Journal of Nonlinear Mathematical Physics 2001, V.8, Supplement, 62–68 Proceedings: NEEDS'99

On Huygens' Principle for Dirac Operators and Nonlinear Evolution Equations

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Abstract

We exhibit a class of Dirac operators that possess Huygens' property, i.e., the support of their fundamental solutions is precisely the light cone. This class is obtained by considering the rational solutions of the modified Korteweg-de Vries hierarchy.

1 Introduction

The starting point of this article is a classical linear problem, namely the Cauchy initial-value problem

$$\begin{cases} \Box\psi + u\psi = 0, \\ (\psi, \partial_0\psi)\Big|_{x^0=0} = (f, g). \end{cases} \tag{1}$$

Here, $\Box = \partial_0^2 - \sum\limits_{i=1}^{n} \partial_i^2$ and $u = u\left(x^0, \ldots, x^n\right)$ is a given potential in n spatial dimensions and time x^0.

With Hadamard [1], we consider the question: For which potentials is the domain of dependence of the hyperbolic initial value problem at an arbitrary point $y = \left(y^0, \ldots, y^n\right)$ only the intersection of the initial data manifold $x^0 = 0$ with the light cone with vertex at y? This question leads to severe nonlinear restrictions on the potential u, which are directly connected to Soliton theory. Operators with the above property are said to obey *Huygens' principle* (in Hadamard's strict sense) or to possess Huygens' property.

The physical significance of Huygens' property is that a disturbance localized in a small neighborhood of a given point would propagate in a small neighborhood of the positive light cone. A minute's thought reveals that such property is crucial for the meaningful transmission of information. For an elementary account see P Günther's delightful paper [2]. Since Hadamard's time it is known that if $u = 0$, then Huygens' property holds if, and only if, the number of spatial dimensions n is odd and greater than 1. In particular, flat-landers do not like to attend classical music concerts!

Hadamard's question has been the subject of intense interest. An early account can be found in the Courant and Hilbert's masterpiece [3], whereas for the more recent results and direct connections with Soliton theory see [4].

Clearly, Hadamard's question should be considered modulo certain trivial transformations. Those trivial transformations are generated by changes of independent variables and by left or right multiplication of the operator by a non-vanishing function. Also, we consider the question only in sufficiently small domains, since the problem is already interesting enough once considered locally.

For a long while it was conjectured that, modulo the trivial transformations, the only potential $u = u\left(x^0, \ldots, x^n\right)$ such that $\Box + u$ is Huygens was the trivial potential $u = 0$. This turned out to be false! The "simplest" counter-example, which was given by Stellmacher [5], holds in dimension 5+1 and consists of the potential $u = -2/\left(x^0\right)^2$. In modern Soliton theory language, the remarkable discovery of Lagnese and Stellmacher [6, 7] goes as follows: Take the (real) Calogero–Moser potentials decaying at infinity that are given by

$$u_k\left(x^0\right) = 2\left(\log \vartheta_k\right)''\left(x^0\right),$$

where ϑ_k is the k-th Adler–Moser polynomial [8] and $()'$ denotes the derivative. Then, $\Box + u_k$ is Huygens in a sufficiently high number of odd spatial dimensions. More precisely, the minimal number of spatial dimensions for which this occurs is $n = 2k + 3$ for generic choices of the parameters in the Adler–Moser polynomials[1].

This connection with solitons is much deeper. See for example [4, 9] and references therein.

2 Main goals and plan

The first goal of the present article is to show that an analogue to Dirac operators of Huygens' property is also an interesting question for Soliton theory. We do that by showing that adding potentials associated to the rational solutions of the modified Korteweg-de Vries hierarchy to (free) Dirac operators in sufficiently high (odd) spatial dimensions we get Huygens operators. This is closely related to the fact that Huygens holds for $\Box + u_k$ as described above.

However, even in this simple context a novel phenomena occurs. The Dirac operators that we construct possess Huygens' property in a smaller number of spatial dimensions than one would in principle anticipate.

The plan of this paper is the following: In Section 3 we recall the definition of Dirac operators and their relevance. In Section 4 we discuss one of the main tools needed in our analysis, namely the Hadamard expansion. Section 5 states and proves the main result. We conclude with a small discussion of the results and suggestions for further research.

3 Dirac operators

Dirac operators appear in Mathematical Physics in the framework of the construction of a relativistic electron theory. In a celebrated paper [10], Dirac defined a Lorentz-covariant Hamiltonian (and consequently first-order in space variables, as Schrödinger equation is first order in time) whose square yields the Klein–Gordon equation.

Let $\{\gamma^0, \ldots, \gamma^n\}$ be a set of matrices obeying the anti-commutation rule $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I$, where $((g^{\mu\nu})) = \mathrm{diag}[1, -1, \ldots, -1]$ denotes the Minkowski tensor. Dirac operators [11] are defined by

$$\mathcal{D} = \gamma^\mu\partial_\mu + v,$$

[1] We have used the privileged time variable x^0 for the sake of simplicity. Similar results hold for the other variables.

where the summation for repeated indices is implied. Throughout this contribution, we shall also adopt the notation $\partial\!\!\!/ = \gamma^\mu \partial_\mu$ and the Pauli–Dirac representation (generalized for higher dimensions) where $\gamma^0 = \mathrm{diag}[\mathbf{1}, -\mathbf{1}]$. Furthermore, we restrict ourselves to the case where v is *a scalar* and assume that it depends on only *one* variable.

4 Hadamard expansion

In this section we shall look for fundamental solutions of Dirac operators. A *fundamental solution* for $\partial\!\!\!/+v$ is the solution of $(\partial\!\!\!/+v)\Phi = \delta_y$, where δ_y denotes Dirac-delta distribution supported at an arbitrary point y in space-time.

Let $\lambda(x,y) = \sqrt{(x^\mu - y^\mu)(x_\mu - y_\mu)}$ denote the Minkowski distance between two points x and y. The light cone, as usual, is given by $\mathcal{C}(y) = \{x \in \mathbb{R}^{n+1} | \lambda(x,y) = 0\}$.

The first important concept to be recalled is the so-called Riesz kernel. It was introduced by M Riesz in order to unify the treatment of elliptic and hyperbolic problems. They are given by distributions in $\mathcal{D}'(\mathbb{R}^{n+1})$ defined first for $\mathbf{Re}(\alpha)$ sufficiently large by

$$\Lambda^\alpha = \begin{cases} N(\alpha)\lambda^\alpha, & \text{if } (x^\mu - y^\mu)(x_\mu - y_\mu) \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Here, the normalizations constant $N(\alpha)$ is given by

$$N(\alpha) = \left[2^{\alpha+n} \pi^{(n-1)/2} \Gamma\left(\frac{\alpha+n+1}{2}\right) \Gamma\left(\frac{\alpha}{2}+1\right) \right]^{-1}.$$

Then, Λ^α is extended analytically for all values of α in the complex plane by means of $\Box\Lambda^\alpha = \Lambda^{\alpha-2}$, which is satisfied for all α with sufficiently large real part. It is not hard to check that $\Lambda^{-n-1} = \delta_y$. See [12].

We shall say that a fundamental solution Φ obeys *Huygens' principle* if $\mathrm{supp}\,\Phi \subset \mathcal{C}(y)$. More generally, we shall say that a distribution depending on a parameter $y \in \mathbb{R}^{n+1}$ satisfies *Huygens' principle* if it is supported on the light cone $\mathcal{C}(y)$.

Adapting Hadamard's seminal idea from the wave operator situation to the one at hand, we look for series expansion for the fundamental solution

$$\Phi = \sum_{m=0}^{\infty} \partial\!\!\!/^{-m} \Theta^{-n+1} s_m = \sum_{m=0}^{\infty} \left\{ \Theta^{-n+1+2m} s_{2m} + \Lambda^{-n+1+2m} s_{2m+1} \right\},$$

where s_m is a matrix coefficient, $\Theta^\alpha = \partial\!\!\!/\Lambda^\alpha$, which we call Dirac kernels, and Λ^α are Riesz kernels. It can be shown that Θ^α satisfies Huygens' property for $\alpha = 0, -2, -4, \ldots$; while Λ^α does it for $\alpha = -2, -4, \ldots$.

Imposing that Φ is a fundamental solution and using the properties of Riesz kernels, we have the following recursion for the coefficients s_m starting with $s_0 = 1$:

$$s_{2m+1} = (\partial\!\!\!/ - v) s_{2m}, \tag{2}$$

$$s_{2m} + \frac{1}{m}(x^\mu - y^\mu)\,\partial_\mu s_{2m} = -(\partial\!\!\!/ + v) s_{2m-1}. \tag{3}$$

In all these cases s_m should be smooth (in particular when $y^\mu \to x^\mu$).

This expansion is unique in the following sense: Let us suppose that Φ and $\widetilde{\Phi}$ are both fundamental solutions such that $(\partial\!\!\!/ + v)\Phi = \delta_y$ and $(\partial\!\!\!/ + v)\widetilde{\Phi} = \delta_y$. So, their difference is a solution of the homogeneous equation $(\not\partial + v)(\Phi - \widetilde{\Phi}) = 0$. Let h be a solution of the homogeneous equation, so $\widetilde{\Phi} = \Phi + h$. As h bears no relation to the support y of the Dirac-delta distribution it is clear that it is immaterial to study the validity of Huygens' principle in terms of Φ or $\widetilde{\Phi}$. Thus, we can fix a fundamental solution and study its Hadamard series.

An important property of this series is that it is unique, provided we require s_m to be smooth. This can be easily seen from equations (2) and (3). Actually the uniqueness for (2) is trivial. In the case of (3) it should be proved. Let s^1 and s^2 be two different solutions of (3). So, the difference $s^1 - s^2 = S_{2m}$ is smooth and satisfies

$$S_{2m} + \frac{1}{m}(x^\mu - y^\mu)\partial_\mu S_{2m} = 0.$$

If we assume regularity near the vertex of the light cone, it is well known that the unique solution to this equation is the trivial one. See, [1], Book 2, Chapter 3, Section 61.

Bearing in mind this fact it is easy to prove the equivalence between Huygens' principle and the truncation of the Hadamard series:

Remark. A Dirac operator in the form $\not\partial + v$ possesses Huygens' property in odd spatial dimension $n \iff$ its Hadamard series truncates at n, i.e, $s_m = 0$ for $m \geq n$.

A proof of this remark is quite simple. In the $\Leftarrow$ direction, it follows from the properties of Dirac and Riesz kernels. In the other direction, it is a consequence of the uniqueness of the Hadamard expansion.

5 Huygens property and the mKdV hierarchy

The modified Korteweg-de Vries (mKdV) equation

$$v_t = 6v^2 v_x - v_{xxx},$$

together with the Miura map, and the Korteweg-de Vries equation have been instrumental in the development of the inverse scattering method in Soliton theory [13, 14]. Associated to the mKdV one can find a full hierarchy of commuting flows which can be obtained by different, though equivalent, methods. Lax pairs, bi-hamiltonian formalism, recursion operators, to cite just a few. The rational solutions of the mKdV have also been subject of a number of studies. We follow [8]. The Adler–Moser polynomials arise by considering the recurrence relation

$$\vartheta'_{k+1}\vartheta_{k-1} - \vartheta_{k+1}\vartheta'_{k-1} = (2k+1)\vartheta_k^2,$$

with the initial condition $\vartheta_0 = 1$ and $\vartheta_1 = x$. It is remarkable (although noticed already by Burchnall–Chaundy [15]) that each ϑ_k is a polynomial in x. Furthermore, ϑ_k depends on $k - 1$ free parameters which can be chosen so that

$$v_k = \partial_x \log(\vartheta_{k+1}/\vartheta_k)$$

is a solution of the different flows of the mKdV hierarchy. See Section 4 of [8] for a proof of this fact and many other interesting related results.

Henceforth, for notational simplicity, we set the time variable $x^0 = t$ and $()'$ will denote derivative w.r.t. t. We also identify the polynomial variable in the Adler–Moser polynomial with $x^0 = t$. So,

$$(\not{\partial}+v_k)(\not{\partial}-v_k) = \Box-\gamma^0 v_k'-v_k^2 = \Box-\begin{pmatrix} v_k'+v_k^2 & 0 \\ 0 & -v_k'+v_k^2 \end{pmatrix}. \tag{4}$$

It follows from the properties of the Adler–Moser polynomials we have $u_k = -v_k' - v_k^2$ and $u_{k+1} = v_k' - v_k^2$. If we consider the fundamental solutions Ψ_k such that $(\Box + u_k)\Psi_k = \delta_y$, the function

$$\Phi = (\not{\partial} - v_k)\begin{pmatrix} \Psi_k & 0 \\ 0 & \Psi_{k+1} \end{pmatrix} \tag{5}$$

is the fundamental solutions of $\not{\partial} + v_k$. So, if Ψ_k and Ψ_{k+1} have Huygens' property in a certain dimension, then Φ as defined above has this property in the same dimension.

We should expect, from the above argument that $\not{\partial} + v_k$ has the Huygens' property in the minimum dimension where $\Box + u_k$ and $\Box + u_{k+1}$ are both Huygens. However, an amazing fact happens and the dimension is reduced by two. We shall refer to this as the *reduction of the dimension*. This property holds for the Dirac operator in the vacuum. In this case, it is Huygens for every odd dimension, unlike the wave operator, which is Huygens just from 3 on. We are now ready to state:

Theorem. *Let ϑ_k denote the k-th Adler–Moser polynomial and*

$$v_k(x_0) \stackrel{\text{def}}{=} \partial_0 \log\left((\vartheta_{k+1}/\vartheta_k)(x_0)\right).$$

Then, $\not{\partial} + v_k$ is Huygens in $n = 2k+3$ spatial dimensions for every $k \geq 0$.

Proof. Let us define $\mu_k = \vartheta_{k+1}/\vartheta_k$, then $v_k = \mu_k'/\mu_k$. It can be shown [8] that $u_k = -\mu_k''/\mu_k$. The operator $\Box + u_k$ is known to be Huygens in dimension $2k+3$. Let Ψ_k be the fundamental solution of $\Box + u_k$. As Ψ_k obeys Huygens' principle, its Hadamard expansion terminates. Let r_k be the last term in the expansion. If we are in the minimal dimension for the validity of this property, then r_k is the coefficient of the Λ^{-2} term. Let us study the behavior of this term:

$$(\Box+u_k)\Psi_k = (\Box+u_k)\left(\cdots + \Lambda^{-2} r_k\right) = \cdots + \Lambda^{-2}\left(\partial_t^2 + u_k\right) r_k.$$

As $(\Box + u_k)\Psi_k = \delta_y = \Lambda^{-n-1}$ and $n \geq 3$ we claim that $(\partial_t^2 + u_k) r_k = 0$.

Using the definition of u_k we write $(\partial_t + v_k)(\partial_t - v_k) = \partial_t^2 + u_k = (\partial_t - v_{k-1})(\partial_t + v_{k-1})$. This gives $r_k = \alpha_k \mu_k + \beta_k/\mu_{k-1}$.

As μ_k is a ratio of two consecutive Adler–Moser polynomials, its asymptotic behavior, in $t \to \infty$, is given by $\mu_k = \mathcal{O}\left(t^{k+1}\right)$.

But the differential equation obeyed by r_k, when $t \to \infty$ (and remembering that $u_k = -\mu_k''/\mu_k \to 0$), is $\left(\partial_t^2\right) r_k \to 0$ or, in other words, $r_k = o\left(t^2\right)$ when $t \to \infty$. Thus, $r_k = \beta_k/\mu_{k-1}$. We define $\Phi_k = (\not{\partial} - v_k)\,\mathrm{diag}[\Psi_k, \Psi_{k+1}]$. The above properties yield that Φ_k is the fundamental solution of $\not{\partial} + v_k$.

We shall study the last term in Φ_k. As Ψ_k and Ψ_{k+1} should be acting in the same spatial dimension n, and they obey Huygens' principle in this dimension, we chose the minimum n such that Ψ_{k+1} is supported in the light cone, $n = 2k+5$, and in this case the last term in the expansion of Ψ_k is $\Lambda^{-4} r_k$. In order to study the behavior of the last term in Φ_k we write:

$$\Phi_k = (\partial\!\!\!/ - v_k)\left(\cdots + \Lambda^{-2}\begin{pmatrix} 0 & 0 \\ 0 & r_{k+1} \end{pmatrix}\right)$$

$$= \cdots - \Lambda^{-2}(\partial_t + v_k) r_{k+1}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \cdots + \Lambda^{-2} \cdot 0.$$

This equation shows that the last term in the Hadamard expansion of Φ_k is Θ^{-2}. As Θ^0 has the Huygens' property, we can lower the dimension by two, to $n = 2k + 3$. ■

6 Discussion

We have shown that upon choosing $v = v_k(x_0)$, where v_k is a rational solution solution of the mKdV hierarchy (with the variable x_0 now replacing the spatial variable x of mKdV), we get that $\partial\!\!\!/ + v_k$ is a Huygens operator. The link between Huygens wave operators and nonlinear evolution equations has been already noticed, and was particularly highlighted in the recent work of Berest. See [16] and references therein. The connection with Dirac operators is plausible once one is aware of the relations between Schrödinger operators and ZS-AKNS operators. The latter are however only connected to Dirac operators in $1 + 1$ dimensions. In this work we have shown that Huygens' property holds for Dirac operators in space-time dimension $n + 1$ for $n = 2k + 3$. This is a nontrivial fact if one tries to use (4), since there we have diagonal wave operators that are Huygens in different minimal number of dimensions.

As a consequence of the Theorem, $\partial\!\!\!/$ is Huygens in dimension 1, $\partial\!\!\!/ + 1/t$ is Huygens in dimension 3 and so on. Just to compare, $\Box$ is Huygens in dimension 3, $\Box - 2/t^2$ is Huygens in dimension 5 and so on.

There are many follow up questions to the present work. To cite just a few, one can consider potentials that are not scalar or that depend on more than one variable. Presumably rational solutions of the AKNS hierarchy would play a role here. We are presently exploring some of those avenues.

Acknowledgements

Both authors were supported by the Brazilian National Research Council, CNPq. JPZ was also supported by PRONEX-FINEP through grant 76.97.1008-00.

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