Uniform (projective) hyperbolicity or no hyperbolicity: A dichotomy for generic conservative maps *

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Introduction

The results presented in this paper exploit a connection between the following a priori loosely related problems: How do Lyapunov exponents of conservative (symplectic or volume preserving) diffeomorphisms depend on the underlying dynamics? How typical is it for Lyapunov exponents to vanish?

We prove that *Lyapunov exponents can be simultaneously continuous at a given diffeomorphism only if the corresponding Oseledets splitting is dominated or else trivial*, almost everywhere.

Trivial splitting means that all Lyapunov exponents are equal to zero. Domination is a property of uniform hyperbolicity on the projective bundle, whose precise definition will be recalled in a while.

As a consequence one gets a surprising dichotomy *for a residual (dense G_δ) subset of volume preserving C^1 diffeomorphisms on any compact manifold: the Oseledets splitting is either dominated or trivial*, almost everywhere.

Analogous results hold *for symplectic C^1 diffeomorphisms*, where the conclusion is even stronger: *domination is replaced by partial hyperbolicity.*

Moreover, there are versions of these statements for continuous cocycles with values in various matrix Lie groups.

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In the sequel we give the precise statements, and ideas of the proofs. For that, we begin by explaining the meaning and significance of the domination property. Complete proofs of these statements will appear in [4].

Acknowledgements. We lectured on these topics on several occasions. Recently, during the Workshop on Dynamical Systems held at the ICTP-Trieste in August 2001, we were able to improve our approach, resulting in substantially stronger statements. Questions asked by colleagues present at the ICTP, especially A Avila, B Fayad, K Khanin, J Mather, J Milnor, C G Moreira, J Palis, E Pujals, M Shishikura, Ya Sinai, F Takens, J–C Yoccoz, and L–S Young, motivated us to write this outline of the arguments, and helped improve the presentation.

Lyapunov exponents

1. Let $f : M \to M$ be a C^1 diffeomorphism preserving the volume μ of a compact Riemannian manifold M. Oseledets theorem [11] states that, for μ-almost every $x \in M$, there exists a splitting

$$T_x M = E_x^1 \oplus \cdots \oplus E_x^k, \quad k = k(x) \in \mathbb{N},$$

of the tangent space, and there exist real numbers $\hat\lambda_1(x) > \cdots > \hat\lambda_k(x)$ such that

$$\|Df^n(x)v_j\| \approx e^{n\hat\lambda_j(x)}\|v_j\|$$

for all $v_j \in E_x^j$ and $1 \le j \le k$, if $|n|$ is large enough. More precisely,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log\|Df^n(x)v_j\| = \hat\lambda_j(x) \quad \text{for all} \quad v_j \in E_x^j \setminus \{0\}\,. \tag{1}$$

The *Oseledets subspaces* E_x^j and the *Lyapunov exponents* $\hat\lambda_j(x)$ depend measurably on x, and they are invariant under the dynamics:

$$\hat\lambda_j(f(x)) = \hat\lambda_j(x) \quad \text{and} \quad E^j_{f(x)} = Df(x)\cdot E_x^j, \quad \text{with} \quad k(f(x)) = k(x).$$

2. In particular, one can always find $m(x) \in \mathbb{N}$ such that

$$\|Df^n(x)v_i\| \ge 2\,\|Df^n(x)v_j\| \quad \text{for all} \quad n \ge m(x) \tag{2}$$

and all norm 1 vectors $v_i \in E_x^i$ and $v_j \in E_x^j$ with $1 \le i < j \le k$. That is, iterates of Df are eventually more expanding along E_x^i than along E_x^j, for any $i < j$.

Let us stress, however, that this is a purely asymptotic statement. The limit in (1) is usually *not* uniform on x. Correspondingly, the value of $m(x)$ in (2) does depend on the point, in a complicated fashion. Most important, in general it is not even possible to choose $m(\cdot)$ bounded over each orbit $o(x) = \{f^\ell(x) : \ell \in \mathbb{Z}\}$.

Domination property

3. We say that *the Oseledets splitting is dominated at* $x \in M$ if $m(\cdot) \in \mathbb{N}$ as in (2) may be chosen uniform over the whole orbit of x. That is, domination at x means that there is $m \in \mathbb{N}$ such that for every iterate $y = f^\ell(x)$, $\ell \in \mathbb{Z}$, we have

$$\|Df^n(y)v_i\| \geq 2\,\|Df^n(y)v_j\| \quad \text{for all} \quad n \geq m \tag{3}$$

and all norm 1 vectors $v_i \in E^i_y$, $v_j \in E^j_y$, and $1 \leq i < j \leq k$.

More generally, we say that the Oseledets splitting is dominated over an invariant set Γ if there exists $m \in \mathbb{N}$ such that (3) is satisfied for every $y \in \Gamma$, $n \geq m$, $1 \leq i < j \leq k(y)$, and v_i, v_j as before. The definition in the previous paragraph corresponds to the case $\Gamma = o(x)$, of course.

4. Geometrically, domination is tantamount to uniform hyperbolicity of the dynamics induced on the *projective bundle*, i.e. the bundle of directions in the tangent space. We explain this with the aid of Figure 1.

From the relation (3) one easily deduces that

$$\|Df^n(y)v_i\| \geq c\,2^{n/m}\,\|Df^n(y)v_j\| \quad \text{for every} \quad n \geq 1 \tag{4}$$

and for all $y \in o(x)$, $1 \leq i < j \leq k$, and norm 1 vectors $v_i \in E^i_y$ and $v_j \in E^j_y$, where the constant $c > 0$ depends only on m and f. Then, given any $y \in o(x)$ and non-zero $v \in T_yM$, let $v = v_1 + \cdots + v_k$ be the splitting of v along Oseledets subspaces, and $1 \leq p \leq q \leq k$ be, respectively, smallest and largest such that $v_p \neq 0 \neq v_q$.

From (4) one gets that, as n increases to infinity the component $Df^n(y)v_p$ becomes much larger than any other $Df^n(y)v_j$. So, $Df^n(y)v$ approaches the direction of $E^p_{f^n(x)}$, exponentially fast. There is also a dual statement for large negative n, with p replaced by q. Clearly, for most vectors $p = 1$ and $q = k$. This means that E^1 is a global hyperbolic attractor and E^k is a global hyperbolic repeller for the dynamics of Df on the projective bundle, with the other E^j's in the role of saddles.

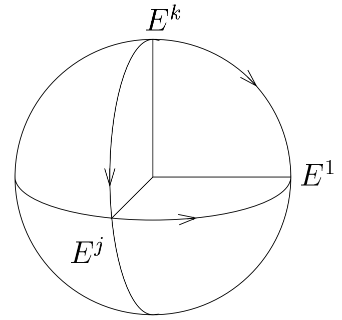

Figure 1: Uniform projective hyperbolicity

5. It is not difficult to find open sets of maps whose Oseledets splitting is dominated, e.g., any small neighborhood of a linear torus automorphism whose eigenvalues have multiplicity 1 and different norms. Notwithstanding, domination is really a very rigid property.

For one thing, if the number $k(y)$ and the dimensions $d_j(y) = \dim E^j_y$ of the Oseledets subspaces are constant on Γ, *domination implies that the splitting is continuous on* Γ, and even admits a continuous extension to the closure. In particular, *the angles between the various Oseledets subspaces are bounded from zero,* uniformly over the invariant set Γ. Actually, this last fact remains true even if the number or the dimensions of the subspaces are variable. That is because we can always partition Γ into a finite number of invariant subsets corresponding to fixed values of k and $d_1, \ldots, d_k$.

Due to this rigidity, it is often possible to exclude a priori the existence of dominated splitting, using topological arguments. One such situation will appear near the end of paragraph **8**, another in paragraph **18**.

A global picture for generic conservative maps

6. $\mathrm{Diff}^1_\mu(M)$ denotes the space of volume preserving C^1 diffeomorphisms on M, endowed with the C^1 topology. Our first main result is

Theorem 1. *For any compact manifold M, there exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1_\mu(M)$ such that, for every $f \in \mathcal{R}$, the Oseledets splitting is dominated or else trivial, at almost every point.*

Later we shall explain how Theorem 1 may be derived from a statement about continuity of Lyapunov exponents. Right now let us discuss the conclusion of the theorem in more detail.

First, let us consider $f \in \mathcal{R}$ to be ergodic. Then the number and dimensions of the Oseledets subspaces are constant μ-almost everywhere. The theorem says that,

- either all Lyapunov exponents vanish μ-almost everywhere,
- or the Oseledets splitting is dominated, also μ-almost everywhere.

In the latter case, the splitting extends continuously to the whole manifold: the system is projectively uniformly hyperbolic on M.

7. In general, let $\mathcal{O}(d_1, \ldots, d_k)$ be the set of points for which the Oseledets splitting exists and involves k subspaces, with dimensions $d_1, \ldots, d_k$, respectively. In particular $\mathcal{O}(d)$ denotes the set of points whose Oseledets splitting is trivial, that is, whose Lyapunov exponents are all zero.

For $k \geq 2$, let $\mathcal{O}_{\mathrm{dom}}(d_1, \ldots, d_k) \subset \mathcal{O}(d_1, \ldots, d_k)$ be the subset of points where the Oseledets splitting is dominated. Theorem 1 says that

$$\mu\big(\mathcal{O}(d_1, \ldots, d_k) \setminus \mathcal{O}_{\mathrm{dom}}(d_1, \ldots, d_k)\big) = 0$$

for all $k \geq 2$ and any choice of $d_1, \ldots, d_k$. Therefore,

$$M = \mathcal{O}(d) \cup \bigcup_{k \geq 2} \bigcup_{d_1, \ldots, d_k} \mathcal{O}_{\mathrm{dom}}(d_1, \ldots, d_k) \tag{5}$$

up to a zero volume set.

$\mathcal{O}_{\mathrm{dom}}(d_1, \ldots, d_k)$ may be written as an increasing union

$$\mathcal{O}_{\mathrm{dom}}(d_1, \ldots, d_k) = \bigcup_{m=1}^{\infty} \mathcal{O}_m(d_1, \ldots, d_k) \tag{6}$$

where each $\mathcal{O}_m(d_1, \ldots, d_k)$ corresponds to fixing the choice of m in (3). The Oseledets splitting is continuous on every $\mathcal{O}_m(d_1, \ldots, d_k)$, and extends continuously to the closure. In general, these extensions need not coincide on the intersections of the closures.

8. As the reader may easily check, for area preserving diffeomorphisms on surfaces, domination is equivalent to uniform hyperbolicity (in the usual sense, no projectivisation). On the other hand, for a residual subset of C^1

diffeomorphisms on any manifold [1], hyperbolic sets have a sort of automatic ergodicity: either they have zero volume or they coincide with the whole ambient space.

Because of this, in dimension 2 the conclusion of Theorem 1 is simpler:

Theorem 2 ([3], partially based on [10]). *For a residual subset of area preserving C^1 diffeomorphisms on any compact surface, both Lyapunov exponents are zero at almost every point or else the diffeomorphism is Anosov.*

It would be nice to know whether this simpler picture remains true in arbitrary dimension, without assuming ergodicity:

Problem 1. Is there a residual set $\mathcal{R}_1 \subset \mathcal{R}$ for which, in the context of (5)-(6), either $\mathcal{O}(d)$ or some $\mathcal{O}_m(d_1, \ldots, d_k)$ has full volume in M ?

Corollary 1 below gives a partial answer, for symplectic maps.

Also, existence of Anosov diffeomorphisms imposes strong topological restrictions on the manifold. In particular, the second alternative in Theorem 2 is possible only if $M = \mathbb{T}^2$.

Problem 2. Which manifolds support diffeomorphisms having a dominated splitting defined on the whole ambient space ?

Continuity of Lyapunov exponents

9. Let $T_x M = E_x^1 \oplus \cdots \oplus E_x^k$ be the Oseledets splitting of $f \in \mathrm{Diff}^1_\mu(M)$ at some point $x \in M$. Let

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_d(x), \quad d = \dim M,$$

be the Lyapunov exponents of f at x, counted with multiplicity: each $\hat{\lambda}_j(x)$ appears exactly $d_j(x) = \dim E_x^j$ times. Formally,

$$\lambda_i(x) = \hat{\lambda}_j(x) \quad \text{if} \quad 1 \leq i - \big[d_1(x) + \cdots + d_{j-1}(x)\big] \leq d_j(x).$$

This defines measurable functions λ_i, $1 \leq i \leq d$, over a full measure subset of the manifold M.

[1] This subset includes *all* C^2 diffeomorphisms [6]. A corresponding statement for domination is false if $d > 3$: there exist C^∞ diffeomorphisms exhibiting invariant sets supporting a dominated splitting and whose volume is neither zero nor full.

Now we consider the *average Lyapunov exponents* of a diffeomorphism f, given by $\lambda_i(f) = \int \lambda_i(x) d\mu(x)$, for $1 \le i \le d$. This defines functions

$$\lambda_i : \mathrm{Diff}^1_\mu(M) \to \mathbb{R}, \qquad 1 \le i \le d.$$

Note that $\lambda_1(x) + \cdots + \lambda_d(x) = 0$, because f preserves volume. Hence,

$$\lambda_1(f) + \cdots + \lambda_d(f) = 0 \quad \text{for every} \quad f \in \mathrm{Diff}^1_\mu(M). \tag{7}$$

10. Another main result, from which we shall deduce Theorem 1, is

Theorem 3. *Suppose $f_0 \in \mathrm{Diff}^1_\mu(M)$ is a continuity point for the map*

$$\mathrm{Diff}^1_\mu(M) \ni f \mapsto (\lambda_1(f), \ldots, \lambda_d(f)) \in \mathbb{R}^d.$$

Then, at almost every point, the Oseledets splitting of f_0 is either dominated or trivial.

Symplectic diffeomorphisms

11. The previous results extend to the symplectic case. Let M be a compact manifold of dimensional $d = 2l$, endowed with a symplectic form ω. Let $\mathrm{Symp}^1_\omega(M)$ be the space of ω-preserving C^1 diffeomorphisms on M. This is a closed subset of the space $\mathrm{Diff}^1_\mu(M)$ of C^1 diffeomorphisms that preserve the volume measure μ induced by the volume form $\omega^l = \omega \wedge \cdots \wedge \omega$.

Theorem 4. *Theorems 1 and 3 remain true when one replaces $\mathrm{Diff}^1_\mu(M)$ by $\mathrm{Symp}^1_\omega(M)$. Actually, in the symplectic context the conclusion is stronger: instead of domination one gets partial hyperbolicity.*

The present notion of partial hyperbolicity is also stronger than usual, in that the central bundle exhibits only zero Lyapunov exponents, and the dimensions of the stable bundle and the unstable bundle are equal:

12. The Oseledets splitting of a symplectic diffeomorphism has a symmetric structure:

$$T_x M = E^s_x \oplus \cdots \oplus E^1_x \oplus \left[E^0_x \oplus \right] E^{-1}_x \oplus \cdots \oplus E^{-s}_x$$

with $\dim E^i_x = \dim E^{-i}_x$ for $1 \le i \le s$ (the dimension of E^0_x may be zero) and Lyapunov exponents

$$\hat\lambda_s(x) > \cdots > \hat\lambda_1(x) > \Big[\hat\lambda_0(x) = \Big] 0 > \hat\lambda_{-1}(x) > \cdots > \hat\lambda_{-s}(x)$$

satisfying $\hat\lambda_i(x) + \hat\lambda_{-i}(x) = 0$ for $1 \le i \le s$. Let

$$E^+_x = E^s_x \oplus \cdots \oplus E^1_x \quad \text{and} \quad E^-_x = E^{-1}_x \oplus \cdots \oplus E^{-s}_x \,.$$

We say that the Oseledets splitting is *partially hyperbolic* at x if it is dominated at x and Df is uniformly expanding along E^+ and uniformly contracting along E^-:

$$\|Df^{-m}|E^+_y\| \le \tfrac{1}{2} \quad \text{and} \quad \|Df^m|E^-_y\| \le \tfrac{1}{2} \tag{8}$$

for any iterate $y = f^\ell(x)$, $\ell \in \mathbb{Z}$, where $m \in \mathbb{N}$ is uniform over the orbit of x.

Proposition 1. *Let f be a symplectic diffeomorphism. If the Oseledets splitting of f is dominated at a point x then it is partially hyperbolic at x.*

This fact was first observed by Mañé [9]. A proof is given in [1], for $\dim M = 4$, and in [4], for the general case.

13. Theorem 4 has the following interesting consequence, that extends conclusions in paragraph **8**.

Corollary 1. *For a residual subset of* $\mathrm{Symp}^1_\mu(M)$*, either the diffeomorphism is Anosov or almost every point exhibits some Lyapunov exponent equal to zero (with even multiplicity).*

Indeed, for $f \in \mathcal{R}$, the set of points whose Lyapunov exponents are all non-zero is a countable union of hyperbolic sets: the union is over the value of m in (8). Hence, restricting to some residual $\mathcal{R}_1 \subset \mathcal{R}$, either that set has zero measure, or the whole manifold is hyperbolic for f. Recall also that $\dim E^0_x$ is always an even number, in the symplectic case.

The semi-continuity argument

14. We are going to deduce Theorem 1 from Theorem 3. For each $1 \le i \le d$, let

$$\Lambda_i(f) = \lambda_1(f) + \cdots + \lambda_i(f)\,.$$

The relation (7) means that $\Lambda_d(f) \equiv 0$. Clearly, $f \mapsto (\lambda_1(f), \ldots, \lambda_d(f))$ is continuous at f_0 if and only if $f \mapsto (\Lambda_1(f), \ldots, \Lambda_{d-1}(f))$ is continuous at f_0.

Proposition 2. *Every $f \mapsto \Lambda_i(f)$, $1 \le i \le d-1$, is upper semi-continuous, both on $\mathrm{Diff}^1_\mu(M)$ and on $\mathrm{Symp}^1_\omega(M)$.*

This proposition is proved as follows. For the largest Lyapunov exponent $\Lambda_1 = \lambda_1$ one uses the relation

$$\lambda_1(f) = \inf_{n\ge 1} \frac{1}{n} \int \log \|Df^n\| \, d\mu \,. \tag{9}$$

The infimum of continuous functions being upper semi-continuous, the conclusion follows.

For $i \ge 2$, one considers the vector bundle V^i over M whose fiber V^i_x is the space of i-forms on $(T_xM)^*$. The derivative Df induces a fibered map $Df^{\wedge i} : V^i \to V^i$, and the largest Lyapunov exponent of $Df^{\wedge i}$ is precisely $\Lambda_i(f)$ (see [8]). Thus we have the relation corresponding to (9) for $Df^{\wedge i}$:

$$\Lambda_i(f) = \inf_{n\ge 1} \frac{1}{n} \int \log \|(Df^{\wedge i})^n\| \, d\mu \,, \tag{10}$$

and semi-continuity follows as before.

Remark 1. A natural choice of a norm in (10) is such that the quantity $\|(Df^{\wedge i}(x))^n\|$ is the supremum of the i-volume of $Df^n(x)(P)$ over i-parallelepipeds $P \subset T_xM$ of unit i-volume. Of course, any other norm would work as well.

Theorem 1 is now an immediate consequence of Theorem 3, Proposition 2, and the well-known fact that the set of continuity points of any semi-continuous function defined on a Baire space contains a residual subset.

The perturbation strategy

15. The proof of Theorem 3 is quite long. Here we only give a glimpse of the strategy to reduce Lyapunov exponents along finite pieces of orbits.

Recall that the Oseledets splitting is dominated at x if condition (3) is satisfied. It is not hard to see that this condition can be reformulated as follows: There exists $m \in \mathbb{N}$ such that for every iterate $y = f^\ell(x)$, $\ell \in \mathbb{Z}$,

$$\|Df^m(y)v\| \ge 2\, \|Df^m(y)w\| \tag{11}$$

for all norm 1 vectors $v \in E^1_y \oplus \cdots \oplus E^i_y$ and $w \in E^{i+1}_y \oplus \cdots \oplus E^k_y$, and all $1 \le i \le k-1$.

Now suppose the Oseledets splitting is not dominated, over a positive measure set of orbits: for some i and for arbitrarily large m there exist iterates y for which (11) does not hold. The basic idea is to take advantage of this fact to, by a small perturbation of the map, cause a vector originally in $E_y^{1,i} = E_y^1 \oplus \cdots \oplus E_y^i$ to move to $E_{f^m(y)}^{i+1,k} = E_{f^m(y)}^{i+1} \oplus \cdots \oplus E_{f^m(y)}^k$, thus "blending" different expansion rates.

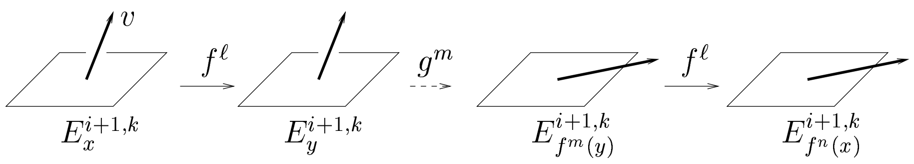

Figure 2: Blending expansion rates

More precisely, given a perturbation size $\epsilon > 0$ one chooses m sufficiently large with respect to ϵ. Then for $n \gg m$ one chooses a convenient $y = f^\ell(x)$, with $\ell \approx n/2$, where domination fails. By composing Df with small rotations near, at most, m iterates of y, one causes the orbit of some $v \in E^{1,i}$ to move to $E^{i+1,k}$. See Figure 2. That is, one constructs a perturbed map g preserving the orbit segment $\{x, \ldots, f^n(x)\}$ such that $Dg^n(x)v \in E_{f^n(x)}^{i+1,k}$. As a result, it is possible to show that

$$\|(Dg^{\wedge j}(x))^n\| \lessapprox \exp\left[n\left(\Lambda_{j-1} + \frac{\lambda_j + \lambda_{j+1}}{2}\right)\right] \ll \exp(n\Lambda_j),$$

where $j = \dim E^{1,i}$.

This local procedure is then repeated for several points x. Using (10) and a tower argument, one proves that $\Lambda_j(f)$ drops under such arbitrarily small perturbations, contradicting continuity.

Linear cocycles over transformations

16. The previous methods also extend to products of deterministic continuous matrices. In this setting one considers a measurable invertible transformation $f : M \to M$ on a compact space M, preserving a Borel probability measure μ. The system should be *aperiodic*, meaning that the set of periodic points of f should have zero μ-measure. For simplicity, here we also take the system to be ergodic.

Then one considers the space of all continuous maps $A : M \to G$, where G is a convenient matrix group. In all that follows we may take $G = SL(d, \mathbb{R})$,

$GL(d,\mathbb{R})$, $Sp(d)$, ... , and we may also replace $\mathbb{R}$ by $\mathbb{C}$ as the field of coefficients. Associated to each A, one considers the *cocycle*

$$F : M \times \mathbb{R}^d \to M \times \mathbb{R}^d, \quad F(x,v) = \big(f(x), A(x)v\big).$$

Note that $F^n(x,v) = \big(f^n(x), A^n(x)v\big)$, where

$$A^n(x) = A(f^{n-1}(x)) \cdots A(f(x))\, A(x) \quad \text{and} \quad A^{-n}(x) = \left[A^n(f^{-n}(x))\right]^{-1},$$

for $n \geq 1$. As before, Oseledets splittings $\{x\} \times \mathbb{R}^d = E_x^1 \oplus \cdots \oplus E_x^k$ and Lyapunov exponents

$$\hat{\lambda}_j(x) = \lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(x)v_j\|, \quad v_j \in E_x^j \setminus \{0\},$$

are well-defined μ-almost everywhere.

17. We prove that for a residual subset of continuous maps $A : M \to G$, either the Oseledets splitting is dominated, and hence continuous, over the whole space M, or the Lyapunov exponents are all equal ($k = 1$) at μ-almost every point. As before, this follows from proving that average Lyapunov exponents can be simultaneously continuous at some A_0 only if the Oseledets splitting of the cocycle associated to A_0 is either dominated or trivial.

The converse is also true: if the Oseledets splitting of A_0 is trivial or dominated then A_0 is a continuity point for all Lyapunov exponents. The proof has two main ingredients. The first one is the robustness of dominated (or trivial) splittings: every C^0 nearby map A has an invariant splitting into subspaces with the same dimensions and uniformly close to the Oseledets subspaces of A_0. The second one is an semi-continuity argument within each of these subspaces, to show that its Lyapunov exponents are close to the corresponding exponents of A_0.

It is interesting to put this conclusion together with a theorem of Ruelle [12] stating that if the Oseledets splitting of A_0 is dominated, and the Oseledets subspaces all have dimension 1, then the Lyapunov exponents vary *smoothly* with the matrix map A near A_0.

Oseledets subspaces of C^0 generic cocycles need not be 1-dimensional:

Example 1. Let $f : M \to M$ have a fixed point p. Let $A_0 : M \to SL(3,\mathbb{R})$ be constant, with one real and two complex conjugate eigenvalues. Assuming the norms of the eigenvalues are not all equal, the Oseledets splitting of A_0

is the (constant) splitting into eigenspaces, and it is dominated. Then every A in a C^0 neighborhood $\mathcal{U}$ of A_0 admits a dominated splitting $F^1 \oplus F^2$, with $\dim F^i_x = i$. In particular, the Lyapunov exponents of A can not be all equal. So, for every $A \in \mathcal{U} \cap \mathcal{R}$ the Oseledets splitting must be dominated, and it must be a refinement of $F^1 \oplus F^2$. Now, assuming $\mathcal{U}$ is sufficiently small, F^2 admits no continuous invariant proper subbundle. That is because $A(p)$ has complex eigenvalues along F^2_p. Hence, if $A \in \mathcal{U} \cap \mathcal{R}$ then its Oseledets splitting must coincide with $F^1 \oplus F^2$ almost everywhere.

But [5] shows that, for appropriate choices of (f, μ), the majority of *Hölder continuous* maps $A \in \mathcal{U}$ (an open dense subset, whose complement has infinite codimension) do have all their Oseledets subspaces with dimension 1.

18. Here is one situation where simple topological reasons prevent the existence of dominated splittings, for a whole C^0 open set of cocycles:

Example 2. Let $f : S^1 \to S^1$ be a homeomorphism and $\alpha : S^1 \to S^1$ be a continuous map with non-zero degree. Let $G = SL(2, \mathbb{R})$. Fix any $B \in G$ and define $A : S^1 \to G$ by $A(x) = BR_{\alpha(x)}$, where R_α is the rotation of angle α. Then the cocycle associated to any map $M \to G$ in a C^0 neighborhood of A admits no invariant continuous subbundle, let alone a dominated splitting.

A simple proof goes as follows. Let $\xi_0 : S^1 \to \mathbb{P}^1$ be any continuous section in the real projective space $\mathbb{P}^1$, and $\xi_1 : S^1 \to \mathbb{P}^1$ be its push-forward:

$$\xi_1(x) = A\big(f^{-1}(x)\big)\xi_0\big(f^{-1}(x)\big) = BR_{\alpha(f^{-1}(x))}\xi_0\big(f^{-1}(x)\big).$$

Then $\deg(\xi_1) = 2\deg(\alpha) + \deg(\xi_0)$ (the factor 2 comes from the fact that we are considering maps to $\mathbb{P}^1$ instead of S^1). This implies that $\xi_1 \neq \xi_0$, hence ξ_0 can not be invariant. These topological arguments extend immediately to a C^0 neighborhood of A.

In spite of this, non-zero Lyapunov exponents may occur in this setting. For example, if f is an irrational rotation, $\alpha = \mathrm{id}$, and $\|B\| > 1$ then, by Herman's subharmonicity argument ([7], see also [2]), the cocycle associated to A has a positive exponent.

References

[1] M.–C. Arnaud, *The generic symplectic C^1 diffeomorphisms of 4-dimensional symplectic manifolds are hyperbolic, partially hyperbolic, or have a completely elliptic point*, preprint Orsay, 2000.

[2] A. Avila, J. Bochi, *A formula with some applications to the theory of Lyapunov exponents*, www.preprint.impa.br, 2001.

[3] J. Bochi, *Genericity of zero Lyapunov exponents*, preprint IMPA 2000, available at www.preprint.impa.br.

[4] J. Bochi, M. Viana, *A sharp dichotomy for conservative systems: zero Lyapunov exponents or projective hyperbolicity*, in preparation.

[5] C. Bonatti, M. Viana, *Lyapunov exponents with multiplicity* 1 *for deterministic products of matrices*, www.preprint.impa.br, 2001.

[6] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lect. Notes in Math. 470 (1975), Springer Verlag.

[7] M. Herman, *Une méthode pour minorer les exposants de Lyapounov et quelques exemples montrant le caractère local d'un théorème d'Arnold et de Moser sur le tore de dimension 2*, Comment. Math. Helvetici 58 (1983), 453–502.

[8] F. Ledrappier, *Quelques propriétés des exposants caractéristiques*, École d'Été de Probabilités de Saint-Flour XII–1982, Lect. Notes in Math. 1097 (1984), Springer Verlag.

[9] R. Mañé, *Oseledec's theorem from the generic viewpoint*, Procs. Intern. Congress Math., Warszawa (1983), vol 2, 1259-1276.

[10] R. Mañé, *The Lyapunov exponents of generic area preserving diffeomorphisms*, Procs. Intern. Conf. Dynam. Syst., Montevideo, Pitman RNMS 362 (1996), 110–119.

[11] V. Oseledets, *A multiplicative ergodic theorem*, Trudy Moscow Math. Obs. 19 (1968), 179–210.

[12] D. Ruelle, *Analyticity properties of the characteristic exponents of random matrix products*, Adv. Math. 32 (1979), 68–80.

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