A gap theorem for hypersurfaces of the sphere with constant scalar curvature one

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Abstract. We consider closed hypersurfaces of the sphere with scalar curvature one, prove a gap theorem for a modified second fundamental form and determine the hypersurfaces that are at the end points of the gap. As an application we characterize the index one hypersurfaces with scalar curvature one in the real projective space.

Key words: scalar curvature, sphere, Clifford torus, index one.

1 Introduction

To state our main result we need some notation.

$x : M^n \to S^{n+1}(1)$ will be a closed (compact without boundary) hypersurface of the unit sphere $S^{n+1}(1)$. We denote by A the linear map associated to the second fundamental form and by $k_1, ..., k_n$ its eigenvalues (principal curvatures of M). We will use the first three elementary symmetric function of the principal curvatures:

$$S_1 = \sum_{i=1}^{n} k_i,\ S_2 = \sum_{i,j} k_i k_j,\ S_3 = \sum_{i<j<l} k_i k_j k_l, \text{ with } i, j, l = 1, ..., n.$$

We will also use the normalized means: *the mean curvature* $H = \frac{1}{n} S_1$ and the *scalar curvature* R, given by $n(n-1)(R-1) = S_2$. Finally, we introduce the first two *Newton tensors* by

$$P_0 = Id,\ P_1 = S_1 Id - A.$$

Clearly P_1 commutes with A and it is also a self-adjoint operator. We will show later (see Remark 2.1) that if $R = 1$ and $S_1 \geq 0$, then all eigenvalues of P_1 are nonnegative, hence we can consider $\sqrt{P_1}$.

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We can now state our gap theorem.

Theorem 1. *Let $x : M^n \to S^{n+1}(1)$ be a closed orientable hypersurface with scalar curvature $R = 1$ (equivalently, $S_2 = 0$). Assume that S_1 does not change sign and choose the orientation such that $S_1 \geq 0$ and that*

$$\|\sqrt{P_1}A\|^2 \leq \text{trace}P_1.$$

Then:
(i) $\|\sqrt{P_1}A\|^2 = \text{trace}P_1$.
(ii) M^n *is either a totally geodesic submanifold or* $M^n = S^{n_1}(r_1) \times S^{n_2}(r_2) \subset S^{n+1}(1)$, *where* $n_1 + n_2 = n$, $r_1^2 + r_2^2 = 1$ *and* $(\frac{r_2}{r_1})^2 = \beta$ *satisfies the quadratic equation:*

$$n_1(n_1 - 1)\beta^2 - 2n_1n_2\beta + n_2(n_2 - 1) = 0.$$

Our theorem was inspired by a similar theorem on minimal submanifolds of the sphere first proved by J. Simons [S] (part *(i)*) and latter completed (part *(ii)*) by S.S. Chern, M. do Carmo and Kobayashi [CdCK] and, independently, by H.B. Lawson [L].

Remark. The condition on the modified second fundamental form in above theorem can not be dropped, as can be seen by the following example: Let $M^6 \to S^7(1)$ be an isoparametric hypersurface with principal curvatures given by

$$\lambda_1 = \lambda_2 = \theta,\ \lambda_3 = \frac{\theta + 1}{1 - \theta},\ \lambda_4 = \lambda_5 = -\frac{1}{\theta}\ \text{ and } \lambda_6 = -\frac{1 - \theta}{1 + \theta},$$

where θ is given by $\theta = \sqrt{\dfrac{13 + \sqrt{165}}{2}}$ (see [Mü]). It is easy to see that M^6 has $R = 1$ and $S_1 > 0$. We would like to thank Luiz Amancio de Sousa Junior for showing us this example.

As an application of Theorem 1, we will present a characterization of index one closed hypersurface with constant scalar curvature one of the real projective space $\mathbb{P}(\mathbb{R})^{n+1}$. For minimal submanifolds this result was obtained recently by M. do Carmo, M. Ritoré and A. Ros [dCRR].

Before giving a formal statement we need some considerations. Hypersurfaces of a curvature one space form with constant scalar curvature one are solutions to a variational problem (see [Re], [Ro], [BC]) whose *Jacobi equation* is

$$T_1 f = L_1 f + \{\|\sqrt{P_1}A\|^2 + \text{trace}P_1\}f = 0.$$

Here $f \in C^\infty(M)$ and L_1 is a second order differential operator given by

$$L_1 f = div(P_1 \nabla f),$$

where ∇f is the gradient of f. Notice that L_1 generalizes the Laplacian. However, differently from the Laplacian, L_1 is not always elliptic. J. Hounie and M.L. Leite, [HL] have proved that if $S_3 \neq 0$ everywhere, then L_1 is elliptic. Of course, from the definition of L_1, it follows that L_1

is elliptic if and only if P_1 is positive definite (or negative definite). For the next theorem we will assume that L_1 is elliptic and P_1 is positive definite. Denote by $Ind(M)$ the *Morse index* of M, i.e., the number of negative eigenvalues of T_1.

Theorem 2. *Let $x : M^n \to \mathbb{P}(\mathbb{R})^{n+1}(1)$ be a closed hypersurface with scalar curvature one. Then $Ind(M) \geq 1$ and if $Ind(M) = 1$, M is the Clifford hypersurfaces obtained by the projection of the Clifford torus of Theorem 1.*

2 Preliminaries

In this section we will present some properties of the r^{th} Newton tensors in M and describe the Clifford hypersurfaces of $\mathbb{P}(\mathbb{R})^{n+1}$.

2.1 The r^{th} Newton tensors

We introduce the r^{th} Newton tensors, $P_r : T_pM \to T_pM$, which are defined inductively by

$$\begin{aligned} P_0 =& \ I, \\ P_r =& \ S_r I - AP_{r-1}, \ r > 1. \end{aligned}$$

It is easy to see that each P_r commutes with A and if e_i an eigenvector of A associated to principal curvature k_i, then

$$P_1(e_i) = \mu_i e_i = (S_1 - k_i)e_i.$$

In [Re], Reilly showed that the P_r's satisfy the following

Proposition 2.1 ([Re], see also [BC] lemma 2.1) *Let $x : M^n \to N^{n+1}$ be an isometric immersion between two Riemannian manifolds and let A be its second fundamental form. The r'th Newton tensor P_r associated to A satisfies:*

1. $\operatorname{trace}(P_r) = (n-r)S_r$,
2. $\operatorname{trace}(AP_r) = (r+1)S_{r+1}$,
3. $\operatorname{trace}(A^2P_r) = S_1S_{r+1} - (r+2)S_{r+2}$.

It follows from (3) that if $S_2 = 0$, $\operatorname{trace}(A^2P_1) = -3S_3$.

Remark2.1 Observe that if $S_2 = 0$ and $S_1 \geq 0$, we have that

$$S_1^2 = |A|^2 + 2S_2 \geq k_i^2.$$

Thus, $0 \leq (S_1^2 - k_i^2) = (S_1 - k_i)(S_1 + k_i)$, what implies that all eigenvalues of P_1 are nonnegative, that is, P_1 is a positive operator. We also remark that if $S_2 = 0$ and P_1 has one eigenvalue equal to zero, then

$$P_1A \equiv 0. \tag{1}$$

In fact, if $\mu_{i_0} = 0$, then $k_{i_0} = S_1$. As $S_1^2 = |A|^2$, we get

$$\sum_{i \neq i_0} k_{i_0}^2 = 0.$$

So $k_i = 0$, for all $i \neq i_0$, hence $P_1 A \equiv 0$.

Associated to each Newton tensor P_r, we define a second order differential operator

$$L_r(f) = \mathrm{trace}(P_r \mathrm{Hess}\, f).$$

If N^{n+1} has constant sectional curvature, it follows from Codazzi equation (see Rosenberg [Ro], p. 225) that L_r is

$$L_r(f) = div_M(P_r \nabla f).$$

Hence L_r is a self-adjoint operator and for any differentiable functions f and g on M^n,

$$\int_M f L_r g dM = \int_M g L_r f dM \tag{2}$$

We observe that for $r = 0$, L_0 is the Laplacian which is always an elliptic operator. For $r > 0$ we have to adjoint some extra condition in order to get that L_r is elliptic. For hypersurfaces of $\mathbb{R}^{n+1}$ with $S_r = 0$, Hounie and Leite, [HL], were able to give a geometric condition that is equivalent to L_r being elliptic. In fact their proof can be generalized to hypersurfaces of the sphere and we have that

Theorem 2.1 ([HL] - Proposition 1.5) *Let M be a hypersurface in $\mathbb{R}^{n+1}$ or S^{n+1} with $S_r = 0$, $2 \leq r < n$. Then the operator $L_{r-1}(\phi) = div(P_{r-1}\nabla\phi)$ is elliptic at $p \in M$ if and only if $S_{r+1}(p) \neq 0$.*

Thus, for hypersurfaces with $S_2 = 0$, L_1 is an elliptic operator if and only if $S_3 \neq 0$. Since $L_1(f) = div_M(P_1 \nabla f)$, if L_1 is elliptic then P_1 is definite and then $S_1 \neq 0$.

Let $a \in \mathbb{R}^{n+2}$ be a fixed vector. Let $x : M \to S^{n+1}(1) \subset \mathbb{R}^{n+2}$ be an isometric immersion with $S_2 = 0$ and let N be its unit normal vector. The functions $f = \langle N, a\rangle$ and $g = \langle x, a\rangle$ satisfy (see [BC], lemma 5.2)

$$L_1(g) = -(n-1)S_1 g \tag{3}$$

and

$$L_1(f) = 3S_3)f. \tag{4}$$

2.2 Clifford hypersurfaces of $\mathbb{P}(\mathbb{R})^{n+1}$.

We are now going to describe some properties of the *Clifford hypersurface* in $\mathbb{P}(\mathbb{R})^{n+1}$. A Clifford torus in $S^{n+1}(1)$ is given by the product immersion of $M = S^{n_1}(r_1) \times S^{n_2}(r_2)$, with $n_1 + n_2 = n$ and $r_1^2 + r_2^2 = 1$, which is a closed hypersurface of $S^{n+1}(1)$. It is easy to see that this immersion is invariant under the antipodal map, hence it induces an immersion of M into $\mathbb{P}(\mathbb{R})^{n+1}$. This hypersurface will be called *Clifford hypersurface*. If $x : S^{n_1}(r_1) \times S^{n_2}(r_2) \to S^{n+1}(1)$ is a Clifford torus, then the unit normal vector at a point $p = (p_1, p_2) \in S^{n_1}(r_1) \times S^{n_2}(r_2)$ is given by

$$N = (-\frac{r_2}{r_1}p_1, \frac{r_1}{r_2}p_2).$$

Thus, the principal curvatures of M are $\frac{r_2}{r_1}$ with multiplicity n_1 and $-\frac{r_1}{r_2}$ with multiplicity n_2. It is easily checked that the scalar curvature of M is equal to one ($S_2 = 0$) if and only if $(\frac{r_2}{r_1})^2 = \beta$ satisfies the quadratic equation:

$$n_1(n_1 - 1)\beta^2 - 2n_1n_2\beta + n_2(n_2 - 1) = 0. \tag{5}$$

We will show in a while that only one of the torus given by 5 yields $S_1 > 0$. Notice that L_1 is an elliptic operator and in order to calculate the index of M, we first observe that in a principal basis, P_1 is a diagonal matrix whose elements are

$$\{(n_1 - 1)\frac{r_2}{r_1} - n_2\frac{r_1}{r_2}\} \text{ with multiplicity } n_1$$

and

$$\{n_1\frac{r_2}{r_1} - (n_2 - 1)\frac{r_1}{r_2}\} \text{ with multiplicity } n_2.$$

Thus,

$$\text{trace}P_1 = (n - 1)S_1 = n_1\frac{r_2}{r_1} - n_2\frac{r_1}{r_2}$$

and

$$\|\sqrt{P_1}A\|^2 = -3S_3 = (n - 1)S_1,$$

where in the last equality we use the special form of P_1 in the Clifford torus. Thus the *Jacobi operator* reduces to

$$T_1(f) = L_1(f) + \{\|\sqrt{P_1}A\|^2 + \text{trace}P_1\}f = L_1(f) + 2(n - 1)S_1f.$$

If $\varphi = const.$, $L_1(\varphi) = 0$ and

$$T_1(\varphi) + 2(n - 1)S_1\varphi = 0.$$

Thus the first eigenvalue of T_1 is negative. Thus $Ind(M)$ is at least 1. Now let us look at the second eigenvalue of T_1. By using the expression of the eigenvalues of P_1 given above, we have that

$$L_1(f) = div(P_1\nabla f) = \{(n_1 - 1)\frac{r_2}{r_1} - n_2\frac{r_1}{r_2}\}\Delta^{n_1}(f) + \{n_1\frac{r_2}{r_1} - (n_2 - 1)\frac{r_1}{r_2}\}\Delta^{n_2}(f),$$

where Δ^{n_i} is the laplacian in $S^{n_i}(r_i)$, $i=1,2$. Thus the second eigenvalue of L_1 is given by

$$\lambda_2 = -\{(n_1-1)\frac{r_2}{r_1} - n_2\frac{r_1}{r_2}\}\nu_2^{\Delta^{n_1}} + \{n_1\frac{r_2}{r_1} - (n_2-1)\frac{r_1}{r_2}\}\nu_2^{\Delta^{n_2}},$$

with $\nu_2^{\Delta^{n_i}}$ is the first nonzero eigenvalue of Δ^{n_i} that corresponds to eigenfunction which are invariant by the antipodal map (see [BGM] chap III, CII). Thus

$$\begin{aligned} \lambda_2 &= -\left[\left\{(n_1-1)\frac{r_2}{r_1} - n_2\frac{r_1}{r_2}\right\}\frac{n_1}{r_1^2} + \left\{n_1\frac{r_2}{r_1} - (n_2-1)\frac{r_1}{r_2}\right\}\frac{n_2}{r_2^2}\right] \\ &= \frac{-1}{r_1^3 r_2^3}\left\{[n_1(n_1-1) - n_1(n-1)r_1^2]r_2^2 + [n_2(n-1)r_2^2 - n_2(n_2-1)]r_1^2\right\} \end{aligned} \tag{6}$$

Observe that

$$S_1 = n_1\frac{r_2}{r_1} - n_2\frac{r_1}{r_2} = \frac{n_1 r_2^2 - n_2 r_1^2}{r_1 r_2}. \tag{7}$$

The fact that $S_2=0$ is equivalent to

$$n(n-1)r_1^4 - 2n_1(n-1)r_1^2 + n_1(n_1-1) = n(n-1)r_2^4 - 2n_2(n-1)r_1 2^2 + n_2(n_2-1) = 0. \tag{8}$$

By using (7) and (8), we have that

$$[n_1(n_1-1) - n_1(n-1)r_1^2]r_2^2 = (n-1)S_1 r_1^3 r_2^3$$

and

$$[n_2(n-1)r_2^2 - n_2(n_2-1)]r_1^2 = (n-1)S_1 r_1^3 r_2^3.$$

Thus,

$$\lambda_2 = -2(n-1)S_1.$$

Since the second eigenvalue of T_1 is given by $\lambda_2 + 2(n-1)S_1$, it is equal to zero. This shows then that the Clifford hypersurfaces of $\mathbb{P}(\mathbb{R})^{n+1}$ have index one.

Remark. Observe that, by equation (7), the condition $S_1 \geq 0$ means that

$$n_1 r_2^2 - n_2 r_1^2 \geq 0.$$

On the other hand, since $\beta = \left(\frac{r_2}{r_1}\right)^2$, the above inequality implies that

$$n_1\beta \geq n_2. \tag{9}$$

The condition $S_2 = 0$ is equivalent to

$$n_1(n_1-1)\beta^2 - 2n_1 n_2\beta + n_2(n_2-1) = 0, \tag{10}$$

and one can easily see that only one solution of (10) is compatible with (9).

3 A gap theorem for hypersurfaces of the sphere with $R = 1$.

In this section we prove a gap theorem for hypersurfaces of the sphere with $R = 1$.

Theorem 3.1 (Theorem 1 of the Introduction) *Let $x : M^n \to S^{n+1}(1)$ be a closed orientable hypersurface with scalar curvature $R = 1$ (equivalently, $S_2 = 0$). Assume that S_1 does not change sign and choose the orientation such that $S_1 \geq 0$ and that*

$$\|\sqrt{P_1}A\|^2 \leq \mathrm{trace}P_1.$$

Then:
(i) $\|\sqrt{P_1}A\|^2 = \mathrm{trace}P_1$.
(ii) M^n is either a totally geodesic submanifold or $M^n = S^{n_1}(r_1) \times S^{n_2}(r_2) \subset S^{n+1}(1)$, where $n_1 + n_2 = n$, $r_1^2 + r_2^2 = 1$ and $(\frac{r_2}{r_1})^2 = \beta$ satisfies the quadratic equation:

$$n_1(n_1-1)\beta^2 - 2n_1n_2\beta + n_2(n_2-1) = 0.$$

Proof. Let us calculate $\frac{1}{2}L_1\|A\|^2$. Since $R = 1$, Remark 2.1 shows that $S_2 = n(n-1)(R-1) = 0$. Thus $\|A\|^2 = (nH)^2 = S_1^2$. Hence,

$$\frac{1}{2}L_1\|A\|^2 = \frac{1}{2}L_1S_1^2 = S_1L_1S_1 + \langle P_1\nabla S_1, \nabla S_1\rangle.$$

From [AdCC](Lemma 3.7), by using that $2S_2 = n(n-1)(R-1) = 0$, we have

$$L_1S_1 = |\nabla A|^2 - |\nabla S_1|^2 + n\|A\|^2 - S_1^2 + 3S_1S_3.$$

Thus,

$$L_1S_1 = |\nabla A|^2 - |\nabla S_1|^2 + (n-1)S_1^2 + 3S_1S_3. \tag{11}$$

Now observe that

$$\|\sqrt{P_1}A\|^2 = \mathrm{trace}P_1A^2 = -3S_3.$$

Then, equation (11) becomes

$$L_1S_1 = |\nabla A|^2 - |\nabla S_1|^2 + (n-1)S_1^2 - S_1\|\sqrt{P_1}A\|^2.$$

Thus,

$$\begin{aligned} \tfrac{1}{2}L_1\|A\|^2 &= S_1L_1S_1 + \langle P_1\nabla S_1, \nabla S_1\rangle \\ &= S_1(|\nabla A|^2 - |\nabla S_1|^2 + (n-1)S_1^2 - 3S_1\|\sqrt{P_1}A\|^2) + \langle P_1\nabla S_1, \nabla S_1\rangle \\ &= S_1(|\nabla A|^2 - |\nabla S_1|^2) + S_1^2((n-1)S_1 - \|\sqrt{P_1}A\|^2) + \langle P_1\nabla S_1, \nabla S_1\rangle. \end{aligned}$$

Since M is compact, we obtain

$$\begin{aligned} 0 &= \frac{1}{2}\int_M L_1\|A\|^2 dM \\ &= \int_M \{S_1(|\nabla A|^2 - |\nabla S_1|^2) + S_1^2((n-1)S_1 - \|\sqrt{P_1}A\|^2) \\ &+ \langle P_1\nabla S_1, \nabla S_1\rangle\}dM. \end{aligned} \tag{12}$$

We recall the following lemma of [AdCC] (lemma 4.1):

Lemma 3.1 (AdCC) *Let M be an n-dimensional compact hypersurface in an (n+1)-dimensional unit sphere S^{n+1}. If the normalized scalar curvature R is constant and $R-1 \geq 0$, then*

$$|\nabla A|^2 - |\nabla S_1|^2 \geq 0. \tag{13}$$

Since $S_1 \geq 0$ and P_1 is positive, we have that

$$\langle P_1 \nabla S_1, \nabla S_1 \rangle = \|\sqrt{P_1}\nabla S_1\|^2 \geq 0. \tag{14}$$

Our hypothesis and inequalities (13) and (14) implies that the right-hand side of (12) is non-negative. Thus we conclude that

$$S_1(|\nabla A|^2 - |\nabla S_1|^2) + S_1^2((n-1)S_1 - \|\sqrt{P_1}A\|^2) + \langle P_1 \nabla S_1, \nabla S_1 \rangle = 0. \tag{15}$$

Since each term in above equation is non-negative, we have

$$S_1((n-1)S_1 - \|\sqrt{P_1}A\|^2) = 0.$$

Observe that when $S_1 = 0$, $\|A\|^2 = 0$ and $\|\sqrt{P_1}A\|^2 = 0$. Thus the first part of the theorem is proved.

Now, let us assume that $\|\sqrt{P_1}A(p)\|^2 = (n-1)S_1(p)$, for all $p \in M$. If $S_1(p) = 0$ for all $p \in M$, since $S_2 = 0$, $\|A\|^2 = 0$ and M is totally geodesic. Let us suppose that there exists a point p_0 in M such that $S_1(p_0) > 0$. So the set $\mathcal{A} \subset M$ where $S_1(p) > 0$ is open and non-void set of M. We claim that P_1 is positive definite in $\mathcal{A}$. In fact, if P_1 has one eigenvalue equal to zero, then by Remark 2.1, $P_1 A \equiv 0$ and since $\|\sqrt{P_1}A(p)\|^2 = (n-1)S_1(p)$, we conclude that $S_1 = 0$, which is a contradiction. On each connected component of $\mathcal{A}$, we have that

$$\langle P_1 \nabla S_1, \nabla S_1 \rangle = 0$$

and

$$|\nabla A|^2 - |\nabla S_1|^2 = 0.$$

Since P_1, is positive definite, the first equation implies that $\nabla S_1 = 0$. This fact implies that $|\nabla A|^2 = 0$, by the second equation, i.e., the second fundamental form of M is covariant constant. It follows that this component $\mathcal{A}$ is a piece of a Cliffod torus, by using the following theorem of H.B. Lawson ([L] - Theorem 4, see also [CdCK] Lemma 3).

Theorem 3.2 [L] *Let M^n be an isometrically immersed hypersurface of S^{n+1}, over which the second fundamental form is covariant constant. Then, up to isometries of S^{n+1}, M^n is an open set of $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$.*

Since along the boundary of $\mathcal{A}$, $\|A\|^2 = S_1^2 = 0$, we conclude that $\partial\mathcal{A} = \emptyset$ and M is a Clifford torus. □

4 Characterization of index one closed hypersurfaces with $R = 1$ in the real projective space

In this section we will assume that the operator L_1 is elliptic and will describe the index of closed hypersurfaces in the real projective space $\mathbb{P}(\mathbb{R})^{n+1}$. In order to do that we are going to use the covering map of S^{n+1} onto $\mathbb{P}(\mathbb{R})^{n+1}$. The following result will be helpful.

Lemma 4.1 *Let $M^n \to S^{n+1}$ is a closed hypersurface with $R = 1$. Then the index of the quadratic form*

$$I(f,f) = -\int_M fT_1 f dM = -\int_M fL_1 f + ((n-1)S_1 - 3S_3)f^2 dM$$

is greater than one.

Proof. First of all observe that for constant functions $f = const.$, we have that

$$I(f,f) = -\int_M fL_1 f + ((n-1)S_1 - 3S_3)f^2 dM = -\int_M ((n-1)S_1 - 3S_3)f^2 dM < 0.$$

Thus $indI \geq 1$.
Suppose that this index is equal to one. Let $\{e_1, ..., e_{n+2}\}$ be an orthonormal basis of $\mathbb{R}^{n+2}$. If we write the normal vector field of the immersion as $N = \sum_{i=1}^{n+2} n_i e_i$, we obtain that

$$L_1(n_i) = 3S_3 n_i, \text{ for all } i = 1, .., n+2.$$

Thus

$$I(n_i, n_i) = -\int_M ((n-1)S_1)n_i^2 dM \leq 0.$$

Since the functions n_i are linearely independent, the index one hypothesis implies that $(n-1)$ of the $n_i's$ have to be null and since $|N| = 1$, after reordering if necessary, we have $n_1 = 1$ and $n_i = 0$ for $i = 2, ..., n+2$. Thus the normal vector field $N = e_1$. This implies that M^n is totally geodesic, contradicting the hypothesis that $S_1 > 0$. We conclude then that $ind(M) > 1$.

The main result of this section is the following characterization of index one closed hypersurfaces of $\mathbb{P}(\mathbb{R})^{n+1}$.

Theorem 4.1 (Theorem 2 of the introduction) *Let $x : M^n \to \mathbb{P}(\mathbb{R})^{n+1}(1)$ be a closed hypersurface with scalar curvature one. Then $Ind(M) \geq 1$ and if $Ind(M) = 1$, M is the Clifford hypersurfaces obtained by the projection of the Clifford torus of Theorem 3.1.*

Proof. The proof is inspired by the proof of the minimal case in [dCRR]. Observe that the index one hypothesis implies that M must be connected. Since, by lemma 4.1, S^{n+1} does not have an index one hypersurface with $R = 1$, x can not lift to an immersion of M into the

S^{n+1}. Thus we get that there exists a connected twofold covering $\widetilde{M} \to M$ and an isometric immersion $\widetilde{x} : \widetilde{M} \to S^{n+1}$ which is locally congruent to the immersion of M in $\mathbb{P}(\mathbb{R})^{n+1}$. An object in $\widetilde{M}$ that corresponds to an object in M will be denoted by the same notation a in M. We have that the immersion $\widetilde{x}$ is such that $R = 1$ and $S_3 \neq 0$. This implies that we can choose the orientation of $\widetilde{M}$ in such way that $S_1 > 0$. If we denote by $\pi : \widetilde{M} \to \widetilde{M}$ the isometric involution induced by the covering, then $\widetilde{x}$ must satisfy:

$$\widetilde{x} \circ \pi = -\widetilde{x}$$

and

$$N \circ \pi = -N,$$

where N is the unit normal vector field of the immersion.

Let λ_1 be the first eigenvalue of the operator

$$T_1(\varphi) = L_1(\varphi) + ((n-1)S_1 + 3S_3)\varphi.$$

We know that its first eigenspace is one-dimensional and generated by a function φ that does not change sign on $\widetilde{M}$. Now, let $\varphi_1 = \varphi \circ \pi$. Since π is an isometry, we get that $T_1(\varphi_1) = \lambda_1 \varphi_1$. This implies that $\varphi = \pm\varphi \circ \pi$. Observe that if $\varphi = -\varphi \circ \pi$, φ has to change sign on $\widetilde{M}$. Thus $\varphi = \varphi \circ \pi$.

From the fact that $Ind(M) = 1$, we obtain that any function $u : \widetilde{M} \to \mathbb{R}$ such that $u \circ \pi = u$ and $\displaystyle\int_{\widetilde{M}} u\varphi d\widetilde{M} = 0$ satisfies

$$I(u,u) = -\int_{\widetilde{M}} \{uL_1u + ((n-1)S_1 + 3S_3)u^2\} d\widetilde{M} \geq 0.$$

Moreover, if such a function u satisfies $I(u,u) = 0$, then u is a Jacobi function, that is,

$$L_1u + ((n-1)S_1 + 3S_3)u = 0.$$

Given $a, b \in \mathbb{R}^{n+2}$, let $\phi_{a,b} : \widetilde{M} \to \mathbb{R}^{n+2}$ defined by

$$\phi_{a,b} = \langle \widetilde{x}, a\rangle \widetilde{x} + \langle N, a\rangle N + \langle \widetilde{x}, b\rangle N.$$

By doing the calculation coordinatewise and using equations (3) and (4) we have that

$$L_1(\widetilde{x}) = -(n-1)S_1\widetilde{x}$$

and

$$L_1(N) = 3S_3N.$$

Thus,

$$L_1(\langle \widetilde{x}, a\rangle \widetilde{x}) = -2(n-1)S_1\langle \widetilde{x}, a\rangle \widetilde{x} - P_1A(a^t),$$
$$L_1(\langle N, a\rangle N) = 6S_3\langle N, a\rangle N - P_1A^2(a^t)$$

and

$$L_1(\langle\widetilde{x},b\rangle N)=[-(n-1)S_1+3S_3]\langle\widetilde{x},b\rangle N-P_1A(b^t),$$

where a^t, b^t are the tangent projection of a and b. This implies that

$$T_1(\phi_{a,b})=-[(n-1)S_1+3S_3][\langle\widetilde{x},a\rangle\widetilde{x}-\langle N,a\rangle N]+X_{a,b}, \tag{16}$$

where $X_{a,b}$ is a tangent vector field. Then,

$$-\int_{\widetilde{M}}\langle T_1(\phi_{a,b}),\phi_{a,b}\rangle d\widetilde{M}$$

$$=\int_{\widetilde{M}}[(n-1)S_1+3S_3][\langle\widetilde{x},a\rangle^2-\langle N,a\rangle^2-\langle\widetilde{x},b\rangle\langle N,a\rangle]d\widetilde{M}.$$

Now, by (2), we have

$$\int_{\widetilde{M}}[(n-1)S_1+3S_3]\langle\widetilde{x},b\rangle\langle N,a\rangle d\widetilde{M}$$

$$=-\int_{\widetilde{M}}\{\langle N,a\rangle L_1(\langle\widetilde{x},b\rangle)-\langle\widetilde{x},b\rangle L_1(\langle N,a\rangle)\}d\widetilde{M}=0.$$

Thus,

$$-\int_{\widetilde{M}}\langle T_1(\phi_{a,b}),\phi_{a,b}\rangle d\widetilde{M}=\int_{\widetilde{M}}[(n-1)S_1+3S_3][\langle\widetilde{x},a\rangle^2-\langle N,a\rangle^2]d\widetilde{M}. \tag{17}$$

Observe that the above expression does not depend on b. We are going to show that for any $a\in\mathbb{R}^{n+2}$, it is possible to choose $b\in\mathbb{R}^{n+2}$ such that $\int_{\widetilde{M}}\varphi\phi_{a,b}d\widetilde{M}=0$. To do this, consider a linear map $F:\mathbb{R}^{n+2}\to\mathbb{R}^{n+2}$ given by

$$F(b)=\int_{\widetilde{M}}\varphi\langle\widetilde{x},b\rangle Nd\widetilde{M}.$$

We claim that F is injective (thus a linear isomorphism). In fact, if $b\neq 0$ is such that $F(b)=0$, one has that (17), with $\phi=\phi_{0,b}=\langle\widetilde{x},b\rangle N$, implies that

$$I(\phi,\phi)=0.$$

Then, $T_1(\phi)=0$. On the other hand, for $a=0$,

$$T_1(\phi)=X_{0,b}=-P_1A(b^t)=0, \tag{18}$$

where b^t is the tangent projection of b along $\widetilde{M}$. Since P_1 is positive definite, (18) says that $A(b^t)=0$ on $\widetilde{M}$, which is the same that $\langle N,b\rangle$ is constant along $\widetilde{M}$. As we have that $N\circ\pi=-N$, we get that $\langle N,b\rangle=0$. This implies that the function $u=\langle\widetilde{x},b\rangle$ satisfies that $\mathrm{Hess}u(X,Y)=\langle X,Y\rangle u$. We need the following result of M.Obata ([O], Theorem A)

Theorem 4.2 *([O]) In order that a complete Riemannian manifold of dimension $n \geq 2$ admit a non-constant function ϕ with* $\mathrm{Hess}\phi(X,Y) = c^2\phi\langle X,Y\rangle$*, it is necessary and sufficient that the manifold be isometric to a sphere $S^n(c)$ of radius $\frac{1}{c}$ in the $(n+1)$ Euclidean space.*

Thus, if u is non-constant, then $\widetilde{M}$ is isometric to a unit sphere and since $\widetilde{M}$ is isometrically immersed in $S^{n+1}(1)$, this implies that $\widetilde{M}$ is totally geodesic. On the other hand, if u is constant, $\widetilde{M}$is totally umbilic. Since $S_2 = 0$, $\widetilde{M}$ is again totally geodesic. In both cases, $S_1^2 = |A|^2 = 0$, which is a contradiction to the fact that $S_1 > 0$. Thus the claim is proved.

Take an orthonormal basis $\{a_1, ..., a_{n+2}\}$ of $\mathrm{I\!R}^{n+2}$. By using the isomorphism F, for any $i = 1, ..., n+2$, it is possible to find $b_i \in \mathrm{I\!R}^{n+2}$ such that $\int_{\widetilde{M}} \varphi\phi_{a_i,b_i} d\widetilde{M} = 0$. Thus each coordinate ϕ_{ij} of ϕ_{a_i,b_i} is such that $\int_{\widetilde{M}} \varphi\phi_{ij} d\widetilde{M} = 0$. Then, $I(\phi_{ij}, \phi_{ij}) \geq 0$. From equation (17), we have

$$
\begin{aligned}
0 &\leq \sum_{i=1}^{n+2} \int_{\widetilde{M}} [(n-1)S_1 + 3S_3][\langle\widetilde{x}, a_i\rangle^2 - \langle N, a_i\rangle^2] d\widetilde{M} \\
&= \sum_{i=1}^{n+2} \int_{\widetilde{M}} [(n-1)S_1 + 3S_3](|\widetilde{x}|^2 - |N|^2) d\widetilde{M} = 0.
\end{aligned}
$$

This implies that $T_1(\phi_{a_i,b_i}) = 0,\ i = 1, ..., n+2$. Hence, $\langle T_1(\phi_{a_i,b_i}), \widetilde{x}\rangle = 0$ and, by equation (16), we obtain that

$$[(n-1)S_1 + 3S_3]\langle\widetilde{x}, a_i\rangle = 0,\ \forall i = 1, ..., n+2.$$

But this is only possible if $(n-1)S_1 + 3S_3 = 0$. Since $\|\sqrt{P_1}A\|^2 = -3S_3 = (n-1)S_1$, theorem (3.1) implies that $\widetilde{M}$ is a Clifford torus. □

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