# Non-Periodic Bifurcations at the Boundary of Hyperbolic Systems

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November 6, 2001

**Abstract**

We prove that diffeomorphisms with tangencies between stable and unstable manifolds of non-periodic points are abundant on the boundary of hyperbolic systems in a probabilistic sense. For these diffeomorphisms all periodic points are hyperbolic and all their stable and unstable manifolds are transverse.

# 1 Introduction

A great deal of understanding of those dynamical systems lying beyond the hyperbolic (Axiom A [17]) domain has been achieved through the following strategy: one considers parametrized families $(\Phi_\theta)_{\theta\in[-1,1]}$ of systems (diffeomorphisms or flows) such that $\Phi_\theta$ is hyperbolic for initial $\theta = 1$, but crosses the boundary of hyperbolic systems at some parameter, say $\theta = 0$.

The problem of understanding typical behavior of $\Phi_\theta$ is, probably, not realistic without some further assumptions about the dynamics at $\theta = 0$. However, in a series of papers that appeared in the seventies and the early eighties, Newhouse, Palis, and Takens were able to develop a detailed theory of the case when the limit set (forward and backward accumulation points of all the orbits) of $\Phi_0$ is finite. See [8, 9, 10, 11]. In this case, for the generic (Baire second category) arc of diffeomorphisms, the first bifurcation corresponds to one of the following two situations:

(NH) *there exists a unique periodic orbit that is non-hyperbolic,* and it is either a saddle-node (one eigenvalue equal to 1), a period-doubling (one eigenvalue equal to $-1$), or a Hopf orbit (two complex conjugate eigenvalues with norm 1);

(NT) all the periodic orbits are hyperbolic, but *there exists a unique non-transverse intersection between some stable and some unstable manifold of periodic orbits*; this intersection is quasi-transverse (codimension 1).

The hypotheses that the limit set be finite was dropped in many subsequent papers, including [3, 4, 6, 12, 13, 14, 15, 16], that deal with the formation and/or destruction of non-trivial hyperbolic sets through homoclinic or heteroclinic bifurcations. Prior to that, there had been the work of Newhouse [7] on coexistence of infinitely many attractors in the unfolding of homoclinic tangencies.

In all these works the authors assume that the bifurcation is of periodic type, that is, of type (NH) or (NT) as above. It is natural to ask whether this is still a generic property of arcs of diffeomorphisms when the limit set at the bifurcation is not finite. However, to present date this question is wide open.

In fact, our results in this work indicate that we have a negative answer to the probabilistic reformulation of the problem. That is, there exists large (in a measure-theoretical sense) subsets of the boundary of the hyperbolic domain formed by diffeomorphisms whose periodic orbits are all hyperbolic, and whose stable and unstable manifolds of periodic orbits are all transverse. The reason hyperbolicity breaks down is the presence of a tangency between the stable manifold and the unstable manifold of some non-periodic orbits.

A formal statement will appear in the main theorem below. Right now, let us give an idea of the construction. We consider a 2-parameter family $(\Phi_{\tilde{A},\theta})$ of diffeomorphisms such that $\Phi_{A,1}$ is hyperbolic, and has a non-trivial attractor. We use the parameter $\theta$ to deform the diffeomorphism near a point in the attractor, so as to create for $\theta = 0$ a (cubic) tangency between some stable and some unstable manifold. Then we prove that the diffeomorfism $\Phi_{\tilde{A},\theta}$ is hyperbolic for intervals of values of $\theta$ accumulating $\theta = 0$.

The other parameter $\tilde{A}$ allows us to control which stable and unstable leaves are involved in the tangency. In fact, the construction, which is based on a Benedicks-Carleson [2, 5] type of argument, involves exclusions: it can be carried out only for a positive Lebesgue measure set $E$ of $\tilde{A}$. It is in this sense that we say that our examples are probabilistically abundant. Then

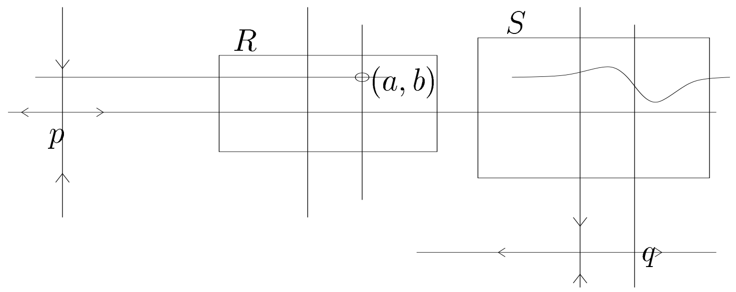



Figure 1: Cubic Tangency

we prove that for all $\tilde{A} \in E$ all the periodic orbits of $\Phi_{\tilde{A}}$ are hyperbolic and all their stable and unstable manifolds are transverse.

# 2 Description of the Problem

Let $\Phi$ be a diffeomorphism on a 2-dimensional manifold $M$ with a non-trivial hyperbolic attractor $\Omega$. Suppose $p$, $q$ are two periodic saddles in $\Omega$ with unstable index 1 and let $A$ be a heteroclinic point associated to $p$ and $q$ (see figure 1).

Up to requiring generic non-resonance conditions on eigenvalues (see [18]), we may suppose $\Phi$ that is linearizable around $p$ and $q$ and both linearizing coordinates domains contain $A$. Let $(x_r, y_r) \in \mathbb{R}^2$ and $(x_s, y_s) \in \mathbb{R}^2$ denoting such coordinates at $p$ and $q$ respectively.

Now fix cubes $R$ and $S$, containing $A$ and $B = \Phi(A)$, such that for $(x_r, y_r) \in R$ we have

$$\Phi(x_r, y_r) = (\lambda \cdot x_r, \sigma \cdot y_s) \in S;$$

where $\lambda > 1, \sigma < 1$. But, for simplicity of notation, we omit subscripts and put only:

$$\Phi(x, y) = (\lambda \cdot x, \sigma \cdot y).$$

Since the idea is to create a cubic tangency between the unstable direction and the stable manifold we pick up the $y$ direction and deform it. That is, we deform $\Phi$ until we get something like:

$$\varphi(x, y) = (\sigma \cdot y + x \cdot (x^2 + y^2), \lambda \cdot x - y \cdot (x^2 + y^2)).$$

More precisely, we denote $\theta$ the angle that the unstable direction makes with the stable direction and we intend to work with deformations with taylor series around $A$ as:

$$\varphi_\theta(x,y) = (\sigma\cdot y + x\cdot(C\cdot\theta + D\cdot x^2 + E\cdot y^2), \lambda\cdot x - y\cdot(C'\cdot\theta + D'\cdot x^2 + E'\cdot y^2)).$$

where the constants $C, C', D, D', E, E'$ are positive real numbers and, for simplifying notation, let we also assume $A = B = 0$. But that can be done not only for $A$ (and $B$) but also for every $\tilde{A}\in R$ so that we have the right to put $\varphi = \varphi_{\theta,\tilde{A}}$

Observe that we can define one such deformation by glueing the above $\varphi_{\theta,\tilde{A}}$ with $\Phi$. For this purpose, let $R'\supset R$ a little big cube and take smooth partitions of the unity $\Psi_1, \Psi_2$ associated to $R'$, $M\setminus R$ respectively. Then we just define:

$$\Phi_{\tilde{A},\theta} = \Psi_1\cdot\varphi_{\theta,\tilde{A}} + \Psi_2\cdot\Phi.$$

Futhermore, we suppose $\Phi$ is strongly dissipative in the sense we make precise later (see Section 10).

Now, we finish the section with the statement of the main theorem.

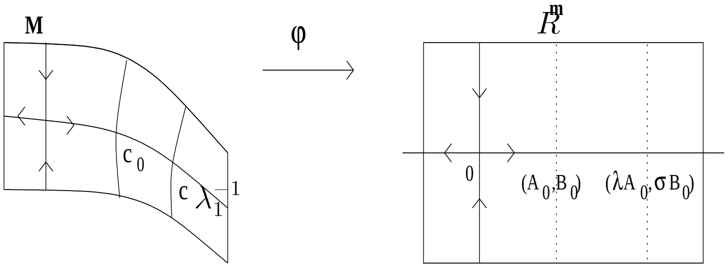
**Theorem A.** *Let $\Phi_{\tilde{A},\theta}$ be as above. Then, there is a positive Lebesgue measure (area) set of values $\tilde{A}$ such that $\Phi_{\tilde{A},0}$ is approached by hyperbolic systems and this set intersects the unstable manifold of $p$ in a subset $E$ with positive Lebesgue measure (lenght). For $\tilde{A}\in E$ the diffeomorphism $\Phi_{\tilde{A},0}$ is on the boundary of hyperbolic systems. Moreover, in the last case, all periodic orbits of $\Phi_{\tilde{A},0}$ are hyperbolic and their stable and unstable manifolds are all transverse.*

The next sections are dedicated to stablish the main theorem above and are as follows: in the Sections 3, 4, 5, 6 define the critical point concept by the use of the contractive approximations; Sections 7, 8, 9 describe the whole process and prove the induction hypotheses; finally, Section 10 computes the remaining parameters after putting so many restrictions.

# 3 Expansion for Orbits Distant from Critical Region

The purpose of this section is to show that pieces of orbits that remains distant from the point $\tilde{A}$ have some kind of uniform expansion on the horizontal direction. Precisely, we look for a $\lambda$-expansion for the vector $(1,0)$ for pieces of orbits outside ( say $\delta$-distant from ) the critical region.

The argument begins by computing an average expansion bigger than one for $\Phi_{A,0}$ and every orbit throghout $R$. Then, fixing $\delta > 0$, there is a neighborhood of $\Phi_{A,0}$ in which the maps $\Phi_{\tilde{A},\theta}$ still have average expansion bounded away from one for orbits outside a $\delta$-region around the heteroclinic point. So that the remaining sections will assume we are always with $(\tilde{A},0)$ close enough to $(A,0)$ in this sense.




**Lemma 3.1.** *as above, for every $\delta > 0$ there is a neighborhood $V$ of $\Phi_{A,0}$ in the $C^2$ topology such that maximal $\Phi_{\tilde{A},\theta}$-orbits, $\Phi_{\tilde{A},0} \in V$, have uniform expansion bounded away from one. Moreover, if we consider a non-maximal orbit then the passage from $R$ to $S$ introduces a small factor $\delta^2$ on the norm and $\frac{1}{\delta^2}$ on the slope of an almost horizontal vector over the orbit.*

First of all, fix $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ eigeinvalues or inverse of eigeinvalues of $p$ and $q$ (see figure 3). Let also $b'$ ( $b$ ) denote cubes transversal to stable ( unstable ) manifold of $p$ ( $q$ ) such that an orbit reaches $R$ ( S ) only if some earlier ( later ) iterate belong to $b'$ ( $b$ ).

Let also $v$ a vector on the tangent space of $b'$ such that one positive iterate of it goes throught a point $(x,y) \in R$.

We intend to estimate the average expansion that an almost horizontal vector $v$ gain walking from $b'$ to $b$.

Without loss of generality, suppose $p = q = 0$.

Let $K'$ ( $K$ ) the time the vector $v$ spend from $b'$ ( $b$ ) to $R$ ( $S$ ).Then,

$K' = \frac{-ln(|y|)}{ln\lambda_1}$

$K = \frac{-ln(|\Phi_1(x,y)|)}{ln\lambda_2}$

where $\Phi_1$ means the first coordinate function of $\Phi$.

Now, to estimate expansion, let $v = v_{b'} = (1, 0)$. Iterating $v_{b'}$ until it enters cube $b$ we conclude that,

$v_b = ({\lambda_2}^{K'} \cdot {\lambda_3}^{K} \cdot (C \cdot \theta + 3 \cdot D \cdot x^2 + E \cdot y^2), {\lambda_2}^{K'} \cdot {\lambda_4}^{-K} \cdot (\lambda_2 - 2 \cdot E \cdot x \cdot y))$

Then $slope(v_b)$ is:

$$\frac{|\lambda_2 - 2 \cdot E \cdot x \cdot y| \cdot |{\lambda_1}^{-1} \cdot y + x \cdot (C \cdot \theta + D \cdot x^2 + E \cdot y^2)|^{\frac{ln(\lambda_3) + ln(\lambda_4)}{ln(\lambda_3)}}}{|C \cdot \theta + 3 \cdot D \cdot x^2 + E \cdot y^2|}$$

And $\| v_b \|^2$ is:

$$\left( \frac{|C \cdot \theta + 3 \cdot D \cdot x^2 + E \cdot y^2|}{|y|^{\frac{ln(\lambda_2)}{ln(\lambda_1)}} \cdot |\lambda_1^{-1} \cdot y + x \cdot (C \cdot \theta + D \cdot x^2 + E \cdot y^2)|} \right)^2 + \left( \frac{|\lambda_2 - 2 \cdot E \cdot x \cdot y| \cdot |{\lambda_1}^{-1} \cdot y + x \cdot (C \cdot \theta + D \cdot x^2 + E \cdot y^2)|}{|y|^{\frac{ln(\lambda_2)}{ln(\lambda_1)}}} \right)^2$$

Observe that a simply analysis on the slope and norm of $v_b$ shows that both are bounded away from zero even for very small $y$. Moreover, the first term of the norm estimative above tends to grow with decreasing of $y$ even if we assume more dissipativeness for the attractor $\Omega$ by demanding increase of $\lambda_1$.

To compute the average expansion put $C_i = ln(\lambda_i)$ and then,

$$\left(\frac{1}{K' + K}\right) \cdot ln(\| v_b \|) \geq C_3 \cdot \frac{ln\left( \frac{|y|^{\frac{C_2}{C_1}} \cdot |{\lambda_1}^{-1} \cdot y + x \cdot (C \cdot \theta + D \cdot x^2 + E \cdot y^2)|}{|C \cdot \theta + 3 \cdot D \cdot x^2 + E \cdot y^2|} \right)}{ln\left( |y|^{\frac{C_3}{C_1}} + |{\lambda_1}^{-1} \cdot y + x \cdot (C \cdot \theta + D \cdot x^2 + E \cdot y^2)| \right)}$$

Now we observe, by comparing exponents, that there is a uniform, in $y$, average expansion.

So maximal pieces of orbits outside some small $\delta$-region around the bifurcation have, as said before, uniform expansion and small slope: the piece of orbit from $b'$ to $b$ always has an average expansion bigger than 1. If the piece of orbit is not a maximal one outside this region, meaning that the orbit begins in $b'$ but do not reach $b$, there is a crossing time trought $R$ not recovering the lost of expansion: this generate, by the form of the map from $R$ to $S$, a decrease of a factor $\delta^2$ on norm and an increase of $\frac{1}{\delta^2}$ on the slope of horizontal vectors. Recall that due to the global foliation pieces of orbits outside this problematic region surely expands and are pressing to horizontal directions ( decrease of slope ).

As in Hénon-like case these establish the driving force lemma which permit the induction process works.

□

Remark: If $\Phi_{\tilde{A},\theta}$ was obtained by partitions of the unity, as said in anterior section,

$\Phi_{\tilde{A}\theta} = \Psi_1 \cdot \varphi_{\theta,\tilde{A}} + \Psi_2 \cdot \Phi$

Than it is easy to see that pieces of orbits throught $R' \setminus R$ still have average expansion bounded away from zero. For this just differentiate the above formulae, remenbering that $\Psi_1 + \Psi_2 = 1$, and computations are as before.

Remark: Before keep going with the computations a comment about the constants involved is need:

1. $M \geq \| D\Phi_z \|$ is the maximum expansion constant.
2. $b \geq \| det(D\Phi_z) \|$ is the determinant of $f$ and for hyperbolic attractors we may think it as an extra parameter which can be reduced as wanted by increasing the attraction rates.
3. $\lambda$ means the expansion obtained inductively.
4. $1 > \beta > \alpha > \delta$ are small numbers defined below and $N$ mean a big time in which we surely have expansion for horizontal vectors.

# 4 Contractive Directions

This section is dedicated to obtain the $C^2$-small variation of the contractive directions. For the sake of notational simplicity sometimes we simply write the diffeomorphism as $\Phi$.

*Definition: We say a point $z_0$ is $\lambda$-expansive up to $n$ meaning*

$\| \omega_\nu(z_0) \| \geq \lambda^\nu$ ; $0 \leq \nu \leq n$ ; $\omega_\nu(z_0) = D\Phi_{z_1}{}^\nu \cdot (1,0)$ , $z_1 = \Phi(z_0)$

That is, $z_0$ is expansive if the horizontal vectors expands until time $n$ in the $z_0$ orbit.

But if $z_0$ is expansive there exist a maximal expansive direction up to $n$ that will be denoted by $f^n$. Clearly,

$\| D\Phi_{z_1}{}^n \cdot f^n \| \geq \lambda^n$

There is also a maximal contractive direction denoted by $e^n$ with $e^n \perp f^n$ and $\| D\Phi_{z_1}{}^n / e^n \| \cdot \| D\Phi_{z_1}{}^n / f^n \| = | detD\Phi_{z_1}{}^n |$.

From now on $b$ means a number so that:

$| detD\Phi_{z_1} | \leq b$; $\forall z_0, \forall \theta \geq 0$

So,

$\| D\Phi_{z_1}{}^n \cdot e^n \| \leq \frac{b^n}{\lambda^n} = (\frac{b}{\lambda})^n$.

Now an lemma that provides $C^0$-exponentially convergence of $e^n$ as $n$ increases. That is, if they exist for all values of $n$ they should approach the stable manifold at $z_0$.

**Lemma 4.1.** *suppose $e^\nu$ and $e^\mu$ exist, then*
$| \, angle(e^\mu, e^\nu) \, | \leq const \cdot (\frac{b}{\lambda^2})^\mu; \; \mu \leq \nu$
$\| \, D\Phi_{z_1}{}^\mu \cdot e^\nu \, \| \leq const \cdot (\frac{Mb}{\lambda^2})^\mu$

Proof: Write $e^{\nu-1} = \xi \cdot e^\nu + \eta \cdot f^\nu \Rightarrow | \, angle(e^{\nu-1}, e^\nu) \, | = | \arctan \frac{\eta}{\xi} \, | \leq \frac{1}{\sqrt{\frac{1}{\eta^2} - 1}}$

But its know that: $| \, \eta \, | \cdot \| \, D\Phi_{z_1}{}^\nu \cdot f^{\nu-1} \, \| \leq \| \, D\Phi_{z_1}{}^\nu \cdot e^{\nu-1} \, \|$ because $D\Phi_{z_1}{}^\nu \cdot e^\nu \perp D\Phi_{z_1}{}^\nu \cdot f^\nu$

Putting all estimates together: $| \, \eta \, | \leq \frac{M \cdot (\frac{b}{\lambda})^{\nu-1}}{\lambda^\nu} = \frac{M}{\lambda} \cdot (\frac{b}{\lambda^2})^{\nu-1}$

Finally,
$| \, angle(e^{\nu-1}, e^\nu) \, | \leq \frac{1}{\sqrt{\frac{1}{[\frac{M}{\lambda} \cdot (\frac{b}{\lambda^2})^{\nu-1}]} - 1}} \leq \frac{1}{\sqrt{\frac{1}{2} \cdot [\frac{M}{\lambda} \cdot (\frac{b}{\lambda^2})^{\nu-1}]^2}} \leq \frac{2M}{\lambda} \cdot (\frac{b}{\lambda^2})^{\nu-1}$

where it was used that $1 \leq \frac{1}{2} \cdot \frac{1}{[\frac{M}{\lambda} \cdot (\frac{b}{\lambda^2})^{\nu-1}]^2}$ which holds for $\nu$ very large.

Observe that we may suppose $\nu \geq N$ with $N$ as big as needed because $e^1 = \ldots = e^N$.

Then
$| \, angle(e^\mu, e^\nu) \, | \leq \sum_\mu^{\nu-1} \frac{2M}{\lambda} \cdot (\frac{b}{\lambda^2})^i \leq (\frac{2M\lambda^2}{\lambda(\lambda^2 - b)}) \cdot (\frac{b}{\lambda^2})^4 = const \cdot (\frac{b}{\lambda^2})^\mu$

And
$\| \, D\Phi^\mu \cdot e^\nu \, \| \leq M^4 \cdot \frac{2M\lambda^2}{\lambda \cdot (\lambda^2 - b)} \cdot (\frac{b}{\lambda^2})^\mu + (\frac{b}{\lambda})^\mu \leq const \cdot (\frac{Mb}{\lambda^2})^\mu$ ; for $\mu$ large.

□

We know for $1 \leq \nu \leq N$: $e^1 = \ldots = e^N$.

For $\nu > N$ the last lemma assure $e^\nu$ and $e^N$ very close to each other. That means, $e^\nu$ is almost vertical. I.e., increasing $N$ $e^\nu$ is as close to $y$-axis as wanted

**Lemma 4.2.** *let $z_1$ $\delta$-distant from $\tilde{A}$ and $\| \, D\Phi^\nu(z_1).v \, \| \geq \lambda$ for $1 \leq \nu \leq n$ then every unitary small slope vetor $v_0$ at $z_1$ satisfies:*
$\| \, D\Phi^\nu(z_1) \cdot v_0 \, \| \geq \frac{1}{2} \cdot \| \, D\Phi^\nu(z_1) \cdot f^v \, \| = \frac{1}{2} \cdot \| \, D\Phi^\nu(z_1) \, \| = \frac{1}{2} \cdot \| \, f_\nu^\nu \, \|$ *; $\nu$ big*

Proof: Put $v_0 = \alpha \cdot e^\nu + \beta \cdot f^\nu$ with $\| \, D\Phi^\mu \cdot f^\nu \, \| \geq \| \, D\Phi^\mu \mu_0 \, \| \geq \lambda^\mu$

But $N$ big implies: $| \, \beta \, | \geq 1 - c$ and $| \, \alpha \, | \leq c$ for all $v_0$ small slope vector and with $c$ small as wanted.

Then,
$\| \, D\Phi^\nu v_0 \, \| = | \, \alpha e_\nu{}^\nu + \beta f_\nu{}^\nu \, | \geq | \, \beta \, | \cdot \| \, f_\nu{}^\nu \, \| - | \, \alpha \, | \cdot \| \, e_\nu{}^\nu \, \| \geq \frac{1}{2} \cdot \| \, f_\nu{}^\nu \, \|$

□

Before next lemma recall some important constants: $\| \, D^2\Phi(x) \, \| \leq K$ and $\| \, D\Phi(x) \, \| \leq M$.

Now if $z_0$ and $\xi_0$ close to each other and $u$ and $v$ are very close vectors on $z_0$ and $\xi_0$ then we expect the same behavior on iteration.

**Lemma 4.3.** *let $z_0, \xi_0 \in R$ very close , ie, $\| z_0 - \xi_0 \| \leq \sigma^n$ and $u$ and $v$ close unitary vectors , ie , $\| u - v \| \leq \sigma^n$*

*Then if $\| D\Phi^\nu(z_1) \cdot u \| \geq \lambda^\nu; 1 \leq \nu \leq n$ :*

*$a) \frac{1}{2} \leq \| \frac{D\Phi^n(z_1) \cdot u}{D\Phi^\nu(\xi_1) \cdot \nu} \| \leq 2$*

*$b) \mid angle(D\Phi^\nu(z_1) \cdot u, D\Phi^\nu(\xi_1) \cdot v) \mid \leq \sigma^{\frac{n}{2}}$*

Proof: compute $\| D\Phi^\nu(z_1) \cdot u - D\Phi^\nu(\xi_1) \cdot v \| \leq \| D\Phi^\nu(z_1) \cdot u - D\Phi^\nu(\xi_1) \cdot u \|$ $+ \| D\Phi^\nu(\xi_1)(u - v) \| \leq \| D\Phi^\nu(z_1) - D\Phi^\nu(\xi_1) \| + \| D\Phi^\nu(\xi_1) \| \cdot \| u - v \|$

Remark that: $\| D\Phi^\nu(z_1) - D\Phi^\nu(\xi_1) \| = \| D\Phi(z_{\nu-1}) \cdot D\Phi^{\nu-1}(z_1) - D\Phi(\xi_1) \cdot D\Phi^{\nu-1}(\xi_1) \| \leq KM^\nu \sum_{i=1}^{\nu} \| z_i - \xi_i \|$ is proved by induction.

Then

$\leq KM^\nu \cdot \sum_{i=1} \nu \| z_i - \xi_i \| + M^\nu \cdot \| u - v \| \leq M^\nu \cdot (\sigma^n \cdot M^\nu \cdot \frac{MK}{M-1} + \sigma^n) \leq CM^{2\nu}\sigma^n$

For $\sigma$ small with respect to $M$ and $\lambda$ :

$\| D\Phi^\nu(z_1) \cdot u - D\Phi^\nu(\xi_1) \cdot v \| \leq CM^{2\nu}\sigma^n \leq \frac{1}{2}\lambda^\nu \leq \frac{1}{2} \cdot \| D\Phi^\nu(z_1) \cdot u \|$

$a)$ follow dividing by $\| D\Phi^\nu(z_1) \cdot u \|$ and $b)$ also.

□

Now we suppose the point is far from the critical region, then for two bounded points up to $n$ and two very close small slope unitary vectors on these points we expect the expansion and angle almost the same:

**Lemma 4.4.** *let $z_0$ $\delta$-distant from $\tilde{A}$ and $\lambda$-expanding up to $n$ and $\| z_\nu - \xi_\nu \| \leq \sigma^\nu$ for $1 \leq \nu \leq n$ bounded points. Then:*

*$a) \frac{1}{2} \leq \frac{\| D\Phi^\nu(z_1) \cdot u \|}{\| D\Phi^\nu(\xi_1) \cdot v \|} \leq 2$*

*$b) \mid angle(D\Phi^\nu(z_1) \cdot u, D\Phi^\nu(\xi_1) \cdot v) \mid \leq const \cdot \left(\frac{b}{\lambda}\right)^{\nu-1}$*

*for any $1 \leq \nu \leq n$ and any unitary small slope vectors $u$ and $v$.*

Proof: Until $N$ big :

$$\begin{cases} u_\mu = D\Phi^\mu(z_1) \cdot u \\ v_\mu = D\Phi^\mu(\xi_1) \cdot v \end{cases}$$

$$D\Phi^\mu(z_1) = \begin{pmatrix} \lambda^N & 0 \\ 0 & \sigma^N \end{pmatrix} = D\Phi^\mu(\xi_1) \ ; \ \mu \leq N$$

First put $v_0 = \alpha_0 \cdot u_0 + \epsilon_0$ ; $\epsilon_0$ vertical $u_0 = u$ , $v_0 = v$ with $\| \epsilon_0 \| \leq c$ and $\mid \alpha \mid \geq 1 - c$

Then, until $N$:

$v_\mu = \alpha_\mu \cdot u_\mu + \epsilon_\mu$ ; $\begin{cases} \| \epsilon_\mu \| \leq C\sigma^\mu \leq C \\ | \alpha_\mu | = \ldots = | \alpha_0 | \geq 1 - c \end{cases}$

These means $v_\mu$ and $u_\mu$ approach to each other for $\mu \leq N$.

Now suppose $v_\mu = \alpha_\mu \cdot u_\mu + \epsilon_\mu$ at $z_{\mu+1}$ and $\xi_{\mu+1}$.

Relate time $\mu$ with time $v \approx k \cdot \mu$ ; $k \in N$ fixed.

At $z_{\mu+1}$ there are $f^{\nu-\mu} = f^{\nu-\mu}(z_{\mu+1})$ and $e^{\nu-\mu} = e^{\nu-\mu}(z_{\mu+1})$.

Recall that the above lemma give $\| u_\nu \| \geq \frac{\lambda^\nu}{2}$ ; $\nu \leq n$ big.

$\| {f^{\nu-\mu}}_{\nu-\mu} \| \geq \| D\Phi^{\nu-\mu}(z_{\mu+1}) \cdot \frac{u_\mu}{\|u_\mu\|} \| \geq \frac{\lambda^\nu}{2 \cdot \|u_\mu\|} \geq \frac{\lambda^\nu}{2 \cdot M^\mu}$

hence,

$\| {e_{\nu-\mu}}^{\nu-\mu} \| \leq \frac{b^{\nu-\mu}}{\frac{\lambda^\nu}{2M^\mu}} = \frac{2 \cdot M^\mu \cdot b^{\nu-\mu}}{\lambda^\nu}$

Also ${u'}_\mu = \frac{u_\mu}{\|u_\mu\|}$ and $f^{\nu-\mu}$ are not so far since :

if ${u'}_\mu = \alpha \cdot f^{\nu-\mu} + \beta \cdot e^{\nu-\mu} \Rightarrow \alpha = \langle {u'}_\mu, f^{\nu-\mu} \rangle \Rightarrow$

$\Rightarrow D\Phi^{\nu-\mu}(z_{\mu+1}) \cdot {u'}_\mu = \alpha \cdot {f_{\nu-\mu}}^{\nu-\mu} + \beta \cdot {e_{\nu-\mu}}^{\nu-\mu} \Rightarrow$

$\Rightarrow \| {f_{\nu-\mu}}^{\nu-\mu} \| \cdot | \alpha | = \| D\Phi^{\nu-\mu} \cdot {u'}_\mu - \beta \cdot {e_{\nu-\mu}}^{\nu-\mu} \| \geq \frac{\lambda^\nu}{2M^\mu} - 1 \cdot \frac{2 \cdot M^\mu \cdot b^{\nu-\mu}}{\lambda^\nu} \Rightarrow$

$\Rightarrow | \alpha | \geq \frac{\frac{\lambda^\nu}{2M^\mu} - \frac{2M^\mu b^{\nu-\mu}}{\lambda^\nu}}{M^{\nu-\mu}} = \frac{1}{2} \cdot \frac{\lambda^\nu}{M^\nu} - \frac{2 \cdot M^{2\mu} \cdot b^{\nu-\mu}}{M^\nu \cdot \lambda^\nu} =$

$\frac{1}{2} \cdot \left(\frac{\lambda}{M}\right)^\nu - 2\left(\frac{b^{1-\frac{1}{k}}}{M^{1-\frac{2}{K}} \cdot \lambda}\right)^\nu \geq \frac{1}{2} \cdot \left(\frac{\lambda}{M}\right)^\nu - 2 \cdot \left(\frac{b^{\frac{1}{2}}}{\lambda}\right)^\nu \geq \frac{1}{5} \cdot \left(\frac{\lambda}{M}\right)^\nu$

Observe that this holds for $\lambda^2 > M \cdot b^{\frac{1}{2}}$ and $\nu$ large ( without increasing $k$ ) or increasing $k$ ( and the above inequality is easily satisfied ).

So $| \langle u_\mu, f^{\nu-\mu} \rangle | \geq \frac{1}{5} \cdot \frac{\lambda^\nu}{M^\nu} \cdot \| u_\mu \| \geq \frac{1}{10} \cdot \frac{\lambda^{\nu+\mu}}{M^\nu}$

This estimate will be usefull because we decompose $\epsilon_\mu = \bar{\alpha}_\mu \cdot u_\mu + \bar{\beta}_\mu \cdot e^{\nu-\mu}$ in terms of $e^{\nu-\mu}$ and

$v_\nu = D\Phi^{\nu-\mu}(\xi_{\mu+1}) \cdot v_\mu = D\Phi^{\nu-\mu}(z_{\mu+1}) \cdot v_\mu + (D\Phi^{\nu-\mu}(\xi_{\mu+1}) - D\Phi^{\nu-\mu}(z_{\mu+1})) \cdot v_\mu$

But $v_\mu = \alpha_\mu \cdot u_\mu + \epsilon_\mu$ , $v_\nu = \alpha_\mu \cdot u_\nu + D\Phi^{\nu-\mu}(z_{\mu+1}) \cdot \epsilon_\mu + (D\Phi^{\nu-\mu}(\xi_{\mu+1}) - D\Phi^{\nu-\mu}(z_{\mu+1})) \cdot v_\mu$

And so , $v_\nu = (\alpha_\mu + \bar{\alpha}_\mu) \cdot u_\nu + [\bar{\beta}_\mu \cdot {e^{\nu-\mu}}_{\nu-\mu} + (D\Phi^{\nu-\mu}(\xi_{\mu+1}) - D\Phi^{\nu-\mu}(z_{\mu+1})) \cdot v_\nu]$

Now we need to estimate $\bar{\alpha}_\mu$ and $\bar{\beta}_\mu$ in order to get $\alpha_\nu$ and $\epsilon_\nu$ :

$| \bar{\alpha}_\mu | = \frac{|\langle \epsilon_\mu, f^{\nu-\mu} \rangle|}{|\langle u_\mu, f^{\nu-\mu} \rangle|}$

And ,

$| \bar{\beta}_\mu | = \frac{|\langle \epsilon_\mu, {u_\mu}^\perp \rangle|}{|\langle {u_\mu}^\perp, e^{\nu-\mu} \rangle|} = \frac{|\langle \epsilon_\mu, {u_\mu}^\perp \rangle|}{|\langle u_\mu, f^{\nu-\mu} \rangle|}$

It follows that,

$| \bar{\alpha}_\mu | \leq \| \epsilon_\mu \| \cdot \frac{10 \cdot M^\nu}{\lambda^{\nu+\mu}}$ and $| \bar{\beta}_\mu | \leq \| \epsilon_\mu \| \cdot \frac{10 \cdot M^{\nu+\mu}}{\lambda^{\nu+\mu}}$

By induction,

$\| D\Phi^{\nu-\mu}(\xi_{\mu+1}) - D\Phi^{\nu-\mu}(z_{\mu+1}) \| \leq K \cdot M^{\nu-\mu} \cdot \sum_{j=\mu+1}^{\nu} \| z_j - \xi_j \| \leq 2K \cdot M^{\nu-\mu} \cdot \sigma^{\mu+1}$ ; if $\sigma$ very small

Now bound $\epsilon_\nu$ :

$\| \epsilon_\nu \| \leq \frac{2 \cdot M^\nu \cdot b^{\nu-\mu}}{\lambda_\nu} \cdot \frac{10 \cdot \|\epsilon_\mu\| \cdot M^{\nu-\mu}}{\lambda^{\nu+\mu}} + 2K \cdot M^{\nu-\mu} \cdot \sigma^{\mu+1} \cdot M^\mu$

Or

$\| \epsilon_\nu \| \leq 2 \cdot K \cdot M^\nu \cdot \sigma^{\mu+1} + \frac{20 \cdot b^{\nu-\mu} \cdot M^\nu}{\lambda^{2 \cdot \nu+\mu}} \cdot \| \epsilon_\mu \|$

Inductively: first $\| \epsilon_\mu \| \leq c\tilde{\sigma}^\mu \leq (\frac{2K}{M} \cdot \sigma^{\frac{k-1}{k}}) \cdot (M \cdot \sigma^{\frac{1}{k}})^{\mu+1}$ if $k$ big and $\mu$ big ($N$ big).

$\| \epsilon_{k^n \cdot \mu} \| \leq (M\sigma^{\frac{1}{k}})^{k^n\mu+1} \cdot [\frac{2k\sigma^{\frac{k-1}{k}}}{M}] \cdot$

$\cdot (1 + 20\bar{\sigma}^{k^n} + 20^2\bar{\sigma}^{k^n+k^{n-1}} + \ldots + 20^n\bar{\sigma}^{k^n+\ldots k})$

Where $\bar{\sigma} = (\frac{b^{\frac{k-1}{k}} \cdot M^{\frac{1}{k}}}{\lambda^{2+\frac{1}{k}} \cdot \sigma^{\frac{k-1}{k^2}}})^\mu$ can be taken small than 1.

Inductively on $\alpha_\mu$ : first $| \alpha | \geq (1-c)$

$| \alpha_{k^n \cdot \mu} | = | \alpha_{k^{n-1} \cdot \mu} + \bar{\alpha}_{k^{n-1} \cdot \mu} | \geq$

$\geq (1-c) - \frac{10C'\lambda^{k+1}}{M^k} \cdot (\frac{M^{k+1}\sigma^{\frac{1}{k}}}{\lambda^{k+1}})^{\mu+1} - \ldots - \frac{10C'\lambda^{k+1}}{M^k} \cdot (\frac{M^{k+1}\sigma^1 k}{\lambda^{k+1}})^{k^n\mu+1}$

where $\| \epsilon_{k^n{}_\mu} \| \leq C' \cdot (M \cdot \sigma^{\frac{1}{k}})^{k^n \cdot \mu+1}$ with small C'.

To finish we need $\frac{M^{k+1}\sigma^{\frac{1}{k}}}{\lambda^{k+1}}$ small than 1 but this happen only if $k$ is not big and $\sigma$ is small. This means that the argument is unusefull for Anosov situation for example. But this problem doesn't exist in our case because we have strong contractiveness, ie , the possibility to reduce $b = \det(D\Phi)$ as wanted (and fix $k = 2$).

□

Let's now estimate how $f^\nu$ depends on $(z, \theta)$.

**Lemma 4.5.** *let $z$ $\lambda$-expansive up to $n$ then for $\nu \leq n$ :*

*a) $\| D_{(\theta,z)}f^\nu \| \leq K^\nu$ and $\| D_{(\theta,z)}g^\nu \| \leq K^\nu$ ; here $g^\nu = \frac{D\Phi^\nu \cdot f^\nu}{\|D\Phi^\nu \cdot f^\nu\|}$*

*b) $\| D_{(\theta,z)}{f_j}^\nu \| \leq K^\nu$ and $\| D_{(\theta,z)}{g_j}^\nu \| \leq K\nu$ ; here ${g_j}^\nu = (D\Phi^*)^{-j} \cdot f^\nu = \frac{(D\Phi^*)^{\nu-j} \cdot g^\nu}{\|D\Phi^\nu \cdot f^\nu\|}$*

Proof: Over the sphere bundle on $M$ define: $F^\nu : T_1(M) \to T_1(M)$

$F^\nu(f) = \frac{(D\Phi^\nu)^* \cdot (D\Phi^\nu) \cdot f}{\|(D\Phi^\nu)^* \cdot (D\Phi^\nu) \cdot f\|}$ ; where * means adjunt operator

By the singularity values theorem for linear maps:

there exist $u_1 \ldots u_n$ ortonormal base, $u_i$ eigeinvectors of $A^*A$ and: $Au_i = \sigma_i \cdot v_i, A^*v_i = \sigma_i \cdot u_i$

Observe also that $max \| Au \|^2 = max\langle A^*Au, u\rangle$ ; for $\| u \| = 1$

Write $u = a_1 \cdot u_1 + \ldots a_n \cdot u_n$ then ${a_1}^2 + \ldots + {a_n}^2 = 1$

And,

$max\langle A^*Au, u\rangle = max({\sigma_1}^2 \cdot {a_1}^2 + \ldots + {\sigma_n}^2 \cdot {a_n}^2)$ ; where $\| u \| = 1$ or ${a_1}^2 + \ldots + {a_n}^2 = 1$

Put $\sigma_1 \geq \ldots \geq \sigma_i \geq \ldots$ , then $max\langle A^*Au, u\rangle \leq {\sigma_1}^2$ maximum for u unitary and attain this value for $u = u_1$.

So that the maximum expanding vector for $D\Phi^\nu$ is the eigeinvector of $(D\Phi^\nu)^*(D\Phi^\nu)$ with maximum eigeinvalue. These means,

$F^\nu(f^\nu) = f^\nu$ ,ie , $f^\nu$ is a fixed point for $F^\nu$.

The above linear algebra facts also provide:

$[(D\Phi^\nu)^*]^{-1} \cdot f^\nu$colinear to $D\Phi^\nu \cdot f^\nu$

$\| [(D\Phi^\nu)^*]^{-1} \cdot f^\nu \| = \| D\Phi^\nu \cdot f^\nu \|^{-1}$

The idea now is to use the implicit function theorem for the fixed point $f^\nu$ to understand its dependence on parameters :

$F^\nu(f) = F^\nu(\theta, z, f)$ ; $f^\nu = f^\nu(\theta, z)$ by the implicit map theorem.

First compute, $\dot{f} \in T(T_1(M)) = f^\perp$:

$\partial_f F(\theta, z, f)\dot{f} = \frac{(D\Phi^\nu)^*(D\Phi^\nu)\dot{f}}{\|(D\Phi^\nu)^*(D\Phi^\nu)f\|} - \frac{\langle (D\Phi^\nu)^*(D\Phi^\nu)f, (D\Phi^\nu)^*(D\Phi^\nu)\dot{f}\rangle}{\|(D\Phi^\nu)^*(D\Phi^\nu)f\|^2} \frac{(D\Phi^\nu)^*(D\Phi^\nu)f}{\|(D\Phi^\nu)^*(D\Phi^\nu)f\|}$

Then

$\| \partial_f F(\theta, z, f^\nu) \| = \| \frac{(D\Phi^\nu)^*(D\Phi^\nu)[f^\nu]^\perp}{\|(D\Phi^\nu)^*(D\Phi^\nu)f^\nu\|} \leq \frac{\|(D\Phi^\nu)^*[{f_\nu}^\nu]^\perp\| \cdot \|D\Phi^\nu[f^\nu]^\perp\|}{\|(D\Phi^\nu)^*(D\Phi^\nu)f^\nu\|}$

Reverting the Linear Algebra argument before:

$\| A \| = max \| Au \| = \| Au_1 \| = \sigma_1 \cdot \| v_1 \| = \sigma_1$

$\| A^* \| = max \| A^*v \| = \| A^*v_1 \| = \sigma_1 \cdot \| u_1 \| = \sigma_1$

That is,

$\| \partial_f F(\theta, z, f^\nu) \| \leq \frac{(\frac{b}{\lambda})^{2\cdot\nu}}{(\lambda)^{2\cdot\nu}} = (\frac{b}{\lambda^2})^{2\cdot\nu}$

Moreover,

$$\partial_{(\theta,z)} F(\theta, z, f) \cdot (\dot{\theta}, \dot{z}) = \frac{D_{(\theta,z)}[({D\Phi_{z,\theta}}^\nu)^* \cdot ({D\Phi_{z,\theta}}^\nu)] \cdot (\dot{\theta}, \dot{z}) \cdot f}{\| ({D\Phi_{z,\theta}}^\nu)^* \cdot ({D\Phi_{z,\theta}}^\nu) \cdot f \|}$$
$$- \frac{\langle ({D\Phi_{z,\theta}}^\nu)^* \cdot ({D\Phi_{z,\theta}}^\nu) \cdot f, D_{(\theta,z)}[({D\Phi_{z,\theta}}^\nu)^* \cdot ({D\Phi_{z,\theta}}^\nu)] \cdot (\dot{\theta}, \dot{z}) \cdot f\rangle}{\| ({D\Phi_{z,\theta}}^\nu)^* \cdot ({D\Phi_{z,\theta}}^\nu) \cdot f \|^2} \cdot$$
$$\cdot \frac{({D\Phi_{z,\theta}}^\nu)^* \cdot ({D\Phi_{z,\theta}}^\nu) \cdot f}{\| ({D\Phi_{z,\theta}}^\nu)^* \cdot ({D\Phi_{z,\theta}}^\nu) \cdot f \|}$$

Observe that the second term is the projection of $D_{(\theta,z)}(\ldots)$ over $\frac{(D\Phi^\nu)^* \cdot (D\Phi^\nu) \cdot f}{\|(D\Phi^\nu)^* \cdot (D\Phi^\nu) \cdot f\|}$.
Implying :

$\| \ \partial_f F(\theta,z,f^\nu) \ \| \leq \ max \frac{\| D_{(\theta,z)}[(D\Phi_{z,\theta}{}^\nu)^* \cdot (D\Phi_{z,\theta}{}^\nu)] \cdot (\dot\theta,\dot z) \cdot f^\nu \|}{\| (D\Phi^\nu)^* \cdot (D\Phi^\nu) \cdot f^\nu \|} \ \leq \ (\frac{M}{\lambda})^{2\nu}$ ; where maximum is over $| \ (\dot\theta,\dot z) \ | = 1$.

Now by implicit map theorem,

$F^\nu(\theta,z,f^\nu(\theta,z)) - f^\nu(\theta,z) = 0$

$\Rightarrow \partial_{\theta,z} F^\nu(\theta,z,f^\nu(\theta,z)) + \partial_f F^\nu(\theta,z,f^\nu(\theta,z)) \cdot D_{(\theta,z)} f^\nu(\theta,z) - D_{(\theta,z)} f^\nu(\theta,z) = 0$

$\Rightarrow \| \ D_{(\theta,z)} f^\nu(\theta,z) \ \| = \| \ \partial_{\theta,z} F^\nu(\theta,z,f^\nu(\theta,z)) \cdot (I - \partial_f F^\nu(\theta, z f(\theta,z))^- 1 \ \| \leq 2(\frac{M}{\lambda})^{2\nu}$

because $\| \ \partial_f F^\nu(\theta, z f^\nu(\theta,z)) \ \|$ is small than $\frac{1}{2}$.

To close $a)$ observe that $g^\nu$ is a fixed point to $(D\Phi^\nu) \cdot (D\Phi^\nu)^*$ and so the same estimates holds for $Dg^\nu$.

To prove $b)$ take derivatives

$D_{(\theta,z)}(D\Phi_{(z,\theta)}{}^j \cdot f_{(\theta,z)}{}^\nu) \cdot (\dot\theta,\dot z) = D_{(\theta,z)}(D\Phi_{(f_{(\theta,z)}{}^{j-1}(z) \ \theta)} \cdot D\Phi_{(z,\theta)}{}^{j-1} \cdot f_{(\theta,z)}{}^\nu) \cdot (\dot\theta,\dot z) =$

$= \ D_{(\theta,z)}(D\Phi_{(f_{(\theta,z)}{}^{j-1}(z),\theta)}) \ \cdot \ (\dot\theta,\dot z) \ \cdot \ D\Phi_{(z,\theta)}{}^{j-1} \ \cdot \ f_{(\theta,z)}{}^\nu \ + \ D\Phi_{(f_{(\theta,z)}{}^{j-1}(z),\theta)} \ \cdot D_{(\theta,z)}(D\Phi_{(z,\theta)}{}^{j-1} \cdot f_{(\theta,z)}{}^\nu) \cdot (\dot\theta,\dot z)$

The second derivate appears:

$D_{(\theta,z)}(D\Phi(f^{j-1}_{(\theta,z)}(z),\theta)) = [D_{(\theta,z)} \cdot D_z](f_{(\theta,z)}) \ |_{(\theta,z)=(\theta f^{j-1}_{(\theta,z)}(z))} \cdot D_{(\theta,z)}(\theta, f^{j-1}_{(\theta,z)}(z))$

But all these are supposed also to be bounded by $M$ so:

$\| \ D_{(\theta,z)}(D\Phi^j_{(z,\theta)} \cdot f^\nu(\theta,z)) \ \| \leq M^{2j} + M \cdot \ \| \ D_{(\theta,z)}(D\Phi^{j-1}_{(z,\theta)} \cdot f^\nu(\theta,z)) \ \|$

By recurrence,

$\| \ D_{(\theta,z)}(F^\nu_j(\theta,z) \ \| \leq M^{2j} + M^{2j-1} + M^{2j-2} = \ldots 2 \cdot M^j (\frac{M}{\lambda})^{2\nu}$

Now $b)$ is easy to establish.

For the complement of $b)$ do the same as above with $(D\Phi^*_{(z,\theta)})^{-j} \cdot f^\nu(\theta,z)$

$\square$

Now a better estimate:

**Lemma 4.6.** *let $z$ $\lambda$-expanding up to $n$. Then for $\nu \leq n$ :*

*$\| \ D_{(\theta,z)} f^\nu \ \| \leq const \cdot \gamma$ ; $\gamma < 1$ as given bellow*

Proof: First for $\nu = 1$:

$\partial_{(\theta,z)} F((\theta,z), f^1) = \frac{D_{(\theta,z)}[(D\Phi)^* \cdot D\Phi] \cdot (\dot\theta \dot z f^1)}{\| D\Phi \cdot f^1 \|^2} - \langle \frac{D_{(\theta,z)}[(D\Phi)^* \cdot D\Phi] \cdot (\dot\theta \dot z f^1)}{\| D\Phi \cdot f^1 \|^2}, f^1 \rangle \cdot f^1$

Using $(\theta,z)$ small a taylor expansion gives that:

$D\Phi_{(z,\theta)}$ be $\| \ (\theta,z) \ \| - C^1$ close to $\begin{pmatrix} 0 & \sigma \\ \lambda & 0 \end{pmatrix}$

It means $\Phi_{(z,\theta)}(x,y)$ to be $\| (\theta, z) \| - C^2$ close to the map $\varphi(x,y) = (y,x)$ for $(\theta, z)$ small.

So that,

$(D\Phi)^* D\Phi = \begin{pmatrix} 0 & \lambda \\ \sigma & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma \\ \lambda & 0 \end{pmatrix} + \epsilon = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} + \epsilon$

With $\| \epsilon \|_{C^1} \leq const \cdot \| (\theta, z) \|$

Because $f^1$ is a horizontal vector we get :

$\| \partial_{(\theta,z)} F((\theta, z), f^1) \| \leq \frac{\| D_{(\theta,z)}[(D\Phi)^* \cdot D\Phi] \cdot (\ldots, \ldots) \cdot f^1 \|}{\| D\Phi \cdot f^1 \|^2} \leq const \cdot \| (\theta, z) \|$

Then the proof of anterior lemma assure: $\| D_{(\theta,z)} f^1 \| \leq const \cdot \| (\theta, z) \|$ as we intend to use induction the above means that the first expanding vector ( and so the contractives ) became almost constant as we approach $(0,0)$.

Now relate $f^{\nu+1}$ and $f^\nu$ : $f^{\nu+1} = \langle f^{\nu+1}, f^\nu \rangle \cdot f^\nu + h^\nu$ , with $h^\nu$ $\nu$-contractive.

Taking derivatives

$D_{(\theta,z)} f^{\nu+1} = \langle f^{\nu+1}, f^\nu \rangle \cdot D_{(\theta,z)} f^\nu + [D_{(\theta,z)}(\langle f^{\nu+1}, f^\nu \rangle)] \cdot f^\nu + D_{(\theta,z)} h^\nu$

Observe that :

$\left\{ \begin{array}{l} | \langle f^{\nu+1}, f^\nu \rangle \leq 1 \\ \| f^{\nu+1} \| \equiv 1 \\ \frac{bM^2}{\lambda^4} = \frac{M^2}{\lambda^3} \cdot \sigma \leq 1 \end{array} \right. . \Rightarrow \langle D\Phi^{\nu+1}, f^{\nu+1} \rangle = 0 \Rightarrow$

$\Rightarrow | \langle D\Phi^{\nu+1}, f^\nu \rangle | \leq const \cdot (\frac{b}{\lambda^2})^\nu \cdot \| D\Phi^{\nu+1} \| \leq const \cdot (\frac{bM^2}{\lambda^4})^\nu$

Puting all togheter,

$\| D\Phi^{\nu+1} \| \leq \| D\Phi^\nu \| + (\gamma^\nu + \| Dh^\nu \|)$ ; with $\gamma < 1$

By induction,

$\| D\Phi^{\nu+1} \| \leq (\gamma^1 + \ldots) + (\| Dh^1 \| + \ldots) + \| D\Phi^1 \| \leq \frac{\gamma}{1-\gamma} + const \cdot \| (\theta, z) \| + \sum_i \| Dh^i \|$

With this special $\gamma$ we only need to bound $\| Dh^i \|$ by something like $\gamma^i$ also in order to have $\| D\Phi^{\nu+1} \| \leq const \cdot \gamma$ ; $\gamma \leq 1$ fixed.

For this purpose write as [19] :

$\begin{pmatrix} \alpha_* & 0 \\ 0 & \Delta_* \end{pmatrix}_{m \times m}$ matrix of $(D\Phi^\nu)^*$ on ortonormal $\{g^\nu, v_2\}$ and $\{f^\nu, u_2\}$

Observe that $\Delta_* = (\Delta_{22})$ matriz of $(g^\nu)^\perp \longrightarrow (f^\nu)^\perp$ and $\alpha_* = \| D\Phi^\nu \cdot f^\nu \|^{-1}$

As lemma before $\| Du_i \|, \| Dv_i \| \leq (const)^\nu$ $(u_1 = f^\nu, v_1 = g^\nu)$

Write also: $h^\nu = h_2 \cdot u_2 \in (f^\nu)^\perp$

Then $h^\nu = (D\Phi^\nu)^* \cdot \bar{h}^\nu$ ; where $\bar{h}^\nu = \bar{h}_2 \cdot v_2$

The coordinates are related by $\Delta_*$ : $h_2 = \Delta_{22} \cdot \bar{h_2} = \| D\Phi^\nu \cdot f^\nu \| (\alpha_* \Delta_{22}) \cdot \bar{h_2}$

As was done in the proof of anterior lemma :

$\left\{ \begin{array}{l} \| \bar{h}^\nu \| \leq (const)^\nu \text{ , } \| Dh^\nu \| \leq (const)^\nu \\ \| D\Phi^\nu \cdot f^\nu \| \leq (const)^\nu \text{ , } \| D(\| D\Phi^\nu \cdot f^\nu \|) \| \leq (const)^\nu \end{array} \right.$

So we bound $Dh_2$ as wanted if bound $\alpha_* \cdot \Delta_{22}$ and $D(\alpha_* \Delta_{22})$.

By chain rule and dissipativeness we bound: $\alpha_* \cdot \Delta_{22} = det(D\Phi^\nu)^* = det(D\Phi^\nu) = \Pi_{i=1}^{\nu} det(D\Phi_{z_i})$ ; over orbit $z_i$.

Remark that the determinant is not small than 1 if we were working with Anosov. Instead, the case of hyperbolic attractor no longer has this difficult because of the strong dissipativeness property. And in this case a straightfoward argument extend this lemma to the second derivate.

□

Once the preceding lemma is established the following is easy to obtain.

**Lemma 4.7.** $\| D_{(\theta,z)} e_i^\nu \| \leq (const \cdot \gamma)^i$ ; $i \leq \nu \leq n$

Proof : As before we prove $f^{j+1} = c_j \cdot f^j + h^j$ ; $\left\{ \begin{array}{l} | c_j | \leq 1 \\ \| Dc_j \| \leq (const \cdot \gamma)^j \\ \| h_j \| \leq (const \cdot \gamma)^j \\ \| Dh_j \| \leq (const \cdot \gamma)^j \end{array} \right.$

It follows , $f^i = \hat{c} \cdot f^\nu + \hat{h}$ with $\| \hat{h} \|, \| D\hat{h} \| \leq (const \cdot \gamma)^i$

Now decompose: $e^\nu = \langle e^\nu, f^i \rangle \cdot f^i + \tilde{h}^i$ ; with $\tilde{h}^i$ $i$-contrative. Then

$De_i^\nu = \langle e^\nu, f^i \rangle \cdot D\Phi_i^i + D(\langle e^\nu, f^i \rangle) \cdot f_i^i + D\tilde{h}_i^i$

Observe that: $\| f_i^i \| \leq const^i$ , $\| D\Phi_i^i \| \leq const^i$

But $\langle e^\nu, f^i \rangle = \langle e^\nu, \hat{h} \rangle \Rightarrow | \langle e^\nu, f^i \rangle |$ , $| D(\langle e^\nu, f^i \rangle) | \leq (const \cdot \gamma)^i$

So it remains to estimate $D\tilde{h}_i^i$ by something like $(const \cdot \gamma)^i$ in order to establish the lemma.

For this purpose write, $\left( \begin{array}{cc} \alpha & 0 \\ 0 & \Delta \end{array} \right)$ , $\alpha = \| f_i^i \|$ and $\Delta : (f^i)^\perp \to (g^i)^\perp$ be the matrix of $D\Phi^i$ with respect to basis $\{f^i, u_2\}$ and $\{g^i, v_2\}$ as before.

Then,

$\tilde{h}_i^i = \frac{\alpha \cdot \Delta \tilde{h}^i}{\| f_i^i \|}$

And a similar argument as lemma before close the proof.

□

Remark: Observe that $\gamma < 1$ depends on $\sigma$ (or $b$) so that it represents the strong contraction, which can be reduced, in hyperbolic attractor case.

# 5 Critical Points

The critical point concept is of fundamental importance for the whole process so this section is intended to make clear how to create them.

Iterating horizontal curves by $\Phi$ the tangent vector become:

$D\Phi_\theta(1,0) = (C \cdot \theta + 3 \cdot D \cdot x^2 + E \cdot y^2) \cdot (1,0) + (\lambda - 2 \cdot D' \cdot x \cdot y) \cdot (0,1)$

Put

$a_\theta(x,y) = C \cdot \theta + 3 \cdot D \cdot x^2 + E \cdot y^2$

In some sense the "critical" situation occurs when $a_\theta(x,y)$ attain its minimum and the vector becomes more vertical than anyone else: this happen when $(x,y) = (0,0)$ with minimum $C \cdot \theta$.

Also,

$da_\theta = (6 \cdot D \cdot x, 2 \cdot E \cdot y) \neq (0,0)$; if $(x,y) \neq (0,0)$

$d^2 a_\theta = \begin{pmatrix} 6 \cdot D & 0 \\ 0 & 2 \cdot F \end{pmatrix}$ is non-degenerate meaning that $a_\theta$ is a morse function.

But the segments parallel to $(1,0)$ are not the right curves in which the critical points should live. Instead suppose curves $\gamma - C^2$ close to them $(\gamma < 1)$. Then, by the closeness, $a_\theta(x,y)$ still have a uniq minimum ( of order $C \cdot \theta$ ) with $d^2 a_\theta(x,y)$ non-degenerate and this minimum is $const \cdot \gamma$ distant from $(0,0)$ ( this correspond to a $\gamma - C^2$ pertubation of $a_\theta(x,y)$ ).

The above still holds if the decomposition is not in $(1,0)$ and $(0,1)$ anymore but in $f^\nu$ and $e^\nu$ because $f^\nu$ is $const \cdot \gamma^\nu - C^2$ close to $(1,0)$ . But in this case the minimum is obtained $const \cdot \gamma^\nu$ distant from $(0,0)$.

This $C^2$ perturbation argument allow us to accurate critical points:

$z^0, z^1, \ldots, z^\nu, \ldots$ with $\| z^\nu - z^{\nu+1} \| \leq const \cdot \gamma^\nu$

But we can also induce critical points over nearly closed curves. Suppose curves $\gamma - C^2$-close to horizontal lines. Then for two $C^2$-close curves (the distance between them much small than their lengths) the above kind of pertubation arguments applies to induce critical aproximation from one to the other. For this observe: close curves expand by a lemma before; close points have $f^\nu$ $\gamma - C^2$ close by the lemma on $C^2$ small variation of $f^\nu$. Simplistically we say this as: almost the same curve and vectors for the above decomposition implies almost the same minimum.

# 6 Contractive Distributions

Now a section devoted to the study of the behavior of the $e^\nu$ trajectories.

Let $e_r^\nu = (r, 1)$ point in direction of the $\nu$-th contractive approximation.

**Lemma 6.1.** *let $z$ $\lambda$-expand up to $n$ and distant from $\tilde{A}$ and $\Phi(\tilde{A})$, then for $\nu \leq n$ :*

*a) $\mid angle(e_r^{\nu-1}, e_r^\nu) \mid \leq (const \cdot \gamma)^\nu$*

*b) $\Gamma^\nu$ the $e_r^\nu$-trajectory of $z$ exist with a fixed length and all $\eta \in \Gamma^\nu$ is bound to $z$.*

Proof: Write $e_r^{\nu-1} = (r_1, 1)$ and $e_r^\nu = (r_2, 1)$. Also put $(1, R_1), (1, R_2)$ colinear to $f^{\nu-1}$ and $f^\nu$ repectively.

By the closeness of $f^{\nu-1}$ and $f^\nu$ :

$\mid angle(e_r^{\nu-1}, e_r^\nu) \mid \leq \mid r_1 - r_2 \mid \leq \mid R_1 - R_2 \mid \leq (const \cdot \gamma)^\nu$; some $\gamma < 1$

This easily prove a).

Now let's establish b). First suppose fixed a region far from $q$.

In this region $e^1$ exist and is always vertical.

Integrate the $e^1$-trajectory $\Gamma^1$ of $z$ in $[-\gamma^{\frac{1}{4}}, \gamma^{\frac{1}{4}}]$ around $z$.

For all $\eta \in \Gamma^1$ we have : $\begin{cases} \parallel \eta - z \parallel \leq 5\gamma^{\frac{1}{4}} \\ \parallel \eta^1 - z^1 \parallel \leq (5\gamma^{\frac{1}{4}})^2 \end{cases}$ . ; because $e^1(z), e^1(\eta)$ contracts and $angle(e^1(z), e^1(\eta))$ is small.

These means $\eta$ expands up to 2 because of boundness with $z$ and also means $\tau \in [\eta - \sigma, \eta + \sigma]$ expands up to 2 for a small $\sigma$ by a lemma before ( $[\eta - \sigma, \eta + \sigma]$ denotes the horizontal segment ).

Put $U^1 = \bigcup_{\eta \in \Gamma^1} [\eta - \sigma, \eta + \sigma]$

Now we integrate the $e^2$-trajectory of $z$ in $[-\gamma^{\frac{1}{4}}, \gamma^{\frac{1}{4}}]$ around $z$ again. This is because $e^i$ is almost constant and $e^i, e^j$ have almost the same inclination ( they get apart to each other in a rate small relative to $\sigma$ below ).

For all $\eta \in \Gamma^2$ we have: $\begin{cases} \parallel \eta - z \parallel \leq 5\gamma^{\frac{1}{4}} & \text{again because } e^1, e^2 \text{ are} \\ \parallel \eta^1 - z^1 \parallel \leq (5\gamma^{\frac{1}{4}})^2 & \text{contractive and almost} \\ \parallel \eta^2 - z^2 \parallel \leq (5\gamma^{\frac{1}{4}})^3 & \text{the same} \end{cases}$ .

Then, as before, $\tau \in U^2 = \bigcup_{\eta \in \Gamma^2} [\eta - \sigma^2, \eta + \sigma^2]$ is expanding up to 3.

By induction the $e^\nu$-trajectory $\Gamma^\nu$ of $z$ exist with a fixed length not depending on $\nu$ and is almost vertical.

I.e. $\eta \in \Gamma^\nu$

$\parallel \eta^\nu - z^\nu \parallel \leq (5\gamma^{\frac{1}{4}})^{\nu+1}$ ; $0 \leq \nu \leq n$

And this means boundness between $\eta$ and $z$ giving, for small unitary slope vector $u$ and $v$ :

$\frac{1}{2}\cdot \parallel D\Phi^{\nu}_{\eta} \cdot v \parallel \leq \parallel D\Phi^{\nu}_{z} \cdot u \parallel \leq 2\cdot \parallel D\Phi^{\nu}_{\eta} \cdot v \parallel$

$\mid angle(D\Phi^{\nu}_{\eta} \cdot v, D\Phi^{\nu}_{z} \cdot u) \mid \leq (5\gamma^{\frac{1}{4}})^{\nu+1}$

Remark: Observe that the accuracy algorithm and the existence of above integral curves depends inevitable on the $\gamma^{\nu} - C^2$ convergence of $e^{\nu}$ which, thereby, reduces to the strong contractiveness property ( and so to hyperbolic attractor case ).

# 7 Induction Process

This section is designed in order to make clear the idea of the whole argument, which is a rather long and elaborated induction.

As [19], we do a simultaneous building of the expansion, critical point, contractive direction. First of all, for times between 1 and $N$ ($N$ big) these concepts are all know.

Then in each step of iteration we accurate all existing critical points as described before and create new ones if needed. So we need first:

*Definition:* $\xi_0$ *p-bound to* $z_0$ *if*

$\parallel z_j - \xi_j \parallel \leq const_k \cdot e^{\beta j}$ ; $1 \leq j \leq p$ ; $z_0 = z_0^{k-1}$ *k-expansive*

Remark that boundness to $z_0^{k+1}$ is also to $z_0^k$ because $const_k$ can be taken decreasing with k.

Inductively we define ( for critical points and points bounded to them ) : returns ( inside some fixed $\delta$-region ) , binding periods ( the time the points remain bounded to each other ) and folding periods ( the time needed to decrease slope and became almost horizontal again ).

So suppose $\xi_0$ bound to critical point $\tilde{z}_0$. If $\tilde{z}_0$ returns while they are bound to each other (i.e., during binding period) in a previous binding period of $\tilde{z}_0$ then say $\xi_0$ has a bound return and binding period of $\tilde{z}_0$ is take as that of $\xi_0$. If this do not happen but yet $\xi_0$ return we say it is a free return and apply a parameter cutting procedure in order to fullfill:

$[BA]$ $\quad d_n(\xi_0) = \parallel \xi_n - z_0 \parallel \geq e^{-\alpha n}$; where $z_0$ is an special critical point obtained at the return $n$.

*Definition: $p_0$ primary binding period:*
$\| \xi_{n+j} - z_j \| \leq C \cdot e^{-\beta j}; 1 < j \leq p_0$ , $p_0$ *maximum*

After the binding period we expect to be in a free time.

*Definition: p binding period if*

1. $p+1$ *free time (so binding period inside must end inside).*

2. $p \leq p_0$ *maximum above*

Recall that free time means neither a return nor part of binding period.
Observe that we intend to establish $p_0 \leq \bar{C} \cdot \log \frac{1}{d_\nu^2(\xi_0)} \leq \bar{C} \cdot \alpha \cdot n$
But $p_0$ and p are not too different:
If $p < p_0$ then $\mu' = p + 2$ is a return with $p'$ binding period: $\mu + p' \geq p_0 \Rightarrow p_0 - p \leq 2 + p' \leq 2 + \bar{C} \cdot \alpha \cdot \mu' \leq 2 + \bar{C} \cdot \alpha \cdot p_0$
But $p_0 \geq N$ is bigger than 2 ( else if $p_0 < N$ trivially $p = p_0$ ) $\Rightarrow p_0 - p \leq [\frac{2}{p_0} + \bar{C} \cdot \alpha] \cdot p_0 \Rightarrow p_0 \cdot [1 - \frac{2}{p_0} - \bar{C} \cdot \alpha] \leq p \leq p_0$.

*Definition:* $l_0 = C' \cdot \log \frac{1}{d_n(\xi_0)}$ *is primary folding period.*

Also as binding period, after an folding period ends we expect to be in a fold-free time ( ie, not return or inside any folding period)

*Definition: $l$ folding period if*

1. $l+1$ *fold-free time (so folding period inside must end inside)*

2. $l \leq l_0$ *maximum above*

Now $l, l_0$ are not too different:
If $l < l_0$ then $\mu' = l + 2$ is a return with $l'$ folding period and $\mu + l' \geq l_0 \Rightarrow l_0 - l \leq 2 + l' \leq 2 + C'\alpha\mu' \leq 2 + C'\alpha l_0$

But $l_0 \geq N$ is bigger than 2 (else if $l_0 < N$ trivially $l = l_0$) $\Rightarrow l_0 - l \leq [\frac{2}{l_0} + C'\alpha] \cdot l_0 \Rightarrow l_0 \cdot [1 - \frac{2}{l_0} - C'\alpha] \leq l \leq l_0$

Remark: We adjust $C$ and $C'$ afterwards so that binding period be greater than folding and so that in free times we stay with "almost horizontal direction".

Recall the usual splitting algorithm:

1. If not at beginning or ending of folding period iterate $\omega_n$ componentwise: $\begin{cases} w_n^* \text{ horizontal} \\ \sigma_n \text{ "vertical", i.e., in the contractive direction} \end{cases}$
2. In a return $n$ split, $\omega_n = \beta_n(1,0) + \alpha_n \cdot e^l(\xi_{n+1})$ and put: $\begin{cases} \sigma_n = D\Phi\sigma_{n-1} + \alpha \cdot e^l(\xi_{n+1}) \\ \omega_n^* = D\Phi\omega_{n-1}^* - \alpha_n \cdot e^l(\xi_{n+1}) \end{cases}$.
3. At the end of the folding period we put back to horizontal component all (now very small) vertical pieces created at the folds:

$$\begin{cases} \omega_n^* = D\Phi\omega_{n-1}^* + \sum \alpha_{n_i} \cdot D\Phi^{l_i} \cdot e^{l_i} \\ \sigma_n = D\Phi\sigma_{n-1} - \sum \alpha_{n_i} \cdot D\Phi^{l_i} \cdot e^{l_i} \end{cases}.$$

Now we adjust $C$ and $C'$ :

$$\begin{aligned} d_n^2 M^{p+1} &\geq \| z_{n+p+1} - z_{p+1} \| \geq \| z_{n+p_0+1} - z_{p_0+1} \| \cdot M^{-(p_0-p)} \\ &> C \cdot e^{-\beta(p_0+1)} M^{-(p_0-p)} \geq C \cdot e^{p \cdot logM - p_0 \cdot [\beta + logM] - \beta} \\ &\geq C \cdot e^{p \cdot [\log M - \frac{(\beta+logM)}{(1-\frac{2}{p_0}-\bar{C}\alpha)}] - \beta} = e^{-p \cdot [\frac{\beta + logM \cdot (\frac{2}{p_0} + \bar{C}\alpha)}{(1-\frac{2}{p_0}-\bar{C}\alpha)}] - \beta} \end{aligned}$$

Taking log:
$p \geq 2 \cdot \frac{(1-\frac{2}{p_0}-\bar{C}\alpha)}{(\beta+\log M)} \cdot \log \frac{1}{d_n} - (1 - \frac{2}{p_0} - \bar{C}\alpha) \cdot (1 + \frac{\log C}{(\beta+\log M)})$
Equivalently,
$C' \leq \frac{(1-\frac{2}{p_0}-\bar{C}\alpha)}{(\frac{\beta}{2}+\log\sqrt{M})} - \frac{(1-\frac{2}{p_0}-\bar{C}\alpha)\cdot(1+\frac{\log C}{(\beta+\log M)})}{\log \frac{1}{d_n}}$
Observe that for $\beta, \alpha, \delta$ small (implying $d_n$ small and $p_0$ big) this is close to $\frac{1}{\log\sqrt{M}}$.
At the end of the folding period we must be back to "almost horizontal directions". For this purpose:

**Lemma 7.1.** *There is an $S$ reducing with the attractor dissipativeness $\gamma$ with:* $\mid slope(\omega^*_\nu) \mid \leq S; \forall \nu$

Proof: Induction,

For $\nu = 0$ or $\nu$ return $\mid slope(\omega^*_\nu) \mid = 0$.

For $\nu$ not a return or ending of folding period the orbit is, at least, $\delta$ far from the critical point ( the worst situation is at the end of the folding period) and clearly we may bound slope.

Let now $\nu$ coincide with the end of a folding period: $\nu = \mu + l$

Then,

$\omega^*_\nu = \beta_\nu \omega^*_l + \alpha_\mu D\Phi^l e$

By induction hypothesis: $\mid slope(\omega^*_l) \mid \leq S$

The slope increase is due to: $\alpha_\mu D\Phi^l e$

To have an increase of a term $\tilde{C}$ we need:

$\mid \alpha_\mu \mid \cdot \parallel D\Phi^l e \parallel \leq \tilde{C} \cdot \parallel \omega^*_l \parallel \cdot \mid \beta_\mu \mid$

Observe that if we know the bound below for $(\frac{b}{\lambda})^l$ we get:

$\left\{ \begin{array}{l} \parallel D\Phi^l e \parallel \leq (\frac{b}{\lambda})^l \\ \parallel \omega^*_l \geq \lambda^l \end{array} \right.$ because $(\frac{b}{\lambda})^l \leq \tilde{C} \lambda^l d^2_\mu \Rightarrow \parallel D\Phi^l e \parallel \leq \frac{\tilde{C}}{const} \parallel \omega^*_l \parallel d^2_\mu$

Then we get the slope increase:

$$\mid \alpha_\mu \mid \cdot \parallel D\Phi^l \parallel \leq const \cdot \parallel \omega^*_{\mu-1} \parallel \cdot \parallel D\Phi^l \parallel \leq const \cdot \frac{\mid \beta_\mu \mid \cdot \parallel D\Phi^l \parallel}{d^2_\mu}$$

$$\leq \frac{const\tilde{C}}{const} \cdot \mid \beta_\mu \mid \cdot \parallel \omega^*_l \parallel$$

But to satisfy this condition:

$l \geq \frac{1}{\log \frac{\lambda}{\sqrt{b}} \log \frac{1}{d_\mu} - \frac{\log \tilde{C}}{\log \frac{\lambda^2}{b}}}$

We already know:

$l \geq C' \cdot (1 - \frac{2}{p_0} - C'\alpha) \log \frac{1}{d_\mu}$

The above will be satisfyed if:

$\frac{1}{\frac{\lambda}{\sqrt{b}}} \cdot \log \frac{1}{d_\mu} - \frac{\log \tilde{C}}{\log \frac{\lambda^2}{b}} \leq C' \cdot (1 - \frac{2}{l_0} - C'\alpha) \log \frac{1}{d_\mu}$

These gives one condition on C' and joining with the other obtained before we get the final inequalities to be fullfilled:

$\frac{1}{\log \frac{\lambda}{\sqrt{b}}} \cdot \frac{1}{(1-\frac{2}{l_0}-C'\alpha)} - \frac{\log \tilde{C}}{\log \frac{\lambda^2}{b} \cdot \log \frac{1}{d_\mu} \cdot (1-\frac{2}{l_0}-C'\alpha)} \leq C' \leq \frac{(1-\frac{2}{p_0}-\bar{C}\alpha)}{(\frac{\beta}{2}+\log \sqrt{M})} - \frac{(1-\frac{2}{p_0}-\bar{C}\alpha)}{\log \frac{1}{d_\mu}} \cdot (1 + \frac{logC}{\beta + \log M})$

Now we argue that such $C'$ exist because if:

decrease $R$ (but keep $M$) $\Rightarrow$ increase $N \Rightarrow$ increase $l_0, p_0$

decrease $\alpha, \beta, \delta$ in this order ( but keep $C$ and so $\bar{C}$) $\Rightarrow$ increase $\frac{1}{d_\mu}$

and also observe that the left term is close to $\frac{1}{\log \frac{\lambda}{\sqrt{b}}}$ and the right one to $\frac{1}{\log \sqrt{M}}$ so the final inequality holds if we require something like $bM < \lambda^2$.

The slope bound is obtained in the same way.

□

Fix $\Theta = \frac{c'}{\log \gamma^{-1}}$. At each step $k$ the critical points have generation $g \leq \Theta k$. Recall that generation for [19] is defined by $W^u$ here we just replace it by $W = \bigcup_{k\geq 0} \Phi^k(R \bigcap \{y = \bar{y}\})$ where $\Phi_{z,\theta}$ bifurcate at $z = (\bar{x}, \bar{y})$ and $R$ stands for the bifurcate cube. That is, fix an initial piece of $W$ denoted by $W_0$ ( where are the low levels critical points ) cutting all the attractor then $W_i = \Phi(W_{i-1}) \setminus W_{i-1}$ is where live the $i$-generation critical points.

So in each iteration we just update all existing critical points $\tau_0$ by the below perturbation argument. We also use "g" as a rule for creation of new ones: create all $z_0$ that is possible obeying $g \leq \Theta k$ bound for generation since $\| z_0 - \tau_0 \| \leq \gamma^{\frac{g}{10}}$ is satisfied. Moreover, we also suppose that a $k$-th critical point has a $\gamma - C^2$ almost horizontal neighbourhood ( denoted $[z_0 - {\rho_0}^{\Theta k}, z_0 + {\rho_0}^{\Theta k}]$ ) of length ${\rho_0}^{\Theta k}$.

And if generation is big assume vectors are expanded by $D\Phi^g$ from $\Phi^{-g}([z_0 - \rho_0^{\Theta k}, z_0 + \rho_0^{\Theta k}])$ to $[z_0 - \rho_0^{\Theta k}, z_0 + \rho_0^{\Theta k}]$.

Observe that these $\rho_0$-neigborhoods allowed to bound the number of critical points by $Const \cdot (\frac{M}{\rho_0})^{\Theta \cdot k}$, exactly as [5].

Now we just iterate as if the dimemsion 1 applying $[BA]$ and $[FA]$ for restricting parameter set.

For all points $\xi_0$ bound to critical points we establish inductively some estimates due to to the cubic form of the map at returns $n$ with binding period $p$ :

1. $d_n(\xi_0)^2 \cdot const \leq \mid \frac{\beta_n}{\|\omega^*_{n-1}\|} - \tilde{\beta}_0 \mid \leq const \cdot d_n(\xi_0)^2$

2. $\mid \alpha_n \mid \leq const \cdot \| \omega^*_{n-1} \|$

Here recall that critical points are minimals so we let $\tilde{\beta}_0$ be such a minimal.

Also we have the comparison formulae during binding period and also show that the lost due to $d_n(\xi_0)^2$ can be recovered at the end of the binding period.

1. $\frac{1}{\tau_1} \leq \frac{(\frac{\|\omega^*_{n+j}\|}{\|\omega^*_n\|})}{\|\omega^*_j\|} \leq \tau_1$ ; $0 \leq j \leq p-1$

2. $\frac{\|\omega^*_{n+p}\|}{\|\omega^*_n\|} \cdot d_n(\xi_0)^2 \geq 1$

And finally recall that we estimate: $p \leq \bar{C}\alpha n$.

Now, once these propertiers have been proved, it follows that $\xi_0$ is expansive:

$\prod_n^{n+p} \frac{\|\omega^*_i\|}{\|\omega^*_{i-1}\|} = (\frac{\|\omega^*_{n+p}\| \cdot d_\nu(\xi_0)^2}{\|\omega^*_n\|}) \cdot (\frac{\|\omega^*_n\|}{d_n(\xi_0)^2 \cdot \|\omega^*_{n-1}\|}) \geq 1$

That's for binding period and after by "driving force":

$\prod_{n+p+1}^{k} \frac{\|\omega^*_i\|}{\|\omega^*_{i-1}\|} \geq \lambda^{k-(n+p+1)}$

Hence,

$\| \omega^*_k \| \geq \lambda^{F_k} \cdot e^{-\alpha \cdot k} = e^{c_0 \cdot F_k - \alpha k}$; $F_k$ meaning total free times.

By [FA] we get the expansion on horizontal: $\| \omega^*_k \| \geq e^{c_1 \cdot k}$

The expansion on the whole vector $\omega_k$ is obtained by a comparison formulae with $\omega^*_k$.

For this we prove :

**Lemma 7.2.** *for every time $\mu$ there are fold-free times $\mu_1 \leq \mu \leq \mu_2$ with*

$\mu_2 - \mu_1 \leq C'\alpha\mu$

*Moreover, if some $\nu > \mu$ is free then even better,*

$\mu_2 - \mu_1 \leq [\frac{C'}{\frac{const[1-\frac{2}{p_0}-\bar{C}\alpha]}{\beta+\log M} - C'}] \cdot (\nu - \mu)$

Proof: If $\mu$ is fold-free take $\mu_1 = \mu = \mu_2$.

Suppose $\mu$ part of folding period. Take $n$ the small return with $\nu$ part of its folding period $l$. Then $\mu_1 = n-1, \mu_2 = n+l+1$ are fold-free times with $\mu_1 \leq \mu \leq \mu_2$ and $\mu_2 - \mu_1 = l \leq C'\alpha n \leq C'\alpha\mu$

For the second observe that,

$\nu - \mu \geq p - l \geq (\frac{const[1-\frac{2}{p_0}-\bar{C}\alpha]}{\beta+\log M} - C') \cdot \log \frac{1}{d_n(z_0)}$

And, so

$\mu_2 - \mu_1 = l \leq C' \log \frac{1}{d_n(z_0)} \leq \frac{C'}{(\frac{const[1-\frac{2}{p_0}-\bar{C}\alpha]}{\beta+\log M} - C')}(\nu - \mu)$

□

**Lemma 7.3.** *for $\mu < \nu$ a consequence of induction is the formulae :*
$\| \omega_\nu^* \| \geq (min \frac{\|\omega_j^*\|}{\|\omega_{j-1}^*\|}) \cdot \| \omega_\mu^* \| \geq (min\ d_j(z_0)^2) \cdot \| \omega_\mu^* \|$ *; minimum taken for* $\mu < j \leq \nu$

Proof: Each return in this period of time introduces a small factor $d_j(z_0)^2$ which is compensated or not depending on the possible ending of binding time.

□

With these two lemmas we are now able to prove the comparison lemma:

**Lemma 7.4.** *inductively we get:* $e^{-[C' \cdot \log M+3] \cdot \alpha \cdot \mu} \leq \frac{\|\omega_\mu\|}{\|\omega_\mu^*\|} \leq e^{[C' \cdot \log M+2] \cdot \alpha \cdot \mu}$

Proof: Using lemma above, $\exists \mu_1 \leq \mu \leq \mu_2$ with $\mu_2 - \mu_1 \leq C' \alpha \mu$ because they are fold-free: $\omega_{\mu_i} = \omega_{\mu_i}^*$. Then
$\| \omega_\mu \| \leq M^{\mu-\mu_1} \cdot \| \omega_{\mu_1} \| \leq M^{C'\alpha\mu} \cdot \| \omega_{\mu_1}^* \| \leq M^{C'\alpha\mu} \cdot e^{2\alpha\mu} \cdot \| \omega_\mu^* \| = e^{[C' \log M+2] \cdot \alpha\mu} \cdot \| \omega_\mu^* \|$
Remark that the last inequality holds by anterior lemma and the same reasoning may be applied to the other inequality.

□

Finally, the expansion on $\omega_k^*$ can be extended to $\omega_k$ :
$\| \omega_k \| \geq e^{-[C' \log M+3]\alpha k} \| \omega_k^* \| \geq e^{(C_1 - [C' \log M+3]\alpha)k}$ ; for $\alpha$ small

# 8 Binding and Loss of Growth on Returns

That's the part in which we are interested in how to obtain the "tangencial position" required in order to apply the 1 dimemsion argument.

*Definition: For some small fixed $\lambda_0$ and for $n$ a free return for $z_0$ we call $1 \leq r \leq n$ a favorable time for $z_0$ if*

1. *$r$ is a fold free time;*
2. *$z_r$ is distant from $\tilde{A}$ and $\Phi(\tilde{A})$;*
3. $d_j(z_r) \geq \lambda_0^{j+1}$ *; $\forall 0 \leq j < n - r$*

**Lemma 8.1.** *let $n$ free return for $z_0$. Then, there are $1 = m_1 < m_2 < \ldots < m_s \le n$ ; with* $\begin{cases} m_{i+1} \le 3m_i \\ n \le 3m_s \end{cases}$ *. such that $n - m_i$ is a favorable time for $z_0$.*

Proof:Let $k$ the last return before $n$ and $\mu = k + l + 1$ the first fold-free time after $k$.

Observe that, in fact, $k + l$ is the last non fold-free time after $k$ and we may assume $n - k \ge 100$ , say, reducing constants like $\delta$.

That means we find easily $1 = m_1 < m_2 < \ldots < m_r = n - \mu$ as lemma requires since $\mu \le n - m_i \le n$.

Also an anterior lemma asserted that fold-free times be quite frequent. Applying to $k$ and $k + p + 1 < n$ free time:

$\mu - k \le \mu_2 - \mu_1 \le (\frac{C'}{const - C'}) \cdot (k + p + 1 - k) \le \frac{C'}{Const - C'} \cdot (n - k)$ ; with $const = const(\beta, M)$

Observe that the inequalities $C'$ must satisfy are of the order

$\frac{1}{\log \frac{\lambda}{\sqrt{b}}} \le C' \le \frac{1}{\log \sqrt{M}}$

So we may reduce $b$ ( and so $C'$ ) and, therefore, we may suppose

$\frac{C'}{const(\beta, M) - C'} \le \frac{1}{2}$

Then we resume all this as $\mu - k \le \frac{1}{2}(n - k)$

Now suppose $m_i \ge \frac{1}{2}(n - k)$ is defined and by induction obtain $m_{i+1}$ as follows:

Invoke the lemma on denseness of fold-free times with $n - 3m_i$ and $k + p + 1 < n$ free time,

$\mu_i - (n - 3m_i) \le \mu_2 - \mu_1 \le \frac{C'}{const - C'} \cdot (k + p + 1 - (n - 3m_i)) \le \frac{C'}{const - C'} \cdot (n - (n - 3m_i)) \le \frac{3C'}{const - C'} \cdot m_i \le \frac{1}{2} m_i$

In direction of c) put $I = [\mu_i, n - 1]$ and observe that: $d_j(z_0) \ge (e^{\bar{C}})^{-|I|}$.

In fact, if $j$ is a return,

$[p \le p_0 \le \bar{C} \log \frac{1}{d_j} \Rightarrow d_j \ge (e^{\bar{C}})^{-p} \ge (e^{\bar{C}})^{-(n-j)}$ ; because $p \le n - j]$

$d_j \ge (e^{\bar{C}})^{-(n-j)} \ge (e^{\bar{C}})^{-|I|}$ ; otherwise, $d_j \ge \frac{1}{2}\delta \ge \frac{1}{4} d_k \ge \frac{1}{4}(e^{\bar{C}})^{-(n-k)} \ge \frac{1}{4}(e^{\bar{C}})^{-2m_i} \ge (e^{\bar{C}})^{-|I|}$

So the proof of this lemma reduces to the next one.

□

**Lemma 8.2.** *let $I = [p, q - 1]$ with $p$ fold-free and $z_p$ distant from critical region. Also suppose $\inf_I d_j \ge (e^{\bar{C}})^{-|I|}$*

*Then $\exists \nu \in [p, \frac{(p+q)}{2} - 1]$ fold-free, $z_\nu$ distant from critical region and $d_{\nu + j} \ge \lambda_0^{j+1}$ ; $\forall j \in [0, q - \nu - 1]$*

Proof: Trivial for small intervals $\mid I \mid \leq \frac{\log \lambda_0^{-1}}{\bar{C}}$:

$d_{p+j} \geq (e^{\bar{C}})^{-|I|} \geq (e^{\bar{C}})^{-\frac{\log \lambda_0^{-1}}{\bar{C}}} \geq \lambda_0$ ;$\forall j \in [0, q-p-1] \Rightarrow d_{p+j} \geq \lambda_0^{j+1}$ ;$\forall j \in [0, q-p-1]$

Observe that $\frac{\log \lambda_0^{-1}}{\bar{C}}$ may be as big as wanted by reducing $\lambda_0$.

For big intervals we use induction: let $m$ such that $4m - 3 \leq \mid I \mid \leq 4m$ and put $J = [p, p + 2m - 1]$.

Also suppose first,

$\inf_J d_j \geq (e^{\bar{C}})^{-|J|}$

By induction there is a good time $\nu \in [p, p + m - 1]$ with:

$d_{\nu+j} \geq \lambda_0^{j+1}$ ;$\forall j \in [0, p + 2m - \nu - 1]$

Also for $j \in [p + 2m - \nu, q - \nu - 1]$ we still have:

$d_{\nu+j} \geq (e^{\bar{C}})^{-|I|} \geq (e^{\bar{C}})^{-4m} \geq (e^{\bar{C}})^{-4(j+1)} = \lambda_0^{j+1} \cdot (\frac{e^{-4\bar{C}}}{\lambda_0})^{j+1} \geq \lambda_0^{j+1}$

So for this case lemma is closed suppose now $d_{\bar{\nu}} < (e^{\bar{C}})^{-|J|}$ and recall that we also have $\mid I \mid > \frac{\log \lambda_0^{-1}}{C}$. Then,

$(e^{\bar{C}})^{-\frac{|I|}{2}} \leq (e^{\bar{C}})^{-\frac{\log \lambda_0^{-1}}{C} \cdot \frac{1}{2}} = \sqrt{\lambda_0} = \frac{\delta}{2} \cdot (\frac{2\sqrt{\lambda_0}}{\delta}) \leq \frac{\delta}{2}$

Meaning that $\bar{\nu}$ is a return time.

As $\bar{C}$ came from anterior lemma ( and binding period definition),

$\frac{2}{(\beta+\log M)} \cdot \log \frac{1}{d_{\bar{\nu}}} \leq \bar{p} \leq \bar{C} \cdot \log \frac{1}{d_{\bar{\nu}}}$ where $\bar{p}$ is binding period of $\bar{\nu}$

Then compute,

$\bar{p} \geq \frac{2}{(\beta+\log M)} \cdot \log \frac{1}{d_{\bar{\nu}}} \geq \frac{2}{(\beta+\log M)} \cdot \log \frac{1}{(e^{\bar{C}})^{-|J|}} = \frac{2\bar{C}}{(\beta+\log M)} \mid J \mid \geq \frac{\bar{C}m}{(\beta+\log M)}$

On the other hand

$\bar{\mu} - \bar{\nu} \leq \frac{C'}{const(\beta M)-C'} \cdot ((\bar{\nu} + \bar{p}) - \bar{\nu}) = \frac{C'}{const-C'} \cdot \bar{p} = \frac{C'}{const-C'} \cdot \bar{C} \log \frac{1}{d_{\bar{\nu}}} \leq \frac{C' \cdot \bar{C}}{const-C'} \cdot \log(e^{\bar{C}})^{|I|} \leq \frac{C' \cdot \bar{C}^2}{const-C'} \cdot 4m$

Where $\bar{\mu} = \bar{\nu} + \bar{l} + 1$ is the first fold-free time after return $\bar{\nu}$.

Also,

$\mid (\bar{\nu} + \bar{p}) - \bar{\mu} \mid \geq \mid (\bar{\nu} + \bar{p}) - \bar{\nu} \mid - \mid \bar{\mu} - \bar{\nu} \mid \geq (\frac{\bar{C}}{(\beta+\log M)} - \frac{4C'\bar{C}^2}{const-C'}) \cdot m$

and the above is a positive fraction of $m$ by reducing $C'$.

Put $L = [\bar{\mu}, \bar{\mu} + \tilde{q}m - 1]$ for some $0 \leq \tilde{q} \leq 1$ such that $L \subset [\bar{\nu}+1, \bar{\nu}+\bar{p}-1]$

It follows from the choice of $L$ and easy computation:

$d_j \geq e^{\alpha(j-\bar{\nu})} \geq (e^{\bar{C}})^{-|L|}$ ;$\forall j \in L$

And this means, by induction, that here is a good $\nu \in [\bar{\nu}, \bar{\nu} + \frac{\bar{q}m}{2} - 1]$ with

$d_{\nu+j} \geq \lambda_0^j$ ;$\nu \leq j + \nu < \bar{\mu} + \bar{q}m$

To finish we just extend this as before to $\nu \leq \nu + j < q$.

$\square$

Remark: The above favorable time argument is completly dependent on reducing the determinant $b$. So, as it is , may work only for hyperbolic attractor case

Once the above lemma on the existence of favorable position is established it is easy to conclude $z_{n-m_i}$ expands until $m_i$

For this take $1 \leq j \leq m_i$ and apply lemma on denseness of fold-free times: there is a fold-free time $k \geq j$ with,

$k - j \leq \mu_2 - \mu_1 \leq \frac{C'}{(const(\beta,M)-C')} \cdot j$

Then,

$\| \omega_{n-m_i}(z_0) \| \cdot \min d_l^2(z_0) = \| \omega^*_{n-m_i}(z_0) \| \cdot \min d_l^2(z_0) \leq \| \omega^*_{n-m_i+k}(z_0) \| =$

$\| \omega_{n-m_i+k}(z_0) \| \leq M^{(k-j)} \| \omega_{n-m_i+j}(z_0) \|$

where the minimum is for $n - m_i + k \geq l \geq n - m_i$

And so,

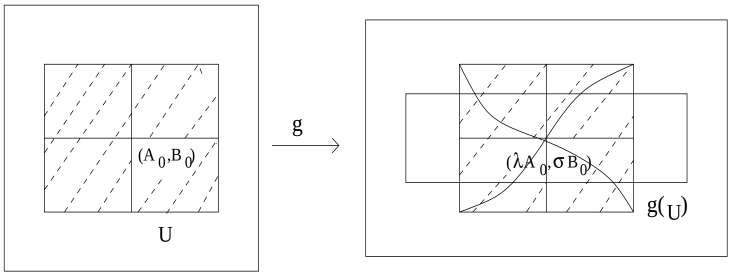
$\frac{\|\omega_{n-m_i+j-1}(z_0)\|}{\|\omega_{n-m_i-1}(z_0)\|} \geq M^{-(k-j)} \cdot \lambda_0^{2k} \geq M^{-\frac{C'}{(const-C')}\cdot j} \cdot \lambda_0^{2k} \geq M^{-\frac{C'}{(const-C')}\cdot j} \cdot \lambda_0^{2(\frac{1+C'}{(const-C')})\cdot j}$

Now observe that $\omega_{n-m_i}(z_0) = \omega^*_{n-m_i}(z_0)$ is almost horizontal because $n - m_i$ is a fold-free time and using a comparison lemma proved before to this vector and $\omega_j(z_{n-m_i})$ :

$\| \omega_j(z_{n-m_i}) \| \geq \frac{1}{2} M^{-\frac{C'}{(const-C')}\cdot j} \cdot \lambda_0^{2(\frac{1+C'}{(const-C')}+1)\cdot j}$

expands with $1 \leq j \leq m_i$

Let's stop for the moment all the computations. The figure below resume all that were done until now:




Remark that $\gamma_0 = [\eta_0 - {\rho_0}^{m_i}, \eta_0 + {\rho_0}^{m_i}]$.

We also know $\eta_0$ expansive up to $m_i$ (lemma of the contractive distributions ) :

1. $\frac{2\cdot\|\omega_n(z_0)\|}{\|\omega_{n-m}(z_0)\|} \geq \| D\Phi^{m_i}_{\eta_0} \cdot \gamma'(\eta_0) \| \geq \frac{1}{2} \frac{\|\omega_n(z_0)\|}{\|\omega_{n-m}(z_0)\|}$ ; with $\gamma'(\eta_0)$ unitary
2. $| \, angle(D\Phi^{m_i}_{\eta_0} \cdot \gamma'(\eta_0), \omega_n(z_0)) \, | \leq (5\gamma^{\frac{1}{4}})^{m_i+1}$

Now a technical lemma:

**Lemma 8.3.** *let $n$ free time and $k < n$ then $\| \omega_n \| \geq e^{const\cdot k} \cdot \| \omega_{n-k} \|$*

Proof: take $n - j \leq n - k$ fold-free and apply lemma of denseness of folding period:
$(n-k)-(n-j) = j-k \leq \mu_2 - \mu_1 \leq (\frac{C'}{const-C'}) \cdot (n-(n-k)) = (\frac{C'}{const-C'}) \cdot k$
then,
$\| \omega_n \| = \| \omega_n^* \| \geq e^{const\cdot j} \cdot \| \omega_{n-j}^* \| \geq e^{const\cdot j} \cdot M^{-(j-k)} \| \omega_{n-k} \| \geq e^{const\cdot k} \cdot M^{-\frac{C'}{(const-C')}\cdot k} \cdot \| \omega_{n-k} \|$
$\square$

This lemma can even be extended to points bound to $z_0$ exactly as [5].

Now we iterate $m_i$ times the curve $\gamma_0 = [\eta_0 - \rho_0^{m_i}, \eta + \rho_0^{m_i}]$ below until time $n$ and conclude it is yet almost horizontal. First put $\gamma = \Phi^{m_i}(\gamma_0)$ and $\eta = \Phi^{m_i}(\eta_0)$.

**Lemma 8.4.** *With the above notations,*

1. *$\gamma$ is almost horizontal meaning $const \cdot \gamma^{\frac{1}{2}} C^2$ horizontal*
2. *from $\gamma_0$ to $\gamma$ we expand, ie. , $\| \gamma'(\Phi(\gamma_0(\xi)) \| = \| D\Phi^{m_i}_{\gamma_0(\xi)} {\gamma'}_0(\xi) \| \geq \| {\gamma'}_0(\xi) \| = 1$ ; $\xi \in \gamma$*

Proof: Applying the last lemma,

$$\| D\Phi^{m_i}_{\xi} {\gamma'}_0(\xi) \| \geq \frac{\| \omega_n(z_0) \|}{\| \omega_{n-m_i}(z_0) \|} \cdot \frac{1}{2} \geq \frac{1}{2} \geq \frac{1}{2} \cdot e^{const\cdot m_i} \geq 1$$

if $m_i$ is large and $\xi$ bound to $\eta_0$ in $\gamma_0$.

For $m$ not big above observe that the expansion come from "driving force lemma" (and because $n$ a return close to hetereclinic point).

But $\omega_n(z_0)$ is almost horizontal:

$| \, angle(D\Phi^{m_i}_{\eta_0} {\gamma'}_0(\eta_0)) \, | \leq (5\gamma^{\frac{1}{4}})^{1+m_i} + S \leq const\gamma^{\frac{1}{2}}$
So that the lemma is proved if the curvature $k$ of $\gamma$ satisfies :
$k \leq const\gamma^{\frac{1}{2}}$
Therefore we compute $k$ :
$k = \frac{\| \langle \gamma', \gamma' \rangle \cdot \gamma'' - \langle \gamma', \gamma'' \rangle \cdot \gamma' \|}{\| \gamma' \|^4} = \frac{|\det(\gamma', \gamma'')|}{\| \gamma' \|^3}$
First of all fix $\Delta_m = \Phi^m(\gamma_0)$. Then by calculus :

$$\begin{cases} \Delta_m = f(\Delta_{m-1}) \\ {\Delta'}_m = D\Phi {\Delta'}_{m-1} \\ {\Delta_m}'' = D\Phi^2({\Delta'}_{m-1})^2 = D\Phi {\Delta''}_{m-1} \end{cases} .$$

Also we get,
$k(\Delta_m) = (\frac{\|{\Delta'}_{m-1}\|}{\|{\Delta'}_m\|})^3 \cdot [\frac{|\det(D\Phi{\Delta'}_{m-1}, D\Phi{\Delta''}_{m-1})| + |\det(D\Phi{\Delta'}_{m-1}, D\Phi^2{\Delta'}^2_{m-1})|}{\|{\Delta'}_{m-1}\|^3}]$
Estimate as:
$k(\Delta_m) \leq K_{m-1}(\frac{|\det(D\Phi{\Delta'}_{m-1}, D\Phi{\Delta''}_{m-1})|}{|\det({\Delta'}_{m-1}, {\Delta''}_{m-1})|} k(\Delta_{m-1}) + L)$

$$\text{with} \begin{cases} K_{m-1} = (\frac{\|{\Delta'}_{m-1}\|}{\|{\Delta'}_m\|})^3 \\ L = | \det(D\Phi \frac{{\Delta'}_{m-1}}{\|{\Delta'}_{m-1}\|}, D\Phi^2(\frac{{\Delta'}_{m-1}}{\|{\Delta'}_{m-1}\|})^2) \, | \end{cases} .$$

Observe that $\frac{|\det(D\Phi{\Delta'}_{m-1}, D\Phi{\Delta''}_{m-1})|}{|\det({\Delta'}_{m-1}, {\Delta''}_{m-1})|}$ measure the area increase of $f$ so is small than $D\gamma$ ( for some fixed constant $D$ ).

On the other hand far from critical region the map have hyperbolic behavior and the sectionally dissipativeness guarantee $L \leq const\gamma^{\frac{1}{2}}$ for some, even big, *const*.

For $\Delta_{m-1}$ close to $p$ we still compute $L \leq const\gamma^{\frac{1}{2}}$ but now from the cubic form of $\Phi$.

Now induction and remark that
$K_{m_i-1} \ldots K_j = (\frac{\|{\Delta'}_j\|}{\|{\Delta'}_{m_i}\|})^3 \leq (e^{const(j-m_i)})^3$
prove $k(\gamma) \leq const\gamma^{\frac{1}{2}}$
□

*Definition: for* $n$ *a return we regress* $m_i$ *in time as above to obtain* $\gamma^{m_i} = [\eta - \rho_0^{g_i}, \eta + \rho_0^{g_i}] \subset [\eta - \rho_0^{m_i}, \eta + \rho_0^{m_i}] \subset \gamma$ *by the above lemma which assert expansion from* $\gamma_0$ *to* $\gamma$*. Now there is a maximum* $k$ *such that* $g_k \leq \Theta n$ *and* $\gamma^{m_k}$ *contains a critical point* $\tau_0$*. Call this* $\tau$ *the binding point of* $z_n$*.*

From now on assume $d_\mu = \| z_n - \tau \| \geq e^{-\alpha n}$

**Lemma 8.5.** *let* $k$ *as in the above definition then*

1. $\| z_n - \eta \| \leq \gamma^{\frac{3}{8}} d_n$

*2.* $\mid angle(\omega^*_{n-1}(z_0), \gamma'(\eta)) \mid \leq \gamma^{\frac{3}{8}} d_n$

*In particular, there is a curve* $\bar{\bar{\gamma}}$ $\gamma - C^2$ *close to horizontal throught* $\tau$ *and* $z_n$ *tangent to* $\gamma^{m_k}$ *at* $\tau$ *and to* $\omega_{n-1}*(z_0)$ *at* $z_n$.

Proof: Recall contractive distributions results:
$\| z_n - \eta \| \leq (const\gamma)^m \cdot (const\gamma^{\frac{1}{2}}) \leq \gamma^{\frac{3}{8}+\frac{g}{10}}$ because $z_{n-m_k}, \eta_0 \in \Gamma^{m_k}$
$\mid angle(\omega^*_{n-1}(z_0), \gamma'(z_0)) \mid \leq (const\gamma^{\frac{1}{4}})^g \leq \gamma^{\frac{3}{8}+\frac{g}{10}} d_n$

As [19] the whole proof reduces to $d_n \geq \gamma^{\frac{2}{10}}$ and the case $g \geq \frac{\Theta n}{3}$ follows imediately from [BA] condition and some fixed value of $\Theta$ above:
$d_n \geq e^{-\alpha n} \geq \gamma^{\frac{g}{10}} \Leftrightarrow g \geq (\frac{10\alpha}{\log \gamma^{-1}})n \Leftarrow C' \geq 10\alpha$

For $g \leq \frac{\Theta n}{3}$ it holds $d_n \geq \rho_0^{3g}$ this is proved by contradiction showing that $\gamma^{m_{k+1}}$ contains a critical point (contradicting choice of $k$ because of favorable time requirements $g_{k+1} \leq 3g_k - 2 \leq \frac{3\Theta n}{3} - 2 \leq \Theta n$ ).

Fix $\mu$ such that $\Theta\mu < g_{k+1} \leq \Theta(\mu+1)$ and put $\tau = \tau^n, \ldots \tau^{\mu-1}$ the decreasing precision of $\tau$.

We claim that there is $\bar{z_0} \in \gamma^{m_{k+1}}$ such that:
$\| \tau^{\mu-1} - \bar{z_0} \| \leq (const\gamma^{\frac{1}{2}})^{g_k}$
$\mid angle(\gamma'^{m_k}(\tau^{\mu-1}), \gamma'^{m_{k+1}}(\bar{z_0})) \mid \leq (const\gamma^{\frac{1}{2}})^{g_k}$

These means that points and tangent vectors are close and allowed, by perturbation arguments as before, induce $\bar{\tau}^{\mu-1} \in \gamma^{m_{k+1}}$ $(\mu-1)$-th critical point. Also observe that a special choice of $\Theta$ guarantee $(const\gamma^{\frac{1}{2}})^{g_k} \leq \frac{1}{2}\rho_0^{\mu-1}$. This $\bar{\tau}^{\mu-1}$ was included as critical point because closeness:
$\| \bar{\tau}^{\mu-1} - \tau^{\mu-1} \| \leq (const\gamma^{\frac{1}{2}})^{g_k} \leq \frac{1}{2}\rho_0^{\mu-1}$

Therefore, $\gamma^{m_{k+1}}$ contains also $\bar{\tau}$ an $n$-th critical point and because we are supposing $d_n < \rho_0^{3g_k}$ :
$\| \bar{\tau} - \eta^{k+1} \| \leq \| \bar{\tau} - \tau \| + \| \tau - z_n \| + \| z_n - \eta^{k+1} \| \leq \frac{1}{2}\rho^{\mu-1} + (const\gamma)^{\mu-1} + \rho^{3g_k} + (const\gamma)^{m_{k+1}}(const\gamma^{\frac{1}{2}}) < \rho^{g_{k+1}}$

because $g_{k+1} \leq 3g_k - 2$ and $g_{k+1} \ll \mu$ by $\Theta$
ie, $\bar{\tau} \in \gamma^{m_{k+1}} = [\eta^{k+1} - \rho_0^{g_{k+1}}, \eta^{k+1} + \rho_0^{g_{k+1}}]$ which is a contradiction.

It remains to establish the above claim in order to prove the lemma.

First put $\tilde{\gamma}_0 = [z_{n-m_{k+1}} - \rho_0^{g_{k+1}}, z_{n-m_{k+1}} - \rho_0^{g_{k+1}}]$ and $\tilde{\gamma} = \Phi^{m_{k+1}}(\tilde{\gamma}_0)$

From the argument of contractive distributions :
$\tilde{\gamma} \ni \tilde{z_0} \longleftrightarrow \bar{z_0} \in \gamma^{m_{k+1}}$ such that $\| \tilde{z_0} - \bar{z_0} \|$ and $\mid angle(\tilde{\gamma'}(\tilde{z_0}), \gamma'^{m_k+1}(\bar{z_0})) \mid$ bound by something like $(const\gamma^{\frac{1}{2}})^{g_{k+1}}$

Therefore, we can prove the claim with $\tilde{z_0}$ replacing $\bar{z_0}$.

Observe that an anterior lemma prove the expansion from $\gamma$ to $\gamma^m$ :

$\| \eta_0^k - \Phi^{-m_k}(\tau^{\mu-1}) \| \leq \| \eta^k - \tau^{\mu-1} \| \leq \rho_0^{g_k} + (const\gamma)^{\mu} < \rho_0^{m_k}$

In order to obtain $\tilde{z_0}$ we construct as done in contractive ditribution section an $m_k$-th contractive lines $\Gamma^{m_k}$ throught $\tau^{\mu-1}$. If we have intersection then we get $\tilde{z_0}$ and estimates of that section prove the claim.

In fact look for intersection between $\Gamma^{m_k}$ and $f^{m_{k+1}-m_k}(\tilde{\gamma_0}) = \tilde{\gamma_1}$

First note that as anterior lemma we can prove $\tilde{\gamma_1}$ is $const\gamma^{\frac{1}{2}} - C^2$ horizontal. Hence take $\tilde{\gamma_2} \supset \tilde{\gamma_1}$ with radius $\geq \rho_0^{g_k}$ around $z_{n-m_k}$. By the above closeness of $\eta_0^k$ and $\tau_0^{\mu-1} = \Phi^{-m_k}(\zeta^{\mu-1})$ and the choosen radius of $\tilde{\gamma_2}$ we see $\Gamma^{m_k}$ intersects $\tilde{\gamma_2}$ in a point $\xi$.

We are left to show $\xi \in \tilde{\gamma_1}$ to finish lemma.

Take $\tilde{z_0} = \Phi^{m_k}(\xi)$ and so,

$\| \tilde{z_0} - z_n \| \leq \| \tilde{z_0} - \tau^{\mu-1} \| + \| \tau^{\mu-1} - \tau \| + \| \tau - z_n \| \leq (const\gamma)^{m_k}(const\gamma^{\frac{1}{2}}) + (const\gamma)^{\mu} + \rho_0^{3g_k} < \rho_0^{g_{k+1}}$

again recall: $g_{k+1} \leq 3g_k - 2$ and $d_n = \| \tau - z_n \| < \rho_0^{3g_k}$

As anterior lemma again, from $\tilde{\gamma_0}$ to $\tilde{\gamma}$ we expands and have almost horizontal curves. Recall the length of $\tilde{\gamma_0}$ as was defined below and deduce $\tilde{\gamma}$ has radius larger then $\rho_0^{g_{k+1}}$ proving $\tilde{z_0} \in \tilde{\gamma}$ and so $\xi \in \tilde{\gamma_1}$ as desired.

□

Now recover more induction hypothesis.

**Lemma 8.6.** *let $n$ be a free return for $z_0$ then, $| \alpha_n | \leq const \cdot \| \omega_{n-1}^* \|$ and $const \cdot d_n^2 \leq | \frac{|\beta_n|}{\|\omega_{n-1}^*\|} - \tilde{\beta_0} | \leq const \cdot d_n^2$*

Proof:

Let $\Delta$ be the $\gamma - C^2$ close to horizontal curve tangent to $\gamma^{m_k\prime}$ and $\omega_{n-1}^*$ given by previous lemma: $\Delta(s) = (x(s), y(s))$

Put $\Psi = \Phi \circ \Delta$ and so $\Psi'(s) = D\Phi \cdot \Delta'(s)$

Computing,

$\Psi'(s) = \alpha(s)e^L + \beta(s)(1,0)$

Write the contractive direction $e^l = (q, 1)$ giving:

$\alpha(s) = [(\lambda - 2D'x(s)y(s))x'(s) - (C'\theta + D'x^2 + 3E'y^2)y'(s)]$

$\beta(s) = [(C\theta + 3Dx^2 + Ey^2)x'(s) + (\sigma + 2Exy)y'(s)] - q(s)[(\lambda - 2D'xy)x'(s) - (C'\theta + D'x^2 + 3E'y^2)y'(s)]$

We may suppose $s$ to be the arc-length of $\Delta$: $\sqrt{x'^2 + y'^2} = 1$

Then, because $\omega_{n-1}^* = (x'(l), y'(l))$ has bounded slope:

$\frac{|\alpha(l)|}{\|\Delta'(l)\|} = \frac{|\alpha_n|}{\|\omega_{n-1}^*\|} = | (\lambda - 2D'xy)x' - (C'\theta + D'x^2 + 3E'y^2)y' | \leq Const$

For the sake of notational simplicity assume $(0,0)$ critical point and critical image.

Putting $|\tilde{\beta}| = \frac{|\beta|}{\|\Delta'\|}$ the other inequality:

$|\tilde{\beta}(s) - \tilde{\beta}(0)| = | x^2 + y^2 | \cdot | \frac{(3Dx^2+Ey^2)x'}{x^2+y^2} + \frac{(2Exy)y'}{x^2+y^2} + \frac{(2qD'xy)x'}{x^2+y^2} + \frac{q(D'x^2+3E'y^2)y'}{x^2+y^2} + \frac{C\theta(x'-x'_0)}{x^2+y^2} + \frac{\sigma(y'-y'_0)}{x^2+y^2} + \frac{\lambda(qx'-q_0x'_0)}{x^2+y^2} + \frac{C'\theta(qy'-q_0y'_0)}{x^2+y^2} |$

Now observe that $\frac{3Dx^2+Ey^2}{x^2+y^2}$ and $\frac{D'x^2+3E'y^2}{x^2+y^2}$ are both bounded and bounded away from 0. Also, because the curve is almost horizontal, $| x' | \geq 1 - C$ not so small and $| y' | \leq C$ small. And $q$ is very small because the contractive direction are almost vertical. The other $| \frac{xy}{x^2+y^2} |$ is easily seem to be bounded for small $(x, y)$.

So that we obtain the desired

$| x^2 + y^2 | \cdot const \leq | \tilde{\beta}(s) - \tilde{\beta}(0) | \leq | x^2 + y^2 | \cdot const$

if we bound small the last four terms.

For this purpose, first observe that $\frac{s^2}{x^2+y^2} \approx 1$ now because our curve is an almost horizontal curve in the $C^2$ sense. So that we may replace $x^2 + y^2$ by $s^2$.

Now we come back to the contruction of $\Delta$ and observe that we may suppose it have zero curvature in $\tau = (0,0)$, ie. , $x''(0) = 0 = y''(0)$. Moreover, may we may even suppose $| x''' |$ , $| y''' |$ small.

This is enough to bound the first two terms above because, by L'hopital,

$\lim_{s\to 0} \frac{y'(s)-y'(0)}{s^2} = \lim_{s\to 0} \frac{y''(s)}{2s} = \lim_{s\to 0} \frac{y'''(s)}{2} = y'''(0)$ small

For the remaining two remark that the following holds if we suppose $q'(0) = 0$ :

$\lim_{s\to 0} \frac{q(s)x'(s)-q(0)x'(0)}{s^2} = \lim_{s\to 0} \frac{q'(s)x'(s)+q(0)x''(0)}{2s} =$

$\lim_{s\to 0} \frac{q''(s)x'(s)+2q'(0)x''(0)+q(s)x'''(s)}{2} = q''(0)x'(0) + q(0)x'''(0)$ small

That may be done by changing the vector field $e^l$ between $\tau$ and $z_n$ over the curve $\Delta$ since only on them we use the fact that these direction are the contractive ones ( all this still keeping the small $C^2$ variation of this vector field ).

Observe that if the critical point has $\tilde{\beta}_0 = 0$ as definition of critical points in [19] then, because it is an $n$-th critical approximation:

$| angle(D\Phi_\tau \cdot \Delta'_\tau, e^l(\tau_1)) | \leq const(\frac{b}{\lambda^2})^l$

Then, $| \tilde{\beta}(0) | \leq const \cdot (\frac{b}{\lambda^2})^l \cdot \| D\Phi\Delta' \| \leq constM \cdot (\frac{b}{\lambda^2})^l$

Also recall $(\frac{b}{\lambda^2})^l \leq \tilde{C}\lambda^l d^2$ and $\tilde{C}$ reduces with $\gamma < 1$.

Implying, $| \tilde{\beta}(0) | \leq (constM\tilde{C})d^2 = const \cdot d^2$

This is the case for the central critical point when the tangency is reached. Until there the critical points are created so that $\tilde{\beta}_0$ is some kind of a minimal and we can only guarantee that the difference is of the order of $d^2$.

$\square$

Even if $n$ is a bound return for $\xi_0$ we establish inductive hypothesis.

**Lemma 8.7.** *Let $n$ be a bound return to critical point $z_0$ and $\xi_0$ bound to $z_0$ up to $n$. Then,* $\frac{|\alpha_n(\xi_0)|}{\|\omega^*_{n-1}(\xi_0)\|} \leq const$ *and* $const \cdot d_n^2 \leq \frac{|\beta_n(\xi_0)|}{\|\omega^*_{n-1}(\xi_0)\|} \leq const \cdot d_n^2$

Proof: Take $k$ minimum with $n-k$ return for $\xi_0$ and binding period of $\xi_0$ at $n-k$ contains $n$. Then $k$ is a free return for $z_0$ and $\xi_{n-k}$ is bound to it up to $k$. By anterior lemma ( extended to points bound to critical points ) we have inductively:

$\frac{|\alpha_n(\xi_{n-k})|}{\|\omega^*_{k-1}(\xi_{n-k})\|} \leq const$ ; $const \cdot d_n^2 \leq \frac{|\beta_k(\xi_{n-k})|}{\|\omega^*_{k-1}(\xi_{n-k})\|} \leq const \cdot d_k^2$

If there are no folding periods endings from $n-k$ to $n$ then we iterate horizontalwise and

$\omega^*_{n-1} = \beta_\nu \omega^*_{k-1} \Rightarrow \frac{\alpha_n}{\|\omega^*_{n-1}\|} = \frac{\alpha_k}{\|\omega^*_{k-1}\|}$ and $\frac{\beta_n}{\|\omega^*_{n-1}\|} = \frac{\beta_k}{\|\omega^*_{k-1}\|}$

Let now returns with folding period ending in $[n-k, n]$ : $\mu_1 < \mu_2 < \ldots \mu_s = n-k$

$\omega^*_{n-1} = \beta_{\mu_1} \omega^*_{n-\mu_1-1} + \alpha_{\mu_1} D\Phi^{n-\mu_1-1} e$

By induction,

$\| \alpha_{\mu_1} D\Phi^{n-\mu_1-1} e \| \leq const \| \omega^*_{\mu_1-1} \| \cdot \| D\Phi^{n-\mu_1-1} e \| \leq const \| \omega^*_{\mu_1-1} \| (\frac{b}{\lambda})^{l_1} M^{n-\mu_1-l_1-1} = const \| \omega^*_{\mu_1-1} \| (\frac{b}{\lambda M})^{l_1} M^{n-\mu_1}$

Now recall:

$l \geq \frac{C'}{2} \log \frac{1}{d^2_{\mu_1}}$ , $n-\mu \leq \bar{C} \log \frac{1}{d^2_{\mu_1}}$ (folding/binding period)

Implying,

$| \alpha_{\mu_1} | \| D\Phi^{n-\mu_1-1} e \| \leq const \| \omega^*_{\mu_1-1} \| (\frac{1}{d^2_{\mu_1}})^{\frac{C'}{2} \log \frac{b}{\lambda M} + \bar{C} \log M} = const \cdot \| \omega^*_{\mu_1-1} \| [d^2]^{\frac{C'}{2} \log \frac{b}{\lambda M} + \bar{C} \log M}$

While $\beta$ can be estimate by: $| \beta_{\mu_1} | \| \omega^*_{n-\mu_1-1} \| \geq const \cdot d^2_{\mu_1} \| \omega^*_{\mu_1-1} \| \lambda^{n-\mu_1-1}$

Now bound the angle:

$| angle(\omega^*_{n-1}, \omega^*_{n-\mu_1-1}) | = | angle(\frac{\omega^*_{n-1}}{\|\beta_{\mu_1} \omega^*_{n-\mu_1-1}\|}, \frac{\omega^*_{n-\mu_1-1}}{\|\omega^*_{n-\mu_1-1}\|}) | \leq \frac{\|\alpha_{\mu_1} D\Phi^{n-\mu_1-1} e\|}{\|\beta_{\mu_1} \omega^*_{n-\mu_1-1}\|} \leq \frac{const \|\omega^*_{\mu_1-1}\| [d]^{-\frac{C'}{2} \log \frac{b}{\lambda M} + \bar{C} \log M}}{const \cdot d^2 \|\omega^*_{\mu_1-1}\| \lambda^{n-\mu_1-1}} \leq \frac{const \cdot d^{\frac{C'}{2} \log \frac{\lambda M}{b} - 2 - \bar{C} \log M}}{\lambda^{n-\mu_1}}$

Remark that, reducing $b$, $C'$ is at least of the order $\frac{1}{\log\frac{\lambda}{\sqrt{b}}}$ meaning the above to be exponencially small and so sumable over $\mu_1 \dots \mu_s$ :

$| \, angle(\omega^*_{n-1}, \omega^*_{k-1}) \, | \leq const \cdot d^2$

Now using formulae below relating $\omega^*_{n-1}$ and $\omega^*_{k-1}$ :

$| \, \frac{\alpha_n}{\|\omega^*_{n-1}\|} - \frac{\alpha_k}{\|\omega^*_{k-1}\|} \, | \leq const \cdot d^2$

$| \, \frac{\beta_n}{\|\omega^*_{n-1}\|} - \frac{\beta_k}{\|\omega^*_{k-1}\|} \, | \leq const \cdot d^2$

We finish by observing that the constants above are series sum of exponentially small numbers ( so also small ) so that we conclude the lemma with the below lemma for time $n-k$.

$\square$

# 9 The Binding Period

The basic goal of this section is to obtain the property of bounded distorsion on binding periods and with this establish all remaining induction hypotheses. The motivation for the lemmas is [5].

First we do a basic result:

**Lemma 9.1.** *$\xi_0$ , $\eta_0$ bound to critical point $z_0$ up to time $q$. Then,*

$\frac{\|\omega^*_\nu(\eta_0)\|}{\|\omega^*_\nu(\xi_0)\|} \leq \exp[\sum_{k=0}^{\infty} const \cdot e^{-(\beta-\alpha)k}]$

$| \, angle(\omega^*_\nu(\eta_0), \omega^*_\nu(\xi_0)) \, | \leq const[\sum_{k=1}^{\infty} e^{-(\beta-\alpha)k}]; \forall \nu \leq q$

Proof: Write

$\frac{\|\omega^*_\nu(\eta_0)\|}{\|\omega^*_\nu(\xi_0)\|} = \frac{\|\omega^*_{\nu-1}(\eta_0)\|}{\|\omega^*_{\nu-1}(\xi_0)\|} \exp[\log \| \bar{D}\Phi_{\eta_\nu} \frac{\omega^*_{\nu-1}(\eta_0)}{\|\omega^*_{\nu-1}(\eta_0)\|} \| - \log \| \bar{D}\Phi_{\eta_\nu} \frac{\omega^*_{\nu-1}(\xi_0)}{\|\omega^*_{\nu-1}(\xi_0)\|} \|]$

Where $^-$ means projection over $(1,0)$ parallel to $e^l$.

Looking for an induction argument we try to estimate:

$| \log \| \bar{D}\Phi_{(x,y)}(u,v) \| - \log \| \bar{D}\Phi_{(\bar{x},\bar{y})}(\bar{u},\bar{v}) \| |$

where:

1. $(x,y) = \eta_\nu$ , $(\bar{x},\bar{y}) = \xi_\nu$ are $\bar{e}^{\beta\nu}$ close to each other.

2. $(u,v) = \frac{\omega^*_{\nu-1}(\xi_0)}{\|\omega^*_{\nu-1}(\xi_0)\|}$ , $(\bar{u},\bar{v}) = \frac{\omega^*_{\nu-1}(\eta_0)}{\|\omega^*_{\nu-1}(\eta_0)\|}$ are unitary vectors with small bounded slope.

Consider some cases:

$1^{st}$ *case* : $\nu$ not a return not an ending of folding period. Then $^-$ is not needed in this case

$|\log \parallel D\Phi_{(x,y)}(u,v) \parallel - \log \parallel D\Phi_{(\bar{x},\bar{y})}(\bar{u},\bar{v}) \parallel| \leq const\frac{\|(x,y)-(\bar{x},\bar{y})\|}{d_\nu(\xi_0)} = const\frac{\|\xi_\nu - \eta_\nu\|}{d_\nu(\xi_0)}$

because log depends on $\parallel \eta_\nu - \xi_\nu \parallel$ and $d_\nu(\xi_0)$ far from zero.

Therefore, in this case the above factor become:

$\exp[\frac{const\|\eta_\nu - \eta_\nu\|}{d_\nu(\xi_0)}]$

The angle estimate:

$| angle(\omega^*_\nu(\eta_0), \omega^*_\nu(\xi_0)) | \leq | angle(D\Phi_{(x,y)}(u,v), D\Phi_{(\bar{x},\bar{y})}(u,v)) |$
$+ | angle(D\Phi_{(\bar{x},\bar{y})}(u,v), D\Phi_{(\bar{x},\bar{y})}(\bar{u},\bar{v})) | \leq const \parallel (x,y) - (\bar{x},\bar{y}) \parallel$
$+ const_2 | angle(\omega^*_{\nu-1}(\eta_0), \omega^*_{\nu-1}(\xi_0)) |$

with $const_2 < 1$ a small number.

And this give a new term for angle bound.

$2^{nd}$ *case*: If $\nu$ is a return time for $z_0$ the angle estimate is trivial so first:
$| \log \parallel \bar{D}\Phi_{(x,y)}(u,v) \parallel - \log \parallel \bar{D}\Phi_{(\bar{x},\bar{y})}(u,v) \parallel|$

But the horizontal component is $\beta$ from last two lemmas:

$$
\begin{aligned}
&| \log | (3x^2+y^2)u + (\sigma+2xy)v - q(\lambda-2xy)u + q(3y^2+x^2)v | \\
&\quad - \log | (3\bar{x}^2+\bar{y}^2)u + (\sigma+2\bar{x}\bar{y})v - \bar{q}(\lambda-2\bar{x}\bar{y})u + \bar{q}(3\bar{y}^2+\bar{x}^2)v || \\
&\leq \frac{\max | D_t[(x_t^2+y_t^2)u + (\sigma+2x_ty_t)v - q_t(\lambda-2x_ty_t)u + q_t(x_t^2+y_t^2)v]}{\min | (x_t^2+y_t^2)u + (\sigma+2x_ty_t)v - q_t(\lambda-2x_ty_t)u + q_t(x_t^2+y_t^2)v |} \\
&\leq \frac{\max | \mu_t[2x_t{x'}_t + 2y_t{y'}_t - {q'}_t(\lambda-2x_ty_t) + 2q_t({x'}_ty_t + x_t{y'}_t)] +}{\min | (x_t^2+y_t^2)u_t + (\sigma+2x_ty_t)v_t - q_t(\lambda-2x_ty_t)u_t + q_t(x_t^2+y_t^2)v_t |} \\
&\quad \frac{+\nu_t[2x_t{x'}_tq_t + 2y_t{y'}_tq_t + {q'}_t(x_t^2+y_t^2) + 2({x'}_ty_t + x_t{y'}_t)] |}{\min | (x_t^2+y_t^2)u_t + (\sigma+2x_ty_t)v_t - q_t(\lambda-2x_ty_t)u_t + q_t(x_t^2+y_t^2)v_t |}
\end{aligned}
$$

where the maximun and the mininum is taken over $0 < t < 1$ and we denote $x_t = x + t(\bar{x} - x)$ , $y_t = y + t(\bar{y} - x)$, $0 \leq t \leq 1$. Hence the last inequality can be bounded by

$$\frac{\max \mid x'[2xu+2qyu+2qxv+2yv]+y'[2yu+2qxu+2qyv+2xv]-}{\min \mid (x^2+y^2)u+(\sigma+2xy)v-q(\lambda-2xy)u+q(x^2+y^2)v \mid}$$

$$\frac{-q'[(\lambda-2xy)u-(x^2+y^2)v] \mid}{\min \mid (x^2+y^2)u+(\sigma+2xy)v-q(\lambda-2xy)u+q(x^2+y^2)v \mid}$$

$$\leq \frac{\max \| (2xu+2qyu+2qxv+2yv, 2yu+2qxu+2qyv+2xv) \| \cdot \| (x',y') \|}{\min \mid (x^2+y^2)u+(\sigma+2xy)v-q(\lambda-2xy)u+q(x^2+y^2)v \mid}$$

$$+\frac{\max \| dq' \| \cdot \| (x',y') \| \cdot \mid (\lambda-2xy)u-(x^2+y^2)v \mid}{\min \mid (x^2+y^2)u+(\sigma+2xy)v-q(\lambda-2xy)u+q(x^2+y^2)v \mid}.$$

Now observe that $(x',y')=(\bar{x}-x,\bar{y}-y)$ and so $\| (x',y') \| \leq e^{-\beta\nu}$ is exponentially small.

Also remark that $(x,y)$ is $e^{-\alpha\nu}$ far from zero and max ( min ) deal with quadratic maps on $t$ ( $x_t=x+t(\bar{x}-x)$ , $y_t=y+t(\bar{y}-y)$ ) for $t\in[0,1]$ compact. Those together means that we can estimate the above by calculus by something like: $\frac{const}{const+\frac{d\nu}{\|\xi_\nu-\eta_\nu\|}}$

Which is exponentially small because $\beta>\alpha$.

Now bound:

$\mid \log \| \bar{D}\Phi_{(\bar{x},\bar{y})}(u,v) \| - \log \| \bar{D}\Phi_{(\bar{x},\bar{y})}(\bar{u},\bar{v}) \| \mid$

$\mid \log \mid (3\bar{x}^2+\bar{y}^2)u+(\sigma+2\bar{x}\bar{y})v-q(\lambda-2\bar{x}\bar{y})u+q(3\bar{y}^2+x^2)v \mid - \log \mid (3\bar{x}^2+\bar{y}^2)\bar{u}+(\sigma+2\bar{x}\bar{y})\bar{v}-q(\lambda-2\bar{x}\bar{y})\bar{u}+q(3\bar{y}^2+x^2)\bar{v} \mid\mid \leq$

$\frac{|((\bar{x}^2+\bar{y}^2)-(\lambda-2\bar{x}\bar{y})q)(u-\bar{u})+((\sigma+2\bar{x}\bar{y})-q(\bar{x}^2+\bar{y}^2)(v-\bar{v}))|}{\min_{0\leq t\leq 1}|((\bar{x}^2+\bar{y}^2)-(\lambda-2\bar{x}\bar{y})q)u_t+((\sigma+2\bar{x}\bar{y})-q(\bar{x}^2+\bar{y}^2))v_t|} \leq$

$\leq \frac{\|((\bar{x}^2+\bar{y}^2)-(\lambda-2\bar{x}\bar{y})q),(\sigma+2\bar{x}\bar{y})-q(\bar{x}^2+\bar{y}^2))\|\cdot\|(u-\bar{u},v-\bar{v})\|}{\min_{0\leq t\leq 1}|((\bar{x}^2+\bar{y}^2)-(\lambda-2\bar{x}\bar{y})q)u_t+((\sigma+2\bar{x}\bar{y})-q(\bar{x}^2+\bar{y}^2))v_t|} \leq$

For this case observe that $(u-\bar{u},v-\bar{v})=\frac{\omega^*_{\nu-1}(\xi_0)}{\|\omega^*_{\nu-1}(\xi_0)\|}-\frac{\omega^*_{\nu-1}(\eta_0)}{\|\omega^*_{\nu-1}(\eta_0)\|}$ and this is bound because, as said before, we expect an inductive argument where these two unitary vector have angle exponentially small by induction hypothesis.

Finally, putting all together,

$\mid \log \| D\Phi_{(x,y)}(u,v) \| - \log \| D\Phi_{(\bar{x},\bar{y})}(\bar{u},\bar{v}) \| \mid \leq \mid \log \| D\Phi_{(x,y)}(u,v) \| - \log \| D\Phi_{(\bar{x},\bar{y})}(u,v) \| \mid + \mid \log \| D\Phi_{(\bar{x},\bar{y})}(u,v) \| - \log \| D\Phi_{(\bar{x},\bar{y})}(\bar{u},\bar{v}) \| \mid$

we find the required bound.

$3^{rd}$ *case*: let $\nu$ coincide with the end of a folding period, ie , $\nu=\mu+l$ then,

$$\begin{cases} \omega^*_\nu(\xi_0)=\beta_\mu(\xi_0)\omega^*_l(\xi_\mu)+\alpha_\mu(\xi_0)D\Phi^l e^l(\xi_\mu) \\ \omega^*_\nu(\eta_0)=\beta_\mu(\eta_0)\omega^*_l(\eta_\mu)+\alpha_\mu(\eta_0)D\Phi^l e^l(\eta_\mu) \end{cases}.$$

Because of induction we thinck the estimate hold for $\omega_l^*(\xi_\mu), \omega_l^*(\eta_\mu)$

Now, following [19], the idea is to use:

$\mid angle(\omega_\nu^*(\xi_0), \omega_\nu^*(\eta_0)) \mid \leq \mid angle(\omega_l^*(\xi_\mu), \omega_l^*(\eta_\mu)) \mid +2\frac{\chi}{\varepsilon}$

$\frac{\|\omega_\nu^*(\xi_0)\|}{\|\omega_\nu^*(\eta_0)\|} \leq \frac{|\beta_\mu(\xi_0)|}{|\beta_\mu(\eta_0)|}\frac{\|\omega_l^*(\xi_\mu)\|}{\|\omega_l^*(\eta_\mu)\|}(1+\frac{\chi}{\varepsilon})$

Or equivalently,

$\log \parallel \omega_\nu^*(\xi_0) \parallel - \log \parallel \omega_\nu^*(\eta_0) \parallel \leq (\log \parallel \omega_{\mu-1}^*(\xi_0) \parallel - \log \parallel \omega_{\mu-1}^*(\eta_0) \parallel ) + (\log \frac{|\beta_\mu(\xi_0)|\cdot\|\omega_l^*(\xi_\mu)\|}{\|\omega_{\mu-1}^*(\xi_0)\|} - \log \frac{|\beta_\mu(\eta_0)|\cdot\|\omega_l^*(\eta_\mu)\|}{\|\omega_{\mu-1}^*(\eta_0)\|}) + \log(1+\frac{\chi}{\varepsilon}) \mid$

$\mid angle(\omega_\nu^*(\xi_0), \omega_\nu^*(\eta_0) \mid = \mid angle \frac{\omega_\nu^*(\xi_0)}{\|\omega_{\mu-1}^*(\xi_0)\|}, \frac{\omega_\nu^*(\eta_0)}{\|\omega_{\mu-1}^*(\eta_0)\|} \mid \leq$

$\leq \mid angle \frac{\beta_\mu(\xi_0)\omega_l^*(\xi_\mu)}{\|\omega_{\mu-1}^*(\xi_0)\|}, \frac{\beta_\mu(\eta_0)\omega_l^*(\eta_{nu})}{\|\omega_{\mu-1}^*(\eta_0)\|} \mid + \frac{2\chi}{\varepsilon}$

Note that in this expression we may apply induction hypothesis for $\mu - 1$ and $l$.

Remark that,

$$\chi = \mid angle \frac{\beta_\mu(\xi_0)\omega_l^*(\xi_\mu)}{\|\omega_{\mu-1}^*(\xi_0)\|}, \frac{\beta_\mu(\eta_0)\omega_l^*(\eta_{nu})}{\|\omega_{\mu-1}^*(\eta_0)\|} \mid \cdot \frac{\frac{|\alpha_\mu(\xi_0)|\cdot\|D\Phi^l e^l(\xi_\mu)\|}{\|\omega_{\mu-1}^*(\xi_0)\|}}{\frac{|\beta_\mu(\xi_0)|\cdot\|\omega_l^*(\xi_\mu)\|}{\|\omega_{\mu-1}^*(\xi_0)\|}} +$$

$$+ \frac{\| \frac{\alpha_\mu(\xi_0)\cdot D\Phi^l e^l(\xi_\mu)}{\|\omega_{\mu-1}^*(\xi_0)\|} - \frac{\alpha_\mu(\eta_0)\cdot D\Phi^l e^l(\eta_\mu)}{\|\omega_{\mu-1}^*(\eta_0)\|}}{\frac{|\beta_\mu(\xi_0)|\cdot\|\omega_l^*(\xi_\mu)\|}{\|\omega_{\mu-1}^*(\xi_0)\|}} +$$

$$+ \mid \frac{\frac{|\beta_\mu(\eta_0)|\cdot\|\omega_l^*(\eta_\mu)\|}{\|\omega_{\mu-1}^*(\eta_0)\|}}{\frac{|\beta_\mu(\xi_0)|\cdot\|\omega_l^*(\xi_\mu)\|}{\|\omega_{\mu-1}^*(\xi_0)\|}} - 1 \mid \cdot \frac{\frac{|\alpha_\mu(\eta_0)|\cdot\|D\Phi^l e^l(\eta_\mu)\|}{\|\omega_{\mu-1}^*(\eta_0)\|}}{\frac{|\beta_\mu(\eta_0)|\cdot\|\omega_l^*(\eta_\mu)\|}{\|\omega_{\mu-1}^*(\eta_0)\|}}$$

Above $\frac{\frac{|\beta_\mu(\eta_0)|\cdot\|\omega_l^*(\eta_\mu)\|}{\|\omega_{\mu-1}^*(\eta_0)\|}}{\frac{|\beta_\mu(\xi_0)|\cdot\|\omega_l^*(\xi_\mu)\|}{\|\omega_{\mu-1}^*(\xi_0)\|}}$ is bound by induction hypothesis and $2^{nd}$ case where $\frac{|\beta_\mu(\eta_0)|}{|\beta_\mu(\xi_0)|}$ was estimated for $\mu$ return. The same for the angle.

But to estimate the difference $\parallel \frac{\alpha_\mu(\xi_0)D\Phi^l e^l(\xi_\mu)}{\|\omega_{\mu-1}^*(\xi_0)\|} - \frac{\alpha_\mu(\eta_0)D\Phi^l e^l(\eta_\mu)}{\|\omega_{\mu-1}^*(\eta_0)\|} \parallel$ we also observe that we could have estimated in case 2 the relation between $\alpha_\mu(\xi_0)$ and $\alpha_\mu(\eta_0)$ as was done for $\beta$'s.

Finally, we just put all together, exactly as [19], and find the bound looked for.

□

Now suppose $z_0$ a critical point which return at time $n$ and stay binding with $\tau_0$:

**Corollary 9.2.** *for* $1 \leq \nu \leq \min\{p, \bar{C}\log\frac{1}{d_n^2}\}$

$\frac{1}{const} \leq \frac{\|\omega^*_{n+\nu}(z_0)\|}{|\beta_n(z_0)|\cdot\|\omega^*_\nu(\tau_0)\|} \leq const$
*Moreover, the same bound works for $\tau_0$ bound to $z_0$ up to $n+\nu$.*

Proof: The above lemma compare:
$\frac{\|\omega^*_\nu(\xi_n)\|}{\|\omega^*_\nu(\tau_0)\|} \leq const$
So that we only need to compare $\| \omega^*_{n+\nu}(\xi_0) \|$ and $\beta_n(\xi_0)\cdot\omega^*_n(\xi_n)$.
But they differ by folding terms recompositions for returns:
$\| \omega^*_{n+\nu}(\xi_0) \| = \| \beta_n\omega^*_\nu(\xi_n) + \alpha_n D\Phi^\nu e^l \|$
$\Rightarrow \frac{\|\omega^*_{n+\nu}(\xi_0)\|}{|\beta_n(\xi_0)|\cdot\|\omega^*_\nu(\xi_n)\|} \leq 1 + \frac{|\alpha_n|\cdot\|D\Phi^\nu e^l\|}{|\beta_n|\cdot\|\omega^*_\nu(\xi_n)\|}$
And recall that when we recovered induction hypothesis for bound return we estimated $\frac{|\alpha_n|\cdot\|D\Phi^\nu e^l\|}{|\beta_n|\cdot\|\omega^*_{\nu-\mu}(\xi_\mu)\|}$ exponentially small.
Now finished upper inequality because :
$\frac{\|\omega^*_{n+\nu}(\xi_0)\|}{|\beta_n(\xi_0)|\cdot\|\omega^*_n(\tau_0)\|} = \left(\frac{\|\omega^*_{n+\nu}(\xi_0)\|}{|\beta_n(\xi_0)|\cdot\|\omega^*_\nu(\xi_n)\|}\right)\cdot\left(\frac{\|\omega^*_\nu(\xi_n)\|}{\|\omega^*_\nu(\tau_0)\|}\right)$
The lower one is in the same way.
□

Now extend inductions hypothesis for bound points.

**Lemma 9.3.** *let $n$ free return for $z_0$ and $\xi_0$ bound to it up to $n$ then*
$| \alpha_n(\xi_0) | \leq const \| \omega^*_{n-1}(\xi_0) \|$ ; $const\cdot d_n^2 \leq \frac{|\beta_n(\xi_0)|}{\|\omega^*_{n-1}(\xi_0)\|} \leq const\cdot d_n^2$

Proof: Just observe that $\frac{|\xi_n - z_n|}{d_n}$ is very small because $\beta > \alpha$
Also $\frac{|angle(\omega^*_{n-1}(\xi_0),\omega^*_{n-1}(z_0))|}{d_n}$ is so small now because $\xi_0$ and $z_0$ are bounded and by comparison lemma before.
Then we get the tangential position requirement to reproduce the lemma before which dealt only with $z_0$.
□

It's easy compare increasing of $\omega$-vectors on bound orbits

**Lemma 9.4.** *$\xi_0$ and $\eta_0$ bound up to $q$ to the same critical point. Then*
$\frac{\|\omega_\nu(\eta_0)\|}{\|\omega_\nu(\xi_0)\|} \leq \exp[const\cdot e^{(\alpha+C'\log M+2)\nu}\sum_{k=1}^{\nu} e^{-(\beta-\alpha)k}]$
*and,*
$| angle(\omega_\nu(\xi_0),\omega_\nu(\eta_0)) | \leq const\cdot e^{(\alpha+C'\log M+2)\nu}\sum_{k=1}^{\nu} e^{-(\beta-\alpha)k}$

Proof: The comparative lemma before asserted $\| \omega_\nu \| \geq e^{-[C'\log M+2]\alpha\nu} \| \omega^*_\nu \|$
So with $\xi = e^{-[C'\log M+2]\alpha\nu}$ we intend to use:

$\frac{\|\omega_\nu(\xi_0)\|}{\|\omega_\nu(\eta_0)\|} \leq \frac{\|\omega^*_\nu(\xi_0)\|}{\|\omega^*_\nu(\eta_0)\|} \cdot (1+\frac{\chi}{\varepsilon})$

$\mid angle(\omega_\nu(\xi_0), \omega_\nu(\eta_0)) \mid \leq \mid angle(\omega^*_\nu(\xi_0), \omega^*_\nu(\eta_0)) \mid + \frac{2\chi}{\varepsilon}$

Where

$\chi = \mid angle(\omega^*_\nu(\xi_0), \omega^*_\nu(\eta_0)) \mid \cdot \frac{\|\omega_\nu(\xi_0)-\omega^*_\nu(\xi_0)\|}{\|\omega^*_\nu(\xi_0)\|} + \frac{\|(\omega_\nu(\xi_0)-\omega^*_\nu(\xi_0))-(\omega_\nu(\eta_0)-\omega^*_\nu(\eta_0))\|}{\|\omega^*_\nu(\xi_0)\|} +$
$+ \mid \frac{\|\omega^*_\nu(\eta_0)\|}{\|\omega^*_\nu(\xi_0)\|} - 1 \mid \cdot \frac{\|\omega_\nu(\eta_0)-\omega^*_\nu(\eta_0)\|}{\|\omega^*_\nu(\eta_0)\|}$

Recall the usual decomposition $\omega_\nu - \omega^*_\nu = \sigma_\nu$

Each term in $\sigma_\nu$ can be estimated:

$\| \alpha_\mu(\eta_0) D\Phi^{\nu-\mu} e(\eta_n) \| \leq const \| \omega^*_{\mu-1}(\eta_0) \| \cdot \| D\Phi^{\nu-\mu} e(\eta_\nu) \| \leq const \frac{|\beta_\mu(\eta_0)|}{d^2_\mu(\eta_0)} \cdot$

$\| D\Phi^{\nu-\mu} e(\eta_\nu) \|$

The last corollary gives

$\| \omega^*_\nu(\eta_0) \| \geq \frac{1}{const} \mid \beta_\mu(\eta_0) \mid \cdot \| \omega^*_{\nu-\mu}(\eta_\mu) \| \geq \frac{1}{const} \mid \beta_\mu(\eta_0) \mid$

It follows that

$\frac{\|\omega_\nu(\eta_0)-\omega^*_\nu(\eta_0)\|}{\|\omega^*_\nu(\eta_0)\|} \leq \sum_\mu const \cdot d^{-2}_\mu(\eta_0) \cdot \| D\Phi^{\nu-\mu} e(\eta_\nu) \|$

Remark that the above holds also for $\xi_0$.

On the other hand we can use comparative lemma before to time $\mu$ and get :

$\mid \alpha_\mu(\xi_0) - \alpha_\mu(\eta_0) \mid \leq const (\sum_{k=1}^{\mu} e^{-(\beta-\alpha)k}) \| \omega^*_{\mu-1}(\eta_0) \|$

Using $C^2$ small variation of contractive directions we can improve:

$\| \alpha_\mu D\Phi^{\nu-\mu} e(\xi_\mu) - \alpha_\mu D\Phi^{\nu-\mu} e(\eta_\mu) \| \leq const \| \omega^*_{\mu-1}(\eta_0) \| \sum_{k=1}^{\mu} e^{-(\beta-\alpha)k}$

Follows easily,

$\frac{\|\sigma_\nu(\xi_0)-\sigma_\nu(\eta_0)\|}{\|\omega^*_\nu(\xi_0)\|} \leq const \cdot e^{\alpha\nu} \sum_{k=1}^{\nu} e^{-(\beta-\alpha)k}$

Also $\| \omega^*_\nu(\xi_0) - \omega^*_\nu(\eta_0) \| \leq \| \omega^*_\nu(\xi_0) \| (\sum_{k=1}^{\nu} e^{-(\beta-\alpha)k}) const$

The above give all terms and factors to bound $\chi$ :

$\chi \leq const \cdot e^{\alpha\nu} \sum_{k=1}^{\nu} e^{(\beta-\alpha)k}$

Finally putting all together the lemma is established.

$\square$

The lemma below is proved in the same way as [19] since all ingredients needed are stated above.

**Lemma 9.5.** *Let $\eta(s)$ be a $\gamma - C^2$ almost horizontal curve with $\eta(l) = \tau_0$ critical point. If $0 < \sigma < 2\delta$ be such that $\| \eta_\nu(s) - \tau_\nu \| \leq const \cdot e^{-\beta\nu}$ ; $0 \leq s \leq \sigma$ and $1 \leq \nu \leq p$*

*Then,*

*a) $\sigma^2 \| \omega_\nu(\tau_0) \| \leq e^{-\beta\nu}$ ; $\forall\ 0 \leq \nu \leq p-1$*

*b) If moreover $\sigma^2 \| \omega_p(\tau_0) \| \leq const^2 \cdot e^{-2\beta p}$ then $\| \eta_{p+1}(s) - \tau_{p+1} \| \leq const \cdot e^{-\beta(p+1)}$*

Now recover the remainig induction hypothesis.

**Lemma 9.6.** *Let $n$ return for critical point $z_0$ and $p$ its binding period then* $p \leq \bar{C} \log \frac{1}{d^2} \leq \bar{C} \alpha n$
*Moreover if $\xi_0$ is bound to $z_0$ up to $n+p$.*
$\frac{\|\omega^*_{n+p}(\xi_0)\|}{\|\omega^*_n(\xi_0)\|} \cdot d_\nu(\xi_0) \geq const \lambda^{\frac{p}{2}} e^{\frac{-\beta p}{2}}$

Proof: Let $\tau_0$ binding point of $\xi_0$ and $s \mapsto \eta(s)$ $\gamma - C^2$ almost constant curve with $\eta(0) = \tau_0$ and $\eta(\sigma) = \sigma_n$. For if $p > \bar{C} \log \frac{1}{d^2}$ then:
$\| \omega_p(\tau_0) \| \sigma^2 \geq \frac{9}{10} \lambda^p d_n^2 \geq e^{-\beta p}$
, because $\frac{9}{10} e^{(\log \lambda + \beta)p} d^2 = \frac{9}{10} d^{2(1-\bar{C}(p+\log\lambda))} \geq 1$ ; $\bar{C}$ small ; $\sigma \approx d_n$
By lemma before this implies $\| \eta_p(\sigma) - \eta_p \| \geq const \cdot e^{-\beta p}$ contradiction.
To close the proof observe that with the above proved inequality and last corollary and with second item of anterior lemma (since $\| \eta_{p+1} - \tau_{p+1} \| \geq const \cdot e^{-\beta(p+1)}$ )
$\| \omega_p(\tau_0) \| \sigma^2 \geq const \cdot e^{-\beta p} \Rightarrow \| \omega^*_p(\xi_n) \| \sigma = (\| \omega^*_p(\xi_n) \| \cdot \| \omega^*_p(\xi_n) \| \sigma^2)^{\frac{1}{2}} \geq const \lambda^{\frac{p}{2}} e^{-\frac{\beta p}{2}}$
$\| \omega^*_{n+p}(\xi_0) \| \geq const \mid \beta_n(\xi_0) \mid \cdot \| \omega^*_p(\xi_n) \| \geq const \| \omega^*_n(\xi_0) \| \lambda^{\frac{p}{2}} e^{-\frac{\beta p}{2}}$
□

# 10 Parameter Dependence

The goal of this section is to show that, even excluding parameters to satisfy conditions [BA] and [FA], there remains a large lebesgue measure set of $(z_0, \theta)$.

First we fix $\theta$ small and thinck the parameter is given only by the $z_0$ point of $\Phi_{(z_0,\theta)}$.

Even considering this way the parameter space is not one dimensional so the strategy, then, is to fix the $x$ coordinate of the point $\tilde{A}$ letting it vary in an interval $I$. That means that the cube $R$ is divided into a family of segments $I$ paralell to unstable direction.

Therefore all now is done for $z_0 \in I$. Moreover, if we find a positive measure subset of $I$ in which we reach the border of the hyperbolic systems, for each $I$, this will give us a positive measure subset of $R$.

So that,
$z_0^i : I \to \Re^2$ unique critical point for $0 \leq i \leq N$.

That is, until a big time $N$ we have just one critical point for $\Phi_{(z_0,\theta)}$ (namely $z_0$) and parameter set keep unchanged. Precisely, the good parameter set is $S_i = I$ for all $i \leq N$.

Clearly we may choose $N$ an escaping situation for $z_0$:
$length(z_0^N(I)) > \sqrt{\delta}$

For times higher than $N$ we just proceed by induction:

for each $k < n$ we suppose defined $S_k$ and, moreover, for $z_0 \in S_k$ and $\tau_0^k$ a $k$-th critical approximation for $\Phi_{(z_0,\theta)}$ there are a time $\nu \in [\frac{k+1}{2}, k+1]$ and a parameter interval $\omega$ with $e^{-const'\nu} \leq length(\omega) \leq e^{-const''\nu}$ such that:

1. $\tau_0^k$ may be defined in $\omega$
2. $\tau_0^k(z)$ for $z \in \omega$ satisfies all induction hypothesis for all times anterior to $k$
3. time $\nu$ is an escape time for $\tau^k : \omega \to \Re^n$

Remark: As [19] we need inductively a parametrized version of algorithm which creates higher generation critical points from older ones. Also we can rewrite all lemmas before for this parametrized kind of assumptions.

In fact, by mean value theorem , the variation of parameter change almost by $const \cdot length(\omega) \leq const \cdot e^{-const''\nu}$ while we just iterate (generation) only by $\Theta\nu$. So what holds for some $a_0$ parameter easily extend for whole $\omega$ with slightly differents factors.

Put $a_0 \in S_{n-1}$ a fixed "parameter" and $z_0$ an $n$-th critical point for this parameter. Also let $\tau_0$ be an $(n-1)$-th critical approximation which originate $z_0$, $\omega$ its parameter interval and $\nu \in [\frac{n}{2}, n]$ be its escape situation for $\tau_0$.

We estimate the derivate of the critical point $z_0$ with the parameter (with the convention that the dot below always means parameter derivative).

**Lemma 10.1.** $\| \dot{z}_0 \| \leq \sum_{i=1}^{g} \gamma^{\frac{i}{20}} + \sum_{i=1}^{n} \gamma^{\frac{i}{3}}$

Proof: Let $\bar{\tau}_0$ the refinement of $\tau_0$ for $\alpha \in \omega$ so we have:
$\| \bar{\tau}_0 - \tau_0 \| \leq const\gamma^{n-1}$ $\| \bar{\tau}_0 - z_0 \| \leq const\gamma^{\frac{g}{10}}$
by construction $\forall a \in \omega$ (see lat remark)

Now parametrize $\xi : J \longrightarrow W_1$ ; $s \longmapsto \xi_a(s)$ ; $\forall\ a \in \omega$ with $\xi_{a_0}(J) = \Phi^{-g}([z_0 - \rho_0^{\Theta n}, z_0 + \rho_0^{\Theta n}])$ and $\tau : J \longrightarrow W_g$ ( $W_g$ the $g$-iterate of horizontal line where lives the generations $g$ points) given by

$\tau_a(s) = \Phi^g(\xi_0(s)) = (x_a(s), y_a(s))$

We have $\| D\Phi_a^g \cdot \xi'_a \| \geq \frac{1}{2}$ expands because this happens to $a_0$, mean value theorem, the bound for length $\omega$ and the fact $g \leq \Theta n$ (see remark above). Moreover we yet get closeness, i.e. :

$\| D\Phi_a^g \xi'_a(s) - D\Phi_{a_0}^g \xi'_{a_0}(s) \|$ very small

This with the fact that $[z_0 - \rho_0^{\Theta n}, z_0 + \rho_0^{\Theta n}]$ represents an $\gamma - C^2$ almost horizontal curve gives for all $a \in \omega$ :

$| \mathit{slope}(D\Phi_a^g \cdot \xi'_a(s)) | \leq \frac{1}{10}$ very small

As a consequence of the expansion $| x'_a(s) | \geq \frac{1}{4}$

$\tau$ is a $g$-th iterate of $\xi$ then $\| d_{(a,x)}y \|$ , $\| d^2_{(a,x)}y \|$ , $\| d^3_{(a,x)}y \| \leq \mathit{const}^g$

Observe that $z_0 = z_0(a) = (x_0(a), y(x_0(a)))$ the critical point over the curve $\tau$ is obtained as a minimal of:

$| \mathit{angle}(D\Phi_{z_0} \cdot (1, y'), f^n) |$ with $f^n$ expansion direction.

But $f^n$ have small $C^2$ variation so, for this $a$ , the critical point satisfies

$| \ddot{x}_0(a) |$ , $\| \ddot{z}_0(a) \| \leq \mathit{const}^2$

The same kind of reasoning may be applied to get:

$\| \ddot{\tau}_0 - \ddot{\bar{\tau}}_0 \| \leq \mathit{const}^q$ because $\| \ddot{\tau}_0 \|$ , $\| \ddot{\bar{\tau}}_0 \| \leq \mathit{const}^g$

Putting all together

$\begin{cases} \| \ddot{\tau}_0 - \ddot{\bar{\tau}}_0 \| \leq \mathit{const}^q \\ \| \tau_0 - \bar{\tau}_0 \| \leq \mathit{const}\gamma^n \end{cases}$ . (by the way $\bar{\tau}_0$ was obtained )

also $\mathit{length}(\omega) \geq e^{\mathit{const}'n}$ by hypothesis meaning that the size of domain not too small and, by Hadamard's lemma ( [5] )

$\| \dot{\tau}_0 - \dot{\bar{\tau}}_0 \| \leq \mathit{const}\gamma^{\frac{n}{2}} \leq \gamma^{\frac{n}{3}}$

Again by Hadamard's lemma ( because $\| z_0 - \bar{\tau}_0 \| \leq \gamma^{\frac{g}{10}}$ , $\| \ddot{z}_0 - \ddot{\bar{\tau}}_0 \| \leq \mathit{const}^g$ )

$\| \dot{z}_0 - \dot{\bar{\tau}}_0 \| \leq \mathit{const}\gamma^{\frac{g}{15}} \leq \gamma^{\frac{g}{20}}$

Finally

$\| \dot{z}_0 \| \leq \| \dot{z}_0 - \dot{\bar{\tau}}_0 \| + \| \dot{\bar{\tau}}_0 - \dot{\tau}_0 \| + \| \dot{\tau}_0 \| \leq \| \dot{\tau}_0 \| + \gamma^{\frac{g}{20}} + \gamma^{\frac{n}{3}}$

$\square$

Let now $z_0$ an $n$-th critical point as above defined over all $\omega$. Define inductively partitions $P_n$ and good parameter sets $S_n$. Say $z_0/\omega$ is a return if $z_0(a)$ return for some parameter $a$. It's free if it do not belong to any anterior binding period of $z_0(a)$. A free return is called essential if $\Delta(\omega) =$

$\{\| z_n(a) - \tau_0 \| \,;\ a \in \omega\}$ contains some interval:

$I_{r,i} = e^{-r} + \frac{e^{-1}}{r^2} \cdot [i-1, i]$; $0 \leq i \leq r^2$, $r >| \log \delta |$; as done in dimension 1.

This part is quite standart. For $n$ an essential return situation that is the case where we cut $\omega$ :

The elements of $P_n$ ( inside $\omega$ ) are the connected components of $\omega_{r,i}$ which correspond to $I_{r,i}$ above and the complement $\tilde{\omega} = \omega \setminus \bigcup_{r,i} \omega_{r,i}$

For others case we have no reason to cut $\omega$.

The good parameter set $S_n$ is obtained from $S_{n-1}$ by deleting $\omega_{r,i}$ if $r > \alpha n$ ( to satisfy [BA] ) and intervals with less free iterates in $[1, n]$ than $(1-\alpha)n$ ( to fullfill [FA]). Also delete intervals without escape situations in $[\frac{n}{2}, n]$ to complete inductions assumptions.

Remark: Some adjustments are needed in definitions below like return, binding periods and folding periods so to make them uniform in all $\omega$ but the changes do not affect any anterior result in a significant way since $\log d_n(z_0(a))$ is, by induction, nearly constant on the whole $\omega$.

Now the comparative lemma which is basic for parameter exclusions estimates together with the distorsion lemma of $\omega_n$ with parameter in free returns. This lemma follows the general principle that if the phase derivative has a sufficiently strong growth then its is comparable with the parameter derivative. In fact, in our case, parameter and phase variables coincide.

**Lemma 10.2.** *for $\omega \in P_n$ an element of the partition and $\omega \in S_n$ a good subset of parameters*

$const \leq \frac{\|\dot{z}_n(a)\|}{\|\omega_n(a)\|} \leq const$

*Moreover, if $n$ is a free time* $| \, angle(\dot{z}_k(a), \omega_n(a)) \, | \leq \gamma^{\frac{1}{4}}$

With this lemma we state an important corollary which recover induction hypothesis.

**Corollary 10.3.** $e^{const' \cdot n} \leq lenght(\omega) \leq e^{const'' \cdot n}$

Proof:Now its know $p \leq \alpha n$ the bound for binding period. This implies, for $\alpha$ small, that exist a free time $\mu \in [\frac{9}{10} n, n]$

By the angle bound on the above lemma and the fact that $\omega_n(a)$ is almost horizontal (together with the lower inequality above ).

$length(\omega) \cdot \inf \| \omega_\mu \| \leq const \cdot length(z_\mu(\omega)) \leq const$

So,

$length(\omega) \leq \frac{const}{\lambda^{\mu}} = const\lambda^{-\mu} \leq const{\cdot}e^{-\frac{9}{10}\log\lambda n} \leq e^{-const''n}$ since $const'' \leq \frac{9}{10}\log\lambda$

For the other estimate put $\mu$ now the time $\omega$ was created and so,
$length(\omega) \cdot \sup \parallel \omega_{\mu}(a) \parallel \geq const \cdot length(z_{\mu}(\omega)) \geq const\frac{e^{-\alpha\mu}}{(\alpha\mu)^2}$
Finally
$length(\omega) \geq \frac{const \cdot e^{-\alpha\mu}}{M^{\mu}(\alpha\mu)^2} = \frac{const}{(\alpha\mu)^2}e^{-(\alpha+\log M)\mu} \geq e^{-const' n}$
since $\mu \leq n$ and $const'$ small.
□

At this point we prove the crucial lemma stating that $\omega_n(a)$ is "almost constant" for $a \in \omega$ if $n$ is a free return time which is a consequence of the way that the $\omega$ parameter interval was created.

**Lemma 10.4.** *For $n$ a free return situation and $\omega \in P_n$ its associated parameter interval it holds,*
$\frac{\|\omega_n(a')\|}{\|\omega_n(a)\|} \leq const$ *and* $| \, angle(\omega_n(a'), \omega_n(a)) \, | \leq const$

Proof: Repeat the comparison lemma on bound orbits estimating $\frac{\|\omega_n^*(a')\|}{\|\omega_n^*(a)\|}$ and $| \, angle(\omega_n^*(a'), \omega_n^*(a)) \, |$ by the distance between the orbits (there this was estimated by $e^{-\beta k}$ ) $\parallel z_k(a') - z_k(a) \parallel$ plus the influence of different parameters $const{\cdot} \mid a' - a \mid$ divided by the distance from critical point (there $e^{-\alpha k}$ was the bound responsable for this ).

So that now the bound is something like
$\sum_{k=1}^{n} \frac{const|a'-a|+const\|z_k(a')-z_k(a)\|}{d_k}$
But $| \, a' - a \, | \leq e^{-const''n}$ and $d_k \geq e^{-\alpha k}$ so we need only bound
$const \sum^{n} \frac{\|z_k(a')-z_k(a)\|}{d_k}$
which in comparison lemma was estimated by $\sum e^{-(\beta-\alpha)k}$.

Case 1 : if $k \leq N$ then we knows that $\omega_k(\omega)$ and $\omega_N(\omega)$ are nearly straight and with expansion from one to the other this together with anterior lemma which says $\dot{z}_k \approx \omega_k$ gives $\parallel z_k(a') - z_k(a) \parallel \leq \lambda^{-(N-k)} \cdot$
$\parallel z_N(a') - z_N(a) \parallel$ and bound terms if $k \leq N$

For the other terms first put
$N = n_1 < \ldots < n_s = n$ free returns before $n$.

Case 2 : if $n_i < k < n_i + p_i$ ; where $p_i$ binding period of $n_i$ by anterior lemma,
$\frac{\|\dot{z}_k\|}{\|\dot{z}_{n_i}\|} \leq const\frac{\|\omega_k\|}{\|\omega_{n_i}\|} = const \parallel D\Phi^{k-n_i-1}\frac{\omega_{n_i+1}}{\|\omega_{n_i}\|} \parallel$
Write as before

$\frac{\omega_{n_i+1}}{\|\omega_{n_i}\|} = \hat{\alpha}_{n_i}(q,1) + \hat{\beta}_{n_i}(1,0) = \hat{\alpha}_{n_i} e + \hat{\beta}_{n_i}(1,0)$

Then

$\frac{\|\dot{z}_k\|}{\|\dot{z}_{n_i}\|} \leq const(\mid \hat{\alpha}_{n_i} \mid \left(\frac{Mb}{\lambda^2}\right)^{k-n_i-1} + \hat{\beta}_{n_i} \cdot \| \omega_{k-n_i-1}(z_{n_i+1}) \|)$

But it was already estimated $\mid \hat{\alpha}_{n_i} \mid = \frac{|\alpha_{n_i}|}{\|\omega_{n_i-1}\|} = \frac{|\alpha_{n_i}|}{\|\omega^*_{n_i-1}\|} \leq const$

and $\mid \hat{\beta_{\mu_i}} \mid \approx const \cdot d^2_{n_i}$

So

$\frac{\|\dot{z}_k\|}{\|\dot{z}_{n_i}\|} \leq const((\frac{Mb}{\lambda^2})^{k-n_i-1} + d^2_{n_i} \| \omega_{k-n_i-1}(z_{n_i+1}) \|)$

But during binding period the points are bounded to each other so we can apply a former lemma: $\sigma^2 \| \omega_{k-n_i-1}(z_{n_i+1}) \| \leq e^{-\beta(k-n_i-1)}$ with $\sigma \approx d_{n_i}$

Put all together we bound $\frac{\|\dot{z}_k\|}{\|\dot{z}_{n_i}\|}$ by something exponentially small like $e^{-\beta(k-n_i-1)}$

Now recall that $n_i$ is a return and use the anterior lemma ( $z_{n_i}(\omega)$ is nearly straight) $\frac{\|z_k(a')-z_k(a)\|}{\|z_{n_i}(a')-z_{n_i}(a)\|}$ exponentially $e^{-\beta(k-n_i-1)}$ small.

Finally this gives the bound for terms of the sum corresponding to this case ( the quotient by $d_k$ just put factor $e^{\alpha k}$ with small affection on $e^{-\beta}$ since $\beta > \alpha$ )

Case 3 : Suppose $n_i + p_i < k \leq n_{i+1}$ is in free times.

Recall first that an anterior lemma asserted $\frac{\|\omega_n\|}{\|\omega_{n-k}\|} \geq e^{constk}$ ; for $n$ free.

Apply this (with $n = n_{i+1}$ ) and last lemma above to get.

$\| z_k(a') - z_k(a) \| \leq const \cdot e^{-const(n_{i+1}-k)} \| z_{n_i+1}(a') - z_{n_i+1}(a) \|$

This bounds iterates outside binding periods.

Now collect all three cases and estimate is obtained just as [19] .

□

The distorsion lemma is now direct consequence.

**Corollary 10.5.** $\frac{\|\dot{z}_n(a')\|}{\|\dot{z}_n(a)\|} \leq const$ *and* $\mid angle(\dot{z}_n(a'), \dot{z}_n(a)) \mid \leq const$
$\forall a', a \in \omega$ *of the partition uniformily*

Now we concentrate ourselfs in the proof of the main theorem.

# 11 Proof of Theorem A

Recall the bound for the number of critical points with generation $g \leq \Theta \cdot n$:

$$Const \cdot (\frac{M}{\rho_0})^{\Theta k} \qquad (1)$$

Observe that, by definition, $\Theta$ reduces with the increase of the dissipativeness, meaning we may consider the number of critical points of each stage increasing in an exponentially small rate.

With the distorsion lemma above we get:

$$\mathrm{leb}(\omega \cap (S_{n-1} \setminus S_n)) \approx \frac{e^{-\alpha \cdot n}}{\mathrm{leb}(z_n(\omega))} \cdot \mathrm{leb}(\omega). \tag{2}$$

But coming back to the last essential return we get (see [5]):

$$\mathrm{leb}(z_n(\omega)) \geq Const \cdot \mathrm{leb}(I_{r,i}) \geq Const \cdot e^{-\frac{1}{r^2}}. \tag{3}$$

Therefore, from Equations (2) and (3), we have

$$\mathrm{leb}(\omega \cap (S_{n-1} \setminus S_n)) \leq Const \cdot e^{-\alpha_0 \cdot n} \cdot \mathrm{leb}(\omega).$$

Now, for all critical point $z_0$

$$S_n = S_{n-1} \setminus \cup_{z_0}(\omega \setminus S_n(z_0)).$$

Thus, using the bound for the number of critical points (1),

$$\mathrm{leb}(S_{n-1} \setminus S_n) \leq Const \cdot e^{-\alpha_1 \cdot n} \cdot \mathrm{leb}(I).$$

Let $S_\infty = \cap_{n \geq N} S_n$ be the set of final good parameters. Then,

$$\frac{\mathrm{leb}(S_\infty)}{\mathrm{leb}(I)} \geq (1 - \sum_{n \geq N} Const \cdot e^{-\alpha_1 \cdot n}) > 0.$$

Recall that all we get until now is a positive measure subset, for each $\theta$ and $x$ fixed, in which we have expansiveness for the future iterates of the vector $(1,0)$ located over the bifurcation point $z_0$. Then we extend this expansion to all others vectors $(1,0)$ but now because, following its orbits, we may have binding period as long as needed in each return by letting the orbit return close to $z_0$. Supposing $\theta > 0$ then we can also have the orbit of some $(1,0)$ returning exactly over $z_0$, since $z_0$ do not anullate the horizontal component, and the found expansion is surely uniformly bounded away from one. With uniform expansion on $(1,0)$ we find approximations of contractive and expansive directions everywhere which converge to the true transversal stable and unstable manifolds.

This guarantee a positive measure subset $E$ of $\tilde{A}$ in which $\Phi_{(\tilde{A},0)}$ has a cubic tangency in $z_0$. Now if $\tilde{A}$ is over the unstable manifold of $p$ then its stable manifold is dense in the invariant subset because this happen to $p$ and because the two manifolds approach to each other in the past. Then the reasoning of [2] prove that for almost all such $\tilde{A}$ $z_n$ is dense on the unstable manifold. And that means that for these $\tilde{A}$ the cubic tangency is not between invariant manifolds of periodic points.

A final comment is that a one dimension version of this theorem may be proved in basically the same ideas (see [1]) but now without any complication involving dissipativeness like infinitly many critical points for example.

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