Huygens' Principle, Dirac Operators, and Rational Solutions of the AKNS Hierarchy

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Abstract

We prove that rational solutions of the AKNS hierarchy of the form $q = \sigma/\tau$ and $r = \rho/\tau$, where (σ, τ, ρ) are certain Schur functions, naturally yield Dirac operators of strict Huygens' type, i.e., the support of their fundamental solutions is precisely the surface of the light cone. This generalizes to the rational solutions of the AKNS hierarchy a result of Lagnese and Stellmacher that concerns the rational solutions of the KdV hierarchy.

1 Introduction

Music, and other familiar features of signal propagation, as we know it in our experience would be very different if we lived in two spatial dimensions. Indeed, one of the striking characteristics of the fundamental solutions to the wave equation

$$\Box\psi \stackrel{\text{def}}{=} \left(\partial_0^2 - \sum_{i=1}^{n} \partial_i^2\right)\psi = \delta_y \ , \tag{1}$$

in $n = 3$ space dimensions is the fact that the support of the fundamental solution $\psi(\cdot; y)$ is the surface of the light cone with vertex at y, and not the union of this surface with the interior part thereof. This implies that a sharp signal of finite duration will not be "heard or felt" by an observer after a sufficiently long ellapsed time. This amazing property is shared by the wave equation for $n = 3, 5, 7, \cdots$ but is not shared for n even or $n = 1$.

J. Hadamard in his celebrated Yale Lectures [30] called the above property *Huygens' minor premise.* He further studied it for perturbations of the wave operator of the form $\Box + u$, where $u = u(x_0, x_1, \cdots, x_n)$. He characterized such property in terms of the truncation of a certain series expansions of the fundamental solutions. However, such simple conditions are very elusive since they hide complicated nonlinear partial differential equations for the perturbation u. In fact, such conditions were so complicated that for a while it was conjectured that for $n = 3, 5, \cdots$ and modulo certain trivial transformations only the D'Alembertian possessed Huygens' property. By trivial transformations we mean the composition of changes of the independent variable, and left or right multiplication of the of the operator by nonzero functions. For an early account, the reader is refered to Courant & Hilbert's master-piece [16].

It came as a striking surprise the discovery by Stellmacher [51, 52] of Huygens' type wave operators of form $\Box + u$ that were not trivially equivalent to $\Box$. However, as most of the fundamental discoveries in mathematical physics, such finding was only the tip of an iceberg. It turned out the potentials found by Stellmacher were actually the first ones of a full family of potentials later explored by Lagnese and Stellmacher [38] that yield Huygens' type operators. The construction of such family on the other hand, was accomplished through a technique, namely Darboux transformations, which was independently used in soliton theory to construct the rational solutions of the Korteweg-de Vries (KdV) equation:

$$\dot{V} = V_{xxx} - 6VV_x \ .$$

The simple fact that the KdV equation possesses large families of solutions that remain rational by its flow is already a deep one which is directly linked with several other miracles that such equation displays. For an early account see [3, 2], whereas a modern approach can be found in [56]. In fact, it turned out that the findings of Lagnese and Stellmacher remained unknown to the integrable system community for a while and were only pointed out much later by R. Schmming [47].

Given the beauty and the importance of Huygens' property for the meaningful transmission of information, it is natural to ask the question: Which other hyperbolic equations of physical relevance share such a property? This has been the subject of attention by several authors, see for example [29, 34, 33, 4] and references therein.

The purpose of the present work is to consider Huygens' property for Dirac operators. Dirac operators are fundamental objects in the relativistic electron theory. More precisely, we consider the fundamental solutions of Dirac operators $\mathcal{D}\Psi = \delta_y$, where as defined in Section 3,

$$\mathcal{D} = \sum_{\mu} \gamma^{\mu}\partial_{\mu} + v \ . \tag{2}$$

We say that $\mathcal{D}$ satisfies *Huygens' property* (in Hadamard's strict sense) or is of *Huygens' type* if for every y the domain of dependence of the distribution $\Psi(\cdot; y)$ is precisely the light cone with vertex in y.

Our main result is that by taking (q, r) certain fields that are also rational solutions of the AKNS hierarchy of nonlinear equations and setting

$$\mathcal{D} = \partial\!\!\!/ - \frac{1}{2}(q+r)I + \frac{1}{2}(q-r)\bar{\gamma} \ ,$$

where I is the identity, $\bar{\gamma}$ is a certain matrix and $\partial\!\!\!/$ is the free-space Dirac operator in an even-dimensional space-time of sufficiently high order, the operator $\mathcal{D}$ is of Huygens' type. Such fields q and r have explicit expression in terms of Schur polynomials.

The AKNS hierarchy of nonlinear evolution equations is an infinite family of commuting Hamiltonian flows generated by zero curvature conditions of certain matrix systems according to the Lax pair formalism. Furthermore, this hierarchy is a completely integrable system and generalizes several important integrable equations including the KdV hierarchy [1]. It includes as particular cases the cubic nonlinear Schrödinger equation, the modified KdV equation, and others.

Our result extends to the context of Dirac operators and the rational solutions of the AKNS hierarchy the result of Lagnese-Stellmacher for the wave operator and the rational solutions of the KdV hierarchy. This extension turns out to be in a highly nontrivial direction since the operators $\mathcal{D}$ so obtained when squared do not yield (in general) diagonal operators. However, for very particular choices of the parameters that enter the fields q and r we obtain as consequence of our result the already known result of Lagnese and Stellmacher.

The chain of unexpected connections does not stop in the link between the rational solutions of the AKNS hierarchy and Dirac operators of Huygens' type. They also manifest themselves in the so-called bispectral property which was introduced by Grünbaum, and analyzed by Duistermaat and Grünbaum in [19]. It turns out that the fields (q, r) that yield Dirac operators of Huygens' type also yield bispectral AKNS operators. This is the subject of Section 2.4.

The plan for this article goes as follows:

We conclude this introduction with an account on fundamental solutions of wave operators and Hadamard's series. This will set up the notation and basic results that will be needed further in the paper. We also review some earlier results for wave operators.

Section 2 is concerned with infinite dimensional integrable systems, in particular with the AKNS hierarchy, which extends and generalizes the KdV and mKdV hierarchies.

Section 3 is concerned with Dirac operators. Section 4 deals with the analytic properties of such operators that are needed in Section 5, which contains the main result mentioned above.

We conclude in Section 6 with a few further remarks about the results of Lagnese and Stellmacher, some examples, and some suggestions for further research.

1.1 Huygens' Principle for Wave Operators

The remaining sections of the present introduction have the objective of serving as a review of earlier results and to fix the notation and definitions that will be used in the remaining of the work.

Let us start by considering a fundamental solution to the wave equation

$$\Box\psi + u\psi = \delta_y \ . \tag{3}$$

Here as before, $\Box = \partial_0^2 - \sum_{i=1}^n \partial_i^2$ and $u = u(x^0, \cdots, x^n)$ is a given potential in n spatial dimensions and time x^0.

We consider the question posed by Hadamard: For which potentials is the support of $\psi(\cdot; y)$ at an arbitrary point $y = (y^0, \cdots, y^n)$ only the light cone with vertex at y? This requirement leads to severe nonlinear restrictions on the potential u. Hadamard's question has been the subject of intense interest. An early account can be found in [16], whereas for the more recent results and direct connections with soliton theory see [5].

1.2 Fundamental Solutions for the Wave Operator

In this subsection we start by sketching the proof that $\Box$ is a Huygens operator if, and only if, n is odd and greater than 1. This will help make the present work self-contained while we introduce key concepts such as Riesz kernels, which will be instrumental in the sequel. Our approach here follows closely Folland's work [24].

Let λ denote the "time-like geodesic distance"

$$\lambda = \begin{cases} \sqrt{(x^0)^2 - \sum_{i=1}^n (x^i)^2}, & \text{if } (x^0)^2 - \sum_{i=1}^n (x^i)^2 \geq 0 \ , \\ 0 & \text{otherwise.} \end{cases}$$

For values of α with positive real part we define the Riesz kernel[1] as the distribution

$$\Lambda^\alpha = N(\alpha)\lambda^\alpha \ , \tag{4}$$

where the numerical normalization factor is given by

$$N(\alpha) = \left[2^{\alpha+n}\pi^{(n-1)/2}\Gamma(\frac{\alpha+n+1}{2})\Gamma(\frac{1}{2}\alpha+1)\right]^{-1} \ . \tag{5}$$

[1]Note that α is an exponent in λ and an index in Λ.

The numerical factor $N(\alpha)$ obeys the important recursion rule

$$(\alpha+2)(\alpha+n+1)N(\alpha+2) = N(\alpha) \ . \tag{6}$$

If the real part of α is sufficiently large, with the help of preceding formula, one proves [24]

$$\Box\Lambda^{\alpha+2} = \Lambda^{\alpha} \ .$$

So, we can extend the definition of Riesz kernels for all complex values of α.

From equation (5) we see that $N(\alpha)=0$ when $\alpha=-2,-4,-6,\cdots$ or $\alpha=-n-1,-n-3,-n-5,\cdots$. In these cases Λ^{α} is supported on the light cone with vertex in the origin.

By an analytic continuation of Λ^{α} to the region $\Re(\alpha)>-n-2$, one finds that $\Lambda^{-n-1}=\delta_0$, so Λ^{-n+1} is the fundamental solution of $\Box$. Λ^{α} is an analytic family of distributions, i.e., for every fixed φ smooth and compactly supported, $\varphi\in C_0^{\infty}$, the function $\alpha\mapsto\int\Lambda^{\alpha}\varphi$ is analytic.

From the aforementioned values of $N(\alpha)$ we conclude that the fundamental solution of $\Box$ is supported on the light cone (and, consequently, $\Box$ has the Huygens property) if, and only if, n is odd and greater than 1.

For a fundamental solution with the Dirac-delta measure supported at an arbitrary $y\neq 0$, we just need to translate the solution, since the D'Alembertian is invariant under translation.

1.3 Hadamard's Series

In order to construct the fundamental solution Φ of the operator $\Box+u$, we shall develop a series expansion in the above defined Riesz kernels,

$$\Phi = \sum_{k=0}^{\infty} \Lambda^{-n+1+2k} w_k \ . \tag{7}$$

Applying the operator $\Box+u$ to Φ and equating to $\Lambda^{-n-1}=\delta_y$ we find the following recursion:

$$\begin{aligned} w_0 &= 1 \ , \\ w_k + \frac{1}{k}\sum_{\mu=0}^{n}(x^{\mu}-y^{\mu})\partial_{\mu} w_k &= -(\Box+u)w_{k-1} \ . \end{aligned} \tag{8}$$

This recursion is known as Hadamard's recursion. Its termination at k, i.e., $w_{k'}=0$, for every $k'>k$, is equivalent to the validity of Huygens' principle for $n=2k+3$, as can be readily seen from the Riesz kernels properties stated in the preceding section. The termination requirement implies a highly nonlinear condition over u, which makes it very difficult to find examples of Huygens' potentials.

1.4 Lagnese and Stellmacher

We shall now quickly recall some results from the work of Lagnese and Stellmacher [38]. Following an idea originated in the study of geometry of surfaces by G. Darboux [17], we consider the factorization

$$\partial_t^2 = \left(\partial_t + \frac{1}{t}\right)\left(\partial_t - \frac{1}{t}\right) .$$

Now, swap the factors, to find a new operator in the variable t:

$$\left(\partial_t - \frac{1}{t}\right)\left(\partial_t + \frac{1}{t}\right) = \partial_t^2 - \frac{2}{t^2} .$$

The central idea is to show that the operator with the above time dependence on the RHS is also of Huygens' type for odd $n \geq 5$. This can be seen from its fundamental solution

$$\left(\Box - \frac{2}{t^2}\right)\left(\Lambda^{-n+1} + \frac{2}{tt_0}\Lambda^{-n+3}\right) = \delta_y .$$

It is easy to see that this process can be carried on in order to find a full hierarchy of Huygens operators. More explicitly, if we have a Huygens operator $\mathcal{L}$ and find a pair of operators $\tilde{\mathcal{L}}$ and ℓ (called a *intertwining* operator) such that the relation

$$\ell\mathcal{L} = \tilde{\mathcal{L}}\ell$$

holds, then $\tilde{\mathcal{L}}$ is a Huygens operator. Lagnese and Stellmacher, in [38], specialized to the study of operators of the form

$$\mathcal{L} = \Box + c(t) = \partial_t^2 + c(t) - \Delta ,$$

where $c(t)$ is continuous in some interval I. We choose $\mu(t)$ as one of the non-zero solutions[2] of

$$\mu_{tt} + c(t)\mu = 0$$

on an interval $I' \subset I$ and define the linear operators

$$l = \partial_t - \frac{\mu_t}{\mu} , \qquad l^* = -\partial_t - \frac{\mu_t}{\mu} ,$$

where l^* is the formal adjoint of l and the sub-index t denotes derivative with respect to the variable t. We write

$$\mathcal{L} = -\Delta - l^*l .$$

As l commutes with Δ we find that

$$l\mathcal{L} = \tilde{\mathcal{L}}l ,$$

where

$$\tilde{\mathcal{L}} = -\Delta - ll^* .$$

[2]There are two linearly independent solutions, but as this recursive procedure is carried on, one of the solutions comes for free from the previous equation. We are interested in the other solution.

We easily see that

$$ll^* = -\partial_t^2 - \frac{\mu_{tt}}{\mu} + \frac{2\mu_t^2}{\mu^2} \ .$$

We define

$$\tilde{c}(t) = \frac{\mu_{tt}}{\mu} - \frac{2\mu_t^2}{\mu^2} \ ,$$

and the new operator can be written

$$\tilde{\mathcal{L}} = \Box + \tilde{c}(t) \ .$$

The process of decomposing an operator into a product of two differential operators and interchanging the order, producing a new one, is called *Darboux transformation*. See [61].

The repetition of this process produces a family of Huygens' potentials,

$$\Box - u_k(x^0) \ . \tag{9}$$

Lagnese and Stellmacher proved in [38] that each iteration of the Darboux transformation increases by two the minimum dimension where the above operator has the Huygens' property. Actually, the operator defined in equation (9) has the Huygens' property if, and only if, n is odd and $n \geq 2k+3$.

The first members of the family, modulo trivial transformations, are

$$\begin{aligned}
u_0 &= 0 \ , \\
u_1 &= \frac{2}{t^2} \ , \\
u_2 &= \frac{6t(2+t^3)}{(1-t^3)^2} \ , \\
u_3 &= \frac{2[1+18\kappa t - 3\kappa^2 t(50-75t^3-2t^9)]}{[t+\kappa(5+5t^3-t^6)]^2} \ ,
\end{aligned}$$

where u_3 has a free parameter κ.

It seems that R. Schimming was the first to remark [47, 48] that these potentials are exactly the same ones found by Adler and Moser [2] in the framework of the rational solutions of the construction of the KdV hierarchy. By rational solutions of the KdV hierarchy we mean those rational functions (in x) that remain rational by the flow of the KdV hierarchy equations.

The operators $\Box - u_k$ are trivially equivalent to operators with potentials given by

$$u_k(t) = -2\frac{d^2}{dt^2}\log\vartheta_k(t) \ , \tag{10}$$

where ϑ_k are known as the k-th Adler-Moser polynomials[3]. They satisfy the following recursion rule

$$\vartheta'_{k+1}\vartheta_{k-1} - \vartheta'_{k-1}\vartheta_{k+1} = (2k+1)\vartheta_k^2 \ , \quad \vartheta_0(t) = 1 \ , \ \vartheta_1(t) = t \ . \tag{11}$$

[3]The first reference we know of such polynomials seems to be in the work of J. Burchnall and T. Chaudy [9].

They have degree $n_k = \frac{1}{2}k(k+1)$. At each step, a new integration constant appears. We fix this constant by the normalization that the coefficient of $t^{n_{k-1}}$ in ϑ_{k+1} is equal to χ_{k+1}. Furthermore, $t = \chi_1$.

For the first few polynomials one finds

$$\begin{aligned}
\vartheta_0 &= 1 \ , \\
\vartheta_1 &= t \ , \\
\vartheta_2 &= t^3 + \chi_2 \ , \\
\vartheta_3 &= t^6 + 5\chi_2 t^3 + \chi_3 t - 5\chi_2^2 \ .
\end{aligned}$$

Changing the variable t for x in each u_k and endowing it with the correct time dependence, u_k given by (10) becomes a rational solutions of the KdV equation

$$\dot{u} = u_{xxx} - 6uu_x \ .$$

This is an important and unexpected link between integrability theory and Huygens' principle. It is subject of extensive investigation. See, for example, [6] and references therein. For a generalization of the above result to non-Minkowskian metric, see [34].

2 Preliminaries on Integrable Systems

In this section we shall briefly review some of the relevant material on infinite dimensional integrable systems that will be used in the text. More specifically we shall concentrate on the so-called Ablowitz-Kaup-Newell-Segur hierarchy of nonlinear evolution equations. Our approach will follow closely the construction of [42]. For other viewpoints see [21, 12, 11].

2.1 The AKNS Hierarchy

The theory of solitons is extremely rich, with connections to a vast number of seemingly unrelated fields. One of its first developments was the remark that the Korteweg-de Vries equation could be interpreted as a compatibility condition for certain linear differential equations, one of which was the Schrödinger equation. For a historical overview the reader should consult [42, 10, 41] and references therein.

A crucial step in the development of soliton theory resulted from the remark that, besides the KdV, a number of other important equations in Mathematical Physics can be expressed as compatibility conditions of linear systems of the form [1, 57]

$$\partial_x \Psi = (kH + qE + rF)\Psi \stackrel{\text{def}}{=} (kH + Q)\Psi \tag{12}$$

and

$$\partial_{t_\ell} \Psi = (k^\ell H + R_1 k^{\ell-1} + \cdots + R_\ell)\Psi \stackrel{\text{def}}{=} R^{(\ell)}\Psi \ , \tag{13}$$

where H, E, and F are given by

$$H = \mathrm{diag}[1, -1]$$

and

$$E = F^\top = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} .$$

If we take $\ell = 1$, we get $\partial_{t_1} q = \partial_x q$ and $\partial_{t_1} r = \partial_x r$, which are trivially solved in terms of translations of q and r. This explains the usual convention of identifying the variable t_1 with x. The first interesting example occurs if we take $\ell = 2$. In this case, the compatibility condition of (12) and (13) implies $R_1 = qE + rF$, $R_2 = -(qrH - q_x E + r_x F)/2$ and

$$\left\{ \begin{array}{rcl} q_{t_2} & = & \frac{1}{2}(q_{xx} - 2q^2 r) \\ r_{t_2} & = & -\frac{1}{2}(r_{xx} - 2r^2 q) \ . \end{array} \right. \tag{14}$$

If we specialize to $q = \pm r^*$, and change the time t to it, we obtain the nonlinear Schrödinger equation. The other equations of the AKNS hierarchy can be found by considering the compatibility conditions of (12) and (13)

$$\partial_{t_\ell} Q - \partial_x R^{(\ell)} + [kH + Q, R^{(\ell)}] = 0 \ , \tag{15}$$

viewed as an identity in k. It can be shown that [42, 23]:

i For each ℓ, there is a sequence $R_1, \ldots, R_\ell$ of matrices whose entries are polynomials in q, r and their derivatives, such that (15) reduces only to the equation associated to the coefficient of k^0.

ii The above mentioned sequence $R_1, \ldots, R_\ell$ is uniquely determined once we assign weight $i+1$ to $\partial_x^i q$ and $\partial_x^i r$ and require R_i to be homogeneous of weight i. Moreover, the sequence of matrix polynomials $\{R^{(\ell)}\}_{\ell=1}^\infty$ so obtained satisfies

$$R^{(\ell+1)} = kR^{(\ell)} + R_{\ell+1} \ .$$

iii The ℓ-th flow of the AKNS hierarchy is the system obtained by comparing the coefficient of k^0 in (15)

$$\partial_{t_\ell} Q - \partial_x R_\ell + [Q, R_\ell] \ . \tag{16}$$

This corresponds to a system of two partial differential equations of the form

$$\left\{ \begin{array}{rcl} q_{t_\ell} & = & A_\ell(q, ..., \partial_x^\ell q; r, ..., \partial_x^\ell r) \ , \\ r_{t_\ell} & = & B_\ell(q, ..., \partial_x^\ell q; r, ..., \partial_x^\ell r) \ . \end{array} \right. \tag{17}$$

iv The flows generated by the nonlinear evolution equations (17) are Hamiltonian and commute with one another.

As an example of flow generated by the hierarchy for $\ell = 3$ we have

$$\begin{cases} q_{t_3} &= \frac{1}{4}(q_{xxx} - 6qrq_x) \ , \\ r_{t_3} &= \frac{1}{4}(r_{xxx} - 6qrr_x) \ . \end{cases}$$

It is immediate to check that $q = \pm r$ gives the modified KdV equation. Furthermore, such reduction is compatible with all the odd flows of the AKNS hierarchy.

We recall a few more facts about the AKNS hierarchy, which will be used in the sequel. We write the coefficient R_l of equation (12) as

$$R_l \stackrel{\text{def}}{=} e_l E + f_l F + h_l H \tag{18}$$

and take $e_0 = f_0 = h_1 = 0$, $h_0 = 1$, $e_1 = q$, and $f_1 = r$. Then, equation (15) is equivalent to

$$e_{l+1} = qh_l + \frac{1}{2}\partial_x e_l \ , \tag{19}$$

$$f_{l+1} = rh_l - \frac{1}{2}\partial_x f_l \ , \tag{20}$$

$$\partial_x h_{l+1} = f_{l+1}q - e_{l+1}r \ , \tag{21}$$

with $0 \leq l \leq n-1$, together with

$$\partial_{t_l} q = \partial_x e_l + 2qh_l \ , \tag{22}$$

$$\partial_{t_l} r = \partial_x f_l - 2rh_l \ . \tag{23}$$

The coefficients e_j, h_j and f_j can now be viewed as polynomials in the variables $\partial_x^l q$ and $\partial_x^l r$, with $l \in \mathbb{Z}_{\geq 0}$. We now recall a result from [60], whose proof will be ommited here:

Lemma 1 ([60]). *For $1 \leq j \leq l-1$, h_{j+1} defined by*

$$h_{j+1} = -\frac{1}{2} \sum_{m+n=j+1 \ m,n\geq 1} e_n f_m + h_n h_m \tag{24}$$

satisfies equation (21) with $l = j$. Moreover, this choice of h_{j+1} is homogeneous of weight $j+1$ if we give $\partial_x^l q$ and $\partial_x^l r$ weight $l+1$.

As a consequence of this lemma, it follows that the AKNS equations can be written down using the recursive formulae (19), (20) with $l = j$ and (24). Furthermore, if we define e_l, f_l, h_l for all values of $l \in \mathbb{Z}_{>0}$, the ℓ-th equation in the AKNS hierarchy can be written as

$$\begin{cases} q_{t_\ell} &= 2e_{\ell+1}(q,\ldots,\partial_x^\ell q; r,\ldots,\partial_x^{\ell-1} r) \\ r_{t_\ell} &= -2f_{\ell+1}(q,\ldots,\partial_x^{\ell-1} q; r,\ldots,\partial_x^\ell r) \ . \end{cases} \tag{25}$$

2.2 Hirota Variables and Schlesinger Transformations

One way of constructing large classes of solutions to the AKNS hierarchy is by means of Schlesinger transformations [42, 22, 36, 35]. In very broad terms, the key idea is to introduce a transformation on the common solution Ψ of equations (12) and (13) of the form $\widetilde{\Psi} = S\Psi$. Such transformation induces a change on the matrices Q and $R^{(\ell)}$ that preserves the form of equation (15). One example is given by

$$\widetilde{R}^{(\ell)} = SR^{(\ell)}S^{-1} + S_{t_\ell}S^{-1} \ . \tag{26}$$

Furthermore, if $\det(S)$ is independent of the AKNS time variables, then the trace of the matrices Q and $R^{(\ell)}$ are preserved. Hence, if we start with Q and $R^{(\ell)}$ of the form given in equation (18) the transformed variables are also of such form. This in turn implies that the transformed fields, which we will denote by $(\widetilde{q}, \widetilde{r})$, satisfy equations of the form (25). In particular, we have

$$\widetilde{R}^{(1)} = kH + \widetilde{q}E + \widetilde{r}F \ .$$

We now recall the notion of Hirota variables for the AKNS hierarchy [42, 32]. Let (q, r) be a solution of all the flows of the AKNS hierarchy and h_j as in Lemma 1. The τ-function is defined as a solution $\tau(x, t_2, t_3, \cdots)$ of

$$h_{j+1} = \frac{1}{2}\partial_{t_j}\partial_{t_1}\log\tau \ , \tag{27}$$

for $j = 1, 2, 3, \cdots$. This definition, at least formallly, makes sense since one can show that $\partial_{t_l}h_{j+1} = \partial_{t_j}h_{l+1}$. See [42]. It is also convenient to introduce σ and ρ in such a way that

$$q = \frac{\sigma}{\tau} \tag{28}$$

and

$$r = \frac{\rho}{\tau} \ . \tag{29}$$

The above defined variables σ, τ, and ρ are called the *Hirota variables* for the AKNS hierarchy.

In the sequel, we shall make use of the Schlesinger transformations induced by gauge transformations of the form

$$S = \begin{bmatrix} -2k - \partial_{t_1}\log(e_1) & e_1 \\ 1/e_1 & 0 \end{bmatrix} \ .$$

This induces at the level of Hirota variables the transformation [4]

$$S : (\sigma, \tau, \rho) \longmapsto (\sigma_+, \tau_+, \rho_+) \stackrel{\text{def}}{=} (\tau, -\rho, -\frac{\rho_{xx}\rho - \rho_x^2}{\tau}) \ . \tag{30}$$

[4] We remark for benefit of readers of [42] that we have deviated a bit from its notation here. To transform from its variables to ours we have to perform the following changes: $\rho \mapsto \sigma$, $\sigma \mapsto -\rho$, $\tau \mapsto \tau$, $q \mapsto r$, $r \mapsto -q$, $x \mapsto ix$, where the variables on the left are the ones used in [42].

and at the level of fields e_1 and f_1

$$S : (e_1, f_1) \longmapsto \left(\widetilde{e}_1, \widetilde{f}_1\right) = \left(-\frac{1}{f_1}, f_{1\,xx} - \frac{f_{1\,x}^2}{f_1} - f_1^2 e_1\right) \ . \tag{31}$$

Schlesinger transformations will be used in the next section to generate families of rational solutions to the AKNS hierarchy, and will also be instrumental in the proof that the corresponding Dirac operators are of Huygens' type.

2.3 Schur Polynomials and Rational Solutions of AKNS

The study of special solutions to infinite-dimensional integrable systems naturally leads to interesting finite-dimensional ones. As an example we can cite the rational solutions of the Korteweg-de Vries hierarchy, which is directly connected to Calogero-Moser systems through the results of Airault-McKean-Moser [3], or the rational solutions of the Kadomtsev-Petviashvili equations [37].

One first example of solutions of the AKNS Hierarchy, which can be viewed as a motivation to the following construction (due to [43, 44]), can be obtained imposing $q = 0$. In this case, equation (19) can be easily solved in order to get $e_l = 0$ for all l. From Lemma 1, we conclude that $h_l = 0$. With this in mind, we easily conclude from equation (20) that

$$f_{l+1} = \frac{1}{(-2)^l} \partial_x^l f_1 \ .$$

The AKNS hierarchy (25) is now reduced to

$$\begin{cases} q_{t_\ell} &= 0 \ , \\ r_{t_\ell} &= \frac{1}{(-2)^{\ell-1}} \partial_x^\ell r \ . \end{cases} \tag{32}$$

The first equation is trivially fulfilled. The second one is solved by the so called Schur polynomials, which we define right now. We introduce an infinite sequence of polynomials $\{\mathcal{Q}_j(y_1, y_2, \cdots, y_j)\}_{j=0}^{\infty}$ defined by the requirement that

$$\sum_{j\geq 0} \lambda^j \mathcal{Q}_j = \exp\left(\sum_{j\geq 1} \lambda^j y_j\right) \ . \tag{33}$$

For convenience, we extend the definition of Schur polynomials as zero for negative indices. Using this convention it is easy to check that

$$\partial_{y_k} \mathcal{Q}_j = \mathcal{Q}_{j-k} \ . \tag{34}$$

We introduce the Wronskians

$$\tau_j^d = \mathrm{W}[\mathcal{Q}_d, \mathcal{Q}_{d-1}, \cdots, \mathcal{Q}_{d-j}] \ , \tag{35}$$

where the derivatives should be interpreted with respect to the first variable y_1. We set $y_1 = x$ and take

$$q = \frac{\tau^d_{j-1}}{\tau^d_j} \ , \ r = -\frac{\tau^d_{j+1}}{\tau^d_j} \ . \tag{36}$$

For convenience, we extend the definition (35) of τ^d_j as follows:

$$\tau^d_j \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } j = -1 \text{ and } d = 0, 1, \cdots , \\ 0 & \text{if } j < -1 \text{ or } d < 0. \end{cases}$$

In order to relate the variables y_k in the Schur polynomials with the time variables t_k of the AKNS hierarchy, we set

$$y_k = \frac{1}{(-2)^{k-1}} t_k \ , \quad k = 1, 2, \cdots \ . \tag{37}$$

This rescaling of the variables is such that $q = 0$ and $r = -\mathcal{Q}_d$ satisfy equations (32) in the new variables y_k.

We remark that the Hirota variables obey the relation $(\sigma, \tau, \rho) = (\tau^d_{j-1}, \tau^d_j, -\tau^d_{j+1})$. Furthermore, we have the following result:

Theorem 1 (R.Sachs [44]). *For every pair of integers d and j, such that $d \geq j > -1$, the pair of fields (q, r) defined by equations (35), (36), and (37) is a rational solution of the AKNS hierarchy.*

The fact that rational solutions of the AKNS hierarchy can be generated by a construction as in Theorem 1 is by now not surprising. Indeed, Sato & Sato [46] characterized the polynomial tau-functions of the KP hierarchy. A large class of rational solutions of the KP can be obtained from Schur polynomials. See [46, 20]. On the other hand, the AKNS hierarchy can be viewed as a reduction of the KP hierarchy. An alternative proof of Theorem 1 can be found in Section 5 of [60].

We can interprete, following [42] the Schlesinger operator as a shift operator in j in the sequence $\{\tau^d_j\}^{\infty}_{j=-\infty}$ (as can be seen in equation (30)), $S\left[\tau^d_j\right] = \tau^d_{j+1}$. If we consider $j = -1$, then $(q, r) = (0, -\mathcal{Q}_d)$, which is a solution of the AKNS. The effect of each sucessive Schleschinger transformation on the pair (q, r), produces another rational solution for the AKNS hierarchy of the form in equation (36). The new solution $(\widetilde{q}, \widetilde{r})$ is related to the previous one by equation (31), i.e.,

$$(\widetilde{q}, \widetilde{r}) = \left(-\frac{1}{r}, r_{xx} - \frac{r_x^2}{r} - r^2 q \right) \ .$$

The odd flows of the AKNS hierarchy reduce to the mKdV flows if we impose $q = r$. Furthermore, the cellebrated Miura transformation $u = q_x \pm q^2$ sends solutions of the mKdV hierarchy into solutions of the KdV hierarchy. More precisely, in the Appendix we shall show the following result:

Theorem 2. *If $y_{2k} = 0$ for all $k \in \mathbb{N}$, then*

$$\frac{\tau_{j+1}^{2j+1}}{\tau_j^{2j+1}}(x, 0, y_3, 0, y_5, \cdots) = (-1)^{j+1}\partial_x \log \frac{\vartheta_{j+1}}{\vartheta_j}(x, \chi_2, \chi_3, \cdots) \ ,$$

where ϑ_j is the j-th Adler-Moser polynomial and for a suitable choice of the free variables $y_3, y_5, \cdots$ and $\chi_2, \chi_3, \cdots$.

The above result will allow us to give an independent proof of the Lagnese-Stellmacher result without resorting to the use of Darboux transformations.

2.4 Bispectrality

One of the main themes in this article is the connection among Huygens property, rational solutions of certain integrable systems, and bispectrality. We shall now discuss briefly the latter.

We recall that a differential operator $L = L(x, \partial_x)$ is called bispectral if there exists a family of eigenfunctions $\Psi(x, k)$ of L that also satisfies a (positive order) differential equation in the spectral variable k of the form

$$B(k, \partial_k)\Psi = \Theta(x)\Psi \tag{38}$$

where B is independent of x, and $\Theta(x)$ is independent of k. In other words, we have eigenfunctions that are simultaneously solutions of spectral problems in the "physical" and the "spectral" variables.

The motivation for the concept of bispectrality stems from problems in signal analysis such as the time-and-band limiting problem of Slepian [50, 49] and the limited-angle tomography problem [26]. For a historical overview, the reader is refered to [28, 62, 27].

The first surprising remark is that bispectrality is deeply related to integrable systems. Indeed, Duistermaat and Grünbaum [19] classified all the bispectral Schrödinger operators

$$L = -\partial_x^2 + u(x) \ . \tag{39}$$

They showed that besides potentials of the form $u(x) = ax + b$ one can divide the bispectral Schrödinger potentials into two families (modulo translations, addition of a constant, and scalings):

- The rational solutions of the Korteweg-de Vries hierarchy vanishing at infinity. By that we mean the rational potentials (decaying at infinity) that remain rational by the flows of the KdV hierarchy associated to the isospectral deformations of the operator (39).
- The rational potentials obtained by successive iterations of Darboux transformations starting from $u = -1/(4x^2)$. These turned out to be rational solutions of the master-symmetries for the KdV hierarchy [39, 63].

The links between bispectrality and integrable systems were furthered in a number of subsequent works. See [31] for a comprehensive account.

Even for the Schrödinger operator case the ties among bispectrality and Huygens property are far from understood. New connections were found for example in [8]. In any case, it is a striking fact that the rational solutions of the KdV hierarchy give nontrivial families of Huygens operators when added to the wave operator. This result in fact is the consequence of lines of research that were develloped separately [38, 3], and was noticed much later in [47, 48].

Although bispectrality for Schrödinger operators is by now fairly well understood, the higher order case and the matrix operator cases are far from settled despite the results in [56, 55, 59, 58]. For the matrix case, a large family of bispectral AKNS potentials was found by in [60]. The main result was that if one considers the rational fields (q, r) as in equation (36) then the corresponding potentials are (at least generically) bispectral. It turns out that, these fields are precisely the ones that we show, in the present work, correspond to Dirac operators of Huygens type. The upshot being that the flows of the AKNS hierarchy naturally preserve this class of Dirac operators of Huygens type in the same way that it naturally preserves bispectrality for the corresponding AKNS operators.

3 Dirac Operators

Dirac operators play an important role in relativistic quantum mechanics. They appear in Mathematical Physics in the framework of the construction of a relativistic electron theory. In a celebrated paper [18], Dirac defined a Lorentz-covariant Hamiltonian (and consequently first-order in space variables, as Schrödinger equation is first order in time) whose square furnishes the Klein-Gordon equation. One of the main achievements of Dirac's introduction of such equation was the prediction of the positron, which were experimentally confirmed shortly thereafter.

The goal of the present section is to define Dirac operators and discuss some of its properties. In order to do that we start with a brief description of Clifford algebras.

3.1 Clifford Algebras

Let $\{\gamma^\mu\}$, $\mu = 0, \cdots, n$, be a set of elements in an associative algebra over the field of real numbers such that

$$\{\gamma^\mu, \gamma^\nu\} \stackrel{\text{def}}{=} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I} \ , \tag{40}$$

where $(g^{\mu\nu}) = \text{diag}[1, -1, -1, \cdots, -1]$ is the Minkowski tensor in $n+1$ dimensions and $\mathbb{I}$ is the identity algebra and shall be henceforth omitted.

The Clifford algebra generated by these elements is the set of all linear combinations of all products in the form

$$(\gamma^0)^{m_0}(\gamma^1)^{m_1}\cdots(\gamma^n)^{m_n} \ , \tag{41}$$

where $m_\mu = 0$ or $m_\mu = 1$ and when all $m_\mu = 0$ the above product should be interpreted as the identity. If n is odd the Clifford Algebra has maximum dimension 2^{n+1}. So, all the products in the form (41) are linearly independent. See [25, 40].

If n is odd we can define a matrix $\bar{\gamma}$ such that

$$\{\bar{\gamma}, \gamma^\mu\} = 0 \qquad \mu = 0, \cdots, n \tag{42}$$

and

$$\bar{\gamma}^2 = 1 \ . \tag{43}$$

We remark that such matrix cannot be defined for n even. Relations (42) and (43) uniquely define, modulo a sign,

$$\bar{\gamma} = (-1)^{(n-1)/4} \gamma^0 \gamma^1 \cdots \gamma^n \ .$$

A particular representation of this algebra can be obtained for the $2^N \times 2^N$ complex matrices, where $N = (n+1)/2$. These $n+1$ matrices are called Dirac matrices. Due to Pauli's fundamental theorem [45, 54] all the possible representations are conjugated.

In parts of this text a particular representation shall be used. In some cases we will use the Pauli-Dirac representation (extended to $n+1$ dimensions):

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \bar{\gamma} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \ .$$

where σ^i are $N/2 \times N/2$ matrices. In dimension 3+1, σ^i are the well celebrated Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\bar{\gamma}$ is known as γ^5.

We also define the Weyl representation, where

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad \bar{\gamma} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \ .$$

The conjugation matrix between Pauli-Dirac and Weyl representations is given by

$$\frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}.$$

3.2 Dirac Operators

As before, we denote the Minkowski tensor by $(g^{\mu\nu}) = \mathrm{diag}[1, -1, \cdots, -1]$. Associated to $g^{\mu\nu}$ we can construct a Clifford Algebra. Dirac operators are defined by [53]:

$$\mathcal{D} = \gamma^\mu \partial_\mu + v \ .$$

From now on summation for repeated indices is implied, from 0 to n for Greek indices and from 1 to n for Latin indices. We shall adopt the notation $\not\partial = \gamma^\mu \partial_\mu$ and restrict ourselves to the case where v is a linear combination of I and $\bar{\gamma}$,

$$v = aI + \bar{a}\bar{\gamma} \ .$$

We call a the *scalar* potential and $\bar{a}$ the *pseudo-scalar* potential.

The attentive reader may ask about the complex unit i that appears on the side of the free Dirac operator $\not\partial$ in many textbooks. As we will shortly see this is immaterial.

It is easy to see that

$$\not\partial^2 = \Box \ . \tag{44}$$

4 Analytic Background

4.1 Huygens' principle revisited

By a fundamental solution to the Dirac operator we mean a solution of

$$(\not\partial + v)\Psi = \delta_y \ ,$$

where δ_y denotes Dirac-delta distribution supported in an arbitrary point y in space-time.

We shall say that a Dirac operator $\not\partial + v$ (and by extension, its fundamental solution) obeys Huygens' principle if Ψ satisfies

$$\text{supp } \Psi \subset \mathcal{C}(y) \ ,$$

for all $y \in \mathbb{R}^n$, where

$$\mathcal{C}(y) = \{x | (x^\mu - y^\mu)(x_\mu - y_\mu) = 0\} \ ,$$

is the light cone with vertex in y.

We recall that the Lorentz group $O(1, n)$ is the subgroup of the group of invertible matrices $GL(n+1, \mathbb{R}^{n+1})$ that preserves the quadratic form given by

$$x \longmapsto (x^0)^2 - \sum_{i=1}^{n} (x^i)^2 \ .$$

The orthocronous Lorentz group $O^+(1, n)$ is the connected component of $O(1, n)$ containing the identity, i.e., that preserves the direction of time. The D'Alembertian is clearly invariant by the action of this group.

4.2 Free Dirac Operators

From equation (44) we can see that $\not\partial\Lambda^\alpha$ is a fundamental solution of the free Dirac operator $\not\partial$ whenever Λ^α is a fundamental solution of wave operator.

This allows the definition of a family of distributions

$$\Theta^\alpha \stackrel{\text{def}}{=} \partial\!\!\!/ \Lambda^\alpha \ , \tag{45}$$

called Dirac kernels, defined in the same way the Riesz kernels Λ^α were defined in Subsection 1.2.

The Riesz kernels Λ^α and Dirac kernels Θ^α defined above have many important properties:

i $\Box\Lambda^\alpha = \Lambda^{\alpha-2}$;

ii $\Box\Theta^\alpha = \Theta^{\alpha-2}$;

iii $\partial\!\!\!/ \Theta^\alpha = \Lambda^{\alpha-2}$;

iv $\partial\!\!\!/^{-k}\Theta^{-n+1} = \Theta^{-n+1+k}$ if k is even;

v $\partial\!\!\!/^{-k}\Theta^{-n+1} = \Lambda^{-n+k}$ if k is odd;

vi $\partial_\mu\Lambda^\alpha = \frac{\Lambda^{\alpha-2}}{\alpha+n-1}(x_\mu - y_\mu)$, $\alpha \neq -n+1$;

vii if $f \in C^1(\mathbb{R}^{n+1})$ and $\alpha \neq -n+1$ then

$$\gamma^\mu\Theta^\alpha\partial_\mu f = \frac{2}{\alpha+n-1}\Lambda^{\alpha-2}(x^\mu - y^\mu)\partial_\mu f - \Theta^\alpha \partial\!\!\!/ f \ .$$

The three first properties are elementary consequences of the definitions of Dirac and Riesz kernels and equation (44). Properties (iv) and (v) are consequences of the three first. With the help of equation (6) we prove (vi).

Let us define

$$\Gamma \stackrel{\text{def}}{=} \gamma^\mu(x_\mu - y_\mu) \ .$$

We have that

$$\Gamma^2 = \lambda^2 \ , \tag{46}$$

so Γ is the square-root of λ^2 (in the same sense $\partial\!\!\!/$ is the square-root of $\Box$).

From property (vi) we have that, for $\alpha \neq -n+1$

$$\Theta^\alpha = \partial\!\!\!/ \Lambda^\alpha = \frac{\Lambda^{\alpha-2}}{\alpha+n-1}\Gamma \ . \tag{47}$$

With this formula in mind and considering the definition of Clifford algebras (40), we prove property (vii).

We are now ready to determine which free Dirac operators satisfy Huygens' property.

Theorem 3. *The free Dirac operator in dimension n obeys the Huygens' principle if, and only if, n is odd.*

Proof. With the help of equation (47) we can easily prove that (for $\alpha \neq -n+1$)

$$\partial\!\!\!/[\Lambda^\alpha (\log \lambda)\Gamma] = (\alpha + n + 1)\Lambda^\alpha \log \lambda + \Lambda^\alpha \ .$$

Since Λ^α is an analytic family of distributions we can take the limit $\alpha \to -n-1$ to get

$$\partial\!\!\!/[\Lambda^{-n-1}(\log \lambda)\Gamma] = \Lambda^{-n-1} \ .$$

Because Λ^{-n-1} is the Dirac-delta distribution (in a space-time of dimension $n+1$), we conclude that

$$\Theta^{-n+1} = \Lambda^{-n-1}(\log \lambda)\Gamma \ ,$$

is the fundamental solution of $\partial\!\!\!/$. Since $\Lambda^{-n-1} = N(-n-1)\lambda^{-n-1}$ and $N(-n-1) = 0$, Θ^{-n+1} is null wherever $\lambda^{-n-1} \log \lambda\Gamma$ is finite, i.e., outside the (surface of the) light cone. □

An important point to be stressed is that in one dimension, the free Dirac operator obeys Huygens' principle, unlike the free wave operator. In higher dimension they have the same behavior. See [13].

4.3 Trivial Transformations

In this section, we define some transformations that send a Dirac operator into another Dirac operator. Those transformations trivially preserve the validity of the Huygens' principle, so potentials generated by those are uninteresting. We define trivial transformations strongly inspired by Hadamard's original definition.

i Change of coordinates:

$\tilde{x}^\mu = f^\mu(x^0, \cdots, x^n)$, $\mu = 0, \cdots, n$, with $\det\big(\partial_\mu f^\nu\big)_{\mu,\nu=0,\cdots,n} \neq 0$.

In this case, the Dirac operator is transformed according to

$$\partial\!\!\!/ \mapsto (\partial\!\!\!/ f^\mu)\partial_\mu \ .$$

ii Left multiplication:

Now we take $\mathcal{D} \mapsto \bar{\mathcal{D}} = \Xi(x)\mathcal{D}$ and $\Psi \mapsto \bar{\Psi} = \Psi\Xi(y)^{-1}$, where $\Xi(x)$ is an everywhere non-singular matrix, $\Xi \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^{N\times N})$.

In order to see that $\bar{\Psi}$ is the fundamental solution of $\bar{\mathcal{D}}$ we write:

$$\bar{\mathcal{D}}\bar{\Psi} = \Xi(x)\mathcal{D}\Psi\Xi(y)^{-1} = \Xi(x)\delta_y\Xi(y)^{-1} = \delta_y \ .$$

iii Factor transformations:

Let ρ be a non-singular matrix,

$$\rho = \rho_\phi I + \rho_\mu\gamma^\mu + \bar{\rho}_\mu\gamma^\mu\bar{\gamma} + \bar{\rho}\bar{\gamma} \ ,$$

where ρ_ϕ, ρ_μ, $\bar{\rho}_\mu$ and $\bar{\rho}$ are differentiable functions. The factor transformation consists in mapping $\mathcal{D} \mapsto \bar{\mathcal{D}} = \rho(x)\mathcal{D}\rho(x)^{-1}$ and $\Psi \mapsto \bar{\Psi} = \rho(x)\Psi\rho(y)^{-1}$.

The second transformation fully justifies the omission of the complex unit i following Dirac matrices.

Some authors use the expression "gauge transformations" instead of "factor transformations." To avoid confusion with the use of the term "gauge" in electromagnetism, we shall adopt the Hadamard's original term.

It is important to stress that the transformations defined above include the change of the electromagnetic gauge, $A_\mu \to A_\mu + \partial_\mu \theta$, where θ defines the new (electromagnetic) gauge.

$$\gamma^\mu(\partial_\mu + iA_\mu + i\partial_\mu \theta) = e^{-i\theta}\gamma^\mu(\partial_\mu + iA_\mu)e^{i\theta} \ .$$

In this case the fundamental solution is transformed

$$\bar{\Psi} = e^{-i\theta(x)}\Psi e^{i\theta(y)} \ .$$

As we should expect, the change of electromagnetic gauge does not interfere with the validity of the Huygens' principle. For a discussion of Huygens' Principle for Dirac operators in the presence of an electromagnetic field, see [14].

4.4 Hadamard Expansions

Adapting Hadamard's seminal idea for the wave operator, we look for a series expansion for the fundamental solution Ψ. Here, Ψ denotes the solution of

$$(\partial\!\!\!/ + v)\Psi = \delta_y \ . \tag{48}$$

We look for series expansion of Ψ:

$$\Psi = \sum_{k=0}^{\infty} \partial\!\!\!/^{-k}\Theta^{\alpha_0}s_k = \sum_{k=0}^{\infty}\left\{\Theta^{\alpha_0+2k}s_{2k} + \Lambda^{\alpha_0+2k}s_{2k+1}\right\} \ , \tag{49}$$

where $s_k = s_k(x, y)$ is a matrix coefficient and α_0 should be taken equal to $-n+1$.

As v and Θ^α are matrices, we need to study how to exchange these two objects. From equation (47), we conclude that

$$v\Theta^\alpha = \Theta^\alpha v^* \ , \tag{50}$$

where

$$v^* \stackrel{\text{def}}{=} aI - \bar{a}\bar{\gamma} \ . \tag{51}$$

Applying $\partial\!\!\!/ + v$ to Ψ we find

$$(\partial\!\!\!/ + v)\Psi = \Lambda^{\alpha_0-2}\Big[s_0 + \frac{2}{\alpha_0+n-1}(x^\mu - y^\mu)\partial_\mu s_0\Big] +$$

$$+ \sum_{k=0}^{\infty}\Theta^{\alpha_0+2k}\Big[-\partial\!\!\!/ s_{2k} + s_{2k+1} + v^* s_{2k}\Big] +$$

$$+\sum_{k=1}^{\infty}\Lambda^{\alpha_0+2k-2}\left[s_{2k}+\frac{2}{\alpha_0+2k+n-1}(x^\mu-y^\mu)\partial_\mu s_{2k}+\partial\!\!\!/ s_{2k-1}+vs_{2k-1}\right] .$$

Equating $(\partial\!\!\!/ + v)\Psi = \Lambda^{-n-1}$ we find *Hadamard's recursion*:

$$\begin{aligned} s_0 &= 1 , \\ s_{2k+1} &= (\partial\!\!\!/ - v^*)s_{2k} , \\ s_{2k}+\frac{1}{k}(x^\mu-y^\mu)\partial_\mu s_{2k} &= -(\partial\!\!\!/ + v)s_{2k-1} . \end{aligned} \tag{52}$$

Theorem 4. *If $v = aI + \bar{a}\gamma$, Ψ is given by equation (49), and satisfies $(\partial\!\!\!/ + v)\Psi = \delta_y$, then the coefficients in the expansion (52) are uniquely determined by the imposing regularity of s_k in the vicinity of the vertex of the light cone, i.e, when $x \to y$.*

Proof. The uniqueness for odd terms is obvious. For even terms, suppose that s^1 and s^2 are solutions of the third equation in the system (52). So, their difference $\bar{s} = s^1 - s^2$ is a solution of

$$\bar{s}+\frac{1}{k}(x^\mu-y^\mu)\partial_\mu\bar{s} = 0 .$$

Consider a ray emanating from the light-cone vertex y and remember that λ measures the geodesic distance between x and y. Studying the behavior of $\bar{s}$ along this ray we find that $\bar{s}$ obeys the equation

$$\bar{s}(\lambda)+\frac{1}{k}\lambda\bar{s}'(\lambda) = 0$$

The only non-trivial solution of this equation is given by $\bar{s} \propto \lambda^{-k}$, that is not well-behaved near $\lambda = 0$. Then, $\bar{s} = 0$ and the theorem is proved. □

Theorem 5. *A Dirac operator in the form $\partial\!\!\!/ + v$ possesses Huygens' property in odd spatial dimension n if, and only if, its Hadamard series truncates at $n-1$, i.e., $s_k = 0$ for $k \geq n$.*

Proof. If the series truncates, we simply need to take n high enough in order to guarantee that all Riesz and Dirac kernels in the expansion have Huygens' property. In the other direction, we take the Hadamard's series of the operator $\partial\!\!\!/ + v$, and from its uniquenes, all the kernels should be of Huygens' type. □

Corollary 1. *Assume that the Hadamard coefficients s_k for the operator $\partial\!\!\!/ + v$ vanish for $k > d$. Then, the operator $\partial\!\!\!/ + v$ is Huygens for any odd dimension exceeding d.*

We state two further results before characterizing Dirac operators.

Theorem 6. *Trivial transformations do not change the validity of Huygens' principle.*

Proof. In Subsection 4.3 we studied how the fundamental solution of the Dirac operator changes whenever we trivially transform the operator. In the first case of trivial transformations, we simply change the coordinates of the solution, so the order of the series is not changed. In the second case, we multiply the Hadamard coefficients on the right by the matrix $\Xi(y)^{-1}$. For the third transformation we see that

$$\rho\Theta^\alpha = \Theta^\alpha\rho^* + \frac{2}{\alpha+n-1}\Lambda^{\alpha-2}(\rho^\mu I + \bar{\rho}^\mu\bar{\gamma})(x_\mu - y_\mu) \ ,$$

where $\rho^* \stackrel{\text{def}}{=} \rho I_\phi - \rho_\mu\gamma^\mu + \bar{\rho}_\mu\gamma^\mu\bar{\gamma} - \bar{\rho}\bar{\gamma}$. Thus, if Ψ is a terminating series, so is $\bar{\Psi}$. □

Remark 1. *It would be very tempting to generalize the third trivial transformation to arbitrary matrices. In order to do that, it is necessary to understand how to exchange the order of ρ and Θ^α. As far as we see, there is no canonical way to do that.*

Theorem 7. *Change of representation of Dirac matrices do not affect Huygens' property.*

Proof. Due to Pauli's Fundamental Theorem [45, 54] all the representations are equivalent. □

Remark 2. *Given $v = a(x^0)I + \bar{a}(x^0)\bar{\gamma}$, the operators $\not{\partial}+v$, $\not{\partial}-v$, $\not{\partial}-v^*$ and $\not{\partial}+v^*$ are trivially equivalent.*

Proof. The identity

$$-\bar{\gamma}(\not{\partial}+v)\bar{\gamma} = \not{\partial} - v \ ,$$

proves the first equivalence. For the second one, we need to define $M = i\gamma^0\bar{\gamma}$ and thus $M^{-1} = M$. A simple calculation proves that

$$M\gamma^0M^{-1} = -\gamma^0 \ , \quad M\gamma^iM^{-1} = \gamma^i \ \text{ and } \ M\bar{\gamma}M^{-1} = -\bar{\gamma} \ .$$

then $M(\not{\partial}+v)M^{-1} = -\gamma^0\partial_0 + \gamma^i\partial_i + v^*$. Proceeding with a spatial inversion, $x^i \to -x^i$, the last operator changes[5] to $-\not{\partial}+v^*$. The other equivalences are elementary consequences of these two. □

As a final example in this section, let's study the validity of Huygens' principle for the *massive* Dirac operator, i.e., $\not{\partial}+m$, when $m \neq 0$ is a constant.

Example 1. *Consider $v = m$. The solution of recursion (52) is given by $s_k = (-m)^k$, so, by Theorem 5, $\not{\partial}+m$ never obeys Huygens' principle, if $m \neq 0$.*

5 Main Result

In this section we shall prove the main result of the present work. In order to that we start with a preliminary lemma:

Lemma 2. *If $f(x)$ is solution of $\Box^p f(x) = 0$ for some positive integer p, then $\not{\partial}+f(x)(I+\bar{\gamma})$ is a Huygens' operator for a sufficiently high dimension n.*

[5]The transformations we just introduced defines a *space-time inversion.* Since a is a scalar, it remains invariant under this kind of transformation, but $\bar{a}$, a pseudo-scalar, changes sign.

Proof. Write $s_k = (I + (-1)^k \bar{\gamma})\tilde{s}_k$, for $k \geq 1$. Solving Hadamard's recursion we see that this definition is consistent and that $\tilde{s}_k$ obeys Hadamard's recursion with $v = 0$, starting from $\tilde{s}_1 = -f$. We solve the recurstion for $\tilde{s}_{2k}$ and find

$$\tilde{s}_{2k}(x,y) = -k \int_0^1 \partial\!\!\!/ \tilde{s}_{2k-1}(y + z(x-y), y) z^{k-1} dz \ ,$$

To prove that we only need to know that

$$(x^\mu - y^\mu)\partial_\mu F(y + z(x-y)) = z\frac{dF}{dz}(y + z(x-y))$$

and integrate by parts. We find the next term

$$\tilde{s}_{2k+1} = \partial\!\!\!/ \tilde{s}_{2k} = -k \int_0^1 \Box \tilde{s}_{2k-1}(y + z(x-y), y) z^k dz \ .$$

For the odd terms we write

$$\tilde{s}_{2k+1}(x,y) = (-1)^{k+1} k! \int_0^1 \cdots \int_0^1 \Box^k f(\xi_k) z_1^k z_2^{k+1} \cdots z_k^{2k-1} dz_1 \cdots dz_k \ , \tag{53}$$

where $\xi_i = y + z_i(\xi_{i-1} - y)$, and $\xi_0 = x$. For $k = p$ the result follows. $\square$

Corollary 2. *If $q(x)$ is a polynomial of degree d, then the Hadamard's series of $\partial\!\!\!/ + q(x^0)(I + \bar{\gamma})$ is truncated at $d+1$, i.e., $s_{d'} = 0$, whenever $d' \geq d+2$.*

Proof. We rewrite equation (53) as

$$\tilde{s}_{2k+1}(x,y) = (-1)^{k+1} k! \int_0^1 \cdots \int_0^1 q^{(2k)}(\xi_k) z_1^k z_2^{k+1} \cdots z_k^{2k-1} dz_1 \cdots dz_k \ ,$$

with $\xi_i = y^0 + z_i(\xi_{i-1} - y^0)$, and $\xi_0 = x^0$. If d is odd, then $\tilde{s}_{d+2} = 0$. If d is even, $\tilde{s}_{d+1}$ does not depend on x^0, $\tilde{s}_{d+2} + \frac{2}{d+2}(x^0 - y^0)\tilde{s}_{d+2} = -\partial\!\!\!/ \tilde{s}_{d+1} = 0$, then $\tilde{s}_{d+2} = 0$. $\square$

Remark 3. *Corolary 2 remains valid for the null polynomial if one considers its degree equals -1.*

Lemma 3. *Schlesinger transformation do not change the number of terms in the Hadamard' series.*

Proof. We shall adopt the Weyl representation. In this case:

$$H = \bar{\gamma} \ , \quad E = \gamma^0 \frac{I - \bar{\gamma}}{2} \ , \quad F = \gamma^0 \frac{I + \bar{\gamma}}{2} \ .$$

From the construction of the Schlesinger transformation, we see that $R^{(1)} = kH + e_1 E + f_1 F$ implies $\widetilde{R}^{(1)} = kH + \widetilde{e}_1 E + \widetilde{f}_1 F$, where as in equation (31)

$$\widetilde{e}_1 = -\frac{1}{f_1} \ , \quad \widetilde{f}_1 = f_{1\,xx} - \frac{f_{1\,x}^2}{f_1} - f_1^2 e_1 \ .$$

Here, all the functions depend on $t = x^0$ and $'$ denotes derivative with respect to this variable. Let's adopt the following notation

$$\widetilde{R}^{(\ell)}(k) = \mathbf{S}_S[R^{(\ell)}(k)] \ .$$

It is easy to see that

$$\mathbf{S}_S\left[-R^{(1)}(0)\right] = -\mathbf{S}_S\left[R^{(1)}(0)\right] \ . \tag{54}$$

We remark that the above formula is valid only for $\ell = 1$. In the sequel, we fix and ommit $k = 0$. Let us consider $t_1 = x^0 \stackrel{\text{def}}{=} t$ and, for $\ell \neq 1$, t_ℓ is constant. We define $\widetilde{R}$ and $\widetilde{R}'$ such that

$$\begin{aligned} S\bar{\gamma}(\bar{\gamma}\partial_t - R^{(1)}) &= (\bar{\gamma}\partial_t - \widetilde{R})\bar{\gamma}S \ , \\ \bar{\gamma}S(\bar{\gamma}\partial_t - R^{(1)}) &= (\bar{\gamma}\partial_t - \widetilde{R}')S\bar{\gamma} \ . \end{aligned}$$

From the first, we get

$$\widetilde{R} = \mathbf{S}_S\left[\bar{\gamma}R^{(1)}\right]\bar{\gamma} \ ,$$

and from the second

$$\bar{\gamma}\widetilde{R}' = \mathbf{S}_S\left[R^{(1)}\bar{\gamma}\right] \ .$$

Using that $E\bar{\gamma} = -\bar{\gamma}E$ and the same for F, we find $\bar{\gamma}R^{(1)} = -R^{(1)}\bar{\gamma}$ and the same for $\widetilde{R}'$. With the help of equation (54), we find that $\widetilde{R}' = \widetilde{R}$. Summing up both equations

$$(S\bar{\gamma} + \bar{\gamma}S)(\bar{\gamma}\partial_t - R^{(1)}) = (\bar{\gamma}\partial_t - \widetilde{R})(\bar{\gamma}S + S\bar{\gamma}) \ .$$

As S is a linear combination of I, H, E and F (see [42]), or in other words of I, $\bar{\gamma}$, γ^0, and $\gamma^0\bar{\gamma}$, we find that

$$(S\bar{\gamma} + \bar{\gamma}S)\bar{\gamma}\gamma^0\gamma^i = \bar{\gamma}\gamma^0\gamma^i(S\bar{\gamma} + \bar{\gamma}S) \ .$$

From this, we conclude

$$(S\bar{\gamma} + \bar{\gamma}S)(\bar{\gamma}\partial_t - R^{(1)} + \bar{\gamma}\gamma^0\gamma^i\partial_i) = (\bar{\gamma}\partial_t - \widetilde{R} + \bar{\gamma}\gamma^0\gamma^i\partial_i)(S\bar{\gamma} + \bar{\gamma}S) \ , \tag{55}$$

where

$$\widetilde{R} = \mathbf{S}_S\left[\bar{\gamma}R^{(1)}\right]\bar{\gamma} \ .$$

The operator $(\bar{\gamma}\partial_t - R^{(1)} + \bar{\gamma}\gamma^0\gamma^i\partial_i)$ is trivially equivalent to $\partial\!\!\!/ + v$ with $R^{(1)} = -\bar{\gamma}\gamma^0 v$. The same holds for $\widetilde{R} = -\bar{\gamma}\gamma^0\widetilde{v}$. So

$$\gamma^0\widetilde{v} = -\mathbf{S}_S\left[\gamma^0 v\right] \ .$$

For $v = aI + \bar{a}\bar{\gamma}$, we have

$$\gamma^0 v = (a - \bar{a})E + (a + \bar{a})F$$

and a similar equation for $\widetilde{v}$. Defining

$$e_1 = a - \bar{a} \ , \tag{56}$$

$$f_1 = a + \bar{a} \ , \tag{57}$$

we find after a Schlesinger transformation

$$\tilde{a} = -\frac{\tilde{f}_1 + \tilde{e}_1}{2} , \tag{58}$$

$$\bar{\tilde{a}} = -\frac{\tilde{f}_1 - \tilde{e}_1}{2} . \tag{59}$$

From equation (55) we see that the Schlesinger transformation is a particular case of trivial transformation, thus, it does not change the order of the Hadamard series. The last two pair of equations give the effect of a Schleshinger transformation in the components of the potential. □

Theorem 8. *If $q(x^0)$ and $r(x^0)$ are solutions of the AKNS given by (36), then*

$$\partial\!\!\!/ - \frac{q+r}{2} I + \frac{q-r}{2}\bar{\gamma} \tag{60}$$

has the Huygens' property in dimension $d+2$, if d is odd, or $d+3$ if d is even.

Proof. Let's consider, in the Hirota variables, $j = -1$. Then, according to (36), $q = 0$ and $r = -\mathcal{Q}_d$, the d-th Schur polynomial. Then,

$$v = \frac{\mathcal{Q}_d(x^0)}{2}(I + \bar{\gamma}) .$$

From Corollary 2, we conclude that the Hadamard's series of $\partial\!\!\!/ + v$ terminates at $d+1$. From Lemma 3, we know that Schlesinger transformation do not change the order of the series. We also know that Schlesinger transformation adds one to the Hirota's variable j index. Sucessives transformations go through $j = 0, \cdots, d$. Then we prove that the Hadamard's series of Dirac operator given by (60) terminates at $d+1$, i.e., $s_{d'} = 0$, $d' \geq d+2$. From Corollary 1 it follows that the operator in equation (60) is Huygens in dimension $d+2$ if d is odd and $d+3$ if d is even. □

6 Final Remarks

6.1 Lagnese and Stellmacher Revisited

The results of Langese and Stellmacher can be reproduced *without the use of Darboux transformations*. As can be seen, Darboux transformations were not used in the proof of Theorem 8. From this theorem, it is possible to reproduce their result.

Lemma 4. *Let $v(x^0)$ be a scalar potential, i.e., $\bar{a} = 0$. If $\partial\!\!\!/ + v$ has the Huygens' property and $u^{\pm}$ are the Miura tranforms of v, i.e., $u^{\pm} = \pm v' + v^2$, then $\Box - u^{\pm}$ has the Huygens' property.*

Proof. From Remark 2, we know that $\partial\!\!\!/ + v$ and $\partial\!\!\!/ - v$ are trivially equivalent. Consider $\Psi^{\pm}$ and $\Phi^{\pm}$, the fundamental solutions of $\partial\!\!\!/ \pm v$ and $\Box - u^{\pm}$ respectively. In the Pauli-Dirac representation

$$(\partial\!\!\!/ - v)(\partial\!\!\!/ + v) = \Box + \gamma^0 v' - v^2 = \begin{pmatrix} \Box - u^- & 0 \\ 0 & \Box - u^+ \end{pmatrix} = (\Box - u^-) \oplus (\Box - u^+) .$$

In a similar way

$$(\partial\!\!\!/ + v)(\partial\!\!\!/ - v) = (\Box - u^+) \oplus (\Box - u^-)\ .$$

Consider the Hadamard series given by (49) and (7) where the coefficients $s_k^\pm$ and $w_k^\pm$ obey recursions (52) and (8) respectively, with $s_k^\pm$ and $w_k^\pm$ depending only on x^0 and y^0. The even coefficients obey the recursion

$$s_{2k}^\pm + \frac{1}{k}(x^0 - y^0)\partial_0 s_{2k}^\pm = -(\partial\!\!\!/ \pm v)(\partial\!\!\!/ \mp v)s_{2k-2}^\pm = -(\Box - u^\pm) \oplus (\Box - u^\mp)s_{2k-2}^\pm\ .$$

We conclude that

$$s_{2k}^\pm = w_k^\pm \oplus w_k^\mp\ .$$

Then $s_k^\pm$ are terminating if, and only if, $w_k^\pm$ are terminating. Recalling that if s_k^+ is terminating, so is s_k^-, we conclude the proof. □

Theorem 9. *If $u_j = -2\partial_0^2 \log \vartheta_j(x^0)$, where ϑ_j is the j-th Adler-Moser polynomial, then $\Box - u_j(x^0)$ is a Huygens type operator in dimension $n = 2j + 3$.*

Proof. Let d be equal $2j + 1$ and set all even flows to zero, i.e., $t_{2k} = 0 = y_{2k}$. From Lemma 5 and from Theorem 2 (changing signs, if necessary) we find that the Dirac operator in equation (60) is

$$\partial\!\!\!/ - \partial_0 \log \frac{\vartheta_{j+1}}{\vartheta_j}(x^0) I\ ,$$

which is Huygens for dimension $n = 2j+3$ by Theorem 8, i.e, $s_{2j+4} = 0$. From the Lemma 4, we have that $\Box - u^\pm$ has the Huygens' property, where $u^\pm$ are the Miura transforms of v. Then

$$\begin{aligned} u^+ &= -2\partial_0^2 \log \vartheta_j\ , \\ u^- &= -2\partial_0^2 \log \vartheta_{j+1} \end{aligned}$$

are both Huygens in $n = 2j + 5$, since $w_{j+2}^\pm = 0$. We conclude that $\Box - u_j$ is of Huygens type in dimension $2j + 3$, exactly the classical Lagnese and Stellmacher result. □

6.2 Ilustrative Examples

In this section, we shall adopt $x^0 = t$ and $y^0 = t_0$. The constants t_k, $k = 2, 3, \cdots$ are arbitrary.

Example 2. *If $d = 3$ and $j = 1$, we have the operator*

$$\partial\!\!\!/ + \frac{3\,t_3 + 2\,t^3}{-3\,t_3\,t + t^4 + 3\,t_2^2} I + 6\frac{t_2\,t}{-3\,t_3\,t + t^4 + 3\,t_2^2}\bar{\gamma}\ .$$

We immediately see that if we impose $t_2 = 0$, we have a scalar potential.

Example 3. *An interesting example is given if we choose $d = 2$ and $j = 2$:*

$$\partial\!\!\!/ + \frac{1}{2}\frac{2+t^2-t_2}{t^2+t_2}I - \frac{1}{2}\frac{t^2-t_2-2}{t^2+t_2}\bar{\gamma}\ .$$

Its fundamental solution is given by

$$\begin{aligned}
\Psi \;=\;& \Theta^{-n+1} - \\
& \Lambda^{-n+1}\left[\frac{1}{2}\frac{2+t^2-t_2}{t^2+t_2}I + \frac{1}{2}\frac{t^2-t_2-2}{t^2+t_2}\bar{\gamma}\right] + \\
& \Theta^{-n+3}\left[2\,\frac{t_0\,t-t_2}{(t^2+t_2)\,(t_0^2+t_2)}I + \frac{(t+t_0)\,(-1+t_2)}{(t^2+t_2)\,(t_0^2+t_2)}\gamma^0 + \frac{(t+t_0)\,(t_2+1)}{(t^2+t_2)\,(t_0^2+t_2)}\gamma^0\bar{\gamma}\right] + \\
& \Lambda^{-n+3}\left[-\frac{-1+t_0\,t}{(t^2+t_2)\,(t_0^2+t_2)}I + \frac{t-t_0}{(t^2+t_2)\,(t_0^2+t_2)}\gamma^0 - \frac{1+t_0\,t}{(t^2+t_2)\,(t_0^2+t_2)}\bar{\gamma} - \frac{t+t_0}{(t^2+t_2)\,(t_0^2+t_2)}\gamma^0\bar{\gamma}\right]\ .
\end{aligned}$$

Hence, the above operator is Huygens in dimension $5 = 2+3$, in agreement with Theorem 8.

6.3 Conclusion and Suggestions for Further Research

We conclude with a short overview of the main ideas presented in this article as well as some suggestions for further research. Our main result was the fact that the fields (q, r) constructed from certain explicit polynomial Hirota variables produce Dirac operators of the form

$$\mathcal{D} = \partial\!\!\!/ - \frac{1}{2}(q+r)I + \frac{1}{2}(q-r)\bar{\gamma} \tag{61}$$

that satisfy Huygens' principle. The fields (q, r) under scrutiny turned out to provide (generically) large classes of bispectral potentials for AKNS systems [60]. We have also shown that certain reductions of the above families by setting $d = 2j+1$ and $t_2 = t_4 = t_6 = \cdots = 0$ in equation (36) lead to AKNS operators associated to the mKdV hierarchy. In this case, we obtain as a corollary our earlier result in [13]. This was obtained taking into account that in this case, $q = r$, and

$$(\partial\!\!\!/ - q)(\partial\!\!\!/ + q) = \begin{bmatrix} \Box - u_{k+1}(x^0) & 0 \\ 0 & \Box - u_k(x^0) \end{bmatrix}, \tag{62}$$

Furthermore, the potentials in (62) are precisely the ones associated to the $k+1$-st and k-th Adler-Moser polynomials. In particular, we were able to obtain an independent proof of the Lagnese-Stellmacher result without resorting to Darboux transformations.

Since Dirac's equation corresponds to physical systems with spin 1/2, one natural avenue ushered by the present work is the study of Huygens property for higher-spin physical systems. As natural candidates we mention Maxwell's equations (spin 1), Harita-Schwinger's equation (spin 3/2), and Einstein's equation (spin 2). Although there are already studies considering Huygens' property for such systems [4], to our knowledge there are no general results for such higher spin systems as the one we present here for Dirac in connection with the rational solutions of the AKNS hierarchy.

Another natural continuation of the present work would be the study of the symmetry group of Dirac operators of Huygens type. In the wave operator case, Berest analyzed in [7] the group of extended symmetries that preserve Huygens property. He showed that the generators of such extended symmetries have a Virasoro type structure and that the KdV flows naturally preserve Huygens principle at the level of the potential $u(x^0)$. As it turns out the rational potentials obtained by Lagnese-Stellmacher belong to the orbit of such flows. A natural conjecture would be that the rational fields (q, r) presented in this article belong to the orbit of iso-Huygens deformations for Dirac operators. The first step in this program would be the characterization of this group of extended symmetries.

Finally, yet another natural follow up of the present article would be the search for Dirac operators of Huygens type depending on more than two fields. Indeed, the representation of Dirac operators leaves room to many other fields besides q and r since it involves matrices of size $2^{(n+1)/2} \times 2^{(n+1)/2}$, where $n+1$ is the dimension of space-time. It would be plausible that such fields are part of a more complex perturbation of the free Dirac operator in an even dimensional space-time.

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A Appendix: From the AKNS to the mKdV

In this appendix we show how to choose the values of d and j, in equation (35) so as to obtain the rational solutions of the mKdV hierarchy associated to the rational solutions of the AKNS hierarchy. Although the result of the present appendix is quite natural, as discussed in [43], we were not able to locate its proof in the literature.

Let q be a rational solution of the mKdV equation vanishing at infinity. Then, since the solutions of the mKdV are sent into solutions of the KdV it follows that $u = \pm q_x + q^2$ is a rational solution of the KdV vanishing at infinity. The full classification of the rational solutions of the KdV decaying at infinity is well known. See [3, 2, 15]. It follows from [3, 2] that the rational solutions of the KdV can be expressed in the form

$$u = -2\partial_x^2 \log \vartheta_k(x) \ ,$$

where ϑ_k is the k-th Adler-Moser polynomials. It is not hard to check that the corresponding rational solution of the mKdV has the form

$$\pm\partial_x \log \frac{\vartheta_{k+1}}{\vartheta_k}(x) \ ,$$

for some k and a suitable choice of parameters for the Adler-Moser polynomials ϑ_{k+1} and ϑ_k.

We start by proving that

Lemma 5. *If $y_{2k} = 0$ for all $k \in \mathbb{N}$ then*

$$\tau_{j+1}^{2j+1} + \tau_{j-1}^{2j+1} = 0 \ .$$

Proof. Define the "infinite vector" $\vec{y} = (y_1, y_2, \cdots)$ and its scaled version as

$$\vec{y}_a = (ay_1, a^2 y_2, \cdots, a^k y_k, \cdots) \ ,$$

for any real valued a. We easily find that

$$\mathcal{Q}_j(\vec{y}_a) = a^j \mathcal{Q}_j(\vec{y}) \ .$$

If we choose $a = -1$ and fix all even terms $y_{2k} = 0$, we have $\vec{y}_a = -\vec{y}$ and, consequently,

$$\mathcal{Q}_j(-\vec{y}) = (-1)^j \mathcal{Q}_j(\vec{y}) \ .$$

From the definition (33), we have

$$\left[\sum_{j\geq 0} \lambda^j \mathcal{Q}_j(-\vec{y})\right] \left[\sum_{j\geq 0} \lambda^j \mathcal{Q}_j(\vec{y})\right] = 1 \ .$$

We equate term-by-term the above Taylor expansion in λ to find, with the help of previous equations, that (we ommit the parameter $\vec{y}$)

$$\mathcal{Q}_0^2 = 1 \ , \tag{63}$$

$$\sum_{p=0}^{j} (-1)^p \mathcal{Q}_p \mathcal{Q}_{j-p} = 0 \quad j = 1, 2, \cdots \ . \tag{64}$$

Let us define the $(j+2) \times (j+2)$ matrix

$$A^{(0)} = \left(a_{n,m}^{(0)}\right) \ ,$$

where $n, m = 0, \cdots, j+1$ and

$$a_{n,m}^{(0)} = \mathcal{Q}_{2j+1-n-m} \ .$$

In this case

$$\tau_{j+1}^{2j+1} = \det A^{(0)} \ .$$

We define

$$A^{(1)} = \left(a_{n,m}^{(1)}\right) \ ,$$

with

$$a_{n,m}^{(1)} = \sum_{p=0}^{j+1-m} (-1)^p a_{n,m+p} \mathcal{Q}_p \ ,$$

for $n, m = 0 \cdots, j+1$. Column m of matrix $A^{(0)}$ is substituted by a linear combination of all the columns, where the weight of the column m is $\mathcal{Q}_0$. So

$$\det A^{(1)} = \mathcal{Q}_0^{j+1} \det A^{(0)} \ .$$

We redefine the left upper block

$$A^{(2)} = \left(a^{(2)}_{n,m}\right) = \left(a^{(1)}_{n,m}\right) \ ,$$

with $n, m = 0, \cdots, j-1$. From properties (63) and (64) we find

$$A^{(1)} = \left(\begin{array}{ccccc} & & & \vdots & * & * \\ & A^{(2)} & & \vdots & * & * \\ & & & \vdots & * & * \\ \cdots & \cdots & \cdots & \cdot & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & 1 \\ 0 & 0 & 0 & \vdots & 1 & 0 \end{array} \right) \ .$$

Consequently

$$\det A^{(1)} = -\det A^{(2)} \ .$$

Again, from property (64), we find

$$a^{(2)}_{n,m} = (-1)^{n+m} \sum_{p=0}^{j-1-n} (-1)^p \mathcal{Q}_{2j+1-n-m-p} \mathcal{Q}_p \ .$$

We construct the matrix $A^{(3)}$ by multiplying the row n by $(-1)^n$ and the column m by $(-1)^m$,

$$a^{(3)}_{n,m} = \sum_{p=0}^{j-1-n} (-1)^p \mathcal{Q}_{2j+1-n-m-p} \mathcal{Q}_p \ ,$$

with $n, m = 0 \cdots, j-1$. So $\det A^{(3)} = \det A^{(2)}$.

Now, define a new matrix

$$\bar{A}^{(0)} = \left(\bar{a}^{(0)}_{n,m}\right) \ ,$$

$$\bar{a}^{(0)}_{n,m} = \mathcal{Q}_{2j+1-n-m} \ ,$$

with $n, m = 0 \cdots, j-1$. We construct the matrix $\bar{A}^{(1)}$ taking linear combinations of the rows in the matrix $\bar{A}^{(0)}$,

$$\bar{a}^{(1)}_{n,m} = \sum_{p=0}^{j-1-n} (-1)^p \bar{a}^{(0)}_{n+p,m} \mathcal{Q}_p \ .$$

We easily find that

$$\bar{A}^{(1)} = A^{(3)} \ ,$$

$$\det \bar{A}^{(0)} = \tau_{j-1}^{2j+1}$$

and

$$\det \bar{A}^{(1)} = \mathcal{Q}_0^{j-1} \det \bar{A}^{(0)} \ .$$

We conclude

$$\tau_{j-1}^{2j+1} = \det \bar{A}^{(0)} = \frac{1}{\mathcal{Q}_0^{j-1}} \det \bar{A}^{(1)} = \frac{1}{\mathcal{Q}_0^{j-1}} \det A^{(3)} =$$

$$= \frac{1}{\mathcal{Q}_0^{j-1}} \det A^{(2)} = -\frac{1}{\mathcal{Q}_0^{j-1}} \det A^{(1)} = -\mathcal{Q}_0^2 \det A^{(0)} = -\tau_{j+1}^{2j+1} \ .$$

This finishes the proof. □

Lemma 6. *If $y_k = 0$ for $k \geq 2$ then*

$$\tau_{j+1}^{2j+1} = (-1)^{j+1}(j+1)\tau_j^{2j+1} \frac{1}{y_1} \ .$$

Proof. From equation (33) we have

$$\mathcal{Q}_j = \frac{y_1^j}{j!} \ .$$

From now on we use the convention that $1/j! = 0$ for $j < 0$. The Hirota variables τ_j^d, given by equation (35), are monomials in y_1 of degree $(d-j)(j+1)$. Thus, we can suppose $y_1 = 1$ without loss of generality. We define the $(j+2) \times (j+2)$ matrix

$$B^{(0)} = \left(b_{n,m}^{(0)} \right)$$

where $n, m = 0 \cdots, j+1$ and

$$b_{m,n}^{(0)} = \frac{1}{(2j+1-m-n)!} \ .$$

We write

$$b_{n,m}^{(1)} = \begin{cases} -(j-n)! b_{n,m}^{(0)} + (j-n-1)! b_{m,n+1}^{(0)} & n = 0, \cdots, j+1 \ , \quad m = 0, \cdots, j-1 \ ; \\ -b_{n,m}^{(0)} & n = 0, \cdots, j+1 \ , \quad m = j \ ; \\ b_{n,m}^{(0)} & n = 0, \cdots, j+1 \ , \quad m = j+1 \ . \end{cases}$$

Then

$$\det B^{(1)} = (-1)^{j+1} \left[\prod_{n=0}^{j-1} (j-n)! \right] \det B^{(0)} \ ,$$

where $B^{(1)} = \left(b_{n,m}^{(1)} \right)$, with $n, m = 0 \cdots, j+1$.

We easily prove that if $n = j+1$ and $m = 0, \cdots, j-1$, then $b_{j+1,m}^{(1)} = 0$. Furthermore $b_{j+1,j}^{(1)} = -1$ and $b_{j+1,j+1}^{(1)} = 0$.

We construct a new matrix,

$$B^{(2)} = \left(b_{n,m}^{(2)} \right)$$

with $n, m = 0, \cdots, j$ and

$$b^{(2)}_{n,m} = \begin{cases} b^{(1)}_{n,m} & n = 0 \cdots, j \ , \quad m = 0, \cdots, j-1 \\ b^{(1)}_{n,m+1} & n = 0 \cdots, j \ , \quad m = j \ . \end{cases}$$

Then

$$\det B^{(2)} = \det B^{(1)} \ .$$

We re-write, after a simple computation

$$b^{(2)}_{n,m} = \begin{cases} \frac{(j-m-1)!(j+1-n)}{(2j+1-n-m)!} & n = 0 \cdots, j \ , \quad m = 0 \cdots, j-1 \ ; \\ \frac{1}{(j-m)!} & n = 0, \cdots, j \ , \quad m = j \ . \end{cases}$$

We define a new matrix $B^{(3)} = \left(b^{(3)}_{n,m}\right)$ with $n, m = 0, \cdots, j$ such that

$$b^{(3)}_{n,m} = \begin{cases} \frac{1}{(j+1-n)} \frac{1}{(j-1-m)!} b^{(2)}_{n,m} & n = 0, \cdots, j \ , \quad m = 0, \cdots, j-1 \ ; \\ \frac{1}{(j+1-n)} b^{(2)}_{n,m} & n = 0, \cdots, j \ , \quad m = j \ . \end{cases}$$

It is immediate to see that

$$\det B^{(3)} = \left[\prod_{n=0}^{j} (j+1-m) \prod_{m=0}^{j-1} (j-1-m)! \right]^{-1} \det B^{(2)} \ .$$

From the definition we have that

$$b^{(3)}_{n,m} = \frac{1}{(2j+1-n-m)!} \ ,$$

with $n, m = 0, \cdots, j$. Then

$$\tau^{2j+1}_{j} = \det B^{(3)} \ .$$

We conclude noting that

$$\tau^{2j+1}_{j+1} = \det B^{(0)} = (-1)^{j+1} \left[\prod_{m=0}^{j-1} (j-n)! \right]^{-1} \det B^{(1)} = (-1)^{j+1} \left[\prod_{m=0}^{j-1} (j-n)! \right]^{-1} \det B^{(2)} =$$

$$= (-1)^{j+1} \frac{(j+1)!}{j!} \det B^{(3)} = (-1)^{j+1} (j+1) \tau^{2j+1}_{j} \ .$$

Relaxing the hypothesis that $y_1 = 1$ we finish the proof. $\square$

With the help of the preceeding lemmas, we can prove

Theorem 2. *If $y_{2k} = 0$ for all $k \in \mathbb{N}$, then*

$$\frac{\tau^{2j+1}_{j+1}}{\tau^{2j+1}_{j}} (x, 0, y_3, 0, y_5, \cdots) = (-1)^{j+1} \partial_x \log \frac{\vartheta_{j+1}}{\vartheta_j} (x, \chi_2, \chi_3, \cdots) \ ,$$

where ϑ_j is the j-th Adler-Moser polynomial and for a suitable choice of the free variables $y_3, y_5, \cdots$ and $\chi_2, \chi_3, \cdots$.

Proof. From the Lemma 5 and from equation (36) we get $q = r$, so we are in the mKdV hierarchy. All the solutions have been classified and are of the form

$$\pm\partial_x \log \frac{\vartheta_{k+1}}{\vartheta_k}(x) \ ,$$

for $k \in \mathbb{N}$. See [2]. So, for each $j \in \mathbb{N}$, there is a k such that

$$\frac{\tau_{j+1}^{2j+1}}{\tau_j^{2j+1}}(x) = \pm\partial_x \log \frac{\vartheta_{k+1}}{\vartheta_k}(x) \ ,$$

for a particular choice of the variables $\chi_2, \chi_3, \cdots$. Let us consider $y_1 \gg y_j$ for $j \geq 2$. In this case,

$$\mathcal{Q}_j \sim \frac{y_1^j}{j!} \ .$$

Then, from Lemma 6 the LHS is assymptotically equal to $(-1)^{j+1}(j+1)/y_1$. To the RHS, consider the assymptotic behavior in $\chi_1 \to \infty$. In this case, $\vartheta_k \sim \chi_1^{k(k+1)/2}$, and then

$$\partial_{\chi_1} \log \frac{\vartheta_{k+1}}{\vartheta_k}(\chi) \sim \frac{k+1}{\chi_1} \ .$$

Remembering that both k and j are positive, we do the right choice in the sign and conclude that $k = j$. Furthermore $\chi_1 = x = y_1$ and as the above aproximation should be exact, there exists suitable χ_k for $k \geq 2$ such that the identity holds. $\square$

References

[1] Mark J. Ablowitz, David J. Kaup, Alan C. Newell, and Harvey Segur. The inverse scattering transform-Fourier analysis for nonlinear problems. *Studies in Appl. Math.*, 53(4):249–315, 1974.

[2] M. Adler and J. Moser. On a class of polynomials connected with the Korteweg-de Vries equation. *Comm. Math. Phys.*, 61(1):1–30, 1978.

[3] H. Airault, H. P. McKean, and J. Moser. Rational and elliptic solutions of the Korteweg-de Vries equation and a related many-body problem. *Comm. Pure Appl. Math.*, 30(1):95–148, 1977.

[4] M. Belger, R. Schimming, and V. Wünsch. A survey on Huygens' principle. *Z. Anal. Anwendungen*, 16(1):9–36, 1997. Dedicated to the memory of Paul Günther.

[5] Yu. Yu. Berest and A. P. Veselov. The Huygens principle and integrability. *Uspekhi Mat. Nauk*, 49(6(300)):7–78, 1994.

[6] Yuri Berest. Hierarchies of Huygens' operators and Hadamard's conjecture. *Acta Appl. Math.*, 53(2):125–185, 1998.

[7] Yuri Yu. Berest. Deformations preserving Huygens' principle. *J. Math. Phys.*, 35(8):4041–4056, 1994.

[8] Yuri Yu. Berest and Igor M. Loutsenko. Huygens' principle in Minkowski spaces and soliton solutions of the Korteweg-de Vries equation. *Comm. Math. Phys.*, 190(1):113–132, 1997.

[9] J. Burchnall and T. Chaudy. A set of differential equations which can be solved by polynomials. *Proc. London Math. Soc.*, 30:401–414, 1929.

[10] Francesco Calogero and Antonio Degasperis. *Spectral transform and solitons. Vol. I.* North-Holland Publishing Co., Amsterdam, 1982. Tools to solve and investigate nonlinear evolution equations, Lecture Notes in Computer Science, 144.

[11] Paolo Casati, Gregorio Falqui, Franco Magri, and Marco Pedroni. Darboux coverings and rational reductions of the KP hierarchy. *Lett. Math. Phys.*, 41(3):291–305, 1997.

[12] Paolo Casati, Gregorio Falqui, Franco Magri, and Marco Pedroni. A note on fractional KdV hierarchies. *J. Math. Phys.*, 38(9):4606–4628, 1997.

[13] F. A. C. C. Chalub and J. P. Zubelli. On Huygens' Principle for Dirac Operators and Nonlinear Evolution Equations. *J. Nonlin. Math. Phys.*, 8, Supplement:62–68, 2001.

[14] Fabio A.C.C. Chalub. On Huygens' principle for Dirac operators associated to electromagnetic fields, IMPA Preprint - 2001; To appear An. Acad. Brasil. Ciênc.

[15] D. V. Choodnovsky and G. V. Choodnovsky. Pole expansion of nonlinear differential equations. *Nuovo Cimento*, 40B:339–353, 1977.

[16] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. II.* John Wiley & Sons Inc., New York, 1989. Partial differential equations, Reprint of the 1962 original, A Wiley-Interscience Publication.

[17] Gaston Darboux. *Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal. Deuxième partie.* Chelsea Publishing Co., Bronx, N.Y., 1972. Les congruences et les équations linéaires aux dérivées partielles. Les lignes tracées sur les surfaces, Réimpression de la deuxième édition de 1915.

[18] P. A. M. Dirac. The Quantum Theory of the Electron. *Proc. Roy. Soc. of London A*, 117:610–624, 1928.

[19] J. J. Duistermaat and F. A. Grünbaum. Differential equations in the spectral parameter. *Comm. Math. Phys.*, 103(2):177–240, 1986.

[20] G. Falqui, F. Magri, M. Pedroni, and J.P. Zubelli. An elementary approach to the polynomial τ-functions of the KP-hierarchy. *Teoret. Mat. Fiz.*, 122(1):23–36, 2000.

[21] Gregorio Falqui, Marco Pedroni, Franco Magri, and Paolo Casati. *Soliton equations, bi-Hamiltonian manifolds and integrability.* Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1997.

[22] H. Flaschka. Construction of conservation laws for Lax equations. Comments on a paper: "On two constructions of conservation laws for Lax equations" [Quart. J. Math. Oxford Ser. (2) **32** (1981), no. 128, 491–512; MR 83e:58043] by G. Wilson. *Quart. J. Math. Oxford Ser. (2)*, 34(133):61–65, 1983.

[23] H. Flaschka, A. C. Newell, and T. Ratiu. Kac-Moody Lie algebras and soliton equations. II. Lax equations associated with $A_1^{(1)}$. *Phys. D*, 9(3):300–323, 1983.

[24] Gerald B. Folland. Fundamental solutions for the wave operator. *Exposition. Math.*, 15(1):25–52, 1997.

[25] John E. Gilbert and Margaret A. M. Murray. *Clifford algebras and Dirac operators in harmonic analysis.* Cambridge University Press, Cambridge, 1991.

[26] F. A. Grünbaum. Band and time limiting, recursion relations, and some nonlinear evolution equations. In W. Schempp R. Askey, T. Koorwinder, editor, *Special Functions: Group Theoretical Aspects and Applications*, pages 271–286. Dordrecht : Reidel, 1984.

[27] F. Alberto Grünbaum. Time-band limiting and the bispectral problem. *Comm. Pure Appl. Math.*, 47(3):307–328, 1994.

[28] F. Alberto Grünbaum. Some bispectral musings. In *The bispectral problem (Montreal, PQ, 1997)*, pages 31–45. Amer. Math. Soc., Providence, RI, 1998.

[29] Paul Günther. *Huygens' principle and hyperbolic equations.* Academic Press Inc., Boston, MA, 1988. With appendices by V. Wünsch.

[30] Jacques Hadamard. *Lectures on Cauchy's problem in linear partial differential equations.* Dover Publications, New York, 1953.

[31] John Harnad and Alex Kasman. *The bispectral problem. Proceedings of the CRM workshop, Montreal, Canada, March 1997.*

[32] R. Hirota. Direct Methods in Soliton Theory. In R. Bullough and P. Caudrey, editors, *Topics in Current Physics*, pages 157–175. Springer-Verlag, New York, 1980.

[33] N. H. Ibragimov. Huygens' principle. In *Certain problems of mathematics and mechanics (on the occasion of the seventieth birthday of M. A. Lavrent′ev) (Russian)*, pages 159–170. Izdat. "Nauka", Leningrad, 1970.

[34] N. Kh. Ibragimov and A. O. Oganesyan. Hierarchy of Huygens equations in spaces with a nontrivial conformal group. *Uspekhi Mat. Nauk*, 46(3(279)):111–146, 239, 1991.

[35] Michio Jimbo and Tetsuji Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. *Phys. D*, 2(3):407–448, 1981.

[36] Michio Jimbo and Tetsuji Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III. *Phys. D*, 4(1):26–46, 1981/82.

[37] I. M. Kričever. On the rational solutions of the Zaharov-Šabat equations and completely integrable systems of N particles on the line. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 84:117–130, 312, 318, 1979. Boundary value problems of mathematical physics and related questions in the theory of functions, 11.

[38] J. E. Lagnese and K. L. Stellmacher. A method of generating classes of Huygens' operators. *J. Math. Mech.*, 17:461–472, 1967.

[39] Franco Magri and Jorge P. Zubelli. Bi-Hamiltonian formalism and the Darboux-Crum method. I. From the KP to the mKP hierarchy. *Inverse Problems*, 13(3):755–780, 1997.

[40] N. G. Marchuk. Dirac γ-equation, classical gauge fields and Clifford algebra. *Adv. Appl. Clifford Algebras*, 8(1):181–225, 1998.

[41] David Mumford. *Tata lectures on theta. I.* Birkhäuser Boston Inc., Boston, MA, 1983. With the assistance of C. Musili, M. Nori, E. Previato and M. Stillman.

[42] Alan C. Newell. *Solitons in mathematics and physics.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1985.

[43] Robert Sachs. Polynomial τ-Functions for the AKNS Hierarchy. In *Theta Functions (Bowdowin, 1987)*, pages 133–141. Amer. Math. Soc., Providence, RI, 1987.

[44] Robert L. Sachs. On the integrable variant of the Boussinesq system: Painlevé property, rational solutions, a related many-body system, and equivalence with the AKNS hierarchy. *Phys. D*, 30(1-2):1–27, 1988.

[45] J. J. Sakurai. *Advanced Quantum Mechanics.* Addison Wesley, 1967.

[46] Mikio Sato and Yasuko Sato. Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold. In *Nonlinear partial differential equations in applied science (Tokyo, 1982)*, pages 259–271. North-Holland, Amsterdam, 1983.

[47] Rainer Schimming. Korteweg-de vries Hierarchie und Huygensschess Prinzip. In *Dresdener Seminar für Theoretishche Phyzik, Sitzungsberichte*, volume 26. 1986.

[48] Rainer Schimming. Laplace-like linear differential operators with a logarithm-free elementary solution. *Math. Nachr.*, 148:145–174, 1990.

[49] D. Slepian and H. O. Pollak. Prolate spheroidal wave functions, Fourier analysis and uncertainty. I. *Bell System Tech. J.*, 40:43–63, 1961.

[50] David Slepian. Some comments on Fourier analysis, uncertainty and modeling. *SIAM Rev.*, 25(3):379–393, 1983.

[51] Karl L. Stellmacher. Ein Beispiel einer Huyghensschen Differentialgleichung. *Nachr. Akad. Wiss. Göttingen. Math. Phys. Kl. Math.-Phys. Chem. Abt.*, 1953:133–138, 1953.

[52] Karl L. Stellmacher. Eine Klasse Huyghenscher Differentialgleichungen und ihre Integration. *Math. Ann.*, 130:219–233, 1955.

[53] Bernd Thaller. *The Dirac equation.* Springer-Verlag, Berlin, 1992.

[54] C. Wetterich. Massless spinors in more than four dimensions. *Nuclear Phys. B*, 211(1):177–188, 1983.

[55] George Wilson. Bispectral commutative ordinary differential operators. *J. Reine Angew. Math.*, 442:177–204, 1993.

[56] George Wilson. Collisions of Calogero-Moser particles and an adelic Grassmannian (with an appendix by I. G. Macdonald). *Invent. Math.*, 133(1):1–41, 1998.

[57] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Ž. Èksper. Teoret. Fiz.*, 61(1):118–134, 1971.

[58] J. P. Zubelli. Differential equations in the spectral parameter for matrix differential operators. *Physica D*, 43:269–287, 1990.

[59] J. P. Zubelli. On a zero curvature problem related to the ZS-AKNS operator. *Journal of Mathematical Physics*, 33(11):3666–3675, 1992.

[60] Jorge P. Zubelli. Rational solutions of nonlinear evolution equations, vertex operators, and bispectrality. *J. Differential Equations*, 97(1):71–98, 1992.

[61] Jorge P. Zubelli. *Topics on wave propagation and Huygens' principle.* Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 1997.

[62] Jorge P. Zubelli. The bispectral problem, rational solutions of the master symmetry flows, and bi-Hamiltonian systems. In *The bispectral problem (Montreal, PQ, 1997)*, pages 139–155. Amer. Math. Soc., Providence, RI, 1998.

[63] Jorge P. Zubelli and D.S. Valerio Silva. Rational solutions of the master symmetries of the KdV equation. *Commun. Math. Phys.*, 211(1):85–109, 2000.