Ures [9]) For every $\varepsilon > 0$ there is an open set of arcs $(\phi_\mu)_\mu$ unfolding at $\mu = 0$ a critical saddle-node horseshoe of Hausdorff dimension less that $1/2+\varepsilon$ such that some $(0, \mu_0]$ is an interval of persistence of tangencies.

Let us observe that a saddle-node horseshoe always has Hausdorff dimension strictly bigger than $1/2$, by [10].

Question 2. Is there a necessary and sufficient condition involving fractal dimensions of the saddle-node horseshoe Λ_0 guaranteeing the existence of an interval J of the form $(0, \mu_0)$ of persistence of tangencies ?

A corresponding question was originally asked by Palis and Takens [22, Section 7], in the context of homoclinic bifurcations. As we explained, in that context the frequency of hyperbolicity is essentially determined by the Hausdorff dimension of the hyperbolic set associated to the tangency. Here, in view of the previous observations, a natural approach would be to consider not only the dimension of the saddle-node horseshoe but also the dimensions of the hyperbolic sets of the circle maps $R_\infty(\sigma, \cdot)$.

Question 3. Does there exist a non-empty open subset of the space $\mathcal{O}(\mathcal{M})$ of arcs $(\phi_\mu)_\mu$ of diffeomorphisms unfolding generically a critical saddle-node 1-cycle such that for any arc in this subset the diffeomorphisms ϕ_μ are non-hyperbolic for all small $\mu > 0$?

This final question should be related to the problem of the density of hyperbolic surface diffeomorphisms in the C^1 topology.

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