New Existence Results for Equilibrium Problems

Alfredo N. Iusem* Wilfredo Sosa†

May 21, 2001

Abstract

We consider equilibrium problems in the framework of the formulation proposed by Blum and Oettli, which includes variational inequalities, Nash equilibria in noncooperative games, and vector optimization problems, for instance, as particular cases. We establish new sufficient and/or necessary conditions for existence of solutions of such problems. Our results are based upon the relation between equilibrium problems and certain auxiliary convex feasibility problems, together with extensions to equilibrium problems of gap functions for variational inequalities. Then we apply our results to some particular instances of equilibrium problems, obtaining results which include, among others, a new lemma of the alternative for convex optimization problems.

Keywords: Equilibrium Problems, Convex Feasibility Problems.

1 Introduction

The problem of interest, which we call *Equilibrium Problem*, abbreviated (EP), is defined as follows.

$$\text{EP: Find } x \in K \text{ such that } f(x,y) \geq 0 \text{ for all } y \in K, \tag{1}$$

where

1. X is a real Hausdorff topological vector space,
2. K is a nonempty closed convex subset of X,

*Instituto de Matemática Pura e Aplicada, Estrada Dona Castorina 110, Jardim Botânico, CEP 22460-320 RJ, Rio de Janeiro, Brazil, e-mail: iusp@impa.br

†Instituto de Matemática y Ciencias Afines, Jirón Ancash 536, Lima 1, Lima, Perú, e-mail: sosa@uni.edu.pe

3. $f : K \times K \to I\!R$ is a function which satisfies the following properties:

   P1: $f(x,x) = 0$ for each $x \in K$,

   P2: $f(x,\cdot) : K \to I\!R$ is convex and lower semicontinuous for all $x \in K$,

   P3: $f(\cdot,y) : K \to I\!R$ is upper semicontinuous for all $y \in K$.

This problem was considered in the past, with slight variations in the assumptions, under various headings, for instance in [10], [2], [17], [8], [7], [18], [19], [21] and [13]. The purpose of these works was to extend results concerning particular problems. For example, the work of Brezis, Nirenberg and Stampacchia extended results concerning variational inequalities, corresponding to the case where $f(x,y) = \langle Ax, y - x\rangle$ and A is a monotone operator (see [10], pp. 296-297). Blum and Oettli [8] have pointed out that this formulation includes, as particular cases, optimization problems, Nash equilibria problems, complementarity problems, fixed point problems and variational inequality problems. We show below that some vector optimization problems are also particular cases of EP. Moreover, EP unifies these problems in a convenient way, in the sense that results obtained for some of these problems can be extended to the general formulation of EP (with suitable modifications, of course).
We describe next six particular cases of EP. The first five of them were discussed in [8]; the sixth comes from [14].

a) Convex minimization problem (CMP): Let $h : X \to I\!R \cup \{+\infty\}$ be a convex, lower semicontinuous and proper function. The convex minimization problem is defined as:

$$\text{Find} \quad x \in K \text{ such that} \quad h(x) \leq h(y) \text{ for all } y \in K. \tag{2}$$

If we take $K := \{x \in X \ / \ h(x) < +\infty\}$ and $f(x,y) := h(y) - h(x)$ for all x, $y \in K$, then f satisfies P1, P2 and P3. Moreover, x is a solution of CMP if and only if x is a solution of EP.

b) Fixed point problem (FPP): Assume that X is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $T : X \to \mathcal{P}(X)$ be an upper semicontinuous point-to-set mapping such that $T(x)$ is a nonempty, convex, weakly compact subset of K for each $x \in K$. The fixed point problem is defined as:

$$\text{Find} \quad x \in K \text{ such that} \quad x \in T(x). \tag{3}$$

If we take $f(x,y) := \max_{\varphi \in T(x)} \langle x - \varphi, y - x\rangle$ for all x, $y \in K$, then f satisfies P1, P2 and P3. Moreover, x is a solution of FPP if and only if x is a solution of EP.

c) Complementarity problem (CP): Let K be a closed convex cone, $K^+ := \{x \in X^* / \langle x, y\rangle \geq 0 \text{ for all } y \in K\}$ its polar cone, where X^* is the topological dual space of X, and let $T : K \to X^*$ be a continuous operator. The complementarity problem is defined as:

$$\text{Find } \; x \in K \text{ such that } \; T(x) \in K^+, \;\; \langle T(x), x\rangle = 0. \tag{4}$$

If we take $f(x, y) := \langle T(x), y - x\rangle$ for all x, $y \in K$, then f satisfies P1, P2 and P3. Moreover, x is a solution of CP if and only if x is a solution of EP.

d) Nash equilibria problem in noncooperative games (NEP): Let I be a finite set (the set of players). For each $i \in I$, consider a nonempty closed convex subset K_i of a real Hausdorff topological vector space Y_i (the strategy set of the $i-$th player). Let $X := \prod_{i\in I} Y_i$, $K := \prod_{i\in I} K_i$. For each $i \in I$ consider a continuous function $f_i : K \to I\!R$ (the loss function of $i-$th player, depending on the strategies of all players) which is convex in each one of its arguments. For each x, $y \in K$, we define $x(y_i)$ as $(x(y_i))_j := x_j$ for all $j \neq i$, and $(x(y_i))_i := y_i$. The Nash equilibria problem in noncooperative games is defined as:

$$\text{Find } \; x \in K, \text{ such that } \; f_i(x) \leq f_i(x(y_i)) \text{ for all } i \in I, \text{ for all } y \in K. \tag{5}$$

If we take $f(x, y) := \sum_{i\in I}(f_i(x(y_i)) - f_i(x))$ for all x, $y \in K$, then f satisfies P1, P2 and P3. Moreover, x is a solution of NEP if and only if x is a solution of EP.

e) Variational Inequality Problem (VIP): Let X be a real Banach space and $T : K \to X^*$ an upper semicontinuous point-to-set mapping, such that $T(x)$ is a compact set in the weak* topology for all $x \in K$. The variational inequality problem is defined as:

$$\text{Find } \; x \in K, \; u \in T(x), \text{ such that } \; \langle u, y - x\rangle \geq 0 \text{ for all } y \in K. \tag{6}$$

If we take $f(x, y) = \max_{u\in T(x)}\langle u, y - x\rangle$ for all x, $y \in K$, then f satisfies P1, P2 and P3. Moreover, x is a solution of VIP if and only if x is a solution of EP.

f) Vector Minimization Problem (VMP): Let $C \subset I\!R^m$ be a closed convex cone, such that both C and its polar cone C^+ have nonempty interior. Consider the partial order in $I\!R^m$ given by

$$x \preceq y \text{ if and only if } y - x \in C,$$

$$x \prec y \text{ if and only if } \ y - x \in \text{int}(C).$$

A function $F : K \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be C-convex if and only if K is convex and $F(tx + (1-t)y) \preceq tF(x) + (1-t)F(y)$ for all $x,\ y \in K$ and for all $t \in (0,1)$ Let $C \subset X = \mathbb{R}^m$ be a closed convex cone with nonempty interior, $C^+ = \{y \in \mathbb{R}^m \ / \ \langle x, y\rangle \geq 0 \text{ for all } x \in C\}$ and $F : K \to \mathbb{R}^m$ a C-convex function. The vector minimization problem is defined as:

$$\text{Find } \ \bar{x} \in K \text{ such that } F(x) \not\prec F(\bar{x}) \text{ for all } x \in K. \tag{7}$$

If we take $f(x,y) := \max_{||z||=1,\ z \in C^+} \langle z, F(y) - F(x)\rangle$, then f satisfies P1, P2 and P3. Moreover x is a solution of VMP if and only if x is a solution of EP.

To our knowledge, the history of existence results for this problem began in 1972, when Ky Fan proved the following famous minimax inequality.

Theorem 1.1 *([13], Theorem 1)*
Let X be a real Hausdorff topological vector space and K a nonempty compact convex subset of X. If a function $f : K \times K \to \mathbb{R}$ satisfies:

- *$f(\cdot, y) : K \to \mathbb{R}$ is upper semicotinuous for each $y \in K$,*
- *$f(x, \cdot) : K \to \mathbb{R}$ is quasiconvex for each $x \in K$,*

then there exists a point $x^* \in K$ such that

$$\inf_{y \in K} f(x^*, y) \geq \inf_{w \in K} f(w, w).$$

Ky Fan's Theorem implies existence of solutions for EP with a compact set K, because in this case the assumptions in the definition of EP imply Ky Fan's hypotheses.

Also in 1972, Brezis, Nirenberg and Stampacchia ([10], Theorem 1) proved a version of Ky Fan's theorem, without requiring compactness of K, but demanding instead the following coercivity assumption on f:
Assumption A1: A function $f : K \times K \to \mathbb{R}$ is said to be 1-coercive in K if there exists a nonempty compact subset L of X and $y_0 \in L \cap K$ such that for every $x \in K \setminus L$, $f(x, y_0) < 0$.

For optimization problems, $h : K \to \mathbb{R}$ is said to be coercive if there exists $\alpha \in \mathbb{R}$ such that the closure of $\{x \in K / h(x) \leq \alpha\}$ is compact. If $f(x,y) := h(y) - h(x)$, then assumption A1 is equivalent to this notion of coercivity.

Given a real Hausdorff topological vector space X, let K and C be convex sets with $C \subset K$. Then $core_K C$, the core of C relative to K, is defined through:

$$a \in core_K C \Longleftrightarrow (a \in C \quad and \quad C \cap]a, y[\neq \emptyset \text{ for all } y \in K \setminus C). \qquad (8)$$

In 1994, Blum and Oettli ([8], Theorem 1) proposed another version of Ky Fan's Theorem, using, instead of A1, the following coercivity assumption:
Assumption A2: A function $f : K \times K \to I\!R$ is said to be 2-coercive in K if there exists a nonempty compact convex set $C \subset K$ such that for every $x \in C \setminus core_K C$, there exists $a \in core_K C$ such that $f(x, a) \leq 0$.

Assumption A2 is also an extension of the above mentioned usual notion of coercivity, in the setting of real reflexive Banach spaces (see [8]).

There are other extensions of Ky Fan's minimax inequality, for instance, those presented by George Allen [2] in 1977 and Chin-Lin Yen [21] in 1981.

In the sequel, we will present new existence results for EP. They are based upon the connection between EP and a given convex feasibility problem, which we develop in section 2, and on an extension of Auslender's gap function for variational inequality problems (see [4]). With these tools, we establish in section 3 necessary and sufficient conditions for the existence of solutions of EP. In the particular cases of convex minimization problems and monotone variational inequalities, these conditions take a simpler form and are, to our knowledge, new. These cases are treated in section 4.

2 Equilibrium problems and convex feasibility problems

In this section we consider the following convex feasibility problem (CFP) as an auxiliary problem for EP (i.e. we assume thorougout this section that K and f satisfy the EP assumptions, presented in section 1):

$$\text{Find} \quad x \in \cap_{y \in K} L_f(y), \text{ where } L_f(y) := \{x \in K \text{ such that } f(y, x) \leq 0\}.$$

Note that for each $y \in K$, $L_f(y)$ is a nonempty, closed and convex subset of K, because $f(y, y) = 0$ for all $y \in K$, the function f is lower semicontinuous and convex in the second argument, and K is closed and convex.

The following lemma was proved by Ky Fan in 1961, and gives a sufficient condition for existence of solutions of CFP.

Lemma 2.1 *([12], Lemma 1) Let Y be an arbitrary subset of a real Hausdorff topological vector space X. For each $y \in Y$, consider a closed subset $C(y)$ of X. If the following two conditions hold:*

- **C1.** *the convex hull of any finite subset $\{x_1, .., x_n\}$ of Y, denoted as $co\{x_1, ..., x_n\}$, is contained in $\bigcup_{i=1}^{n} C(x_i)$,*
- **C2.** *$C(x)$ is compact for at least some $x \in Y$,*

then $\bigcap_{y \in Y} C(y) \neq \emptyset$

We need in the sequel the following variation of Lemma 2.1.

Lemma 2.2 *If we replace, in Lemma 2.1, condition C2 by the following condition:*

C3. *The closure $\overline{co}Y$ of the convex hull of Y is compact,*
then the conclusion of Lemma 2.1 still holds, i.e. $\bigcap_{y \in Y} C(y) \neq \emptyset$.

Proof: For each $y \in Y$ we define $D(y) := \overline{co}K \cap C(y)$. Consider any finite subset $\{ x_1, .., x_n \}$ of Y. We get from C1 that $co\{x_1, ..., x_n\} \subset \cup_{i=1}^{n} C(x_i)$ and that $co\{x_1, ..., x_n\} \subset \overline{co}Y$. It follows from these facts that

$$co\{x_1, ..., x_n\} \subset \overline{co}Y \cap \cup_{i=1}^{n} C(x_i),$$

implying

$$co\{x_1, ..., x_n\} \subset \cup_{i=1}^{n} \overline{co}Y \cap C(x_i) = \cup_{i=1}^{n} D(x_i).$$

On the other hand, $D(y)$ is compact for each $y \in Y$. Then, by Ky Fan's Lemma (Lemma 2.1), we have $\cap_{y \in Y} D(y) \neq \emptyset$, implying $\cap_{y \in Y} C(y) \neq \emptyset$. □

Remark 2.3 *Define $U_f(y)$ as*

$$U_f(y) := \{x \in K / f(x, y) \geq 0\} \text{ for all } y \in K.$$

Note that for each $y \in K$, $U_f(y)$ is a nonempty closed set, because $f(y, y) = 0$, f is upper semicontinuous in the first argument and K is a nonempty closed set. Moreover, the family $\{U_f(y)\}_{y \in K}$ satisfies C1. Even if f satisfies properties P1, P2 and P3, in the strong topology, we cannot apply Lemma 2.2 to the family $\{U_f(y)\}_{y \in K}$, because $\{U_f(y)\}_{y \in K}$ is not weakly closed in general, nor to $\{L_f(y)\}_{y \in K}$, because in general this family does not satisfy condition C1. But if f satisfies properties P1, P2 and P3 in the weak topology and K is nonempty, closed, convex and bounded, then we can use Lemma 2.2 and conclude that EP has solutions.

The connection between EP and CFP is established in the following result.

Lemma 2.4 *The solution set of CFP is a subset of the solution set of EP.*

Proof: Let x be a solution of CFP and take any $y \in K$. Define $w_t := ty + (1-t)x$ for each $t \in (0,1)$. Thus, $0 = f(w_t, w_t) \leq tf(w_t, y) + (1-t)f(w_t, x)$ for each $t \in (0,1)$, but $f(w_t, x) \leq 0$ for each $t \in (0,1)$, because $x \in \cap_{y \in K} L_f(y)$ and $w_t \in K$. Then, $0 \leq tf(w_t, y)$ for each $t \in (0,1)$, so that $0 \leq f(w_t, y)$ for each $t \in (0,1)$. Taking limits as $t \to 0$ we conclude, using P3, that $0 \leq f(x, y)$. □

We remark that the converse of the previous lemma does not hold. For example, consider $X = I\!R$, fix $a \in I\!R$, $a > 0$, take $K = [0, a]$ and define

$$f(x,y) := \begin{cases} 0 & \text{if } x = 0 \\ -\frac{y}{x} + 1 & \text{if } x > 0 \ . \end{cases}$$

Then the solution set of EP is $\{0, a\}$, but $\cap_{y \in K} U_f(y) = \cap_{y \in [0,a]} [y, a] = \{a\}$. A significant property which immediately guarantees equality of the solution sets of EP and CFP is the following.

P4: For all x, $y \in K$, if $f(x, y) \geq 0$ then $f(y, x) \leq 0$.

Property P4 was introduced by Biancchi and Schaible [7] as a form of pseudomonotonicity. They pointed out that P4 is equivalent to:

P4': For all $x, y \in K$, if $f(x, y) > 0$ then $f(y, x) < 0$.

We mention that P4 is an extension of the notion of pseudomonotonicity introduced by Karamardian in [16]. Moreover, when the function f is derived from complementarity problems or variational inequality problems and the operator is pseudomonotone in the sense of Karamardian, then f automatically satisfies P4. Henceforth the same occurs when the operator is monotone. P4 also holds automatically for convex optimization problems. Under P4, we have the following immediate result.

Proposition 2.5 *If f satisfies P1, P2, P3 and P4, then EP and CFP have the same solution set.*

Proof: The solution set of CFP is included in the solution set of EP by Lemma 2.4. The reciprocal inclusion follows easily from P4. □

Observe that if f satisfies P1, P2, P3 and P4, then the family $\{L_f(y)\}_{y\in K}$ satisfies C1. In connection with this property, we conjecture that if CFP and EP (with f satisfying P1, P2 and P3) have the same nonempty solution set, then f satisfies P4.

At this point, it is worthwhile to show that the generalization given by the EP formulation over the particular cases (a)–(f) above is genuine, in the sense that there are EP problems which do not fit the format of such particular cases. The following result provides such an example with a quadratic f which does satisfy P1 through P4, in a finite dimensional vector space.

Proposition 2.6 *Consider $X = K = \mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by $f(x, y) := y^tAy + x^tBy - x^tAx$, where $B \in \mathbb{R}^{n\times n}$ is skew-symmetric, and $A \in \mathbb{R}^{n\times n}$ is positive semidefinite, satisfying $x^tAx > 0$ for some $x \in \mathbb{R}^n$. Then*

1. *f satisfies P1, P2, P3 and P4,*
2. *there exists no $h : \mathbb{R}^n \to \mathbb{R}$ such that $f(x, y) = h(y) - h(x)$,*
3. *There exists no $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ such that $f(x, y) = \max_{u\in T(x)}\langle u, y - x\rangle$ for all $x,\ y \in \mathbb{R}^n$,*
4. *There exists no $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ such that $f(x, y) = \max_{u\in T(x)}\langle x - u, y - x\rangle$ for all $x,\ y \in \mathbb{R}^n$.*

Proof:

1. P1 follows from the fact that $x^tBx = 0$ for all $x \in \mathbb{R}^n$, P2 holds because A is positive semidefinite, P3 follows from the definition of f and P4 follows from the fact that $f(x, y)+f(y, x) = 0$ for all $x, y \in \mathbb{R}^n$.
2. The statement holds because f is not separable.
3. We proceed by contradiction. Suppose that there exists $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ such that $f(x, y) = \max_{u\in T(x)}\langle u, y - x\rangle$ for all $x, y \in \mathbb{R}^n$. Pick up $x \in \mathbb{R}^n$ be such that $x^tAx > 0$. Taking first $y = 2x$ and then $y = 3x$, we get $3x^tAx = \max_{u\in T(x)}\langle u, x\rangle$ and $4x^tAx = \max_{u\in T(x)}\langle u, x\rangle$, implying $x^tAx = 0$, which contradicts the fact that $x^tAx > 0$.
4. Follows using the same argument as in the previous item.

□

3 Existence Results

In this section we introduce an extension of Auslender's gap function, introduced in [4] for variational inequality problems. We use this extension in order to obtain necessary and sufficient conditions for existence of solutions of EP. Throughout this section, both f and K are assumed to satisfy the hypotheses of EP, presented in section 1.

Definition 3.1 *The gap function $g_f : X \to \mathbb{R} \cup \{+\infty\}$ is defined as*

$$g_f(x) := \begin{cases} \sup_{y \in K} f(y,x) & \text{if } x \in K \\ +\infty & \text{otherwise.} \end{cases}$$

Lemma 3.2 *The gap function g_f is nonnegative, convex and lower semicontinuous. Moreover, if CFP has solutions then the gap function g_f is proper (i.e. there exists $x \in X$ such that $g_f(x) \in \mathbb{R}$).*

Proof: Nonnegativeness follows from P1. Convexity follows from the convexity of $f(y,\cdot)$ for all $y \in K$. Closedness follows from the lower semicontinuity of $f(y,\cdot)$ for all $y \in K$. Finally, let x be a solution of the CFP, so that $f(y,x) \leq 0$ for all $y \in K$. It follows that $g_f(x) = \sup_{y \in K} f(y,x) = 0 \in \mathbb{R}$ □

Let $m := \inf_{x \in X} g_f(x)$ and $M := \{x \in X / g_f(x) = m\}$. Note that M is the set of minimizers of the gap function, and m is the minimum value of the gap function, when $M \neq \emptyset$. We present next our first existence result for EP.

Theorem 3.3 *CFP has solutions if and only if $M \neq \emptyset$ and $m = 0$. Moreover M is the solution set of CFP.*

Proof: $\Rightarrow$) Take $x \in M$. Note that $\sup_{y \in K} f(y,x) = 0$, because $m = 0$. It follows that $f(y,x) \leq 0$ for all $y \in K$ and so x is a solution of CFP.
$\Leftarrow$) Let x be a solution of CFP. Then $f(y,x) \leq 0$ for all $y \in K$, implying that $g_f(x) \leq 0$. By Lemma 3.2, $g_f(x) \geq 0$. It follows that $g_f(x) = 0$, so that $m = 0$ and $x \in M$. □

We mention that when CFP does not have solutions, it may happen that either $M \neq \emptyset$ (in which case $m > 0$) or $m = 0$ (in which case $M = \emptyset$). For an example of the first kind, take $K = [0, +\infty)$ and $f(x,y) := -\frac{y}{x} + 1$ if

$x > 0$, $f(0, y) = 0$, in which case $m = 1$ and $M = K$; for an example of the second kind, take $K = \mathbb{R}$ and $f(x, y) = e^y - e^x$, in which case $m = 0$ and $M = \emptyset$.

Definition 3.4 *For each nonempty subset K of X, we define the asymptotic approximation of K by as the familiy of sets $A(K)$ given by:*

$$A(K) := \{\{K_n\}/K_n \subset core_K K_{n+1} \neq K_{n+1} \text{ for all } n \in \mathbb{N}, \cup_{n\in\mathbb{N}} K_n = K\},$$

where "core" is defined by (8).

We need the following two technical lemmas for our next existence result.

Lemma 3.5 *Consider EP with $A(K) \neq \emptyset$. For each $\{K_n\} \in A(K)$, the sequence $\{g_f^n\}$, where $g_f^n : X \to \mathbb{R} \cup \{+\infty\}$ is defined by*

$$g_f^n(x) := \begin{cases} \sup_{y\in K_n} f(y, x) & \text{if } x \in K \\ +\infty & \text{otherwise,} \end{cases}$$

is a nondecreasing sequence of convex and lower semicontinuous functions. Moreover, $g_f(x) := \lim_{n\to+\infty} g_f^n(x)$ for all $x \in X$, with g_f as in Definition 3.1.

Proof: The result follows from the definitions of g_f and g_f^n. □

Remark 3.6 *Consider EP and a sequence of nonempty closed convex set $\{K_n\} \in A(K)$. For each $n \in \mathbb{N}$ and $y \in K$, define $L_f(n, y) := \{x \in K_n/f(y, x) \leq 0\}$. Note that $\cap_{y\in K_n} L_f(n, y) \subset \{x \in K_n/f(x, y) \geq 0$ for all $y \in K_n\}$, i.e. each solution of the convex feasibility problem restricted to K_n is a solution of the equilibrium problem restricted to K_n.*

Lemma 3.7 *Let K be a nonempty convex set and $f : K \times K \to \mathbb{R}$ such that $f(x.\cdot) : K \to \mathbb{R}$ is a convex function. If for some $n \in \mathbb{N}$ and some $x \in \cap_{y\in K_n} L_f(n, y)$ there exists $y \in core_K K_n$ such that $f(x, y) \leq 0$, then $f(x, y) \geq 0$ for all $y \in K$.*

Proof: If $x \in \cap_{y\in K_n} L_f(n, y)$, then, by Remark 3.6, we have that $f(x, w) \geq 0$ for all $w \in K_n$. Thus, we only need to show that $f(x, w) \geq 0$ for all $w \in K \setminus K_n$. Take any $w \in K \setminus K_n$. By definition of $core_K K_n$, there exists

$t \in (0,1)$ such that $z := tw + (1-t)y \in K_n$. It follows that $0 \leq f(x,z) \leq tf(x,w) + (1-t)f(x,y) \leq tf(x,w)$, and so $0 \leq f(x,w)$. $\square$

We introduce next a new assumption, which is in fact, as we will show below, the sufficient condition in our next existence result.

Assumption A3: A function $f : K \times K \to I\!R$ satisfies A3 if there exists a sequence of closed convex sets $\{K_n\} \in A(K)$ such that

1. for all $n \in I\!N$, $\cap_{y \in K_n} L_f(n,y) \neq \emptyset$,
2. $M = \{x \in X / g_f(x) \leq m\}$ is nonempty,
3. there exist $i,\ j \in I\!N$ such that $\{x \in X / g_f^i(x) < m\} \subset core_K K_j$.

Before proceeding to our existence result, we establish some cases in which our A3 is guaranteed to hold.

Remark 3.8 *Assume that X is finite dimensional and $h : X \to I\!R \cup \{+\infty\}$ is a proper, convex and lower semicontinuous function such that $h(x) = \lim_{n \to +\infty} h_n(x)$ for all $x \in X$, where $\{h_n\}$ is a nondecreasing sequence of proper, convex and lower semicontinuous functions defined in X with values in $I\!R \cup \{+\infty\}$. It is known that if h has a nonempty and compact sublevel, then for n large enough h_n has a nonempty and compact sublevel (see [11], Lemma 1).*

Proposition 3.9 *Consider EP with X finite dimensional, K an unbounded closed and convex set and f satisfying P1, P2, P3 and P4. If M is nonempty and compact, then f satisfies assumption A3.*

Proof: For each $n \in I\!N$, define $K_n := \{x \in K / ||x|| \leq n\}$ (since K is unbounded, without loss of generality we assume $K_n \neq \emptyset$ for all $n \in I\!N$). Then $\{K_n\} \in A(K)$ is a sequence of closed convex sets. Item 1 of assumption A3 follows from Lemma 2.4. Item 2 holds because M is a nonempty set. Finally, item 3 follows from Remark 3.8. $\square$

Now, we proceed to establish our second existence result for EP.

Theorem 3.10 *If f satifies A3, then EP has solutions.*

Proof: For each $n \in I\!N$ take $x^n \in \cap_{y \in K_n} L_f(n,y)$. If there exists $n \in I\!N$ such that $x^n \in core_K K_n$, then by Lemma 3.7 x^n is solution of EP. On the other hand, if for all $n \in I\!N$, $x^n \in K_n \setminus core_K K_n$, then by item 2 of assumption A3, we have that M is nonempty. Suppose that $m > 0$. By

item 3 of assumption A3, we have that there exist $i,\ j \in I\!N$ such that $\{x \in X / g_f^i(x) < m\} \subset core_K K_j$. Take now $n = \max\{i, j\}$. Then $0 = g_f^n(x^n) \geq g_f^i(x^n) \geq m$, contradicting the fact that $m > 0$. So, $m = 0$, and thus the statement follows applying Theorem 3.3 and Lemma 2.4. □

Next we consider the case of EP with pseudomonotone f in reflexive Banach spaces, i.e. X is a real reflexive Banach space, and f satisfies P4, i.e. for all $x,\ y \in K$, $f(x,y) \geq 0$ implies $f(y,x) \leq 0$, in addition to P1-P3 (where the semicontinuities in P2 and P3 are understood to hold with respect to the strong topology of X). For this case, we will provide a sufficient and necessary condition for the existence of solutions of EP. We will use the following notation: $K_n := \{x \in K / ||x|| \leq n\}$ for each $n \in I\!N$, $L_f(n,y) := \{x \in K_n / f(y,x) \leq 0\}$ for all $y \in K$, $U_f(n,y) := \{x \in K_n / f(y,x) \geq 0\}$ for all $y \in K$.

Observe that $\cap_{y \in K_n} L_f(n,y)$ is the solution set of the CFP restricted to the set K_n. Similarly $\cap_{y \in K_n} U_f(n,y)$ is the solution set of EP restricted to the set K_n. Moreover $\cap_{y \in K_n} L_f(n,y) = \cap_{y \in K_n} U_f(n,y) \neq \emptyset$ for each $n \in I\!N$, if we assume, without loss of generality, that $K_n \neq \emptyset$ for all $n \in I\!N$.

Now, we introduce another assumption, which exploits the geometry of reflexive Banach spaces, and which will be proved to be the announced necessary and sufficient condition for existence of solutions of EP.

P5: For all sequence $\{x^k\} \subset K \setminus \{0\}$ satisfying:

1. $\lim_{k \to +\infty} ||x^k|| = +\infty$,
2. $\frac{x^k}{||x^k||}$ converges weakly to a point x such that $f(y, y+x) \leq 0$ for all $y \in K$,

there exists $\{u^k\} \subset K$ such that, for large enough n:

a) $||u^n|| < ||x^n||$,

b) $f(x^n, u^n) \leq 0$.

The following result gives relations between the coercivity assumptions given by Blum and Oettli, and by Brezis, Niremberg and Stampacchia, introduced in section 1, with assumption P5.

Proposition 3.11 *Take $f : K \times K \to I\!R$.*

i) *If f is 1-coercive in K, then f satisfies P5,*

ii) *If f satisfies P1, P2, P3, P4 and is 2-coercive in K, then f satisfies P5.*

Proof: **i)** By definition of 1-coercivity in K, there exists a compact set C and $y \in K \cap C$ such that $f(x,y) < 0$ for all $x \in K \setminus C$. Consider $\{x^k\} \subset K \setminus \{0\}$ with $||x^k|| \to +\infty$ and $\frac{x^k}{||x^k||} \rightharpoonup x$, satisfying $f(y, x+y) \leq 0$ for all $y \in K$. Taking $u^k = y$ for all $k \in I\!N$, we obtain (a) and (b) in the definition of P5.
ii) By the definition of 2-coercivity, there exists $x \in K$ such that $f(y,x) \leq 0$ for all $y \in K$. Taking $u^k = x$ for all $k \in I\!N$, we easily obtain (a) and (b) in the definition of P5. □

The converse of the previous proposition is not true. For (i), consider $K = I\!R$ and $f(x,y) = h(y) - h(x)$, where $h : K \to I\!R$ is defined by

$$h(x) := \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

for (ii) consider $K = [0, +\infty)$ and $f(x,y) = y - x$. Note that in both cases the function f satisfies P1, P2, P3, P4 and P5, but the first one does not satisfy assumption A1 (so that f is not 1-coercive) and the second one does not satisfy assumption A2 (so that f is not 2-coercive). This fact is consistent with our next result, which says that P5 is both sufficient and necessary for existence of solutions of EP, and the well known fact that neither 1-coercivity nor 2-coercivity are necessary for such existence, though both are sufficient, as discussed in section 1.

Theorem 3.12 *Property P5 is a necessary and sufficient condition for the existence of solutions of EP.*

Proof: **Necessary condition.** Consider $\{x^k\} \subset K \setminus \{0\}$ with $||x^k|| \to +\infty$ and $\frac{x^k}{||x^k||} \rightharpoonup x$ satisfying $f(y, x+y) \leq 0$ for all $y \in K$. Let w be a solution of EP. Taking $u^k = w$ for all $k \in I\!N$, we obtain (a) and (b) in the definition of P5.
Sufficient condition. For each $k \in N$ take $x^k \in \cap_{y \in K_n} L_f(n, y)$. By Lemma 2.2 such an x^k exists. Now we consider two cases:
i) If $||x^k|| = k$ for all $k \in I\!N$, fix any $y \in K$ and take $m > ||y||$. Then $f(y, x^k) \leq 0$ for all $k \geq m$, implying that for all $k \geq m$:

$$f\left(y, \left(1 - \frac{1}{||x^k||}\right)y + \frac{1}{||x^k||}x^k\right) \leq \left(1 - \frac{1}{||x^k||}\right)f(y,y) + \frac{1}{||x^k||}f(y, x^k) \leq 0. \tag{9}$$

Since $(1-\frac{1}{||x^k||})y+\frac{1}{||x^k||}x^k \rightharpoonup y+x$, by P2, lower semicontinuity of $f(y,\cdot)$ and (9), we have that $f(y,y+x) \leq 0$. By P5 there exists $\{u^k\} \subset K$ such that for large enough n, $||u^n|| < ||x^n||$ and $f(x^n,u^n) \leq 0$. The first inequality implies that $u^n \in core_K K_n$. By Lemma 3.7, u^n is a solution of EP.
ii) If there exists $n \in I$ such that $||x^n|| < n$, then $x^n \in core_K K_n$ and by Lemma 3.7, x^n is a solution of EP. □

As a first application of Theorem 3.12, we show that P5 can be weakened, in the presence of an additional condition, which we call P6, and which holds trivially in finite dimensional spaces.
P6: There exists $y \in K$ such that for any unbounded sequence $\{x^n\} \subset K \setminus \{0\}$, satifying that $\{f(y,x^n)\}$ is bounded above, there exists a strong cluster point of $\{\frac{x^n}{||x^n||}\}$.

Property P6 is related to an assumption introduced by Baiocchi and others (see [5]), called *compatibility*, which is satisfied in many applications concerning unilateral problems in mechanics. For more information about it, see [5] and [3].

Corollary 3.13 *Let EP be such that P6 hold. Let* $V := \{x \in X / f(z, z+x) \leq 0$ *for all* $z \in K\}$ *If* $V = \{0\}$*, then EP has at least one solution.*

Proof: The statement follows directly from Theorem 3.12 because P5 holds vacuously (i.e. there exists no sequence $\{x^n\} \subset K \setminus \{0\}$ satisfying items (1) and (2) of P5). □

4 Applications

In this section we apply the existence results, established in section 3, to convex minimization problems and pseudomonotone variational inequalities. We start with the first case, and consider the notation of (a) in section 1. We take $K := dom(h)$ and $f(x,y) := h(y) - h(x)$ for all x, $y \in K$ and the resulting EP with these K and F has the same solution set as the convex minimization problem. Also, this f satisfies P4, in addition to P1, P2 and P3. Next we look at the particular form of the gap function of Definition 3.1 in section 3 for this case.

Proposition 4.1 *The following statements hold:*

1. *The gap function of Definition 3.1 reduces to*

$$g_f(x) := \begin{cases} h(x) - \inf_{y \in K} h(y) & \text{if } x \in K := dom(h) \\ +\infty & \text{otherwise,} \end{cases}$$

2. *if h is bounded below, then $m := \inf_{x \in X} g_f(x) = 0$,*

3. *if $\emptyset \neq M := \{x \in X / g_f(x) = m\} \neq X$, then h is bounded below.*

Proof:

1. The statement follows from the definitions of f and g_f.

2. If $\alpha = \inf_{y \in K} h(y)$ then $\alpha \in I\!R$ and $m = \alpha - \alpha = 0$.

3. We proceed by contradiction. Suppose that h is unbounded below. Then $\inf_{y \in K} h(y) = -\infty$, so that $g_f(x) = +\infty$ for all $x \in X$. Thus, we have that $m = +\infty$ and $M = X$, contradicting the fact that $M \neq X$.

□

The first item in the next theorem strenghtens the result of Theorem 3.3 for the case of convex minimization problem; the second one is a sort of converse of Corollary 3.13: it shows that when M is nonempty and bounded (in which case EP certainly has solutions) then $V = 0$, with the notation of the corollary.

Theorem 4.2 *Let X be a real reflexive Banach space. Then*

1. *If $X \neq M \neq \emptyset$, then the set M of minimizers of the gap function g_f is the solution set of the convex minimization problem,*

2. *if $M \neq \emptyset$ and M is bounded then $V := \{x \in X / h(z + x) \leq h(z)$ for all $z \in K\} = \{0\}$.*

Proof:

1. By Proposition 4.1, we have that $m = 0$, and by Theorem 3.3 we have that M is the solution set of EP.

2. Take $x \in V$, so that $h(z + x) \leq h(z)$ for all $z \in K$. It follows that $z + x \in K$ for all $z \in K$, and thus $h(z + 2x) = h((z + x) + x) \leq h(z + x) \leq h(z)$ for all $z \in K$, implying that $2x \in V$. Using the same argument we have that $nx \in V$ for all $n \in I\!N$. Now take $w \in M$, so that $m \leq h(w + nx) - \inf_{y \in K} h(y) \leq h(w) - \inf_{y \in K} h(y) = m$ for all $n \in I\!N$, and therefore $w + nx \in M$ for all $n \in I\!N$. The boundedness of M implies that $x = 0$.

□

Next we apply Theorem 3.12 to finite dimensional convex optimization problems. We get a sort of lemma of the alternative, which, to our knowledge, is new.

Theorem 4.3 *Let X be a finite dimensional space, C a nonempty, closed and convex subset of X and $h : X \to \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous function. If the following problem*

$$\textit{(AP): find } x \in X \textit{ such that } ||x|| = 1 \textit{ and } h(x+y) \leq h(y) \textit{ for all } y \in C,$$

does not have solutions, then the problem $\min_{x \in C} h(x)$ *has solutions.*

Proof: Define $K = dom(h) \cap C$. If $K = \emptyset$ then the statement holds trivially, because $h(x) = +\infty$ for all $x \in C$. If $K \neq \emptyset$, then define $f(x,y) := h(y) - h(x)$ for all $x,\ y \in K$. Then f satisfies P1, P2, P3 and P4. Now, if AP does not have solutions then P5 holds vacuously, and therefore the result follows from Theorem 3.12. □

Next we apply the results of section 3 to pseudomonotone variational inequalities in finite dimensional spaces. Thus, we consider the notation of (e) in section 1, with $X = \mathbb{R}^n$. We remark that the uppersemicontinuity of T is understood in the sense of Berge, [6], i.e. for each $x \in Dom(T)$ and each open set $U \subset \mathbb{R}^n$ such that $T(x) \subset U$, there exists an open set $V \subset \mathbb{R}^n$ such that $T(y) \subset U$ for all $y \in V$. We assume that T is pseudomonotone in the sense of Karamardian [16], i.e. $\sup_{u \in T(x)} \langle u, y - x \rangle \geq 0$ for all $x,\ y \in Dom(T)$ implies $\langle v, y - x \rangle \geq 0$ for all $v \in T(y)$. As in (e) of section 1, we define $f(x,y) := \max_{u \in T(x)} \langle u, y - x \rangle$ for all $x,\ y \in K$, and we know that VIP and EP have the same solution set. We remark that f trivially satisfies P4, in addition to P1, P2 and P3. Also, the function g_f of section 3 becomes in this case

$$g_f(x) := \begin{cases} \sup_{y \in K} \sup_{u \in T(y)} \langle u, x - y \rangle & \text{if } x \in K \\ +\infty & \text{otherwise.} \end{cases}$$

Definition 4.4 *Given a nonempty set U of $\mathbb{R}^n$, we define the recession cone U_∞ of U as:*

$$U_\infty := \{x \in X / \textit{ there exists } t_n \downarrow 0 \textit{ such that } x^n \in U,\ \ t_n x^n \to x\}.$$

It is well known that if U is convex, then $U_\infty = \{x \in X / y + tx \in U$ for all $t > 0\}$, where y is any point of U (see e.g. Proposition 2.2.1 in [15], Volume 1).

The following theorem establishes the equivalence of three statements related to the set of minimizers of the gap functions, the set of solutions of the variational inequality problem and a certain subset of the recession cone of K. The equivalence of the third statement, which we believe is new, follows from our Theorem 3.12.

Theorem 4.5 *The following statements are equivalent:*

1. *The solution set of VIP is nonempty and bounded,*
2. *$M \cap K$ is nonempty and bounded.*
3. *$K_\infty \cap \{w \in X / \langle u, w\rangle \leq 0$ for all $u \in T(K)\} = \{0\}$.*

Proof: **(1 $\Rightarrow$ 2)** Follows from Theorem 3.3, because M is the solution set of CFP. But, by Proposition 2.5, EP and CFP have the same solution set, because the function f for VIP, as defined in Example (e) of section 1, satisfies P4. On the other hand, as proved in [8], EP and VIP have the same solution set, so that M is indeed the solution set of VIP.
(2 $\Rightarrow$ 3) Take $z \in K_\infty \cap \{w \in X / \langle u, w\rangle \leq 0$ for all $u \in T(K)\}$ and $x \in M \cap K$. Then $\langle u, x + tz - y\rangle = \langle u, x - y\rangle + t\langle u, z\rangle \leq \langle u, x - y\rangle$ for all $u \in T(K)$ all $y \in K$, and all $t > 0$. Thus $m \leq g_f(x + tz) \leq g_f(x) = m$ for all $t > 0$. Since $z \in K_\infty$, it holds that $x + tz \in K$ for all $t > 0$, and therefore $x + tz \in M \cap K$ for all $t > 0$. But by statement (2), $M \cap K$ is bounded, implying $z = 0$.
(3 $\Rightarrow$ 1) The nonemptiness of the solution set of EP (which is the same of VIP) follows from Theorem 3.12, because P5 is vacuously satisfied. Suppose that the solution set is unbounded and let x be a solution. Then there exists $z \in I\!R^n \setminus \{0\}$ such that $x + tz$ are solutions of VIP for all $t > 0$. This implies that $z \in K_\infty$ and $\langle u, x + tz - y\rangle \leq \max_{u \in T(y)} \langle u, x + tz - y\rangle \leq 0$ for all $y \in K,\ u \in T(K),\ t > 0$. Therefore $\langle u, z\rangle \leq 0$ for all $u \in T(K)$, which is a contradiction, because $K_\infty \cap \{w \in X / \langle u, w\rangle \leq 0$ for all $u \in T(K)\} = \{0\}$, by statement (3). □

References

[1] S. Adler, D. Goeleven, M. Théra, Recession mappings and noncoercive variational inequalities, Nonlinear Analysis, Theory, Methods and Applications 26 (1996) 1573-1603.

[2] G. Allen, Variational inequalities, complementarity problems and duality theorems, Journal of Mathematical Analysis and Applications 58 (1977) 1-10.

[3] A. Auslender, Noncoercive optimization problems, Mathematics of Operations Research 21 (1996) 769-782.

[4] A. Auslender, Résolution numérique d'inégalités variatonnelles, RAIRO R2 (1973) 67-72.

[5] C. Baiocchi, G. Buttazo, F. Gastaldi, F. Tomarelli, General existence theorem for unilateral problems in continuum mechanics, Archive for Rational Mechanics and Analysis 100 (1988) 149-189.

[6] C. Berge, Espaces Topologiques et Fonctions Multivoques, Dunod, Paris, 1959.

[7] M. Bianchi, S. Schaible, Generalized monotone bifunctions and equilibrium problems, Journal of Optimization Theory and Applications 90 (1996) 31-43.

[8] E. Blum, W. Oettli, ¿From optimization and variational inequalities to equilibrium problems, The Mathematics Student 63 (1994) 123-145.

[9] H. Brezis, Analyse Fonctionelle: Théorie et Applications, Masson, Paris, 1983.

[10] H. Brezis, L. Nirenberg, G. Stampacchia, A remark on Ky Fan's minimax principle, Bolletino della Unione Matematica Italiana 6 (1972) 293-300.

[11] J.P. Crouzeix, Pseudomonotone variational inequality problems: existence of solutions, Mathematical Programming 78 (1997) 305-314.

[12] K. Fan, A generalization of Tychonoff's fixed point theorem, Mathematische Annalen 142 (1961) 305-310.

[13] K. Fan, A minimax inequality and applications, in O. Shisha (Ed.), Inequality III, Academic Press, New York, 1972, pp. 103-113.

[14] L.M. Graña Drummond, personal communication.

[15] J.-B. Hiriart-Urruty, C. Lemaréchal, Convex Analysis and Minimization Algorithms, Springer, Berlin, 1993.

[16] S. Karamardian, S. Schaible, Seven kinds of monotone maps, Journal of Optimization Theory and Applications 66 (1990) 37-46.

[17] L.D. Muu, W. Oettli, Convergence of an adaptive penalty scheme for finding constraint equilibria, Nonlinear Analysis, Theory, Methods and Applications 18 (1992) 1159-1166.

[18] W. Oettli, A remark on vector-valued equilibria and generalized monotonicity, Acta Mathematica Vietnamita 22 (1997) 215-221.

[19] W. Oettli, D. Schläger, Generalized vectorial equilibrium and generalized monotonicity, to be published in Functional Analysis and Applications.

[20] J. Van Tiel, Convex Analysis, an Introductory Text, John Wiley, New York, 1984.

[21] C. Yen, A minimax inequality and its application to variational inequalities, Pacific Journal of Mathematics 97 (1981) 477-481.