Endogenous Collateral: Arbitrage and Equilibrium without Bounded Short Sales

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Abstract

We study the implications of the absence of arbitrage in an two period economy where default is allowed and assets are secured by collateral choosen by the borrowers. We show that non arbitrage sale prices of assets are submartingales, whereas non arbitrage purchase prices of the derivatives (secured by the pool of collaterals) are supermartingales. We use these non arbitrage conditions to establish existence of equilibrium, without imposing bounds on short sales. The nonconvexity of the budget set is overcome by considering a continuum of agents. Our results are particularly relevant for the collateralized mortgage obligations(CMO) markets.

Keywords: Endogenous Collateral; Non Arbitrage.
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1 Introduction

1.1 Motivation

Housing mortgages stand out as the most clear and most common case of collateralized loans. In the past, these mortgages were entirely financed by commercial banks who had to face a serious adverse selection problem in addition of the risks associated with concentrating investments in the housing sector. More recently, banks have managed to pass these risks to other investors. The collateralized mortgage obligations (C.M.O.) developped in the eighties and nineties constitute the most elaborate mechanism of spreading risks of investing in the housing market. These obligations are derivatives backed by a big pool of mortgages which was split into different contingent flows.

Collateralized loans were first addressed in a general equilibrium setting by Dubey, Geanakoplos and Zame [7]. Collateral was modelled by these authors as a bundle of durable goods, purchased by a borrower at the time assets are sold and surrendered to the creditor in case of default. Clearly, in the absence of other default penalties, in each state of nature, a debtor will honor this commitments only when the debt does not exceed the value of the collateral. Similarly, each creditor should expect to receive the minimum between his claim and the value of the collateral. This pionnering work studied a two-period incomplete markets model with default and exogenous collateral coefficients and discussed also the endogenization of these coefficients, within a menu of finitely many stricly positive possible values.

Later, Araujo, Orrillo and Páscoa [3] made the first attempt at modelling C.M.O. markets and established existence of an equilibrium where the borrowers choice of the collateral is only restricted by the requirement that the value of the collateral, per unit of asset and at the time when it is constituted, must exceed the asset price by some arbitrarily small amount exogenously fixed. Under this requirement the loan can only finance up to some certain fraction of the value of the house. The model clearly captured the spreading features of the C.M.O. markets by assuming that lenders do not trade directly with individual borrowers, but rather buy obligations backed by a weighted average of the collaterals chosen by individual borrowers, with the individual sales serving as weights. However, the model suffered from an important drawback which was the exogenous bound on short sales due to the above exogenous lower bound on the difference between the value of the collateral and the asset price. It is hard to accept the existence of an exogenous uniform upper bound on the fraction of the value the house that can be financed by a loan.

1.2 Results and Methodology

It is well known that in incomplete markets with real assets equilibrium might not exist without the presence of a bounded short sales condition (see Hart [11] for a counter-example and Duffie and Shafer [8] on generic existence). In a model with exogenous collateral this bounded short sales condition does not need to be imposed arbitrarily but it follows from the fact that collateral must be constituted at the exogenously given coefficients. An important question is whether existence of equilibria may dispense any bounded short sales conditions in a model with endogenous collateral. Presumably, the fact that the borrower holds and consumes the collateral may discourage him from choosing the collateral so low that default would become a sure event. We try to explore this fact to show that, in fact, defaulting in every state is incompatible with the first order conditions governing the optimal choice of the collateral coefficients. ¿From here we derive an argument establishing that equilibrium levels of the collateral coefficients backing the C.M.O. are bounded away from zero and, therefore, equilibrium aggregate short sales are bounded.

Allowing borrowers to choose their collateral bundles introduces a non-convexity in the budget set, which is overcome by considering a continuum of agents. This large agents set is actually a nice set up both for the huge pooling of individual mortgages and for the spreading of risks across many investors, that occur in C.M.O. markets. However, for a continuum of agents, having established that aggregate short sales are endogenously bounded does not imply that the short sales allocation is uniformly bounded. To handle this difficulty we appeal to an assumption on preferences that requires the product of consumption and marginal utility to tend to infinity as consumption grows unboundedly. Under equicontinuity of utility functions (and their first derivatives), this assumption can be used to show that short sales are endogenously uniformly bounded as desired to prove existence using a multi-dimensional version of Fatou's lemma applied to a sequence of equilibria of auxiliary economies whose collaterals are bounded from below by decreasing coefficients.

1.3 Arbitrage and Pricing

The existence argument demanded a study of the nonarbitrage conditions for asset pricing in the context of a model where purchases of the C.M.O. and sales of individual assets yield different returns. These nonarbitrage conditions play a crucial role in the study of the first order conditions and in asserting that collateral coefficients will not be chosen too low. This nonarbitrage analysis was absent in the earlier work by Araujo, Orrillo and Páscoa [3], where short sales were exogenously bounded.

Our analysis of the nonarbitrage conditions is close to the study made by Jouini and Kallal [13] in the presence of short sales constraints. In fact, the individual promises of homeowners are assets that can not be bought by these agents and the C.M.O. bought by investors is an asset that can not be short sold by these agents in the same initial period. These sign constraints determine that purchase prices (of the C.M.O.) follow supermartingales, whereas sale prices (of homeowners promises) follow submartingales. The nonarbitrage conditions identify three components in these prices: a base price common to all assets, a spread that depends on the future default and a tail due to the sign constraints. We also show that the price of the minimal cost superhedging strategy is the supremum over all discounted expectations of the claim, with respect to every underlying probability measure (and similarly, the price of a maximal revenue subhedging strategy is instead the infimum over those expectations, in the spirit of the Cvitanic and Karatzas [5] and El Karoui and Quenez [9] approaches to pricing in incomplete markets).

As in Araujo, Orrillo and Páscoa [3], equilibrium asset prices received by borrowers include a personalized spread which is a discounted expected value of future default, with respect to some endogenously determined measure on states, common to all borrowers. Debtors more prone to default are penalized by selling assets at lower prices. Similarly, the C.M.O. price consists of the primitive asset base price reduced by subtracting the discounted expected value of default suffered, with respect to the same endogenously determined measure. This pricing formula is actually motivated by the nonarbitrage conditions, where prices also include these type of spreads, but the similarity is only partial. Actually, equilibrium prices are martingales with respect to an endogenously determined measure common to all agents, whereas nonarbitrage prices can be expressed as sub or super martingales for some consumer-specific measure. These two results are easily seen to be compatible. By absence of arbitrage, a given vector of equilibrium prices of assets and derivatives can be written as sub and super, respectively, martingales with respect to certain consumer-specific measures that depend on the chosen collateral coefficients.

While the formulation in terms of martingales and a common measure is crucial to show that markets clear and aggregate default given by debtors matches aggregate default suffered by creditors in each auxiliary economy, the formulation in terms of sub and super martingales for consumer-specific measures has the merit of explaining not just the spread but also the base price as a discounted expectation. The latter allows us to show that collateral coefficients can not be set too low, otherwise both net returns and net income from short sales would vanish, leaving the positive marginal utility from collateral as the only term in the Kuhn-Tucker condition on the choice of these coefficients.

1.4 Relation to Other Equilibrium Concepts

We close the paper with a discussion of the efficiency properties of the equilibria in C.M.O. markets. We show that an equilibrium allocation is undominated by allocations that are feasible and provide income across states through the same given equilibrium spot prices, although may be financed in the first period in any other way (possibly through transfers across individuals). This results extends usual constrained efficiency results to the case of default and endogenous collateral. An implication is that the no-default equilibrium, the exogenous collateral equilibrium or even the endogenous collateral equilibrium with a bounded short sales are concepts imposing further restrictions on the welfare problem and should be expected to be dominated by the proposed equilibrium concept.

In this paper we simplify the mixing of individual promises by assuming that each C.M.O. mixes the promises of all sellers a certain primitive asset. Since the collateral choice personalizes the asset the resulting derivative represents already a significative mixing across assets with rather different default profiles. Further work should address the composition of derivatives from different primitive assets and certain chosen subsets of debtors. We do not deal also with the case of default penalties entering the utility function and the resulting adverse selection problems. The penalty model was extensively studied by Dubey, Geanakoplos and Shubik [6], extended to a continuum of states and infinite horizon by Araujo, Monteiro and Páscoa [1, 2] and combined with the collateral model by Dubey, Geanakoplos and Zame [7]. Our default model differs also from the bankruptcy models where agents do not honor their debts only when they have no means to pay them, or more precisely, when the entire financial debt exceeds the value of the endowments that creditors are entitled to confiscate (see Araujo and Páscoa [4]).

The paper is organized as follows. Section 2 presents the basic model of default and collateral choice. Sections 3 and 4 address arbitrage and pricing. Section 5 presents the definition of equilibrium and the existence result. Section 6 contains the existence proof and Section 7 discusses the efficiency properties. A mathematical appendix contains some results used in the existence proof.

2 Model of Default and Collateral Choice

We consider an economy with two periods and a finite number S of states of nature in the second period. There are L physical durable commodities traded in the market and J real assets that are traded in the initial period and yield returns in the second period. These returns are represented by a random variable $R : S \mapsto \mathbb{R}^{JL}$ such that the returns from eachasset are not trivially zero. In this economy each sale of asset j (promise) must be backed by collateral. This collateral will consist of goods that depreciate at some rate Y_s depending on the state of nature $s \in S$ that occurs in the second period.

Each agent in the economy is a small investor whose portfolio is $(\theta, \varphi) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+$, where the first and second components are the purchase and sale quantities of assets respectively.

Each seller of assets chooses also the collateral coefficient for the different assets that he sells and we suppose that there exist anonymous collateral coefficients which will be taken as given by each buyer of assets. For each asset j denote by $M_j \in \mathbb{R}^L_+$ the choice of collateral coefficients. The anonymous collateral coefficients will be denoted by $C \in \mathbb{R}^{JL}_+$ and will be be taken as given. The collateral bundle choosen by borrower will be $M\varphi$ and his whole first period consumption bundle is $x_o + M\varphi$.

Denote by $x_s \in \mathbb{R}^l_+$ the consumption vector in state of the world s. Agents endowments are denoted by $\omega \in \mathbb{R}^{(S+1)L}_{++}$. Let π_1 and π_2 be the purchase and sale prices of assets, respectively. Then, the budget constraints of each agent will be the following

$$p_o x_o + p_o M\varphi + \pi_1\theta \leq p_o\omega_o + \pi_2\varphi \tag{1}$$

$$p_s x_s + \sum_{j=1}^{J} D_{sj}\varphi_j \leq p_s\omega_s + \sum_{j=1}^{J} N_{sj}\theta_j + \sum_{j=1}^{J} p_s Y_s M_j \varphi_j + p_s Y_s x_o, \ \ \forall s \in S \tag{2}$$

Here $D_{sj} = \min\{p_s R^j_s, p_s Y_s M_j\}$ and $N_{sj} = \min\{p_s R^j_s, p_s Y_s C_j\}$ are what he will paid and receive with the sale and purchase of one unit of asset j. Now we will represent equations (1) and (2) in matrix form:

$$p \cdot (x - \tilde{\omega}) \leq A\Psi \tag{3}$$

where $x = (x_o, x_1, \ldots, x_S)$, $\tilde{\omega} = (\omega_o, \omega_1 + Y_1 x_o, \ldots, \omega_S + Y_S x_o)$, $\Psi = (\theta, \varphi)'$ and

$$A = \begin{bmatrix} -\pi_1 & \pi_2 - p_o M \\ N_1 & p_1 Y_1 M - D_1 \\ N_2 & p_2 Y_2 M - D_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ N_S & p_S Y_S M - D_S \end{bmatrix}$$

In other words for $i \neq 1: A_{ij} = N_{i-1j}$ when $j \leq J$ and $A_{ij} = p_{i-1}Y_{i-1}M_j D_{i-1j}$ when $j \in \{J+1, .., 2J\}$. Now we will define arbitrage in our context, assuming that agents preferences are monotonic.

3 Arbitrage and Collateral

Let us start by defining arbitrage opportunities in a nontrivial context where $p_o >> 0, \forall s$ and $C_j \neq 0, \forall j$. Monotonicity of preferences determines already that the commmodity arbitrage opportunities derived from zero spot prices have to be ruled out:

Definition 1 *We say that there exist arbitrage opportunities if* $\exists\ (M, \Psi) \in I\!R_+^{(2+L)J}$ *such that*

$$A_{(M)}\Psi > 0 \tag{4}$$

or also when $\pi_1^j = 0$ *or* $p_o M_j - \pi_2^j = 0$ *for some* j*.*

The case when $\pi_1^j = 0$ creates arbitrage opportunities since $C_j \neq 0$ and $p_s >> 0, \forall s$, imply $N_{sj}\theta_j > 0, \forall s$. The case when $p_o M_j \pi_2^j = 0$ creates also arbitrage opportunities since it implies that $M_j \neq 0$ and even if $p_s Y_s M_j = D_{sj}$ for every s there would be unbounded utility gains from consumption of $M_j \varphi_j$ by choosing unbounded short sales of asset j. All trading strategies that satisfy (3) and do not satisfy (4) we called admissible and denote by Θ the set of admissible trading strategies. Now we will characterize the arbitrage free prices.

Theorem 1 *There are no arbitrage opportunities if and only if there exists* $\beta \in I\!R_{++}^{S+2J}$ *such that for each* $j = 1, 2, .., J$

$$\pi_1^j = \sum_{s=1}^{S} \beta_s p_s R_s^j \sum_{s=1}^{S} \beta_s (p_s R_s^j - p_s Y_s C_j)^+ + \beta_{S+j} \tag{5}$$

and

$$\pi_2^j = \sum_{s=1}^{S} \beta_s p_s R_s^j \sum_{s=1}^{S} \beta_s (p_s R_s^j - p_s Y_s M_j)^+ + (p_o M_j \sum_{s=1}^{S} \beta_s p_s Y_s M_j) - \beta_{S+J+j} \tag{6}$$

Proof:

Construct the following matrix:

$$\hat{A} = \begin{bmatrix} -\pi_1 & \pi_2 - p_o M \\ N_1 & p_1 Y_1 M - D_1 \\ N_2 & p_2 Y_2 M - D_2 \\ . & . \\ . & . \\ N_S & p_S Y_S M - D_S \\ I & 0 \\ 0 & I \end{bmatrix}$$

where I is the $J \times J$ identity matrix and 0 is the $J \times J$ null matrix.
By the absence of arbitrage $\nexists\, y \in I\!R^{2J}$ such that $\hat{A}y > 0$. In fact

$$\exists y : \hat{A}y > 0 \Leftrightarrow \exists y : Ay > 0,\ y \geq 0 \text{ or } Ay = 0,\ y > 0$$

$$\Leftrightarrow \exists y\ :\ Ay > 0,\ y \geq 0 \text{ or } \pi_1^j = 0 \text{ or } \pi_2^j - p_o M_j = 0 \text{ for some } j.$$

Then by the Stiemke's lemma we have that $\exists\ \delta = (\delta_0, \delta_1, .., \delta_{S+2J}) \in I\!R_{++}^{S+2J+1}$ such that:

$$\hat{A}'\delta = 0$$

Taking $\beta_i = \frac{\delta_i}{\delta_0}$ for $i = 1, .., S + 2J$, is easy to assert the necessity.
To check sufficency, assume that π_2^j has the proposed form. Then we have:

$$\pi_2^j = \sum_{s=1}^{S} \beta_s \min\{p_s R_s^j, p_s Y_s M_j\} + (p_o M_j - \sum_{s=1}^{S} \beta_s p_s Y_s M_j) - \beta_{S+J+j}$$

Then

$$p_o M_j - \pi_2^j = \sum_{s=1}^{S} \beta_s \left[p_s Y_s M_j \min\{p_s R_s^j, p_s Y_s M_j\} \right] + \beta_{S+J+j} > 0$$

Then for any $y \in I\!R_+^J$, we have

$$\sum_{j=i}^{J} (p_o M_j - \pi_2^j) y_j > 0$$

So $\nexists (M, \Psi)$ satisfying (4).■

Observe that these prices have three components: the first component is similar to the default free prices ('the present value of future promised returns'). The second component is a spread that can be written as a discounted expected value of the part in the return that will not be honoured in case of default. The third component is an additional correction factor due to the fact that purchase and short-sales have differents return coefficients. Recognizing this fact, we decomposed each asset into two differents assets, one that can not be bought and one that can not be sold. The resulting sign constraints determine the presence of the tails β_{S+j} and β_{S+J+j} in the formulas. Moreover the sale price has a component representing the cost of collateral depreciation.

Remarks

- From (6) also we have:

$$p_o M_j - \pi_2^j \geq \beta_{S+J+j}$$

  Since short-sales lead to nonnegative net yields in the second period (once we add to returns the depreciated collateral) and also to consumption of the collateral bundle in the first period, nonarbitrage requires the net coefficient of short-sales in the first period budget constraint to be positive.

- If we had considered the collateral as being exogenous, we would have concluded that there are no arbitrage opportunities if and only if there exists $\beta \in I\!R_{++}^{(S+2J+1)}$ such that

$$\pi^j = \sum_{s=1}^{S} \beta_s D_{sj} + \beta_{S+j} = (p_o - \sum_{s=1}^{S} \beta_s p_s Y_s) C_j + \sum_{s=1}^{S} \beta_s D_{sj} - \beta_{S+J+j} > 0$$

  Then

$$(p_o - \sum_{s=1}^{S} \beta_s p_s Y_s) C_j = \beta_{S+j} + \beta_{S+J+j} > 0,\ \forall j \in J \ \text{ and } p_o C_j - \pi^j > 0,\ \forall j \in J.$$

  For more details on the implications of the absence of arbitrage in the exogenous collateral model see Fajardo [10].

In contrast with the fundamental theorem of asset pricing in frictionless financial markets, we can obtain an alternative result for the default model with collateral where discounted asset prices are no longer martingales with respect to some equivalent probability measure. This result is presented in the next section.

4 Pricing

4.1 A Pricing Theorem

Let $I\!R$ be the real line and $\overline{I\!R} = I\!R \cup \{-\infty, +\infty\}$ the extended real line. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X = I\!R^S$. We say that $f : X \mapsto I\!R$ is a positive linear functional if $\forall\ x \in X^+,\quad f(x) > 0$, where $X^+ = \{x \in X / P(x \geq 0) = 1 \text{ and } P(x > 0) > 0\}$. The next result follows in spirit of the result in Jouini and Kallal [13].

Let $\overline{\pi}_2^j = \pi_2^j - p_o M_j < 0,\ \forall j$ which will be refered to as the net sell price and let $\overline{D}_{sj} = D_{sj} - p_s Y_s M_j,\ \forall j$ and $\forall s$.

Denote by $\iota(x)$ the smallest amount necessary to get at least the payoff x for sure by trading in the underlying defaultable assets. Then no investor is willing to pay more than $\iota(x)$ for the contingent claim x. The specific expression for ι is given by

$$\iota(x) = \inf_{(\theta,\varphi)\in\Theta} \{\pi_1\theta - \overline{\pi}_2\varphi > 0 \Big/ G(\theta,\varphi) \geq x \ \ a.s.\}$$

where

$$G(\theta,\varphi) = \sum_{j=1}^{J} [N_j\theta^j - \overline{D}_j\varphi^j]$$

Theorem 2 *i) There are no arbitrage opportunities if and only if there exist probabilities β_s^*, $s = 1, .., S$ equivalent to P and a positive γ such that the normalized (by γ) purchase prices are supermartingales and the normalized (by γ) net sale prices are a submartingale under this probability.*

ii) Let $\mathcal{Q}^*$ be the set of β^* obtained in (i) and Γ be the set of positive linear functionals ξ such that $\xi|_{\mathcal{M}} \leq \iota$, where $\mathcal{M}$ is a convex cone representing the set of marketed claims. Then there is a one-to-one correspondence between these functionals and the equivalent probability measures β^* given by:

$$\beta^*(B) = \sum_{s=1}^{S} \beta_s^* 1_B(s) = \xi(1_B) \text{ and } \xi(x) = E^*(\frac{x}{\gamma})$$

where E^* is the expectation taken with respect to β^*

iii) For all $x \in \mathcal{M}$ we have

$$[-\iota(-x), \iota(x)] = cl\{E^*(\frac{x}{\gamma}) : \beta^* \in \mathcal{Q}^*\}$$

Proof:

(i) Let $\beta_o = \sum_{s=1}^{S} \beta_s$ and $\beta_s^* = \frac{\beta_s}{\beta_o}$ in theorem 1, we obtain:

$$\pi_1^j/\beta_o = \sum_{s=1}^{S} \beta_s^* p_s R_s^j - \sum_{s=1}^{S} \beta_s^* (p_s R_s^j - p_s Y_s C_j)^+ + \beta_{s+j}/\beta_o$$

and

$$\pi_2^j/\beta_o = \sum_{s=1}^{S} \beta_s^* p_s R_s^j - \sum_{s=1}^{S} \beta_s^* (p_s R_s^j - p_s Y_s M_j)^+$$

$$+(p_o M_j/\beta_o - \sum_{s=1}^{S} \beta_s^* p_s Y_s M_j) - \beta_{S+J+j}/\beta_o$$

Now take $\gamma = 1/\beta_o$. Then to be supermartingales and submartingales π_1^j and $\pi_2^j - p_o M_j$ must be respectively ($\geq$) and ($\leq$) than their expected returns on the second period. ¿From the above equations we have inmediatly the result.

Now if there is a probability measure and a process γ such the normalized prices are sub and supermartingales, we have

$$E^* \left(\sum_{j=1}^{J} [N^j \theta^j - \overline{D}^j \varphi^j] \right) \leq \gamma[\pi_1 \theta - \overline{\pi}_2 \varphi]$$

Then there can not exists arbitrage opportunities.

(ii) Given $\beta^* \in \mathcal{Q}^*$ define $\xi(x) = E^*(\frac{x}{\gamma})$, then

$$\xi(x) = \sum_s \left(\frac{x_s}{\gamma}\frac{\beta_s^*}{P_s}\right)$$

it is a continuous linear functional. Since β^* is equivalent to P and taking the infimum over all supereplicating strategies :

$$E^*(x) \leq E^*\left(\sum_{j=1}^{J}[N^j\theta^j - \overline{D}^j\varphi^j]\right) \leq \gamma[\pi_1\theta - \overline{\pi}_2\varphi]$$

we have $\xi \in \Gamma$.
Now take $\xi \in \Gamma$ and define $\beta^*(B) = \sum_{s=1}^{S}\beta_s^* 1_B(s) = \xi(1_B)$. Since S is finite, β^* is equivalent to P.

Now since $\xi(1_S) = 1$, we have $\beta^*(S) = 1 = \sum_{s=1}^{S}\beta_s^*$, so β^* is a probability.

(iii) By part (ii) take a $\xi \in \Gamma$ then $\forall x \in \mathcal{M}$

$$\xi(x) \leq \iota(x) \Rightarrow -\xi(-x) \leq \iota(x)$$

then replacing x by $-x$ we have

$$\xi(x) \geq -\iota(-x)$$

Hence

$$cl\{\xi(x)/\xi \in \Gamma\} \subset [-\iota(-x), \iota(x)]$$

For the converse, $-\iota(-x) = \iota(x)$ the proof is trivial. Then we suppose that $-\iota(-x) < \iota(x)$. Now it is easy to see that ι is l.s.c. and sublinear. Then the set $K = \{(x,\lambda) \in \mathcal{M} \times I\!R : \lambda \geq \iota(x)\}$ is a closed convex cone. Hence $\forall \epsilon > 0$ we have that $(x, \iota(x) - \epsilon) \notin K$. Applying the strict separation theorem we obtain that there exist a vector ϕ and there exists real number α such that= $\phi \cdot (x, \iota(x) - \epsilon) < \alpha$ and $\phi \cdot (x, \lambda) > \alpha \ \ \forall (x,\lambda) \in K$. Then we can rewrite these inequalities as:

$$\phi_o \cdot x + \phi_{S+1}(\iota(x) - \epsilon) < \alpha$$

$$\phi_o \cdot x + \phi_{S+1}\lambda > \alpha \quad \forall (x, \lambda) \in K$$

where $\phi_o = (\phi_1, ..., \phi_S)$ and, since K is a convex cone, we must have $\alpha < 0$. This implies $\phi_o \cdot x + \phi_{S+1}(\iota(x) - \epsilon) < 0$ and $\phi_o \cdot x + \phi_{S+1}\lambda \geq 0 \;\; \forall (x, \lambda) \in K$. Hence $\phi_{S+1} > 0$ and we can define $\nu(x) = -\frac{\phi_o}{\phi_{S+1}} \cdot x$. It is easy to see that ν is a continuous linear functional and $\nu(x) \leq \iota(x)$, $\forall x \in \mathcal{M}$, since $(x, \iota(x)) \in K$. Also $\nu(x) > \iota(x) - \epsilon$. Now for all $x \in X_+$, we have $\nu(-x) \leq \iota(-x) \leq 0$, so $\nu(x) \geq 0$. With an analoguous argument, we obtain $\nu'(x) \in \Gamma$ such that $\nu'|_{\mathcal{M}} \leq \iota$ and

$$-\iota(-x) \leq \nu'(x) \leq -\iota(-x) + \epsilon$$

Since $\{\nu \in \Xi / \nu|_{\mathcal{M}} \leq \iota\}$ is a convex set and $\{\nu(x)/\nu|_{\mathcal{M}} \leq \iota \;,\; \nu \in \Gamma\}$ is an interval we obtain the inclusion.■

Remarks

- The normalized purchase prices are "strict" supermartingales and the normalized net sale prices are "strict" submartingales. Having strict inequality we not consider the posibility of some assets being martingales. In fact, on one hand, for $C_j \neq 0$ and $p_s >> 0$, $\forall s$, we showed that $\beta_{S+J+j} > 0$ (see theorem 1) and, on the other hand the presence in the utility function of collateral bundle desired by the borrower, allowed us to assert that $\beta_{S+J+j} > 0$ (see theorem 1 also). Jouini and Kallal [13] in a more abstract model showed that the presence of short sale constraints is responsable for the weak inequality which still can accomodate the martingale case.

- We can assert that $p_o > \sum_s \beta_s p_s Y_s$. To see this, rewrite the budget constraint as:

$$p_o \tilde{x}_o + \pi_1 \theta \leq p_o \omega_o + \pi_2 \varphi$$

$$p_s x_s + \sum_{j=1}^{J} D_{sj} \varphi_j \leq p_s \omega_s + \sum_{j=1}^{J} N_{sj} \theta_j + \sum_{j=1}^{J} p_s Y_s \tilde{x}_o$$

where

$$\tilde{x}_o = x_o + M\varphi \geq 0$$

Now there are arbitrage opportunities if there exists $\Psi = (\tilde{x}_o, \theta, \varphi) > 0$ such that $B\Psi \geq 0$, where B is given by:

$$B = \begin{bmatrix} -p_o & -\pi_1 & \pi_2 \\ p_1 Y_1 & N_1 & -D_1 \\ p_2 Y_2 & N_2 & -D_2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ p_S Y_S & N_S & -D_S \\ 1 & 0 & -p_o M \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

In fact

$$-p_o \tilde{x}_o - \pi_1 \theta + \pi_2 \varphi \geq 0 \Rightarrow -p_o x_o - \pi_1 \theta + (\pi_2 - p_o M)\varphi \geq 0$$

But $x_o \geq 0$, then $-\pi_1 \theta + (\pi_2 - p_o M)\varphi \geq 0$ and in analogous way we obtain

$$N_s \theta + (p_s Y_s M - D_s)\varphi \geq 0$$

So we will analize the case in which $x_o > 0$ or $\theta > 0$ or $\varphi > 0$ and $B\Psi = 0$:

- $\theta_j > 0$ for some j implies $\pi_j^1 = 0$ and $N_{sj} = 0, \forall s$, which is impossible since $C_j > 0$ and $p_s >> 0, \forall s$.
- $\varphi_j > 0$ for some j implies $\pi_j^2 - p_o M_j = 0$ and $p_s Y_s M_j - D_{sj} = 0, \forall s$ (implying $M_j \neq 0$). But the consumer derives utility from collateral consumption and therefore he would like to let φ_j grow unboundedly.
- If $x_{ol} > 0$ for some l, we have by the first equation in the matrix product $B\Psi = 0$:

  $$p_o x_o - \pi_1 \theta + (\pi_2 - p_o M)\varphi \geq 0$$

  But $\theta = \varphi = 0$, then $p_{ol} = 0$, but $p_s Y_s x_o > 0$ determinig unbounded returns in the second period.

Hence by Stiemke's lemma, there is $\delta = (\delta_0, \delta_1, .., \delta_{S+2J+1}) \in I\!R^{S+2J+2}_{++}$ such $\delta' B = 0$, then:

$$p_o = \sum_{s=1}^{S} \beta_s p_s Y_s + \beta_{S+1} > \sum_{s=1}^{S} \beta_s p_s Y_s$$

$$\pi_1^j = \sum_{s=1}^{S} \beta_s N_{sj} + \beta_{S+1+j}$$

$$\pi_2^j = \sum_{s=1}^{S} \beta_s D_s^j + \beta_{S+1} M_j - \beta_{S+J+1+j}$$

where $\beta_s = \frac{\delta_s}{\delta_0}$, that is we obtain the same characterization for the purchase and sale prices as in Theorem 1 together with a new insight on the relation between p_o and $\sum_s \beta_s p_s Y_s$.

- Our definition of maximal willingness to pay $\iota(x)$ is in the spirit of the super replication approach of El Karoui and Quenez [9] and Cvitanić and Karatzas [5] to pricing in incomplete markets. We consider as superhedging strategies the defaultable assets.
  Theorem 2, (ii) establishes a one to one correspondence between linear pricing rules, bounded from above by $\iota(x)$, and measures β^*, considered in the sub and supermartingale pricing formulas
  Our result (iii) implies

$$[\inf_{\beta^* \in \mathcal{Q}^*} E^*(\frac{x}{\gamma}), \sup_{\beta^* \in \mathcal{Q}^*} E^*(\frac{x}{\gamma})] = [-\iota(-x), \iota(x)]$$

- The term β_o can be interpreted as a discount factor on riskless borrowing. In fact, let

$$\overline{A} = \begin{bmatrix} N_1 & p_1 Y_1 M - D_1 \\ N_2 & p_2 Y_2 M - D_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ N_S & p_s Y_S M - D_S \end{bmatrix}$$

  If there exists $\hat{\theta} \in I\!R^{2J}$ such that

$$\overline{A}\hat{\theta} = (1, .., 1)$$

Then $\beta_0 = \hat{\theta}' \cdot (\hat{\pi}_1, \hat{\pi}_2)$ is the discount on a riskless borrowing, where $\hat{\pi}_1^j = \pi_1^j - \beta_{S+j}$ and $\hat{\pi}_2^j = p_o M_j + \beta_{S+J+j} - \pi_2^j$, $\forall\, j = 1, .., J$.

4.2 Example

Consider $J = 1$ and $L = 1$ and two possible states of nature s_1 and s_2, with $R_{s_1} > R_{s_2}$. Now from theorem 2:

$$\pi_1 \geq \beta_{s_1} \min\{R_{s_1}, Y_{s_1} C\} + \beta_{s_2} \min\{R_{s_2}, Y_{s_2} C\} \tag{7}$$

and

$$\pi_2 - M \geq \beta_{s_1}[\min\{R_{s_1}, Y_{s_1} M\} - Y_{s_1} M] + \beta_{s_2}[\min\{R_{s_2}, Y_{s_2} M\} - Y_{s_2} M] \tag{8}$$

Now if C, M have adequate values, we have a set of probability measures, then we can find an upper bound for the price of any contingent claim x:

$$\max_{(\beta_{s_1}, \beta_{s_2}) \in \mathcal{Q}} [x(s_2) + (x(s_1) - x(s_2)) \frac{\beta_{s_1}}{\beta_{s_1} + \beta_{s_2}}]$$

Where $\mathcal{Q} = \{(\beta_{s_1}, \beta_{s_2}) \in I\!R^2_{++}$ satisfying (7) and (8)$\}$, similarly for the lower bound.

5 Equilibria

In this section borrowers (sellers of assets) will choose the collateral coefficients. We assume that there is a continuum of agents $H = [0, 1]$ modeled by the Lebesgue probability space $(H, \mathcal{B}, \lambda)$. Each agent h is characterized by his endowments ω_h and his utility U^h. Each agent will buy and sell in the initial period J assets that will be backed by a collateral and in the second period will receive the respective returns.

The allocation of the commodities is an integrable map $x : H \to I\!R_+^{(S+1)L}$. The assets purchase and sale allocations are represented by two integral maps; $\theta : H \to I\!R_+^J$ and $\varphi : H \to I\!R_+^J$, respectively.

As we have mentioned each borrower h will choose the collateral coefficients for each portfolio sold .The allocation of collateral coefficients chosen by borrowers is described by the function $M : H \to \mathbb{R}^J_+$. Each buyer of assets (lender) will take an anonymous collateral coefficient vector $C \in \mathbb{R}^{JL}_+$ as given. This anonimity holds since lenders do not trade directly with borrowers. Let $x^h_{-o} = (x^h_1, \ldots, x^h_S)$ be the commodity consumption in the several states of the world in the second period.

Asset prices are assumed to consist of a base price (common to the purchase and sale prices) and also a spread (varying across sellers in the case of the sale price). Let

$$\pi_1 = q - \sum_s \gamma_s g_{1s} \tag{9}$$

and

$$\pi_2 = q - \sum_s \gamma_s g_{2s} \tag{10}$$

where $g_{1s} = (p_s R_s - p_s Y_s C)^+$ and $g_{2s} = (p_s R_s - p_s Y_s M)^+$. Here q is understood as a base price, whereas $\sum_s \gamma_s g_{1s}$ and $\sum_s \gamma_s g_{2s}$ are spreads proportional to the dishonoured part. The state prices γ_s are common to all agents and taken as given together with the base price q. In the context of C.M.O. markets, π_1 is the vector of prices for the C.M.O.s, whose returns are given by N_s in each state s.

Then the individual problem is

$$\max_{(x^h, \theta^h, \varphi^h, M^h) \in B^h} U^h(x^h_o + M^h \varphi^h, x^h_{-o}) \tag{11}$$

where B^h is the budget set of each agent $h \in H$ given by:

$$B^h(p, C, q, \gamma) = \big\{ (x, \theta, \varphi, M) \in \mathbb{R}^{L(S+1)+2J+JL} : \text{(1) and (2)}$$

$$\text{hold for } \pi_1 \text{ and } \pi_2 \text{ given by (9) and (10)} \big\}$$

Equivalently, the budget set of each agent could be parametrized by (p, C, π_1, q, γ), where the parameters (q, γ) define π_2 according to equation (10).

Remark

- The prices considered above are close to the non arbitrage valuation established in Theorem 1 and even compatible with it for state prices common to all agents provided that correction factors satisfy:

$$(p_o - \sum_{s\in S} \beta_s p_s Y_s) M_j \beta_{S+J+j} = \beta_{S+j}$$

However, in general, equilibrium prices given by (9) and (10) will have non arbitrage representations according to (5) and (6) for " state price" vectors β^h that vary across agents (as β^h depends on the return matrix and therefore on the choice M^h of collateral coefficients).

Definition 2 *An equilibrium is a vector* $((p, \pi_1, \pi_2, C), (x^h, \theta^h, \varphi^h, M^h)_{h\in H})$ *such that:*

-
$$(x^h, \theta^h, \varphi^h, M^h)$$
solves problem (11)

-
$$\int_H \left(x_o^h + \sum_{j\in J} M_j^h \varphi_j^h \right) dh = \int_H \omega_o^h dh$$
$$\int_H x^h(s) dh = \int_H \left(\omega^h(s) + \sum_{j\in J} (Y_s M_j^h \varphi_j^h + Y_s x_o^h) \right) dh \qquad (12)$$

-
$$\int_H (\theta^h - \varphi^h) dh = 0 \qquad (13)$$

-
$$\int_H M_j^h \varphi_j^h dh = C_j \int_H \theta_j^h dh \ \ \forall\, j \in J \qquad (14)$$

-
$$\int_{h\in\mathcal{S}_s^j} (p_s R_s^j p_s Y_s C_j)^+ \theta_j^h dh = \int_{h\in\mathcal{G}_s^j} (p_s R_s^j - p_s Y_s M_j^h)^+ \varphi_j^h dh \ \ \forall s \in S,\ \forall j \in J \qquad (15)$$

Where $\mathcal{S}_s^j = \{h \in H : p_s R_s^j > p_s Y_s C_j\}$ is the set of agents that suffered default in state of nature s on asset j and $\mathcal{G}_s^j = \{h \in H : p_s R_s^j > p_s Y_s M_j^h\}$ is the set of agents that give default in state of nature s on asset j. Note that $\mathcal{S}_s^j$ is equal to H or ϕ, since $p_s R_s^j$ and $p_s Y_s C_j$ do not depend on h.

Some Remarks

- Equations (12) and (13) are the usual market clearing conditions. Equation (14) says that in equilibrium the anonymous collateral coefficientC_j is anticipated as the weighted average of the collateral coefficients allocation M_j.

- Equation (15) says that, in equilibrium aggregate default suffered must be equal to aggregate default given, for each state and each promise. As in Araújo, Orrillo and Páscoa [3] this condition implies that

$$\int_H N_{sj} \theta_j^h dh = \int_H D_{sj}^h \varphi_j^h dh$$

  That is, aggregate actual yields must be equal to aggregate actual payments.

- The above equilibrium concept portraits equilibria in housing mortgages markets where individual mortgages are backed by houses and then huge pools of mortgages are split into C.M.O.s backed by the respective pool of houses.

  In our anonymous and abstract setting, any agent in the economy may be simultaneously a homeowner and an investor buying a C.M.O.. The above equilibrium concept assumes implicitly the existence of one or several financial institutions that buy the pool of mortgages from the consumers at prices π_2^h and issue the C.M.O.s, selling them back to the consumers at prices π_1. These financial institutions make zero profits in equilibrium since $\int \pi_2^h \varphi^h dh = \pi_1 \int \theta^h dh$, by Walras law.
  To simplify, we mix promises of different sellers of a same asset but do not mix different assets into collateralized securities. This simplification does not hurt the interpretation of the above equilibrium as a

C.M.O. equilibrium, since different sellers of a same asset end up selling personalized assets due to different choices of collateral. A more elaborate version of a C.M.O. model should allow for the mix of different primitive assets and for the strategic choice of the mix of assets and debtors by the issuer of the C.M.O..Putting together in a same model the price-taking consumers and investments banks composing the derivatives strategically may be a difficult task, since the latter would have to anticipate the Walrasian response of the former.

We will now fix our assumptions on preferences.
Assumption (P) : preferences are time and state separable, monotonic, representable by a smooth strictly concave utility function u^h satisfying:

i) Inada's condition

ii) $\frac{\partial u^h(z)}{\partial z_{0l}} z_{0l} \to \infty$ for any l, when $\min_l z_{0l} \to \infty$

iii $\{u^h\}_{h\in H}$ and $\{Du^h\}_{h\in H}$ are equicontinuous.

Theorem 3 *If consumers's preferences satisfy assumption (P) and the endowments allocation ω belongs to $L^\infty(H, \mathbb{R}_{++}^{(S+1)L})$, then there exists an equilibrium where borrowers choose their respective collateral coefficients.*

6 Proof of the Existence Theorem

6.1 Outline of the proof

First, we will study economies where collateral coefficients are required to be greater or equal than some exogenously given lower bound, to be more precisely we require

$$M_{jl} \geq \delta \ \ \forall j, l \tag{16}$$

This condition will be relaxed later. It is easier to show existence of equilibria for an economy satisfying condition (16). We can start by establishing existence of equilibria in economies where not only (16) holds but also bundles, portfolios and collateral coefficients are bounded from above. For these truncated economies we can use a generalized game approach. As the upper bound on bundles, portfolios and collateral coefficients tends to infinity, the corresponding sequence of equilibria exhibits nice asymptotic properties. The nonarbitrage conditions must be satisfied beyond a certain order, otherwise the short sales would require a collateral exceeding the available resources. Using these nonarbitrage conditions it is possible to appeal to Fatou's lemma and establish existence in an economy satisfying condition (16).

Then, we let the lower bound on the collateral coefficients to go to zero and study the asymptotics of the associated sequence of equilibria. Once again we invoke the nonarbitrage conditions to assert no agent will choose the collateral so low that he ends up defaulting in every state. ¿From here we deduce that the equilibrium anonymous collateral coefficient backing the C.M.O. does not tend to zero. This allows us to bound aggregate short sales and actually bound uniformly the short sales allocations (using assumtion (P)) as required to apply again Fatou's lemma.

If we had tried to apply Fatou's lemma directly to a sequence of equilibria of truncated economies where condition (16) was not guaranteed we would have faced a major difficulty since the nonarbitrage conditions would not hold along the sequence, even for high orders. In fact, aggregate short sales might grow unboundedly if the collateral coefficients backing the C.M.O. would go to zero at the same time.

6.2 Economies with Collateral Bounded from Below

Let us denote by $\mathcal{E}^\delta$ the economy $((U^h, \omega^h)_{h\in H}, R^j, Y)$ under condition (16). Notice that condition (16) does not imply bounded short-sales, in contrast with the condition in Araujo, Orrillo and Páscoa [3] which required $p_o M_j - q_j \geq \epsilon$ for some j and some a priori given $\epsilon > 0$.

In fact the latter implies, using the first period budget constraint, that $\varphi_j^h \leq (ess\sup_{h,l} \omega_{ol}^h)/\epsilon$, whereas the former implies only that feasible short-sale allocations satisfy $\int_H \varphi^h dh \leq \frac{\int_H \omega_{ol}^h dh}{\delta}$ (that is, the mean short-sale is bounded, but the short-sale allocation is not necessarily uniformly bounded). An equilibrium for the economy $\mathcal{E}^\delta$ a vector $((p, \pi_1, \pi_2, C), (x^h, \theta^h, \varphi^h, M^h)_{h\in H})$ such that:

- $(x^h, \theta^h, \varphi^h, M^h)$ maximizes utility under constraints (1), (2) and (16), for π_1 and π_2 given by (9) and (10).

- Equations (12) through (15) are satisfied.

Proposition 1 *If consumers's preferences satisfy assumption (P) and the endowments allocation ω belongs to $L^\infty(H, I\!R_{++}^{(S+1)L})$, then the economy $\mathcal{E}^\delta$ has an equilibrium where borrowers choose their respective collateral coefficients.*

Proof of Proposition

1 First we show that equilibrium exists when bundles and portfolios are bounded from above and then we examine the asymptotic behavior of the sequence of truncated equilibria, as these upper bounds tends to infinity. Let us in this proof denote the lower bound δ on collateral coefficients by $1/m$.

Truncated Economy
Define a sequence of truncated economies $(\mathcal{E}_n^m)_n$ such that the budget set of each agent h is

$B_n^h(p, C, q, \gamma) := \{(x_n^h, \theta_n^h, \varphi_n^h, M_n^h) \in [0, n]^{L(S+1)+(2+L)J} : (1), (2) \text{ and } (16) \text{ hold } \}$

We assume that $C \in [1/m, n]^{LJ}$.

Generalized Game

For each $n \in \mathbb{N}$ we define the following generalized game played by the continuum of consumers and $S+1+JL$ additional players; where $S+1$ are auctioneers and the other JL players are also fictitious agents. Denote this game by $\mathcal{J}_n$ which is described as follows:

- Each consumer $h \in H$ maximizes U^h in the constrained strategy set $B_n^h(p, q, C, \gamma)$.
- The auctioneer of the first period chooses $(p_o, q, \gamma) \in \triangle^{L+J+S-1}$ in order to maximize

$$p_o \int_H (x_o^h + \sum_j M_j^h \varphi_j^h - \omega_o^h) dh + q \int_H (\theta^h - \varphi^h) dh + \gamma \int_H g^b(\theta^h, \varphi^h, M^h) dh$$

- The auctioneer of state s of the second period chooses $p_s \in \triangle^{L-1}$ in order to maximize $p_s \int_H (x_s^h - Y_s(\sum_j M_j^h \varphi_j^h + x_o^h) - \omega_s^h) dh$.

- Each of the remaining JL fictitious agents chooses $C_{jl} \in [0, n]$ in order to minimize $(C_{jl} \int_H \theta^h dh - \int_H M_{jl}^h \varphi_j^h dh)^2$.

This game has an equilibrium in mixed strategies (see lemma 8 in appendix) and, by Liapunov's Theorem, there exists a pure strategies equilibrium (see lemma 9 in appendix).

Lemma 1 *An equilibrium in pure strategies of the generalized game $\mathcal{J}_n$ is an equilibrium for the truncated economy $\mathcal{E}_n^m$ for n large enough.*

Proof:

Let $z = (x^h, \theta^h, \varphi^h, M^h) : H \to [0, n]^{L(S+1)+2J+LJ}$, (p_o, q, γ), p_s and C be an equilibrium in pure strategies for $\mathcal{J}_n$.

Then by definition of equilibrium one has the following:

$$p_o(x_o^h - \omega_o^h + M^h \varphi^h) + q(\theta^h - \varphi^h) + \gamma g^h \leq 0,$$

$$p_s(x_s^h - \omega_s^h - Y_s M^h \varphi^h) \leq R_s \theta^h - D_s^h \varphi^h, \forall s$$

and

$$u^h(z^h) \geq u^h(z'), \forall z' \in B_n^h(p, q, C, \gamma)$$

Integrating the budget constraint of the first period we have

$$p_o \int_H (x_o^h - \omega_o^h + M^h \varphi^h) dh + q \int_H (\theta^h - \varphi^h) dh + \gamma \int_H g^h dh \leq 0,$$

Now the optimality conditions of the auctioneers' problems imply that

$$\int_H (x_o^h - \omega_o^h + M^h \varphi^h) dh \leq 0 \tag{17}$$

$$\int_H (\theta^h - \varphi^h) dh \leq 0 \tag{18}$$

$$\int_H g^h dh \leq 0 \tag{19}$$

$$\int_H (x_s^h - \omega_s^h - Y_s M^h \varphi^h - Y_s x_o^h) dh \leq 0 \tag{20}$$

Moreover,

$$C_j \int_H \theta_j^h dh = \int_H M_j^h \varphi_j^h dh \, \forall j \in J$$

Now

$$\int_{\mathcal{D}_s^j} (p_s R_s^j - Y_s M_j^h) \varphi_j^h dh = \int_{\mathcal{S}_s^j} (p_s R_s^j - Y_s C_j) \theta_j^h dh_j$$

(See Araújo, Orrillo and Páscoa [3] for a proof) and therefore (see the remarks on definition 2) we have

$$\int_H (R_s \theta^h - D_s^h \varphi^h) dh = 0, \forall s$$

Then, after integrating the budget constraint of the second period, we obtain

$$p_s \int_H (x_s^h - \omega_s^h - Y_s M^h \varphi^h - Y_s x_o^h) dh = 0, \forall s \in S \tag{21}$$

since the utility function is strictly increasing. For n larger enough, we must have $p_{ol} > 0, \forall l \in L$. Otherwise, every consumer would choose $x_{ol}^h = n$ and we would have contradicted (17) But when $p_{ol} > 0$ we must have

$$\int_H (x_{ol}^h - \omega_{ol}^h + (M^h \varphi^h)_l) dh = 0 \; \forall l \in L \tag{22}$$

since the aggregate budget constraint of the first period is a null sum of non positive terms and therefore a sum of null terms.

¿From (22) follows that $\int_H (M^h \varphi^h)_l) dh$ is bounded by the aggregate endowment in period 0 and hence $\int_H (Y_s M^h \varphi^h)_l dh < \infty$. Then, for n larger enough, we must have $p_{sl} > 0, \forall (s,l) \in S \times L$. Otherwise, every consumer would choose $x_{sl}^h = n$ and we would have contradicted (20). Therefore equality holds in (20). In similar way, $q_j > 0, \forall j \in J$, otherwise, each consumer would choose $\theta_j^h - \varphi_j^h = n$ contradicting (18) But when $q_j > 0, \forall j \in J$ we must have $\int_H (\theta_j^h - \varphi_j^h) dh = 0$. ■

Asymptotics of truncated equilibria

Now let $\{(x_n^{mh}, \theta_n^{mh}, \varphi_n^{mh}, M_n^{mh})_{\{h \in H\}}) \in ([0,n]^{S(L+1)+2J+LJ})^H, p_n^m, \pi_n^m, C_n^m\}$ be the sequence of equilibria corresponding to $\mathcal{E}_n^m$.

Lemma 2 *$\{\int_H (x_n^{mh}, \theta_n^{mh}, \varphi_n^{mh}, M_n^{mh}\varphi_n^{mh})dh\}$ is a bounded sequence.*

Proof:

By definition of equilibrium, $\int_H x_{no}^{mh}dh \leq \int_H \omega_o^h dh$ and $\int_H M_n^{mh}\varphi_n^{mh}dh \leq \int_H \omega_o^h dh$.
So

$$\int_H x_{ns}^{mh}dh < \int_H (\omega_s^h + 2Y_s\omega_o^h)dh, \forall s \in S. \tag{23}$$

For each $l \in L$ the following holds

$$\int_H M_{lnj}^{mh}\varphi_n^{mh}dh = C_{lnj}^m \int_H \theta_{jn}^{mh}dh \tag{24}$$

and therefore

$$C_{jnl}^m \int_H \theta_{nj}^{mh}dh \leq \int_H \omega_{ol}^h dh, \forall l \in L \tag{25}$$

Hence, $\int_H \theta_{nj}^{mh}dh = \int_H \varphi_{nj}^{mh}dh$ is bounded, since $\lim_{n\to\infty} C_{jnl}^m > 1/m$ for any l and any j.■

Let us define for each m, $\rho_{slj}^{mn} := p_{sl}^{mn}Y_sC_{jl}^{mn}$. Without lost of generality, the sequence $\{\rho^{mn}\}_n$ admits $\max_{s,l,j} R_{sl}^j$ as an upper bound, since the demand correspondence remains unchanged when ρ_{slj}^{mn} exceeds this bound. Now we are going to modify the problem of each consumer. The key consists of reformulating the consumer's problem in terms of total collateral instead of the collateral backing the sale of each unit of asset. $c^{mh} \in I\!R_+^{LJ}$ be the total collateral by the sale of assets φ^{mh}, $c_j^{mh} := M_j^{mh}\varphi_j^{mh} \in I\!R_+^L, \forall j \in J$.

Lemma 3 *For each m the sequence of allocations $\{x_n^m, \theta_n^m, \varphi_n^m, c_n^m, (p_{sn}^m R_s^j \varphi_{jn}^m - p_{sn}^m c_n^m)\}$ is uniformly bounded.*

Proof:

We rewrite the consumer's problem in the following way:
Given (p, q, γ, ρ), then each agent h chooses $(x^h, \theta^h, \varphi^h, c^h)$ in order to maximize $U^h(x_o^h + c^h, x_{-o}^h)$ subject to

$$p_o(x_o^h - \omega_o^h) + q(\theta^h - \varphi^h) + p_o c^h + \sum_{s,j} \gamma_s [p_s R_s^j \varphi_j^h - p_s c_j^h]^+$$

$$- \sum_{s,j} \gamma_s [p_s R_s^j - \rho_s]^+ \theta_j^h \leq 0 \tag{A},$$

$$p_s x_s^h + \sum_j \min\{p_s R_s^j \varphi_j^h, p_s c_j^h\} \le p_s(\omega_s^h + Y_s x_o) + \sum_j \min\{p_s R_s^j, \sum_l \rho_{lsj}\}\theta_j^h + p_s c^h \tag{B}$$

We also rewrite other equilibrium conditions replacing $M_j^{mh}\varphi_j^{mh}$ by c_j^{mh}. Let $z_n^{mh} = (x_n^{mh}, \theta_n^{mh}, \varphi_n^{mh}, c_n^{mh})$
Now, by Lemma 2 , the sequence z_n^{mh} satisfies the hypothesis of the weak version of Fatou's Lemma. Therefore $\exists z$ integrable such that

$$z^{mh} \in cl\{z_n^m(h)\} \text{ for a.e } h$$

This imply that z^{mh} is budget feasible at
$(p^m, q^m, \gamma^m, \rho^m) = \lim_{n\to\infty}(p^{mn}, q^{mn}, \gamma^{mn}, \rho^{mn})$, passing to a subsequence if necessary .

Claim 3.1 $(x^{mh}, \theta^{mh}, \varphi^{mh}, c^{mh})$ maximizes U^h at the cluster point of $(p^{mn}, q^{mn}, \gamma^{mn}, \rho^{mn})$.

Proof:

Suppose that it is not optimal,i.e; $\exists \overline{z}^{mh} \in B_1^h(p^m, q^m, \gamma^m, \rho^m)$ such that $v^h(\overline{z}^{mh}) > v^h(z^{mh})$. Then by applying the lower hemi-continuity of the budget set of the above redefined problem, parametrized on ρ^m (see lemma 10 in appendix), $\exists \overline{z}_n^{mh} \in B_1^h(p^{mn}, q^{mn}, \gamma^{mn}, \rho^{mn})$ and $\overline{z}_n^{mh} \to \overline{z}^{mh}$. Now, for $n \ge n_o$ one has $\overline{z}_n^{mh} \in B_{1n}^h(p^{mn}, q^{mn}, \gamma^{mn}, \rho^{mn})$. Since v^h is continuous, the following holds

$$v^h(\overline{z}_n^{mh}) > v^h(z_n^{mh}), \forall n \ge n_1$$

Therefore for $n \ge \max\{n_o, n_1\}$, z_n^{mh} is not optimal in the truncated economy $\mathcal{E}_n^m$, a contradiction. We have established Claim 3.1.■

Individual optimality at the cluster points implies that $p_{sl}^{mn} \nrightarrow 0$ ($s = 0, 1, ..., S$; $l = 1, ..., L$) and $\pi_{1j}^{mn} \nrightarrow 0$ ($j = 1, ..., J$). It follows immediately that $x_{ol}^{hmn}, \theta_j^{hmn} \le (ess\sup_{h,l}\omega_{ol}^h)/(\min_{l,j}\left\{\lim_{n\to\infty} p_{ol}^{mn}, \lim_{n\to\infty}\pi_{1j}^{mn}\right\})$.

To show that the short sales allocation is also uniformly bounded we have to use the non arbitrage conditions. For n large enough the non-arbitrage conditions established in Theorem 1 must be verified in equilibrium. In fact, by the market clearing equation of first period commodities we have $\int \varphi_n^m \le m \int \omega_o$ and therefore if the non-arbitrage conditions were violated

$\int \varphi_{nj}^{m}$ would become equal to n , violating the inequality above for n large enough.

Claim 3.2 $(\varphi_{n}^{m})_{n}$ is uniformly bounded.

Proof:

Suppose that there is a sequence n of agents for which $\varphi_{j}^{mn} \rightarrow \infty$. Notice first that $\lambda_{s}^{mn} \nrightarrow \infty$ $(s = 1, ..., S)$. In fact, $\lambda_{s} = u_{sl}'/p_{sl}$, where for any (s, l), $p_{sl}^{mn} \nrightarrow 0$, and $\lambda_{s}^{mn} \rightarrow \infty$ would imply $u_{sl}' \rightarrow \infty, \forall_{l}$, and therefore $x_{sl}^{mn} \rightarrow 0, \forall_{l}$, but endowments in any state s are bounded from below and the additional income is nonnegative.

Now let us examine the behavior of λ_{o}^{mn}. Notice that

$$\lambda_{o} = \sum_{l} u_{ol}' + \sum_{s,l} \lambda_{s} p_{sl} Y_{sl} + \sum_{l} \eta_{l}$$

where the non-negativity multiplier η_{l} is equal to zero when $x_{ol} \neq 0$. The constraint $M_{jl} \geq 1/m$, for any asset j and any commodity l, implies that $M_{j}^{mn} \varphi_{j}^{mn} \rightarrow \infty$ and therefore $u_{olmn}' \rightarrow 0$ for each l. Then $\lambda_{o}^{mn} \rightarrow \infty$ only if $\eta_{l}^{mn} \rightarrow \infty$ for some l. Let us evaluate $\sum\limits_{l \in \mathcal{B}^{h}} \eta_{l}^{h}$, where $\mathcal{B}^{h} = \left\{ l : \eta_{l}^{h} \rightarrow \infty \right\}$

Now $\sum\limits_{l \in \mathcal{B}^{h}} \eta_{l}^{h} = \sum\limits_{l \in \mathcal{B}^{h}} p_{ol} \lambda_{o}^{h} - \sum_{s} \lambda_{s}^{h} \sum\limits_{l \in \mathcal{B}^{h}} p_{sl} Y_{sl} - \sum\limits_{l \in \mathcal{B}^{h}} u_{ol}'$ where

$$\lambda_{o}^{h} = \frac{1}{\sum\limits_{l \notin \mathcal{B}^{h}} p_{ol}} \left(\sum_{l \notin \mathcal{B}^{h}} u_{ol}' + \sum_{s} \lambda_{s}^{h} \sum_{l \notin \mathcal{B}^{h}} p_{sl} Y_{sl} + \sum_{l \notin \mathcal{B}^{h}} \eta_{l} \right)$$

and therefore

$$\sum_{l \in \mathcal{B}^{h}} \eta_{l}^{h} = \frac{\sum\limits_{l \in \mathcal{B}^{h}} p_{ol}}{\sum\limits_{l \notin \mathcal{B}^{h}} p_{ol}} \left(\sum_{l \notin \mathcal{B}^{h}} u_{ol}' + \sum_{s} \lambda_{s}^{h} \sum_{l \notin \mathcal{B}^{h}} p_{sl} Y_{sl} + \sum_{l \notin \mathcal{B}^{h}} \eta_{l} \right) - \sum_{s} \lambda_{s}^{h} \sum_{l \in \mathcal{B}^{h}} p_{sl} Y_{sl} - \sum_{l \in \mathcal{B}^{h}} u_{ol}' \quad (26)$$

Hence $\sum\limits_{l \in \mathcal{B}^{h}} \eta_{l}^{h} \rightarrow \infty$ only if $\frac{\sum_{l \in \mathcal{B}^{h}} p_{ol}}{\sum_{l \notin \mathcal{B}^{h}} p_{ol}} \rightarrow \infty$ or $\sum\limits_{l \notin \mathcal{B}^{h}} u_{ol}' \rightarrow \infty$. The former is impossible since $p_{ol}^{mn} \nrightarrow 0$ $(l = 1, ..., L)$ and the latter was also already ruled out. We have established that $\lambda_o^{mn} \nrightarrow \infty$.

Using now the non-arbitrage conditions for n large enough, there exists a β^{mn} that satisfy (5) and (6)

$$p_o^{mn}x_o^{mn} + \left[\sum_s \beta_s^{mn}(p_s^{mn}Y_sM^{mn} - D_s^{mn}) + \beta_2^{mn}\right]\varphi^{mn} +$$

$$(\sum_s \beta_s^{mn}N_s^{mn} + \beta_1^{mn})\theta^{mn} = p_o^{mn}w_o^{mn}$$

where $\beta_1^{mn} = (\beta_{S+1}^{mn}, \ldots, \beta_{S+J}^{mn})$ and $\beta_2^{mn} = (\beta_{S+J+1}^{mn}, \ldots, \beta_{S+2J}^{mn})$. This implies that $\beta_s^{mn}(p_s^{mn}Y_sM_j^{mn} - D_{sj}^{mn})\varphi_j^{mn}$and $\beta_2^{mn}\varphi^{mn}$ are uniformly bounded.
Now $\frac{\partial L}{\partial \varphi}\varphi = 0$ is equivalent to

$$u_o'M\varphi - \lambda_o\left[\beta_2 + \sum_s \beta_s(p_sY_sM - D_s)\right]\varphi + \sum_s \lambda_s(p_sY_sM - D_s)\varphi = 0$$

Since $\lambda_o^{mn} \nrightarrow \infty$ it follows that $u_{omn}'M^{mn}\varphi^{mn} \nrightarrow \infty$ as well contradicting assumption (P), which requires

$$u_{olmn}'(x_o^{mn} + M^{mn}\varphi^{mn})M_l^{mn}\varphi^{mn} \rightarrow \infty$$

This establishes that the sequence of short sales allocations is also uniformly bounded. We have established Claim 3.2.■

Then, by (1) $\{\sum_j p_o^{mn}M_{jln}^{mh}\varphi_{jn}^{mh}\}$ becomes also a uniformly bounded sequence, implying that $\{M_{jln}^{mh}\varphi_{jn}^{mh}\}$ is uniformly bounded since p_{ol}^{mn} is bounded away from zero, for any l. Hence, from (2), $\{x_{sln}^{mh}\}$ is also uniformly bounded. All these facts imply that the sequence $(x_n^m, \theta_n^m, \varphi_n^m, c_n^m, (p_s^{mn}R_s^j\varphi_{jn}^m - p_s^{mn}c_{mn})^+)$ is uniformly bounded. This completes the proof of lemma 3.□

We can now continue the proof of existence of equilibria for the economy $\mathcal{E}^m$ using the strong version of Fatou's lemma (see Apendix):
$\int_H x^{mh}dh = \lim_{n\to\infty}\int_H x_n^{mh}dh$, $\int_H \theta^{mh}dh = \lim_{n\to\infty}\int_H \theta_n^{mh}dh$,
$\int_H \varphi^{mh}dh = \lim_{n\to\infty}\int_H \varphi_n^{mh}dh$ and

$$\int_H c_j^{mh}dh = \lim_{n\to\infty}\int_H c_{jn}^{mh}dh = \lim_{n\to\infty}\int_H M_{jn}^{mh}\varphi_{jn}^{mh}dh \tag{27}$$

Defining

$$M_j^{mh} = \begin{cases} \frac{1}{\varphi_j^{mh}} c_j^{mh} \in I\!R_+^L, & \varphi_j^{mh} \neq 0 \\ \text{anything} , & \varphi_j^{mh} = 0 \end{cases}$$

we have that

$$\int_H M_j^{mh} \varphi^{mh} dh = \int_H \frac{c_j^{mh}}{\varphi_j^{mh}} \varphi_j^{mh} dh = \int_H c_j^{mh} dh$$

Therefore by using (27) one has

$$\int_H M_j^{mh} \varphi_j^{mh} dh = \lim_{n\to\infty} \int_H M_{jn}^{mh} \varphi_{jn}^{mh} dh$$

And so

$$\int_H^{mh} \varphi^{mh} dh = \lim_{n\to\infty} \int_H M_n^{mh} \varphi_n^{mh} dh$$

Thus all markets clear in the $\mathcal{E}^m$.
We complete the proof of the proposition with the following lemma

Lemma 4 *There exists vector $C^m \in I\!R_+^{JL}$ such that*

(a) $C_j^m \int_H \theta_j^{mh} dh = \int_H M_j^{mh} \varphi_j^{mh} dh \in I\!R_+^L, \forall j \in J$

(b) $\int\limits_H (p_s^m R_s^j - p_s^m C_j^m)^+ \theta_j^{mh} = \int\limits_H (p_s^m R_s^j - p_s^m M_j^{mh})^+ \varphi_j^{mh}, \forall j \in J$

Proof:
Define $\overline{C}_{lj}^m = \frac{\rho_{slj}^m}{p_{sl}^m}$, where $\rho_{slj}^m = \lim_{n\to\infty} \rho_{slj}^{mn} = \lim_{n\to\infty} p_{sl}^{mn} C_{lj}^{mn}$. Then

$$\overline{C}_{jl}^m \int_H \theta_j^{mh} dh = \frac{\rho_{slj}^m}{p_{sl}^m} \int_H \theta_j^{mh} dh = \lim_{n\to\infty} \frac{\rho_{slj}^{mn}}{p_{sl}^{mn}} \lim_{n\to\infty} \int_H \theta_{jn}^{mh} dh = \lim_{n\to\infty} C_{lj}^{mn} \int_H \theta_{jn}^{mh} dh$$

$$= \lim_{n\to\infty} \int_H M_{jln}^{mh} \varphi_{jn}^{mh} dh = \int_H M_{jl}^{mh} \varphi_j^{mh} dh$$

Now, $\int_H (p_s^m R_s^j - p_s^m M_j^{mh})^+ \varphi_j^{mh} dh = \lim_{n\to\infty} \int_H (p_s^{mn} R_s^j \varphi_{jn}^{mh} - p_s^{mn} c_{jn}^{mh})^+ dh$.
By definition of $\overline{C}^m$ one has

$$p_s^m \overline{C}_j^m = \lim_{n\to\infty} \sum_l p_{sl}^m C_{lj}^{mn} = \lim_{n\to\infty} p_{sn}^m C_j^{mn}$$

Then we have

$$\int_{\mathcal{S}_s^j} (p_s^m R_s^j - p_s^m \overline{C}_j^m)^+ \theta_j^{mh} = \int_{\mathcal{D}_s^j} (p_s^m R_s^j - p_s^m M_j^{mh})^+ \varphi_j^{mh}, \forall j \in J,$$

as desired. ■

6.3 Equilibrum in Economies without Lower Bound on Collateral

Proof of Theorem 3

By Proposition 1 we know that equilibrium exist in an economy $\mathcal{E}^m$ where collateral coefficients are bounded from bellow by $1/m$.

Now let $m \to \infty$ and examine the asymptotic properties of the sequence of equilibria for $\mathcal{E}^m$.

Lemma 5 $p_{sl}^m \nrightarrow 0\ \forall_{s,l}$

Proof:

Let A_{sj} be equal to $R_{sj}\theta_j$ when $p_s R_{sj} \leq p_s Y_s C_j$ and equal to $Y_s C_j \theta_j$ otherwise. Then the bundle $\omega_s^h + Y_s x_o + \sum_j A_{sj}$ is bounded from below, away from zero, in each coordinate. Income in each state of the second period is the value of this bundle plus an additional income equal to $p_s Y_s M\varphi - \sum_j D_{sj}\varphi_j \geq 0$. Since preferences are time and state separable and monotonic, for any s and any l we have $p_{sl}^m \nrightarrow 0$. In fact, even in the presence of an unbounded increase in income, possibly offsetting the increase in x_{sl} for an inferior good, the expenditure in some commodity would have to grow unboundedly and therefore $\|x_{sl}^{hn}\| \to \infty$ for every h,implying that the feasibility equations would be violated for m sufficiently large. This completes the proof of this lemma.□

Lemma 6 $C_j^m \not\rightarrow 0$ *as* $m \to 0$.

Proof:

Let $\mathfrak{S}_j^{hm} = \{s \in S : p_s^m R_s^j > p_s^m Y_s M_j^{mh}\}$ be the set of states where the agent gives default in promise j and let $\left(\mathfrak{S}_j^{hm}\right)'$ be it's complement

Claim: $\left(\mathfrak{S}_j^{hm}\right)' \neq \emptyset \;\; \forall\, h, j, m$

Proof:

Let us start by defining an auxiliary optimization problem for each consumer. Consumers will now choose a vector $\tilde{\beta} = (\beta_1, \ldots, \beta_S, \beta_{S+J+1}, \ldots, \beta_{S+J}) \in I\!R_{++}^{S+J}$ satisfying

$$q - \sum_s \gamma_s g_{2s} = \sum_{s=1}^{S} \beta_s p_s R_s^j - \sum_{s=1}^{S} \beta_s (p_s R_s^j - p_s Y_s M_j)^+ + (p_o M_j - \sum_{s=1}^{S} \beta_s p_s Y_s M_j) - \beta_{S+J+j} \tag{28}$$

The auxiliary problem is

$$\max_{(x^h, \theta^h, \varphi^h, M^h, \tilde{\beta}^h) \in B_2^h(p,C,q,\gamma)} u^h(x_o^h + M^h \varphi^h, x_{-o}^h) \tag{29}$$

where

$$B_2^h(p, C, q, \gamma) = \Big\{ (x, \theta, \varphi, M, \tilde{\beta}) : \text{(1), (2) and (28) hold for}$$

$$\pi_1 \text{ given by (9) and } \pi_2 \text{ given by (6)} \Big\}$$

Clearly the original problem (11) and the auxiliary problem (29) have the same solutions.

Suppose now that we impose, in addition, in problem (29) that $\tilde{\beta}$ must be equal to a value that satisfies equation (28), when (p, C, q, γ, M^h) are fixed at their equilibrium values, for the economy $\mathcal{E}^m$. Clearly this reformulated problem has the same solutions as problem (29), for the above equilibrium values of auxiliary parameters (p, C, q, γ). In this reformulated auxiliary problem, in equilibrium, constraint (28) has a zero multiplier and, therefore, the first order condition on M_{jl} becomes:

$$\frac{u'_{ol}}{\lambda_o^m} - \sum_{s \notin \mathfrak{S}_j^{hm}} \beta_s^m p_{sl}^m Y_s + \sum_{s \notin \mathfrak{S}_j^{hm}} \frac{\lambda_s^m}{\lambda_o^m} p_{sl}^m Y_s \leq 0 \tag{30}$$

Hence $(\mathfrak{S}_j^{hm})' \neq \emptyset$ throughout the sequence, that is, $\forall_{m,h} \exists_s : p_s^m Y_s M_j^{hm} \geq p_s^m R_{sj}$. This establishes the claim.$\square$

Let $T_{sj}^m = \left\{z \in R_+^l : p_s^m Y_s z \geq p_s^m R_{sj}\right\}$ and $T_j^m = \bigcup_{s=1}^{S} T_{sj}^m$. Then, for each m, $\forall_h$, $M_j^{hm} \in T_j^m$, implying that $C_j^m \in \overline{conT_j^m}$. Notice that $0 \notin \overline{conT_j^m}$ for each m. Define the corresponding sets at the cluster point $(p_s)_{s=1}^S \gg 0$: $T_{sj} = \left\{z \in I\!R_+^l : p_s Y_s z \geq p_s R_{sj}\right\}$ and $T_j = \bigcup_{s=1}^{S} T_{sj}$. We must have the cluster point C_j of the sequence C_j^m belonging also to $\overline{con}T_j$ which does not contain the origin, hence $C_j \neq 0$. This completes the proof of lemma 6.■

Therefore, $\int \varphi_j^{hm} dh$ is bounded, for any asset j. By feasibility (as in the proof of lemma 2) it follows that $\int (x^{hm}, \theta^{hm}, M^{hm}\varphi^{hm})dh$ is also a bounded sequence. The weak version of Fatou's lemma can be applied (as in the proof of lemma 3) to show the existence of an integrable map z such that $z^h \in cl\left\{x^{hm}, \theta^{hm}, \varphi^{hm}, M^{hm}\varphi^{hm}\right\}$ for a.e. h and

$$\int z^h dh \leq \lim_{m\to\infty} \int (x^{hm}, \theta^{hm}, \varphi^{hm}, M^{hm}\varphi^{hm})dh.$$

As in the proof of claim(3.1) in lemma(3), z^h is an optimal choice of agent h at the cluster point of the sequence $(p^m, C^m, q^m, \gamma^m)$. This implies that $p_{ol}^m \nrightarrow 0$ $(l = 1, ..., L)$, $\pi_{1j}^m \nrightarrow 0$ $(j = 1, ..., J)$ and that the non-arbitrage conditions hold at these cluster prices. It follows immediately that $x_{0l}^{hm}, \theta_j^{hm} \leq (ess\sup_{h,l} \omega_{ol}^h)/(\min_{l,j}\left\{\lim_{m\to\infty} p_{ol}^m, \lim_{m\to\infty} \pi_{1j}^m\right\})$.

Lemma 7 $(\varphi^m)_m$ *is uniformly bounded*

Proof:
Let us use the non-arbitrage conditions and suppose that there is a sequence m of agents for which $\varphi_j^m \to \infty$. Notice first that $\lambda_s^m \nrightarrow \infty$ $(s = 1, ..., S)$ since $u'_{sl} \to \infty, \forall_l$ would imply $x_{sl}^m \to 0, \forall_l$, but endowments in any state s are bounded from below and the additional income is nonnegative. Now let us examine the behavior of λ_o^m. Let us consider two cases:

First, suppose $\theta^m \neq 0$ for infinitely many m. Then, $\lambda_o^m \nrightarrow \infty$. In fact, $\frac{\partial L^m}{\partial \theta} = 0$ requires $\lambda_o^m \pi_1^m = \sum_{s=1}^S \lambda_s^m N_s^m$ where $\pi_1^m \nrightarrow 0$, $\lambda_s^m \nrightarrow \infty$ $(s = 1, ..., S)$ and $N_s^m \nrightarrow \infty$. Second, suppose $\theta^m = 0$ except for finitely many m. In this case, first period income $p_o^m \omega_o^m + \pi_2^m \varphi^m - \pi_1^m \theta$ would tend to ∞ and therefore $\lambda_o^m \to 0$, unless $\pi_2^m \to 0$. When $\pi_2^m \to 0$, we have

$p_o^m M_j^m - \pi_{2j}^m \to \xi > 0$ where ξ is a uniform positive lower bound on $p_o^m M_j^m$ (which exists due to the fact that $0 \notin \overline{con} T_j$ and $p_{ol}^m \nrightarrow 0$ for any l). Hence, $\pi_2^m \to 0$ would imply φ_j^m uniformly bounded by $(ess \sup_{h,l} \omega_{ol}^h)/\xi$, a contradiction. We have established that $\varphi_j^m \to \infty$ implies $\lambda_o^m \nrightarrow \infty$.

We complete the proof applying an argument similar to the one used to bound uniformly the short-sales allocations in the proof of claim (3.2). By the non-arbitrage conditions we have, for m large enough:

$$p_o^m x_o^m + \left[\sum_s \beta_s^m (p_s^m Y_s M^m - D_s^m) + \beta_2^m\right] \varphi^m + (\sum_s \beta_s^m N_s^m + \beta_1^m)\theta^m = p_o^m w_o^m$$

Which implies that $\beta_s^m (p_s^m Y_s M_j^m - D_{sj}^m)\varphi_j^m$ and $\beta_2^m \varphi^m$ are uniformly bounded. Now $\frac{\partial L}{\partial \varphi}\varphi = 0$ is equivalent to

$$u_o' M\varphi - \lambda_o \left[\beta_2 + \sum_s \beta_s (p_s Y_s M - D_s)\right] \varphi + \sum_s \lambda_s (p_s Y_s M - D_s)\varphi = 0$$

since $\lambda_o^m \nrightarrow \infty$ it follows that $u_{0m}' M^m \varphi^m \nrightarrow \infty$ as well contradicting assumption (P), which requires $u_{olm}'(x_o^m + M^m\varphi^m) M_l^m \varphi^m \to \infty$. This establishes that the sequence of short sales allocations is also uniformly bounded and completes the proof of lemma 7.■

We can now apply the strong version of Fatou's lemma to guarantee that markets clear, following the procedure already used earlier at the end of the proof of Proposition 1.□

The above proof of Theorem 3 shows that there is a one to one correspondence between equilibria, as defined in section 6, and reformulated equilibria where each consumer takes as given commodity prices p and the purchase price of derivative π_1, and faces a pricing formula for the sale price of primitive assets, which is precisely the non-arbitrage valuation formula (6) for some vector $\tilde{\beta} = (\beta_1, \ldots, \beta_S, \beta_{S+J+1}, \ldots, \beta_{S+2J}) \in I\!R_{++}^{S+J}$ taken as given. This vector $\tilde{\beta}$ takes the role of 'state prices and may vary across individuals, since the choice of the collateral coefficients has personalized the asset return matrix. The reformulated individual problem is

$$\max_{(x^h,\theta^h,\varphi^h,M^h)\in B_3^h} U^h(x_o^h + M^h\varphi^h, x_{-o}^h) \tag{31}$$

where B_3^h is the budget set of each agent $h \in H$ given by:

$$B_3^h(p, C, \pi_1, \tilde{\beta}) = \big\{(x, \theta, \varphi, M) \in I\!R^{L(S+1)+2J+JL} : (1) \text{ and } (2)$$
$$\text{hold for } \pi_2 \text{ given by } (6)\}$$

Definition 3 *A reformulated equilibrium for $\mathcal{E}$ is a vector $((p, \pi_1, \pi_2, C), (x^h, \theta^h, \varphi^h, M^h)_{h\in H})$ such that:*

- *For each agent h, $(\pi_1, (\pi_2^h)_{h\in H})$ satisfy equations (5) and (6) for some $\beta^h \in I\!R^{S+2J}_{++}$.*
- *$(x^h, \theta^h, \varphi^h, M^h)$ solves problem (31) for $\tilde{\beta}^h = (\beta_1^h, \ldots, \beta_S^h, \beta_{S+J+1}^h, \ldots, \beta_{S+2J}^h)$.*
- *Equations (12), (13), (14) and (15) are satisfied.*

Remark
In the reformulated model we do not have in general the usual Arrow-Debreu contingent claims Walras Law (for the personalized state prices), unless the correction factors happen to cancel out in the aggregate, which is not necessarily the case. To see this, derive the following equation from the definition of matrix $\hat{A}$:

$$(p_o(x_o - \tilde{\omega}_o), .., p_S(x_S - \tilde{\omega}_S), \theta, \varphi) = \hat{A}\Psi$$

Recall that $\tilde{\omega}_s = \omega_s + Y_s x_o$, $\forall s$ and $\tilde{\omega}_o = \omega_o$. Therefore

$$\delta \cdot (p_o(x_o - \tilde{\omega}_o), .., p_S(x_S\tilde{\omega}_S), \theta, \varphi) = 0$$

Then for each agent

$$p_o(x_o^h - \tilde{\omega}_o^h) + \sum_{s\in S} \beta_s^h p_s(x_s^h - \tilde{\omega}_s^h) + \sum_{j\in J} \beta_{S+j}^h \theta_j^h + \sum_{j\in J} \beta_{S+J+j}^h \varphi_j^h = 0$$

Now summing and subtracting: $\sum_{j\in J} p_o M_j^h \varphi_j^h$ and $\sum_{j,s} \beta_s^h p_s Y_s M_j^h \varphi_j^h$. Now integrating over all $h \in H$, we obtain that:

$$p_o \int_H \left[(x_o^h - \tilde{\omega}_o^h) + \sum_{s\in S} \beta_s^h p_s(x_s^h - \tilde{\omega}_s^h - Y_s M^h \varphi^h) \right] = 0 \Leftrightarrow$$

$$\int_H \sum_j \beta^h_{S+j}\theta^h_j dh = \int_H \sum_j \left[(p_o - \sum_s \beta^h_s p_s Y_s) M^h_j - \beta^h_{S+J+j} \right] \varphi^h_j dh$$

In other words a pseudo Walras Law must be satisfied if and only if the aggregate corrections in the derivative prices must be equal to the aggregate corrections in the basic securities. This not necessarily the case for the above reformulated equilibria.

7 Efficency

In this section we prove that an equilibrium allocation is constrained efficent among all feasible allocations that provide income across states through the same spot prices (the given equilibrium prices). In comparison with the equilibrium obtained by Araujo, Orrillo and Páscoa [3], we can say that our equilibrium is Pareto superior, since we are not impossing any kind of bounded short sale.

As in the work of Magill and Shafer [15], we compare the equilibrium allocation with one feasible allocation whose portfolios do not necessarily result from trading competitively in asset markets. That is, in alternative allocations agents pay participation fees which may differ from the market portfolio cost. Equivalently, we allow for transfers across agents which are being added to the usual market portfolio cost.

Proposition 2 *Let* $((\overline{x}, \overline{\theta}, \overline{\varphi}, \overline{M}), \overline{p}, \overline{\pi}_1, \overline{\pi}_2, \overline{C})$ *be an equilibrium. The allocation* $(\overline{x}, \overline{\theta}, \overline{\varphi}, \overline{M})$ *is efficient among all allocations* (x, θ, φ, M) *for which there are transfers* $T^h \in I\!R$ *across agents and a vector* $C \in I\!R^{JL}_+$*, such that*

(i) $\int_H (x^h_o + M^h\varphi^h) dh = \int_H \omega^h_o dh$, $\int_H x^h_s = \int_H (\omega^h_s + Y_s M^h \varphi^h + Y_s x^h_o) dh$, $\int_H \theta^h dh = \int_H \varphi^h dh$

(ii)

$$\overline{p}_s(x^h_s - \omega^h_s - Y_s x^h_o) + \sum_{j\in J} \min\{\overline{p}_s R^j_s, \overline{p}_s Y_s M^h_j\}\varphi^h_j$$

$$= \sum_{j\in J} \min\{\overline{p}_s R^j_s, \overline{p}_s Y_s C_j\}\theta^h_j + \sum_{j\in J} \overline{p}_s Y_s M^h_j \varphi^h_j, \quad \forall s, \quad a.e. \ h$$

(iii) $\overline{p}_o(x_o^h + M^h\varphi^h - \omega_o^h) + \overline{\pi}_1\theta^h - \overline{\pi}_2\varphi^h + T^h = 0$

(iv) $\int_H T^h dh = 0$

(v) $C_j \int_H \theta^h dh = \int_H M_j^h \varphi_j^h, \ \ \forall j$

where the equilibrium prices are given by

$$\overline{\pi}_1 = \overline{q} - \sum_s \overline{\gamma}_s \overline{g}_{1s}$$

and

$$\overline{\pi}_2 = \overline{q} - \sum_s \overline{\gamma}_s \overline{g}_{2s}$$

Proof:

Suppose not, say $(x, \theta, \varphi, M, C)$ together with some transfer fraction T satisfies (i) through (v); $u^h(x_o^h + M^h\varphi^h, x_{-o}^h) \geq u^h(\overline{x}_o^h + \overline{M}^h\overline{\varphi}^h, \overline{x}_{-o})$ for $a.e$ h and $u^h(x_o^h + M^h\varphi^h, x_{-o}^h) > u^h(\overline{x}_o^h + \overline{M}^h\overline{\varphi}^h, \overline{x}_{-o})$ for h in some positive measure set G of agents. Then, for $h \in G$, the first period constraint must be violated, that is,

$$\overline{p}_o(x_o^h + M^h\varphi^h - \omega_o^h) + \overline{\pi}_1\theta^h - \overline{\pi}_2\varphi^h > 0 \tag{32}$$

Now remember that

$$g_s^h = (\overline{p}_s R_s - \overline{p}_s Y_s M^h)^+ \varphi^h - (\overline{p}_s R_s - \overline{p}_s Y_s C)^+ \theta^h$$
$$= (\overline{p}_s R_s - D_s^h)\varphi^h - (\overline{p}_s R_s - N_s)\theta^h$$

By continuity of preferences and monotonicity we can take $G = H$, without loss of generality. Then $\int_H g_s^h dh > 0$ for some s, by (32) and (i), implying $\int_H R_s\theta^h dh > \int_H D_s^h\varphi^h dh$. Now, by (ii),

$$\overline{p}_s \cdot \int_H (x_s^h - \omega_s^h - Y_s(M^h\varphi^h + x_o^h))dh = \int_H R_s\theta^h dh - \int_H D_s^h\varphi^h dh$$

where the right hand side is strictly positive, contradicting $\int_H (x_s^h - \omega_s^h - Y_s(M^h\varphi^h + x_o^h))dh = 0$.■

The above weak constrained efficiency property is in the same spirit as properties found in the incomplete markets model without default (see Magill and Shafer [15]) and also in the exogenous collateral model (without utility penalties) of Dubey, Geanakoplos and Zame [7]. As in these models, it does not seen to be possible to show that equilibrium allocations are undominated when prices are no longer assumed to be constant at the equilibrium levels. However equilibria with default and endogenous collateral, as proposed in this paper, is Pareto superior to the no-default equilibria, to the exogenous collateral equilibria and even to the bounded short-sales endogenous collateral equilibria of Araujo, Orrillo and Páscoa [3], since our equilibria is free of any of the constraints which are used in the definition of these equilibrium concepts (that is, absence of default, exogeneity of collateral and bounded short-sales).

8 Conclusions

In this paper we have obtained a no arbitrage characterization of the prices of collateralized promises, where the collateral coefficients are choosen by borrowers as in Araújo, Orrillo, Páscoa [3]. We also obtained a pricing result consistent with the observation made by Jouini and Kallal [13] for the case of short sale constraints, more precisely we have shown that our buy and net sell prices are supermartingale and submartingales, respectively, under some probability measures. For these probabilities we have found lower and upper bounds for the prices of derivatives written in terms of the primitive defaultable assets. Finally using the nonarbitrage characterization of asset prices we have shown the existence of equilibrium in the model where borrowers choose the collateral coefficients, without imposing uniform bounds on short-sales (thus avoiding a major drawback of the work by Araújo, Orrillo and Páscoa [3]) and we have shown also that this equilibrium is constrained efficient.

9 Appendix

9.1 Mathematical Preliminarities

- Let $C(K)$ the Banach space of continuous functions on the compact metric space K. Let $L^1(H, C(K))$ be the Banach space of Bochner integrable functions whose values belong to $C(K)$. For $z \in L^1(H, C(K))$,

$$||z||_1 := \int_H \sup_K |Z^h| dh < \infty$$

Let $\mathcal{B}(K)$ denotes the set of regular measures on the Borelians of K. The dual space of $L^1(H, C(K))$ is $L^\infty_\omega(H, \mathcal{B}(K))$, the Banach space of essentially strong bounded weak $*$ measurable functions from H into $\mathcal{B}(K)$. We say that $\{\mu_n\} \subset L^\infty(H, \mathcal{B}(K))$ converges to $\mu \in L^\infty_\omega(H, \mathcal{B}(K))$ with respect to the weak * topology on the dual $L^1(H, C(K))$, if

$$\int_H \int_K z^h d\mu_n^h dh \to \int_H \int_K z^h d\mu^h dh \, , \forall f \in L^1(H, C(K))$$

- We will use in this work the following lemmas (in m-dimension).
  Fatou's lemma (Weak Version)
  Let $\{f_n\}$ be a sequence of integrable functions of a measure space $(\Omega, \mathcal{A}, \nu)$ into $I\!R^m_+$. Suppose that $\lim_{n\to\infty} \int_\Omega f_n d\nu$ exists. Then there exists an integrable function $f : \Omega \mapsto I\!R^m_+$ such that:

  1. $f(w) \in cl\{f_n(w)\}$ for a.e w, and
  2. $\int_\Omega f d\nu \leq \lim_{n\to\infty} \int_\Omega f_n d\nu$

  Fatou's lemma (Strong version)
  If in addition the sequence $\{f_n\}$ above is uniformly integrable, then the inequality in 2. holds as an equality.

9.2 Extended Game

We will extend the generalized game defined in section 6.2 by allowing for mixed strategies both in portfolios and collateral. Remember that, for each player a mixed strategy is a probability distribution on his set of pure strategies. In this case the set of measures on the Borelians of $K_n = [0, n]^J \times [0, n]^J \times [1/m, n]^{LJ}$. We denote by $\mathcal{B}$ the set of mixed strategies of each consumer. Since we are not interested in a mixed strategies equilibrium, per se, we will extend the previous game to a game $\overline{\mathcal{J}}_n$ over mixed strategies (that we call extended game) whose equilibria: 1) exist 2) can be purified and 3) a pure version is an equilibrium for the original game. First, before extending the game to mixed strategies, let us rewrite the payoffs of the fictitious agents replacing consumption bundles by the following function of portfolios and collateral:

$$d^h(\theta^h, \varphi^h, M^h) = \text{ arg } \max\{u^h : x^h \in [0, n]^{L(S+1)} \text{ satisfies } (1), (2) \text{ and } (16)\}$$

That is, function d^h solves the utility maximization problem for a given portfolio (θ^h, φ^h) and a given collateral bundle M^h. By the maximum theorem, d^h is continuous. Secondly, we extend the payoffs to mixed strategies.

(i) Each consumer $h \in H$ chooses $(x^h, \mu^h) \in [0, n]^{L(S+1)} \times \mathcal{B}$ in order to maximize $\int_{K_n} U^h(x_o^h + M^h\varphi^h, x_{-o}^h)d\mu^h$ subject to the constraints:

$$p_o(x_o^h - \omega_o^h) + \int_{K_n} [\pi_1\theta^h + (p_oM^h - \pi_2)\varphi^h]d\mu^h \leq 0$$

$$p_s(x_s^h - \omega_s^h - Y_s x_o^h) \leq \int_{K_n} \sum_j (p_s N_s^j \theta_j^h - D_{js}^h \varphi_j^h + p_s Y_s M_j^h \varphi_j^h)d\mu^h \text{ for } s \in S$$

(ii) The auctioneer of the first period chooses $(p_o, q, \gamma) \in \triangle^{L+J+S-1}$ in order to maximize

$$p_o \int_H \int_{K_n} [d_o^h(\theta^h, \varphi^h, M^h) + \sum_j M_j^h \varphi_j^h - \omega_o^h]d\mu^h dh +$$

$$q \int_H \int_{K_n} (\theta^h - \varphi^h)d\mu^h dh + \sum_s \gamma_s \int_H \int_{K_n} g_s(\theta^h, \varphi^h, M^h)d\mu^h dh$$

(iii) The auctioneer of state s in the second period chooses $p_s \in \triangle^{L-1}$ in order to maximize

$$p_s \int_H \int_{K_n} [d_s^h(\theta^h, \varphi^h, M^h) - \sum_j Y_s M_j^h \varphi_j^h - \omega_s^h - Y_s d_o^h(\theta^h, \varphi^h, M^h)]d\mu^h dh$$

(iv) Each of the remaining JL fictitious agents chooses $C_{jl} \in [1/m, n]$ in order to minimize

$$\left(\int_H \int_{K_n} [C_{jl}\theta_j^h - M_{jl}^h \varphi_j^h]d\mu^h dh\right)^2$$

Lemma 8 *$\overline{\mathcal{J}}_n$ has an equilibrium, possibly in mixed strategies over portfolio and collateral together.*

Proof:

The existence argument in Ali Khan [14] can be modified to allow for some atomic players. Consumer's best response correspondences ν^h are convex-valued and upper semicontinuous on the strategies of fictitious agents.

Now, define the correspondence:

$$\alpha(p, \pi, C) = \{f \equiv (x, \mu) \in ([0, n]^{L(S+1)} \times \mathcal{B})^H : f(h) \in \nu^h(p, \pi, C)\}$$

Which is also convex value and upper semicontinuous . The best response correspondences $\mathcal{R}^i$ of the $r = S + 1 + JL$ fictitious agents are convex valued and upper semicontinuous on the profile of consumers' probability measures on $K_n = [0, n]^J \times [0, n]^J \times [1/m, n]^{LJ}$ (with respect to the weak * topology on the dual of $L^1(H, C(K_n)$). The profiles set is compact for the same topology and Fan - Glicksberg fixed point theorem applies to $\alpha \times \prod_{i=1}^{r} \mathcal{R}^i$. ■

Lemma 9 *$\overline{\mathcal{J}}_n$ has an equilibriumin pure strategies.*

Proof:

In this part Liapunov's theorem will be fundamental. First, notice that the payoffs of the atomic players in $\overline{\mathcal{J}}_n$ depend on the profile of mixed strategies $(\mu^h)_h$ only through finitely many e indicators of the form ($e = L + S + SL + JL$).

$$\int_H \int_{K_n} Z_e^h(\theta^h, \varphi^h, M^h) d\mu^h dh \text{ where } Z_e \in L(H, C(K_n))$$

Secondly, let $E^h(p, \pi, C) = \prod_2 \nu^h(p, \pi, C)$ and $Z = (Z_1, \ldots, Z_e)$. Now,

$$\int_{K_n} Z^h(\theta^h, \varphi^h, M^h) dE^h(p, \pi, C) = conv \int_{K_n} Z^h(\theta^h, \varphi^h, M^h) d(extE^h(p, \pi, C))$$

where the integral on the left hand side is the set in $I\!R^e$ of the all integrals of the form $\int_{K_n} Z^h(\theta^h, \varphi^h, M^h) d\mu^h$, for $\mu^h \in E^h(p, \pi, C)$. The integral on the right hand side is defined endogenously. The equality above follows by linearity of the map

$$\mu^h \mapsto \int_{K_n} Z^h(\theta^h, \varphi^h, M^h) d\mu^h$$

Then, Theorem I.D.4 in Hildenbrand [12] implies

$$\int_H \int_{K_n} Z^h(\cdot) dE^h(p, \pi, C) dh = \int_H \int_{K_n} Z^h(\cdot) d(extE^h(p, \pi, C)) dh$$

Then, given a mixed strategies equilibrium profile $(\mu^h)_h$, there exists $(\theta^h, \varphi^h, M^h)$ such that the Dirac measure at $(\theta^h, \varphi^h, M^h)$ is an extreme point of E^h (evaluated at the equilibrium levels of the variables chosen by the atomic players) and $(\theta^h, \varphi^h, M^h)_h$ can replace $(\mu^h)_h$ and keep all equilibrium conditions satisfied, without changing the equilibrium levels of the variables chosen by the atomic players but replacing the former equilibrium bundles by $d^h(\theta^h, \varphi^h, M^h)$.$\blacksquare$

9.3 Lower Hemi-continuity of the Budget Correspondence

Define the following correspondences:

$$B_1^h(p, q, \gamma, \rho) = \{(x^h, \theta^h, \varphi^h, c^h) : \text{ A and B are satisfied } \}$$

and for all $n \in \mathbb{N}$

$$B_{1n}^h(p, q, \gamma, \rho) = \{(x^h, \theta^h, \varphi^h, c^h) \in [0, n]^{(L(S+1)+2J+LJ} : \text{ A and B are satisfied } \}.$$

Lemma 10 *The budget correspondence B_1^h is lower hemi-continuous at any (p, q, γ, ρ) strictly positive, provided that $\omega^h >> 0$.*
(to be used in lemma 3, where A and B are defined)

Proof:
We define $B_o^h(p, q, \gamma, \rho)$ to be the interior of $B_1^h(p, q, \gamma, \rho)$, that is; the following holds:

$$p_o x_o^h + q\theta^h - q\varphi^h + p_o c^h + \sum_s \gamma_s [p_s R_s \varphi^h - p_s c^h]^+ - \sum_s \gamma_s [p_s R_s - \rho_s]^+ \theta^h < p_o \omega_o^h,$$

$$p_s x^h_s + \sum_j \min\{p_s R^j_s \varphi^h_j, p_s c^h_j\} < p_s(\omega^h_s + Y_s x^h_o) + \sum_j \min\{p_s R^j_s, \sum_l \rho_{lsj}\}\theta^h + p_s c^h$$

Let $x^h = 0, \theta^h = 0, \varphi^h = 0$ and c^h_j such that $p_o c^h_j \le p_o \omega^h_o$. It is easy to verify that these variables thus chosen satisfy the budget constraint of agent h with strict inequality. So, $B^h_o(p,q,\gamma,\rho) \neq \phi$. Let $\lim_{n\to\infty}(p^n, q^n, \gamma^n, \rho^n) = (p,q,\gamma,\rho)$ and $(x^h, \theta^h, \varphi^h, c^h) \in B^h_o(p,q,\gamma,\rho)$. Then for every $\{(x^h_n, \theta^h_n, \varphi^h_n, c^h_n)\}$ such that $\lim_{n\to\infty}(x^h_n, \theta^h_n, \varphi^h_n, c^h_n) = (x^h, \theta^h, \varphi^h, c^h)$ and for n large enough, the following holds

$$p^n_o x^h_{no} + q^n \theta^h_n - q^n \varphi^h_n + p^n_o c^h_n + \sum_s \gamma^n_s [p^n_s R_s \varphi^h_n - p^n_s c^h_n]^+ - \sum_s \gamma^n_s [p^n_s R_s - \rho^n_s]^+ \theta^h_n < p^n_o \omega^h_o,$$

$$p^n_s x^h_{ns} + \sum_j \min\{p^n_s R^j_s \varphi^h_{nj}, p^n_s c^h_{nj}\} < p^n_s(\omega^h_s + Y_s x^h_{no}) + \sum_j \min\{p^n_s R^j_s, \sum_l \rho^n_{lsj}\}\theta^h_n + p^n_s c^h_n$$

Thus $(x^h_n, \theta^h_n, \varphi^h_n, c^h_n) \in B^h_o(p^n, q^n, \gamma^n, \rho^n)$ for n large enough, which implies that B^h_o is lower hemi-continuous. Then the result follows from Hildenbrand [12], pag. 26, fact 4. ■

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