COMPLEX BOUNDS FOR MULTIMODAL MAPS: BOUNDED COMBINATORICS

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ABSTRACT. We prove the so called complex bounds for multimodal, infinitely renormalizable analytic maps with bounded combinatorics: deep renormalizations have polynomial-like extensions with definite modulus. The complex bounds is the first step to extend the renormalization theory of unimodal maps to multimodal maps.

1. INTRODUCTION

Renormalization. A multimodal maps is a smooth map $f\colon I \to I$ preserving the boundary of I with a finite number of critical points, all of them local maximum or local minimum. Let f be an infinitely renormalizable map in respect to a critical point p. This means that there are nested intervals P^k, $p \in P^k$ and an increasing sequence of natural numbers N_k such that

- $f^{N_k}(P^k) \subset P^k$.
- $f^{N_k}(\partial P^k) \subset \partial P^k$.
- The intervals $f^i(P^k)$ and $f^j(P^k)$ have disjoint interiors for $i \neq j$, $0 \leq i, j < N_k$.
- The interval P^k is maximal with these properties.

The map f^{N_k} restricted to P^k is the k-th renormalization of f with respect to p, and it will be called $R^k_p(f)$. We say that f has **bounded combinatorics** with respect to p if N_{k+1}/N_k is bounded. The concept of polynomial-like map was introduced in [3]. Sullivan, in [12] showed that the existence of the polynomial-like extensions (for a precise definition of polynomial-like extension, see definition 5.2) of the renormalizations is a useful tool for understanding the dynamics of the infinitely renormalizable maps. We will prove

Theorem 1 (Complex Bounds)**.** *Let f be an analytic multimodal map defined in a neighborhood of the interval I such that all critical points have even criticality. Suppose that f is infinitely renormalizable with respect to a critical point p, with bounded combinatorics. Then there exists ϵ such that for sufficiently large k, the k-th renormalization with respect to p, $R^k_p(f)$, has a polynomial-like extension $R^k_p(f)\colon U \to V$, with $mod(V \setminus U) \geq \epsilon$. Furthermore, $diam(V) \leq C|P^k|$ and the filled-in Julia set of $R^k_p(f)$ is contained in a Poincare neighborhood (see* [2]*) $D_\beta(P^k)$, where β does not depend on k.*

Complex bounds, a kind of compactness result, have a lot of applications in the study of the infinitely renormalizable maps: when f is a quadratic-like maps, the proof of the local connectivity of the Julia set involves the complex bounds, and so do the non existence of invariant line fields supported in the Julia set, the

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convergence of the renormalization operator in the set of infinitely renormalizable maps and the hyperbolicity of this operator in an appropriate space. We expect this theory do be generalized at least for infinitely renormalizable maps with bounded combinatorics.

Complex bounds type results have now a long history. This kind of results was introduced by Sullivan ([12]), in the study of the renormalization operator for unimodal maps with bounded combinatorics. Nowadays, there are a large number of related results. For instance: for infinitely renormalizable unimodal maps with unbounded combinatorics, there are results by M. Lyubich ([9]) and M. Lyubich & M. Yampolsky ([10]), by J. Graczyk & G. Swiatek ([4]), and G. Levin & S. van Strien ([8]). In the thesis [5], there is a proof for infinitely renormalizable bimodal (two critical points) maps with bounded combinatorics in the Epstein class, using the sector lemma (introduced by Sullivan: see [2] for details), but the outline of the proof seems to be incomplete. We will use the methods introduced in [10]. This paper is part of my thesis at IMPA.

2. Topological Results

Assume that all intervals under consideration are closed.

A multimodal map f is renormalizable with respect to a critical point p if there exist $n > 1$ and an interval P^1 such that $P^1, f(P^1), \dots f^{n-1}(P^1)$ have disjoint interiors and $f^n(P^1) \subset P^1$. The smallest n with this property is the period of the renormalization. If there is not an interval containing properly P^1 with the same property, then $f^n(\partial P^1) \subset \partial P^1$. The map f^n restricted to P^1 is again a multimodal map. This map is called a *renormalization* of f. Such renormalization could be renormalizable with respect to p again and so on. If this process never finish we say that f is **infinitely renormalizable** with respect to p. Hence we can construct the intervals P^k as in the introduction.

Let $\mathbb{I}_p^k = \{c \colon f'(c) = 0,\ c \in \cup_{i<N_k} f^i(P^k)\}$. Since $\mathbb{I}_p^{k+1} \subset \mathbb{I}_p^k$, $\mathbb{I}_p^k$ is constant for large k. Denote $\mathbb{I}_p = \mathbb{I}_p^k$, where k is large, the set of critical points **involved** in the renormalizations with respect to p.

Assume that there are no wandering intervals for f (for example, if the critical points are non-flat and f is C^2: see pg. 267 in [2]). It is well-known that $max\{|f^i(P^k)|\}_{i<N_k}$ goes to zero. In particular for large k, there is at most one critical point in each $f^i(P^k)$. We say that an interval L is symmetric with respect to a critical point c if f is monotone in each connect component J of $L \setminus \{c\}$ and $f(J) = f(L)$. Let J be an small interval which contains only one critical point c. The symmetrization of J is an interval $L \supset J$ which is a symmetric interval with respect to c and $f(I) = f(L)$. For large k, P^k is symmetric with respect to p.

For large k, if $q \in f^j(P^k) \cap \mathbb{I}_p$ and $r \in f^{j+i \bmod N_k}(P^k) \cap \mathbb{I}_p$, where $i > 0$ is the smallest number such that $f^{j+i \bmod N_k}(P^k) \cap \mathbb{I}_p \neq \phi$, we say that r is the **successor** of q at level k and q is the **predecessor** of r at level k. Denote by q_k' the successor of q at level k.

For $q \in f^j(P^k) \cap \mathbb{I}_p$ denote by the corresponding capital letter Q_0^k the symmetrization of $f^j(P^k)$. Let $n_k^r = j - i \bmod N_k$, with $r \in f^j(P_0^k)$ and $q \in f^i(P_0^k)$, where q is the predecessor of r at level k. Then $\{f^i(Q_0^k)\}_{i<n_k^{q'_k},\ q\in\mathbb{I}_p}$ is a family of intervals with disjoint interior. Note that, for large k, the boundary of P^k contains a periodic point and its symmetric with respect to p. Moreover, the orbit of this periodic point does not contain critical points. This implies that $f^{n_k^r}(Q_0^k) \subset R_0^k$,

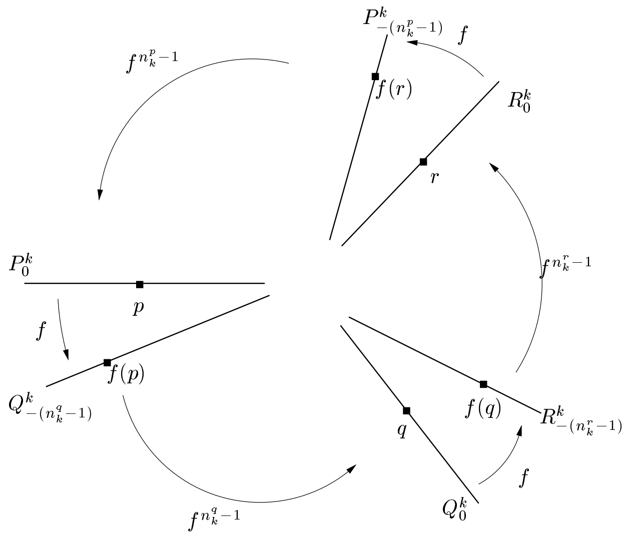

FIGURE 1. Intervals of the k-th renormalization, with 3 critical points involved.

where r is the successor of q at level k, and that the boundary of Q^k_0 contains a point of this periodic orbit.

Note that to prove the complex bounds (see the precise statement in Theorem 1) for f with respect to p, it is sufficient prove the complex bounds for a renormalization with respect to p. Replacing f for a deep renormalization with respect to p, we can assume, without loss of generality, that f satisfies the **standard conditions**:

- f is a composition of unimodal maps with non-flat critical points and $\{f(c)\colon f'(c) = 0\} = \{f(q)\}_{q\in\mathbb{I}_p}$.
- The interval Q^k_0 is symmetric with respect to q.
- For large k and for all $q \in \mathbb{I}_p$, Q^k_0 contains exactly one critical point.
- For large k, there exists a periodic orbit such that Q^k_0 contains in its boundary a point of this orbit, for all $q \in \mathbb{I}_p$.
- For every q, $f^{n^r_k}(Q^k_0) \subset R^k_0$, where r is the successor of q at level k. Moreover $r \in f^{n^r_k}(Q^k_0)$.
- The intervals in $\{f^i(Q^k_0)\}_{i<n^{q'}_k,\ q\in\mathbb{I}_p}$ have disjoint interior.

We say that a set W is a **nice set** if it is an union of closed intervals, each interval contains a unique critical point, the critical points of f are contained in the interior of W and $f^n(\partial W) \cap intW = \phi$, for all $n > 0$.

Let W be a nice set and D be the set of points not in W but whose positive orbit intersects the interior of W. The entry map associates to each $x \in D$ the point $f^n(x)$ in the interior of W with smallest n. It is easy to see that for each connect component J of D, n is constant and the image of J by the first entry map is a connect component of W. Moreover $J, f(J), \dots, f^{n-1}(J)$ are disjoint connect components of D.

Lemma 2.1. *Let c be a critical point. Then there exists $q \in \mathbb{I}_p$ such that $f(c) = f(q)$ and the connect component J of $f^{-1}(f(Q^k_0))$ which contains c is a symmetric* *interval J with respect to c and $f(J) = f(Q^k_0)$. In particular, the union of connect components of $f^{-1}(f(\cup_{q\in\mathbb{I}_p} Q^k_0))$ which contains a critical point is a nice set.*

Proof. If the lemmas does not holds then there exists critical points in $J \setminus \{c\}$. Let $\tilde{c}$ be the closest one of c. $f(\tilde{c}) \neq f(q)$, otherwise there will be another critical point between $\tilde{c}$ and c. But this implies $f(r) \in f(Q^k_0)$, where $r \in \mathbb{I}_p \setminus \{q\}$, which is absurd. ☐

Remark 2.1. *Denote by $\mathcal{N}^k$ the nice set defined in the previous lemma. Let $\mathcal{A}^k = \{f^i(Q^k_0)$ s.t. $0 < i \leq n^{q'_k}_k$, $q \in \mathbb{I}_p\}$. It is easy to see that $\cup_{J\in\mathcal{A}_k} J \cap \mathcal{N}^k = \cup_{q\in\mathbb{I}_p} f^{n^{q'_k}_k}(Q^k_0)$. Hence $f^{n^r_k - i}(Q^k_0) \subset R^k_{-i}$, $i \leq n^r_k - 1$, where r is the successor of q at level k and R^k_{-i} is a domain of the first entry map to $\mathcal{N}^k$. Clearly $f(R^k_{-(i+1)}) = R^k_{-i}$. When we write R^k_{-i}, assume that $i \leq n^r_k - 1$.*

We say that an interval J has the property $*_k$ if there is at most one interval in $\mathcal{A}^k$ in the interior of J. Note that the closure of J contains at most 3 intervals in A^k.

For $x \in I$ the intersection number of x with a family of intervals is the number of intervals in the family which contains x. The intersection number of a family of intervals is the maximal number of intersection with a point in I.

Lemma 2.2. *Let J be an interval such that f^n is monotone in J and $\{f^i(J)\}_{i\leq n}$ is a family of intervals with disjoint interior such that $f^i(J)$ intersects an unique interval in the family $\mathcal{A}^k$ which is contained in $f^i(J)$. Let $L \supset J$ be such that f^n is monotone in L and $f^n(L)$ satisfies $*_k$. Then the intersection number of the family of intervals $\{f^i(L)\}_{i\leq n}$ is at most three.*

Proof. If the intersection number is at least 4, the interior of $f^i(L)$ will contain at least two intervals H_1 and H_2 in the family $\mathcal{A}^k$, for some $i \leq n$. Since $q'_k \in f^{n^{q'_k}_k}(Q^k_0)$, for all $q \in \mathbb{I}_p$, the intervals $f^{n-i}(H_1), f^{n-i}(H_1)$ belong to $\mathcal{A}^k$, which is absurd, since $f^n(L)$ satisfies $*_k$. ☐

The previous lemma will allows us to use the real Koebe lemma (Theorem 3.1 and 3.3 in chapter IV in [2]) without do considerations about the number of intersections of the families of intervals involved.

Lemma 2.3. *Let U be an interval satisfying $*_k$ such that $Q^k_{-i} \subset U$, with $i < n^q_k - 1$. Then there exists an interval $\tilde{U}$ which contains $Q^k_{-(i+1)}$ such that the map f is monotone in $\tilde{U}$ and $f(\tilde{U}) = U$. Furthermore, $\tilde{U}$ satisfies $*_k$.*

Proof. Let $\tilde{U}$ be the maximal interval such that $Q^k_{-(i+1)} \subset \tilde{U}$, $f(\tilde{U}) \subset U$ and f is monotone in $\tilde{U}$. We claim that $f(\tilde{U}) = U$. Otherwise there exists a critical point c to f in the boundary of $\tilde{U}$ such that $f(c)$ is in the interior of U. Then, by lemma 2.1, the interior of $f(\tilde{U})$ contains $f(R^k_0)$, $r \in \mathbb{I}_p$, in the left or right side of Q^k_{-i}. This proves the first statement. The second statement is obvious, since f restricted to $\tilde{U}$ is monotone and $r'_k \in f^{n^{r'_k}_k}(R^k_0)$, for all $r \in \mathbb{I}_p$. ☐

Lemma 2.4. *Let U be an interval satisfying $*_k$ which contains $R^k_{-(n^r_k-1)}$, where q is the predecessor of r at level k. Then there exists $\tilde{U}$ such that $Q^k_0 \subset \tilde{U}$ and each connect component of $\tilde{U} \setminus Q^k_0$ is mapped monotonically to the connect component of $U \setminus R^k_{-(n_k-1)}$ which contains $f(\partial Q^k_0)$. Furthermore, $\tilde{U}$ satisfies $*_k$.*

Proof. Similar to previous lemma. □

Definition 2.1. *Let U be an interval such that $Q_0^k \subset U$. The* ***pullback of** U **along the** k**-cycle of renormalization*** *exists if there are intervals $\tilde{U}_q$, U_r, V_r for each $r \in \mathbb{I}_p$, such that:*

- *$U_q = U$. For $r \in \mathbb{I}_p$, $R^k_{-(n_k^r-1)} \subset V_r$ and $f^{n_k^r-1}(V_r) = U_r$. Furthermore, the map $f^{n_k^r-1}$ is monotone in V_r.*
- *The interval U_r is the symmetric interval with respect to r which contains R_0^k and each connect component of $U_r \setminus R_0^k$, is mapped monotonically in a connect component of $V_c \setminus C^k_{-(n_k^c-1)}$. Here c is the successor of r at level k.*
- *The interval $\tilde{U}_q$ is the symmetric interval with respect to q which contains Q_0^k such that each connect component of $\tilde{U}_q \setminus Q_0^k$ is mapped monotonically in a connect component of $V_c \setminus C^k_{-(n_k^c-1)}$. Here c is the successor of q at level k.*

Let M_q^k be the maximal interval which contains Q_0^k satisfying the property $*_k$.

Lemma 2.5. *If U is an interval satisfying $Q_0^k \subset U \subset M_q^k$, then the pullback of U along the k-cycle of renormalization exists. Furthermore, $\tilde{U}_q \subset M_q^k$, where $\tilde{U}_q$ is as in the previous definition.*

Proof. Immediate consequence of the previous lemmas. □

The δ-neighborhood of an interval J is $\delta\text{-}J = \{x\colon dist(x,J) \leq \delta|J|\}$. Here $dist(x,J) = \inf\{|x-y|\colon y \in J\}$. The mirror image of interval J near a critical point q, which is not in J, is the other interval $\tilde{J}$ near q such that $f(J) = f(\tilde{J})$.

Remark 2.2. *In the previous lemma, suppose $\epsilon\text{-}Q_0^k \subset U$. Then, by the real Koebe lemma and the quasi symmetry in the critical points, one gets $\delta(\epsilon) - Q_0^k \subset \tilde{U}_q$. Furthermore $\tilde{U}_q \subset Q_0^{k-1}$, since the critical points to f^{N_k} in $\tilde{U}_q$ are inside Q_0^k and $Q_0^{k-1} \setminus Q_0^k$ contains in both connect components intervals in the form $f^i(R_0^k)$, $r \in \mathbb{I}_p$, or mirror images of these intervals. Each one of these intervals contains at least one critical point for f^{N_k}.*

Lemma 2.6. *Let $s_k = sup\{|Q^k_{-i}|$ s.t. $q \in \mathbb{I}_p\}$. Then s_k tends to 0.*

Proof. This follows of the non existence of wandering intervals. □

3. Real Bounds

An interval L is δ-small with respect to level j if $L \subset Q_0^{j-1}$ and $|L|/|Q_0^j| \leq \delta$, for some $q \in \mathbb{I}_p$. The interval J_1 is δ-deeply contained in J_2, $J_1 \subset_\delta J_2$, if $\delta\text{-}J_1 \subset J_2$. The intervals J_1 and J_2 are C-commensurable if $1/C \leq |J_1|/|J_2| \leq C$. Let $P(f)$ be the poscritical set $\{f^i(c)\colon f'(c) = 0\}$. Denote by $Fol_\ell(x) = |x|^\ell$ the folding function with criticality ℓ. Let $A_J\colon J \to I$ be an affine map which maps the interval J in $I = [-1,1]$. When we write $J = [a,b]$, $a \neq b$, we do not assume $a < b$.

We are going to prove real estimates. In [5] and [6], J. Hu obtains real estimatives for infinitely renormalizable multimodal maps (with weaker smooth assumptions), similar to lemmas 3.11 and 3.3. However, to use Lyubich-Yampolsky approach we need other estimates (see lemma 3.9). For completeness we present the full argument. In this section f is a C^3 multimodal map satisfying the standard conditions.

Definition 3.1. *One of the points in the boundary of Q^k_{-i} is periodic and the other is eventually periodic. Let $A_{Q^k_{-i}}\colon Q^k_{-i} \to I$ be the affine map which maps the periodic point in -1. For $q \in \mathbb{I}_p$, define $\phi^k_q\colon I \to I$ and $\psi^k_q\colon I \to I$ by*

$$\phi^k_q = A_{Q^k_0} \circ f^{n^q_k - 1} \circ A^{-1}_{Q^k_{-(n^q_k-1)}} \tag{1}$$

$$\psi^k_q = A_{R^k_{-(n^r_k-1)}} \circ f \circ A^{-1}_{Q^k_0}. \tag{2}$$

Here the critical point r is the successor of q at level k.

*The maps ϕ^k_q are the **monotone parts** of the k-th renormalization. The **partial** monotone parts are the diffeomorphisms $A_{Q^k_0} \circ f \circ A^{-1}_{Q^k_{-i}}$. The maps ψ^k_q are the **folding parts** of the renormalization.*

Let us to begin with an obvious lemma:

Lemma 3.1. *Consider $g = h_1 \circ Fol_\ell \circ h_2$, where h_1, h_2 are C^1 diffeomorphisms in a neighborhood of zero. There exists C such that if J, M are sufficiently small intervals such that J contains the critical point c of g and $g(J) \subset M$ then $|D(A_M \circ g \circ A_J^{-1})(x)| \leq C|x - \tilde{c}|^{\ell-1}$. Here $\tilde{c}$ is the critical point of $A_M \circ g \circ A_J^{-1}$.*

Proof. Use that h_1 and h_2 are bi-Lipchitz in a neighborhood of zero. □

The following lemma will be used full-time, sometimes without explicit mention:

Lemma 3.2 (Theorem 8.1 in [11]). *For all $l > 1$, $C_1, C_2 > 0$, and $n_0 \in \mathbb{N}$, there exists δ such that if*

- *The multimodal map $g\colon J \to J$ satisfies $max_J|Dg| \leq C_1$.*
- *For a critical point c of g, $|Dg(x)| \leq C_2|x - c|^{\ell-1}$, for all $x \in J$.*
- *There is $x_0 \in J$ with $|x_0 - c| \leq \delta$ and $|f^n(x_0) - c| \leq \delta$, $n \leq n_0$.*

Then there is an hyperbolic attractor which contains c in its immediate basin.

Note that critical points in $\mathbb{I}_p$ can not be attracted by a periodic orbit, because $\mathbb{I}_p = \mathbb{I}^k_p$ for large k.

Lemma 3.3 (Real Bounds: see [5], [6]). *For large k, the following holds:*

- *There exists $C > 0$ such that $1/C \leq |D\phi^k_q| \leq C$, $q \in \mathbb{I}_p$. The same control of derivatives holds for the partial monotone parts.*
- *There exists $K > 0$ such that $|D\psi^k_q(x)| \leq K|x - c|^{\ell-1}$, where c is the critical point of ψ^k_q and ℓ is the criticality of f in q.*
- *There are λ_1 and λ_2 such that $0 < \lambda_1 < |Q^{k+1}_0|/|Q^k_0| < \lambda_2 < 1$.*

Proof. We will use the smallest interval trick. Let $Z = f^i(C^k_0)$ be the smallest interval in $\{f^i(C^k_0)\colon c \in \mathbb{I}_p,\ 0 \leq i < n^{c'_k}_k\}$. Observe that δ-$Z \subset I$, where δ depends on $max f'$. Taking $\delta \leq 1/2$, the interval δ-Z satisfies $*_k$. Consider the maximal interval X which contains $f(C^k_0)$ such that f^{i-1} is monotone in X and $f^{i-1}(X) \subset \delta$-$Z$. We claim that $f^{i-1}(X) = \delta$-Z. Otherwise, there exists $a \in \partial X$ such that $f^j(a)$ is a critical point for f, $0 < j < i - 2$. By lemma 2.1 and remark 2.2, $f^{j+1}(X) \setminus f^{j+2}(C^k_0)$ contains an interval in $\mathcal{A}^k$. This is a contradiction. By the real Koebe lemma and the quasi symmetry at the critical points, there exists $\tilde{\delta}$ such that $\tilde{\delta}$-C^k_0 satisfies $*_k$. Use lemmas 2.3 and 2.4 many times and the real Koebe lemma to conclude that there is ϵ such that ϵ-Q^k_0 satisfies $*_k$. In particular, M^k_q is a ϵ-neighborhood of Q_0^k. By lemma 2.5, real Koebe lemma and the quasi symmetry at the critical points, one gets control of distortion in the monotone parts of the k-cycle and that the pullback of M^k along the k-cycle is a $\delta(\epsilon)$-neighborhood of Q_0^k. Moreover, this neighborhood is contained in Q_0^{k-1}, by remark 2.2. Hence $|Q_0^k|/|Q_0^{k-1}| < \lambda_2 < 1$. Since $|C_0^k|$ goes to zero, by lemma 3.1, one gets the second statement. In particular, if Q_0^k is small with respect to Q_0^{k-1} then Q_0^k contains an attractor which attracts q, because the combinatorics is bounded and by the previous lemma. But this is absurd, since $\mathbb{I}_p^k$ is constant for large k. □

Lemma 3.4. *The map $f^{n_k^q}$ has negative Schwartzian derivative in $Q^k_{-(n_k^q-1)}$, $q \in \mathbb{I}_p$ and large k.*

Proof. Let $C_2 = maxSf$. Near of q, we have $Sf(x) \le -C_1/(x-q)^2$. By the real bounds

$$Sf^{n_k^q}(x) = Sf(f^{n_k^q-1}(x))|Df^{n_k^q-1}(x)|^2 + \sum_{t<n_k^q-1} Sf(f^t(x))|Df^t(x)|^2 \tag{3}$$

$$\le |Df^{n_k^q-1}(x)|^2(-\frac{C_1}{|Q_0^k|^2} + \sum_{t<n_k^q-1} Sf(f^t(x))|Df^{n_k^q-1-t}(f^t(x))|^{-2}) \tag{4}$$

$$\le \frac{|Df^{n_k^q-1}(x)|^2}{|Q_0^k|^2}(-C_1 + C_3C_2 \sum_{0<t\le n_k^q-1} |Q^k_{-t}|^2) \tag{5}$$

$$\le \frac{|Df^{n_k^q-1}(x)|^2}{|Q_0^k|^2}(-C_1 + C_3C_2 max\{|Q^k_{-t}|\}_{0<t\le n_k^q-1}) \tag{6}$$

We learn this argument in [7]. □

Lemma 3.5. *For every $C_1 \in \mathbb{N}$, there exists C_2 such that , for all ℓ with $Q^k_{-\ell} \subset R_0^j$, $j < k$ and $\ell < C_1 N_j$, we have:*

$$\frac{|R_0^j|}{|Q^k_{-\ell}|} \le C_2 \frac{|Q_0^j|}{|Q_0^k|} \tag{7}$$

C_2 ***does not depend on*** j,ℓ ***or*** k.

Proof. Because $\ell < C_1 N_j$, $Q^k_{-\ell}$ will return to Q_0^k after a bounded number of iterations of monotone and folding parts of the j-th renormalization. In particular, by real bounds, the derivative of map

$$A_{Q_0^j} \circ f^\ell \circ A^{-1}_{R_0^j}$$

is bounded away from infinity. This proves the lemma. □

Lemma 3.6. *The point $f^i(q)$ cuts $R^k_{-(n_k^r-i)}$ in commensurable parts, where q is the predecessor of r at level k.*

Proof. Assume by contradiction that $f^i(q)$ is very close to a point $a \in \partial R^k_{-(n_k^r-i)}$ with $f^{n_k^r-i}(a) \in \partial R_0^k$. By the real bounds for the partial monotone parts of the k-th renormalization, $f^{n_k^r}(q)$ is very close to ∂R_0^k with respect to level k. But $f^{n_k^r}(q)$ returns to R_0^{k+1} after a bounded number of iterates of f^{N_k}. Since R_0^{k+1} is deeply inside R_0^k and by the control of derivatives of f^{N_k} in R_0^k, this is a contradiction. □

Lemma 3.7. *Let J be a interval in the family $\{f^i(Q_0^k)\}_{0<i<n_k^{q'_k}}$. There exists a critical point c for f^{N_k} which cuts J in commensurable parts.*

Proof. Let $f^j(q) \in \partial f^j(Q_0^k)$. Let b be the closest point of $f^j(q)$ in $f^j(Q_0^k)$ such that $f^{N_k-j}(b) = q$. Note that if b is close to $f^j(q)$, using lemma 3.3 and that $f^{N_k}(b) = f^j(q)$, there will be an attractor in the interval $[f^j(q), b]$ which attracts a critical point. This is an absurd. The point b cannot be close to the periodic point in $\partial f^j(Q_0^k)$, otherwise, by the real bounds, $f^{N_k-j}(b) = q$ would be close to the boundary of Q_0^k, which is impossible. □

Lemma 3.8. *There exists δ such that if $i \leq n_k^{q'_k}$ and $f^i(Q_0^k) \subset R_0^{k-1}$, for some $q, r \in \mathbb{I}_p$, then $|f^i(Q_0^k)|/|R_0^{k-1}| \geq \delta$.*

Proof. Note that $f^{N_k-i}(f^i(Q_0^k))$ is commensurable with Q_0^{k-1}. By the real bounds, the map

$$A_{Q_0^{k-1}} \circ f^{N_k-i} \circ A^{-1}_{R_0^{k-1}}$$

has bounded derivative. This proves the lemma. □

Lemma 3.9. *Let L_q^k be the maximal interval L which contains Q_0^k and f^{N_k} has no critical points in $L \setminus Q_0^k$. Then there exists δ, which does not depend on k, and intervals S_q^k , T_q^k such that, for large k*

1. *These intervals are nested: $Q_0^k \subset_\delta S_q^k \subset L_q^k \subset_\delta T_q^k \subset Q_0^{k-1}$.*
2. *The interval S_q^k is the pullback of T_q^k along the k-cycle and this pullback can be extended to the δ-neighborhood of T_q^k which is also contained in Q_0^{k-1}.*

Proof. Let M_q^k be as in lemma 2.5. In the proof of lemma 3.3 we proved that M_q^k contains a δ-neighborhood of Q_0^k. Let $T_p^k = M_q^k \cap Q_0^{k-1}$. Let S_p^k be the pullback of T_p^k along the k-cycle. By remark 2.2 and lemma 2.5, $Q_0^k \subset_\delta S_p^k \subset T_p^k$. Let J be a connect component of $T_p^k \setminus Q_0^k$. There are two cases: if J is a connect component of $Q_0^{k-1} \setminus Q_0^k$ then J clearly contains an interval in the family $\mathcal{A}^k$ or a mirror image of one. Otherwise J is a connect component of $M_q^k \setminus Q_0^k$ and it contains an interval in the family $\mathcal{A}^k$, by the maximality of M_q^k. Note that this interval contained in J is commensurable with Q_0^k(lemma 3.8) and it contains a critical point which cuts it in commensurable parts(lemma 3.7). Hence $L_q^k \subset_\delta T_q^k$, since $L_q^k \setminus Q_0^k$ does not contains critical points for f^{N_k}. Replacing T_q^k, if necessary, by a smaller interval between T_q^k and L_q^k, we obtain the second statement of ii. □

Lemma 3.10. *Let u be the periodic point in ∂Q_0^k. There exist δ such that if J is the maximal interval of monotonicity of f^{N_k} which contains u, then $J \subset_\delta f^{N_k}(J)$.*

Proof. Note that $J \setminus Q_0^k$ is the connect component of $L_q^k \setminus Q_0^k$ which contains u. Hence J is commensurable with Q_0^k and $J \setminus Q_0^k \subset_\delta f^{N_k}(J) \setminus Q_0^k$, for some δ, by lemma 3.9.i. Let c be the boundary point of J inside Q_0^k. Surely $[u, c] \subset [u.f^{N_k}(c)]$. It is sufficient to prove that the interval $[c, f^{N_k}(c)]$ is commensurable with Q_0^k. Indeed, if $[c, f^{N_k}(c)]$ is very small, by the real bounds and lemma 3.2 there will be an attractor which contains a critical point in its immediate basin. This is absurd, since $\mathbb{I}_p^k$ is constant for large k. □

Lemma 3.11 (Bounded geometry: see [5],[6]). *The geometry of the intervals Q_i^k is bounded in the following sense:*

- *If $Q^k_{-j} \subset C^{k-1}_{-l}$, then the distance between Q^k_{-j} and ∂C^{k-1}_{-l} is commensurable with $|Q^k_{-j}|$.*
- *Suppose $R^k_{-i}, Q^k_{-j} \subset C^{k-1}_{-l}$, with $\partial R^k_{-i} \cap \partial Q^k_{-j} = \phi$. Then the space between these intervals is commensurable with their length.*
- *In particular, $\sum_{q\in\mathbb{I}_p} \sum_{i<n^k_q} |Q^k_{-i}|$ goes exponentially fast to zero.*

Proof. Observe that Q^k_{-j} is not small with respect to C^{k-1}_{-l}. Otherwise, by the real bounds and bounded combinatorics, there will be an attractor in Q^k_{-j} which attracts the critical point q. Suppose that a point in the boundary of Q^k_{-j} is close to the boundary of C^{k-1}_{-l}. By the control on the derivatives in the monotone and folding parts of the $(k-1)$-th renormalization, such point will return to Q^k_0 after a large number of iterations of the monotone and folding parts of the $(k-1)$-th renormalization, which is absurd, because the combinatorics is bounded. To prove the second statement, let $a_1 \in \partial R^k_{-i}$ and $a_2 \in \partial Q^k_{-j}$ be such that $dist(R^k_{-i}, Q^k_{-j}) = dist(a_1, a_2)$. Let t_1, t_2 be the minimal times such that $f^{t_1}(R^k_{-i}) \subset R^k_0$ and $f^{t_2}(Q^k_{-i}) \subset R^k_0$. By the lemma 3.9, the real Koebe lemma and the quasi symmetry at the critical points, for every D^k_{-i}, $d \in \mathbb{I}_p$, there is in each side of D^k_{-i} an interval J commensurable with D^k_{-i} such that f^a is monotone in J. Here, a is the time of first entry of D^k_{-i} to R^k_0. Hence if the space between R^k_{-i} and Q^k_{-j} is small and $t = max\{t_1, t_2\}$, then $f^t(a_1)$, $f^t(a_2)$ are very close distinct points in $R^{k-1}_0 \setminus R^k_0$. Moreover $f^{t+1}(a_1)$, $f^{t+1}(a_2)$ are distinct points in the orbit of the periodic point contained in ∂R^k_0. By the control of derivatives of $(k-1)$-renormalization and bounded combinatorics, the interval $[f^{t+1}a_1, f^{t+1}a_2]$ is mapped monotonically into itself by the map f^{N_k}, which is absurd, since the negative Schwartzian derivative of $f^{N_{k-1}}$ in $f(R^{k-1}_0)$ implies that an attractor in $[f^{t+1}a_1, f^{t+1}a_2]$ contains a critical point in its immediate basin. □

Remark 3.1. *We can assume $f \in C^2$ with non flat critical points, or even weaker smoothness assumptions, and substitute the lemma 3.4 by a theorem in pg. 268 of* [2]*, which claims that a no repelling periodic orbit with large period must have a critical point in its immediate basin.*

Remark 3.2. *Let $U_q = T^k_q, U_r, V_r, \tilde{U}_q = S^k_q$ be as in definition 2.1. For each $r \in \mathbb{I}_p$, there is a neighborhood $\tilde{V}_r$ of V_r which is mapped monotonically by $f^{n^r_k-1}$ to a δ-neighborhood of U_r, for some δ, by lemma 3.9.ii, the real Koebe lemma and the quasi symmetry in the critical points. Reducing δ a little bit we obtain distortion control in $\tilde{V}_r$, by real Koebe lemma. Hence, since $\sum_{i<n^r_k} |R^k_{-i}|$ goes exponentially fast to zero, so do $\sum_{i<n^r_k} |f^i(\tilde{V}_r)|$.*

Lemma 3.12. *There is δ such that, for all $c \in \mathbb{I}_p$, $P(f) \cap \delta\text{-}Q^k_0 = P(f) \cap Q^k_0$.*

Proof. If the statement does not holds then there exists $f^i(c) \in I \setminus Q^k_0$, for some $c \in \mathbb{I}_p$, very close to Q^k_0 with respect to level k, hence $f^i(c) \in Q^{k-1}_0$. After a bounded number of iterations of $f^{N_{k-1}}$, $f^i(c)$ is mapped inside Q^{k+1}_0. But this is impossible, because $f^i(c)$ is very close to a point in ∂Q^k_0, which never returns to Q^k_0. □

4. Monotone Pullbacks

In the rest of this paper, f is a multimodal map satisfying the standard conditions, analytic in a neighborhood of I, whose critical points have even criticality.

In particular, for each $q \in \mathbb{I}_p$, there are univalent maps $\tilde{h}_q$ and h_q defined in neighborhoods of $f(q)$ and q, respectively, such that these maps are real in real points and $f = \tilde{h}_q^{-1} \circ Fol_\ell \circ h_q$ near to q. Here $Fol_\ell(x) = x^\ell$, where ℓ is even and may vary with q. Assume that if J is a small interval such that $q \in J$, then $h_q(J) \subset \mathbb{R}^+$. If g is a inverse branch of Fol_ℓ, we say that $h_q \circ g \circ \tilde{h}_q^{-1}$ is an **inverse branch** of f near to $f(q)$. Consider the inverse branches $Fol_\ell^+ : \mathbb{C}_{\mathbb{R}^+} \to \mathbb{C}$ and $Fol_\ell^- : \mathbb{C}_{\mathbb{R}^+} \to \mathbb{C}$ such that $Fol_\ell^+(1) = 1$ and $Fol_\ell^-(1) = -1$. Define the **real inverse branches** of f near to $f(q)$ by

$$f_q^+ = h_q \circ Fol_\ell^+ \circ \tilde{h}_q^{-1} \tag{8}$$

$$f_q^- = h_q \circ Fol_\ell^- \circ \tilde{h}_q^{-1} \tag{9}$$

Fix a level k_0 of renormalization deep enough such that for each $r \in \mathbb{I}_p$, $R_0^{-(n^r_{k_0}-1)}$ is deeply contained in the closure of the domain of f_q^+ and f_q^-, for all $r \in \mathbb{I}_p$, where q is the predecessor of r at level k_0. Let δ_1, δ_2 be such that

- The map f restricted to $B_{\delta_1}(Q^{k_0}_{-i})$ is univalent and $B_{\delta_2}(Q^{k_0}_{-(i-1)}) \subset f(B_{\delta_1}(Q^{k_0}_{-i}))$, for all i such that $0 < i \leq n^q_{k_0} - 1$ and $q \in \mathbb{I}_p$

We say that the interval J is in the level k_0 if $J \subset Q^{k_0}_{-i}$, for some $q \in \mathbb{I}_p$, $0 \leq i \leq n^q_{k_0} - 1$.

Definition 4.1. *Let J be an interval in the level k_0 such that f^n is monotone in J, for some n. Let $z \in \mathbb{C}$. We say that the* ***backward orbit*** *z_{-i}* ***of*** *z* ***along the orbit of*** *J* ***by*** *f^n is well defined if for $i \leq n$, z_{-i} satisfies the following properties:*

- *$z_0 = z$. If $f^{n-i}(J) \subset Q^{k_0}_{-l}$, $l < n^q_{k_0} - 1$, then $z_{-i} \in B_{\delta_2}(Q^{k_0}_{-l})$ and $z_{-(i+1)}$ is the unique point in $B_{\delta_1}(Q^{k_0}_{-(l+1)})$ such that $f(z_{-(i+1)}) = z_{-i}$.*
- *If $f^{n-i}(J) \subset Q^{k_0}_{-(n^q_{k_0}-1)}$, then z_{-i} is contained in the domain of f_q^+ and f_q^-. Furthermore*

$$z_{-(i+1)} = \begin{cases} f_q^+(z_{-i}) & \text{if } f_q^+(f^{n-i}(J)) = f^{n-(i+1)}(J), \\ f_q^-(z_{-i}) & \text{if } f_q^-(f^{n-i}(J)) = f^{n-(i+1)}(J). \end{cases} \tag{10}$$

We say that z_{-n} has the ***itinerary*** *of J by f^n. Let $B \subset \mathbb{C}$ such the backward orbit z_{-i} is well defined for each $z \in B$. Then the set $\{z_{-n} \colon z \in B\}$ is the* ***complex pullback*** *of B until J by f^n.*

For an interval J, let $B_\delta(J)$ be the set of complex numbers whose distance to J is at most δ. Let $\mathbb{C}_J = \mathbb{C} \setminus (\mathbb{R} \setminus J)$. We will denote by $D_\theta(J)$ the Poincare's neighborhood of J with internal angle θ (see pg. 485 in [2], or [12]). Denote by $D(J)$ the Poincare's neighborhood with internal angle $\pi/2$. Given $z \in \mathbb{C}$ and an interval $J = [a_0, a_1]$, let $(\widehat{z, a_i, J})$ be the angle between the segment $[a_i, z]$ and the real ray beginning at a_i which does not contains J. Let $(\widehat{z, J}) = min\{(\widehat{z, a_1, J}), (\widehat{z, a_2, J})\}$. The ϵ-sector with vertex a_i is the set of complex numbers z such that $(\widehat{z, a_i, J}) \leq \epsilon$.

Conditions which allow us to make pullbacks of points along the orbit of an interval will be the main interest in this section. In Epstein maps, the classical tool is the Schwartz lemma: Poincare neighborhoods are mapped by Epstein maps to inside Poincare neighborhoods with same internal angle (see [12]). The following results will allow us to use the same principle to analytic maps (compare with [1]).

Lemma 4.1 (Almost preserved Poincare neighborhood)**.** *There exists K with the following property. There exists $l_0(\theta_0)$ such that if J is an interval in the level k_0 satisfying*

- *f^n is monotone in J.*
- *$\sum_{i \leq n} |f^i(J)| \leq l_0$.*

Then, provided that $\theta \leq \theta_0$, if $z \in D_\theta(f^n(J))$, the backward orbit of z along the orbit of J is well defined and $z_{-n} \in D_{\tilde{\theta}}(J)$, with $\tilde{\theta} = \theta exp(K \sum_{i \leq n} |f^i(J)|)$.

Proof. Since the maps $f_q^{+,-}$ are compositions of univalents maps which map the real line to the real line and the maps $Fol_\ell^{+,-}$ belong to the Epstein class, we can use lemma 5.2 (see also 5.3) in pg. 487 in [2]. □

Lemma 4.2 (Universal distortion control)**.** *For all δ_0 and θ_0 there exist C and ϵ_0 with the following properties. Let $h\colon D_\theta(\delta\text{-}M) \to \mathbb{C}_{M'}$ be an univalent map, $h(M) = M'$, with $\theta \leq \theta_0$ and $\delta \geq \delta_0$. Then*

1. *If $z \in D_\theta(M)$ and the interval J is contained in M then*

$$\frac{dist(h(z), h(J))}{|h(J)|} \leq C \frac{dist(z, J)}{|J|} \tag{11}$$

2. *Provided $\epsilon < \epsilon_0$, $\widehat{(z, J)} \leq \epsilon$ implies $(\widehat{h(z), h(J)}) \leq \alpha = O(\epsilon)$.*

Proof. Use the control on the first and second derivatives given by complex Koebe lemma. □

The following four lemmas are a straightforward modification of statements contained in [10].

Lemma 4.3. *Let $C \geq 0$, $\theta_0 > 0$. There exists $l_0(\theta_0)$ with the following property. Let J and J' be intervals such that*

- *$|J| \leq l_0$.*
- *J contains $f(q)$ and $1/C \leq |J_1|/|J_2| \leq C$, where J_1,J_2 are the connect components of $J \setminus \{f(q)\}$.*
- *J' is the maximal symmetric interval with respect to q such that $f(J')$ is contained in J.*

If $\theta < \theta_0$ and g is an inverse branch of f near to $f(q)$ then $g(D_\theta(J)) \subset D_{\tilde{\theta}}(J')$. The angle $\tilde{\theta}$ depends on θ and C. Note that the inverse branch g ***does not need*** *to be real.*

Lemma 4.4. *If ϵ is small enough, the following holds. Let $J = [a, b]$ be a small interval near to q and $z \in \mathbb{C}$ near to q. Then $(f(z), \widehat{f(a)}, f(J)) \leq \epsilon$ implies $(\widehat{z, b, J}) > \epsilon$.*

Lemma 4.5. *There exists l_0, $C > 0$ and ϵ_0 such that if*

- *The interval J is contained in L, which is a maximal interval of monotonicity of f^n. L is at level k_0.*
- *$\sum_{i \leq n} |f^i(L)| \leq l_0$.*
- *z is a point with the same itinerary of J by f^n and $(\widehat{f^i(z), f^i(J)}) \leq \epsilon \leq \epsilon_0$, for all $i < n$.*

Then $z \in D_\alpha(L)$, where $\alpha = (\pi/2 + O(\epsilon)) exp(C \sum_{i \leq n} |f^i(L)|)$.

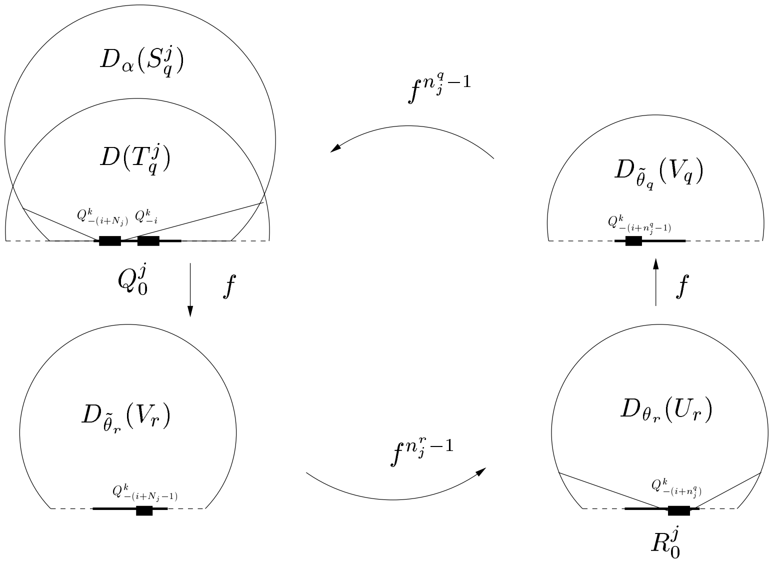

FIGURE 2. lemma 5.1. In the picture, two critical points, q and r, are involved in the renormalization.

Lemma 4.6. *For any ϵ, δ and l_0, with l_0 small enough, there exists $K(\epsilon, \delta, l_0)$ with the following property. If the interval J and z satisfy*

- *There exists an interval L in the level k_0 which contains J, such that f^n is monotone in L. Moreover δ-$f^n(J) \subset f^n(L)$.*
- $\sum_{i \leq n} |f^i(L)| \leq l_0$.
- *z is contained in $\mathbb{C}_{f^n(L)}$, $(\widehat{z, f^n(J)}) \geq \epsilon$ and $dist(z, f^n(J)) \leq K(\epsilon, \delta, l_0)|f^n(J)|$.*

Then the backward orbit of z along the orbit of J is well defined and

$$\frac{dist(z_{-n}, J)}{|J|} \leq C(\delta, \epsilon)\frac{dist(z, f^n(J))}{|f^n(J)|} \tag{12}$$

Moreover, fixing ϵ and δ, K tends to infinity when l_0 goes to zero.

Proof. If z is close to $f^n(J)$, use lemma 4.2.i; otherwise, capture z in a Poincare neighborhood whose diameter is commensurable with $dist(z, f^n(J))$ and use lemma 4.1. For details, see [10]. □

The proof of the following lemma is easy. Assume that q is a local maximum (other case is similar).

Lemma 4.7. *There exists ϵ with the following properties. Let $z \in \mathbb{C}$ be a point near to q, $q \in \mathbb{I}_p$.*

- *If $(f(z), \widehat{f(q), f(q)} + \mathbb{R}^+) < \epsilon$ then $min\{(z, \widehat{q, q + \mathbb{R}^+}), (z, \widehat{q, q + \mathbb{R}^-})\} > \epsilon$.*
- *If*
  - $(f(z), \widehat{f(q), f(q)} + \mathbb{R}^-) < \epsilon$.
  - $min\{(z, \widehat{q, q + \mathbb{R}^+}), (z, \widehat{q, q + \mathbb{R}^-})\} < \epsilon$.

  Then either $f_q^+(f(z)) = z$ or $f_q^-(f(z)) = z$.

5. Complex Bounds

Let $z \in \mathbb{C}$. We say that $z_{-\ell}$ is well defined if the backward orbit of z along the orbit of $Q^k_{-\ell}$ by f^ℓ is well defined.

Recall that the pullback of T^k_p along the k-cycle of renormalization exists. Hence let $U_p = T^k_p, U_r, V_r, \tilde{U}_p = S^k_p$, $r \in \mathbb{I}_p$, be as in definition 2.1. The most important step to prove the complex bounds using the Lyubich-Yampolsky approach is the following lemma:

Main Lemma 5.1 (At most linear growth of distances)**.** *There exists C with the following property. Provided that $k \geq k_0(a)$, if $z \in D(T^{k-a}_q) \cap \mathbb{C}_{U_q}$, then $z_{-(n^q_k-1)}$ is well defined and*

$$\frac{dist(z_{-(n^q_k-1)}, Q^k_{-(n^q_k-1)})}{|Q^k_{-(n^q_k-1)}|} \leq C\frac{dist(z, Q^k_0)}{|Q^k_0|} \tag{13}$$

We say that ℓ is a **return** to R^j_0 if $Q^k_{-\ell} \subset R^j_0$. The time ℓ is a **return to level j** if ℓ is a return to R^j_0 for some $r \in \mathbb{I}_p$.

In this section, we assume that the levels j and k are deep enough. We say that z_{-i} **ϵ-jumps** if $(\widehat{z_{-i}, Q^k_{-i}}) > \epsilon$. Fix a and assume that $z \in D(T^{k-a}_q)\setminus\mathbb{R}$ (for $z \in U_q$, the proof of main lemma follows of the results in section 3).

Remark 5.1. *Let $U_q = T^j_q, U_r, V_r, \tilde{V}_r, \tilde{U}_q = S^j_q$ and $\delta > 0$ be as in remark 3.2. Because $\sum_{i<n^r_j} |f^i(\tilde{V}_r)|$ goes exponentially fast to zero, fixing θ, the complex pullback of $D_\theta(U_r)$ until $D_{\tilde{\theta}}(V_r)$ by $f^{n^r_j-1}$ can be extended to a larger neighborhood $D(\delta\text{-}U_r)$,for $j \geq j(\theta)$, by lemma 4.1. Hence we can use lemma 4.2 with $M = U_r$ and appropriate inverse branchs h_t of f^t, $t \leq n^r_j - 1$, such that $h_t(z) = z_{-t}$, where z is inside $D_\theta(\delta\text{-}U_r)$ and z_{-t} is the backward orbit of z along the orbit of V_r by $f^{n^r_j-1}$.*

Lemma 5.1 (Tour along the j-cycle)**.** *There exist C and ϵ_0 with the following property. Assume that ℓ_0 is a return to Q^j_0, $j \geq k-a$; $z_{-\ell_0}$ is well defined and $z_{-\ell_0} \in D(T^j_q)$. Let $\ell_1 = min\{n^q_k - 1, \ell_0 + N_j\}$. Then $z_{-\ell_1}$ is well defined and*

- **a.** *If ℓ is a return to R^j_0, for $\ell_0 \leq \ell \leq \ell_1$, then $dist(z_{-\ell}, Q^k_{-\ell}) \leq C|R^j_0|$.*
- **b.** *Let $\epsilon \leq \epsilon_0$. If $\ell_0 + N_j \leq n^q_k - 1$ then one of the following statements holds:*
  - *$z_{-(\ell_0+N_j)}$ is contained in $D(T^j_q)$.*
  - *For some return ℓ to level j, $\ell_0 \leq \ell \leq \ell_0 + N_j$, $z_{-\ell}$ ϵ-jumps.*
- **c.** *Let $\epsilon \leq \epsilon_0$. Suppose that $z_{-\ell}$ does not ϵ-jump in returns to level j, $\ell_0 \leq \ell \leq \ell_1$. Then $z_{-\ell}$ does not α-jump, for each ℓ with $\ell_0 \leq \ell < \ell_1$, where $\alpha = O(\epsilon)$.*

Proof. (compare with the proof of lemma 5.3 in [10]) The pullback of T^j_q along the j-cycle exists. Hence let $U_q = T^j_q, U_r, V_r, \tilde{U}_q = S^j_q$, $r \in \mathbb{I}_p$, be as in definition 2.1. The complex pullback of $D_{\theta_q}(T^j_q)$, $\theta_q = \pi/2$, until V_q by $f^{n^q_j-1}$ is inside $D_{\tilde{\theta}_q}(V_q)$, with $\tilde{\theta}_q \sim \pi/2$, by lemma 4.1. By remark 3.2 and lemma 3.6, $f(c)$ cuts V_q in commensurable parts, where c is the predecessor of q at level j. Hence , by lemma 4.3, the complex pullback of $D(V_q)$ by f along the orbit of c is contained in $D_{\theta_c}(U_c)$, where θ_c does not depend on j. By induction, assume that the complex pullback of $D(T^j_q)$ along the j-cycle until U_r is inside $D_{\theta_r}(U_r)$. Again the complex pullback of $D_{\theta_r}(U_r)$ until V_r by $f^{n^r_j-1}$ is inside $D_{\tilde{\theta}_r}(V_r)$, $\tilde{\theta}_r \sim \theta_r$, by lemma 4.1.

The complex pullback of $D_{\theta_r}(V_r)$ by f along the orbit of d is contained in $D_{\theta_d}(U_d)$, where d is the predecessor of r at level j, by remark 3.2, lemma 3.6 and lemma 4.3. Hence the complex pullback of $D(T^j_q)$ along all j-cycle is inside $D_\alpha(S^k_q)$, for some α. Observe that if $Q^k_{-\ell} \subset R^j_0$, for some $r \in \mathbb{I}_p$, then $z_{-\ell}$ and $Q^k_{-\ell}$ are inside $D_{\theta_r}(U_r)$, which has diameter commensurable with $|R^j_0|$ (because $U_r \subset R^{j-1}_0$). This proves **a**. Assume $\ell_0 + N_j \leq n^q_k - 1$ and suppose that $z_{-(\ell_0+N_j)}$ is not inside $D(T^j_q)$. Every point $z \in D_\alpha(S^j_q) \setminus D(T^j_q)$ satisfies $\widehat{(z,J)} \geq \epsilon$, for all interval J inside S^j_q, for a definitive ϵ, since S^j_q is δ-deeply contained in T^j_q. This proves **b**. To prove **c**, suppose that $z_{-\ell}$ does not ϵ-jump in returns to level j, $\ell_0 \leq \ell \leq \ell_1$. If $Q^k_{-\ell} \subset R^j_0$ then $z_{-\ell} \in D_{\theta_r}(U_r)$. By lemma 4.2.ii and remark 5.1, z_{-i} does not α-jump for each $\ell \leq i \leq min\{\ell + n^r_j - 1, \ell_1\}$, with $\alpha = O(\epsilon)$. □

The concept of 'good return' was introduced in [10]:

Lemma 5.2 (Good returns). *Assume that ℓ_0 is a return to Q^j_0, $z_{-\ell_0}$ is well defined and $z_{-\ell_0} \in D(T^j_q)$. If there exists ℓ satisfying*

- *$z_{-\ell}$ is well defined,*
- *ℓ is a return to level j, $j \geq k - a$, with $\ell_0 \leq \ell \leq N_{j+1}$,*
- *z_ℓ ϵ-jumps.*

then provided $k \geq k_0(a)$, $z_{-(n^q_k-1)}$ is well defined and

$$\frac{dist(z_{-(n^q_k-1)}, Q^k_{-(n^q_k-1)})}{|Q^k_{-(n^q_k-1)}|} \leq C(\epsilon)\frac{|Q^j_0|}{|Q^k_0|} \tag{14}$$

Furthermore, if ℓ is the smallest time satisfying the above properties (in this case we say that ℓ is a ***good return*** *to level j) then $z_{-\ell} \in D_{\theta(\epsilon,k-j)}(Q^k_{-\ell})$*

Proof. Let ℓ be the smallest time satisfying the above properties. We have $Q^k_{-\ell} \subset R^j_0$, for some $r \in \mathbb{I}_p$. Then $dist(z_{-\ell}, Q^k_{-\ell}) \leq C|R^j_0|$, by lemma 5.1 (use 5.1.b successively, and finally 5.1.a). Since N_{j+1}/N_j is bounded, by lemma 3.5 there exists C such that $|R^j_0|/|Q^k_{-\ell}| \leq C|Q^j_0|/|Q^k_0|$. Hence

$$\frac{dist(z_{-\ell}, Q^k_{-\ell})}{|Q^k_{-\ell}|} \leq C\frac{|Q^j_0|}{|Q^k_0|} \tag{15}$$

Since $z_{-\ell}$ ϵ-jumps and $|Q^j_0|/|Q^k_0| \leq C(k-j)$, by real bounds, $z_{-\ell}$ is contained in $D_{\theta(k-j,\epsilon)}(Q^k_{-\ell})$. By remark 3.2, there exists an interval of monotonicity $\tilde{V}_q$ for $f^{n^q_k-1-\ell}$ which contains $Q^k_{-(n^q_k-1)}$ and such that $f^{n^q_k-1-\ell}(L)$ contains δ-$Q^k_{-\ell}$, for some δ. Moreover $|Q^j_0|/|Q^k_0| \leq K$, where K depends only on a. By lemma 4.6

$$\frac{dist(z_{-(n^q_k-1)}, Q^k_{-(n^q_k-1)})}{|Q^k_{-(n^q_k-1)}|} \leq C(\epsilon)\frac{dist(z_{-\ell}, Q^k_{-\ell})}{|Q^k_{-\ell}|} \tag{16}$$

□

Lemma 5.3 (Inductive step). *Let α be as in lemma 5.1.c, ϵ small enough and $j \geq k-a$. Provided that $k \geq k_0(a)$, the following holds. Suppose that $z_{-N_j} \in D(T^j_q)$ and $z_{-\ell}$ does not α-jump for $\ell < N_j$. If $N_{j+1} \leq n^q_k - 1$, then one of the following statements holds:*

1. *There exist ℓ with $N_j \le \ell \le N_{j+1}$ such that*
   - *$z_{-\ell}$ is well defined;*
   - *ℓ is a return to level j.*
   - *$z_{-\ell}$ ϵ-jumps.*
2. *$z_{-N_{j+1}}$ is well defined and $z_{-N_{j+1}} \in D(T_q^{j+1})$.*

Proof. (compare with the proof of corollary 5.5 in [10]) Let L be the maximal interval of monotonicity of $f^{N_{j+1}}$ which contains $Q^k_{-N_{j+1}}$. Since $Q^k_{-N_{j+1}}$ is contained in Q_0^{j+1}, the interval L is contained in L_q^{j+1}. In particular, $\sum_{i\le N_{j+1}} |f^i(L)|$ is very small. Assume that (i) does not hold. By lemma 5.1.c, $z_{-\ell}$ never $O(\epsilon)$-jumps for $\ell \le N_{j+1}$. Hence, by lemma 4.5, $z_{-N_{j+1}} \in D_{(\pi/2+O(\epsilon))\beta}(L)$, with $\beta \sim 1$. Since L_q^{j+1} is δ-deeply contained in T_q^{j+1}, then $D_{(\pi/2-O(\epsilon))\beta}(L) \subset D(T_q^{j+1})$, if ϵ is small enough. □

For a deep level ℓ, let $\mathbb{I}_p = \{c_i^\ell\}_{i\le \#\mathbb{I}_p}$, where $c_1^\ell = q$ and c_i^ℓ is the predecessor of c_{i-1}^ℓ at level ℓ.

*Proof of **Main Lemma 5.1**.* (compare [10], subsection 5.2) Let b, with $k-a \le b \le k$, be such that $z \in D(T_q^b) \setminus D(T_q^{b+1})$. In other words, z is in the 'scale' of level b. If $z \in D(U_q)$, we are done by lemma 4.2.i and remark 5.1. Otherwise $dist(z, Q_0^k)/|Q_0^k| \sim |Q_0^b|/|Q_0^k|$. Note that $n_k^q - 1 = CN_b + \sum_{t<\tilde{t}} n_b^{c_t^b} + (n_b^r - 1)$, where $r = c_{\tilde{t}}^b$ and $\tilde{t} \le \#\mathbb{I}_p$. If C is zero, using arguments as in the proof of lemma 5.1.a, $z_{-(n_k^q - n_b^r)}$ is well defined and it is contained in $D_{\theta_r}(U_r)$, where U_r and θ_r are as in the proof of 5.1.a (replacing j by b). In particular $dist(z_{-(n_k^q-n_b^r)}, Q^k_{-(n_k^q-n_b^r)}) \le C|R_0^b|$. Since $n_k^q - n_b^r \le N_{b+1}$, by lemma 3.5 $|R_0^b|/|Q^k_{-(n_k^q-n_b^r)}| \le C|Q_0^b|/|Q_0^k|$. By lemma 4.2.i(recall remark 5.1), we obtain the lemma. Otherwise, suppose that z_ℓ ϵ-jumps for some $\ell \le N_b$, where ℓ is a return to level b. Then we are done, by lemma 5.2. If ϵ-jumps in a return to level b does not happen then $z_{-N_b} \in D(T_q^b)$, by lemma 5.1.b. Let i be the deepest level such that $n_k^q - 1 = CN_i + \sum_{t<\widehat{t}} n_i^{c_t^i} + (n_i^r - 1)$, with C distinct of zero, $r = c_{\widehat{t}}^i$ and $\widehat{t} \le \#\mathbb{I}_p$. Note that C is bounded. Suppose, by induction, that $z_{-N_j} \in D(T_q^j)$, $j < i$, and z_ℓ does not α-jump, for $\ell \le N_j$. By lemma 5.3, either there is a return to level j between N_j and N_{j+1}, which ϵ-jumps, or $z_{-N_{j+1}} \in D(T_q^{j+1})$ and moreover z_ℓ does not $O(\epsilon)$-jump for $\ell \le N_{j+1}$, by lemma 5.1.c. If there are not good returns between the times 0 and N_i then $z_{-N_i} \in D(T_q^i)$. Then either there are good returns to level i between N_i and CN_i, and we can use lemma 5.2, or $z_{-CN_i} \in D(T_q^i)$. Using arguments as in the proof of 5.1.a, $z_{-(n_k^q-n_i^r)}$ is well defined and it is inside $D_{\theta_r}(U_r)$, where U_r and θ_r are as in the proof of lemma 5.1.a (replacing j by i). In particular $dist(z_{-(n_k^q-n_i^r)}, Q^k_{-(n_k^q-n_i^r)}) \le C|R_0^i|$. Since $n_k^q - n_i^r \le N_{i+1}$, one gets $|R_0^i|/|Q^k_{-(n_k^q-n_i^r)}| \le C|Q_0^i|/|Q_0^k|$, by lemma 3.5. By lemma 4.2.i(recall remark 5.1), we obtain the lemma. □

Definition 5.1. *We say that $z \in \mathbb{C}$ has **complex pullbacks along the k-cycle** if given, for each $q \in \mathbb{I}_p$, arbitrary inverse branches g_q defined near to $f(q)$, there exists a sequence $z_{-\ell}$, $\ell \le N_k$ which satisfies*

- *If $f^{N_k-\ell}(P_0^k) \subset Q_0^k$, $t \le n_k^q - 1$, then $z_{-(\ell+t)}$, $t < n_k^q - 1$, is the backward orbit of $z_{-\ell}$ along the orbit of $Q^k_{-(n_k^q-1)}$ by $f^{n_k^q-1}$.*

- *If $f^{N_k-\ell}(P_0^k) \subset Q^k_{-(n_k^q-1)}$, then $z_{-\ell}$ is contained in the domain of g_r and $g_r(z_{-\ell}) = z_{-(\ell+1)}$. Here r is the predecessor of q at level k.*

The following lemma say that a point which never ϵ-jumps has a real itinerary:

Lemma 5.4. *Assume that z has complex pullbacks along the k-cycle. Let $z_{-\ell}$ be a complex pullback such that*

- *If $f^{N_k-\ell}(P_0^k) \subset Q^k_{-j}$, for some $q \in \mathbb{I}_p$, then $(\widehat{z_{-\ell}, Q^k_{-j}}) \leq \epsilon$, for ϵ small enough.*
- *If $f^{N_k-\ell}(P_0^k) \subset Q_0^k$, for some $q \in \mathbb{I}_p$, Then $dist(z_{-\ell}, Q_0^k) \leq C|Q_0^k|$.*

There exists $k_0(C)$ such that if $k \geq k_0$ then z_{-N_k} has the itinerary of a point in ∂P_0^k by f^{N_k}.

Proof. Consider i such that $f^{N_k-i}(P_0^k)$ is contained in $Q^k_{-(n^q-1)}$. Since $(\widehat{z_{-i}, Q^k_{-(n_k^q-1)}}) \leq \epsilon$, z_{-i} is inside a ϵ-sector of angle ϵ which has as its vertex a point in $\partial Q^k_{-(n_k^q-1)}$. We claim that this vertex is the point b such that $\{b\} = f(\partial R_0^k)$, where r is the predecessor of q at level k. Otherwise, $z_{-(i+1)}$ ϵ-jumps, by the first statement in lemma 4.7. Moreover, by the second statement in lemma 4.7, the inverse branch g, defined near to $f(r)$, such that $g(z_{-i}) = z_{-(i+1)}$ must be real. Let $R_0^k = [x_1, x_2]$. By lemma 4.4, if $z_{-(i+1)}$ is contained in the ϵ-sector whose vertex is x_a, then $z_{-(i+1+t)}$, $t < n_k^r$, is contained in the ϵ-sector whose vertex $\tilde{x}_a$ satisfies $f^t(\tilde{x}_a) = x_a$. Using the previous arguments along all k-cycle, the lemma follows. □

Lemma 5.5. *Let $f(z) = z^{2n}$. Let P be a symmetric interval M such that $f(P) \subset M$, with $|M| \leq C|f(P)|$. Then there exist $C_1(C,n)$, $C_2(C,n)$ such that*

$$\frac{dist(z,P)}{|P|} \leq C_1\left(\frac{dist(f(z),M)}{|M|}\right)^{\frac{1}{2n}} + C_2 \tag{17}$$

Definition 5.2. *The renormalization R_p^k has a* ***polynomial-like extension*** *if there exist simply connected domains $\tilde{D}_p$ and D_q, for each $q \in \mathbb{I}_p$, satisfying*

- *The critical point q is the unique critical point of f in D_q. Moreover, if $q'_k \neq p$, $f^{n_k^{q'_k}}: D_q \to D_{q'_k}$ is a ramified covering.*
- *If q is the predecessor of p at level k, then $f^{n_k^p}: D_q \to \tilde{D}_p$ is a ramified covering. Moreover the closure of D_p is contained in the interior of $\tilde{D}_p$.*

In particular $f^{N_k}: D_q \to \tilde{D}_p$ is a polynomial-like map.

The α-sector supported on a interval J is the set of complex number which satisfies $(\widehat{z, P_0^k}) \geq \alpha$.

Proof of Theorem 1. (compare [10], subsection 5.3) By the Main Lemma and lemma 5.5, if $dist(z, P_0^k)/|P_0^k| \leq C$ then $z \in \mathbb{C}\setminus\mathbb{R}$ has complex pullbacks along the k-cycle. Moreover, if ℓ_r is such that $f^{N_k-\ell_r}(P_0^k)$ is contained in Q_0^k, then $dist(z_{-\ell_r}, Q_0^k) \leq \tilde{C}|Q_0^k|$. Hence $z_{-\ell_r} \in D(T_r^{a_k})$, where $k - a_k$ is bounded. Furthermore, for points satisfying $dist(z, P_0^k)/|P_0^k| = C$, since C is large enough, one has

$$\frac{dist(z_{-N_k}, P_0^k)}{|P_0^k|} \leq \frac{1}{10}\frac{dist(z, P_0^k)}{|P_0^k|} \tag{18}$$

Hence, let D be a disc large enough such that (18) holds in ∂D. Then the pullback of $\partial D \cap (\mathbb{C} \setminus \mathbb{R})$ along the k-cycle is contained in $D \setminus \mathbb{R}$. Using arguments as in the proof of lemma 5.1.a, the complex pullbacks of T_p^k are inside $D_\theta(S_p^k)$. Hence, taking a larger disc, if necessary, the pullback of $V = D \cap \mathbb{C}_{T_p^k}$ is compactly contained in $V \cap \mathbb{C}_{L_p^k}$ (clearly the pullback is contained in $V \cap \mathbb{C}_{S_p^k}$, but this could not be true for the closure of the pullback). One gets a polynomial-like extension. Now, it is necessary to control the modulus of the polynomial-like extension. By lemma 2.4 in [10], it is sufficient to prove that the ring $V \setminus K(R_p^k(f))$ has the modulus bounded below, where $K(R_p^k(f))$ is the filled-in Julia set of $R_p^k(f)$. We claim that $K(R_p^k(f))$ is contained in a sector supported on L_p^k. Indeed, take $z' \in K(R_p^k(f))$. Let $z = f^{N_k}(z')$. If $z \in T_p^k$, one gets that $z' \in D_\theta(S_p^k)$. Assume that $z \in \mathbb{C} \setminus \mathbb{R}$. Denote $z_{-\ell} = f^{N_k - \ell}(z')$. If there exist $t_0 < t$, j and $r \in \mathbb{I}_p$ satisfying the following properties

- $R^k_{-t_0}$ is contained in R^j_0.
- $z_{-(\ell_r + t_0)}$ is contained in $D(T_r^j)$.
- R^k_{-t} is contained in Q^j_0, for some $q \in \mathbb{I}_p$. Moreover $t \le N_{j+1}$.
- $(\widehat{z_{-(\ell_r+t)}, R^k_{-t}}) \ge \epsilon$.
- t is minimal with the above properties.

then we capture $z_{-(\ell_r+t)}$ in $D_\gamma(R^k_{-t})$, by lemma 5.2. Using arguments as in the proof of lemma 5.1.a, we obtain that $z' \in D_{\tilde{\gamma}}(P_0^k)$. In the other case, $z_{-\ell}$ never $O(\epsilon)$-jumps for all ℓ. Then, by lemma 5.4, z' has the itinerary of a point in ∂P_0^k. Let L be the interval of monotonicity of f^{N_k} which contains this point. Clearly $L \subset L_p^k \subset_\delta T_p^k$. Since L_p^k is contained in P_0^{k-1} and N_k/N_{k-1} is bounded, $\sum_{i \le N_k} |f^i(L)|$ goes exponentially fast to zero. Hence By lemma 4.5, $z' \in D_{(\pi/2 + O(\epsilon))\beta}(L)$, where $\beta \sim 1$. We captured the Julia set in three Poincare neighborhoods: two of these neighborhoods are based on intervals contained in L_p^k which contain points in ∂P_0^k. The other one is $D_{max\{\tilde{\gamma}, \theta\}}(L_p^k)$. Hence the Julia set is contained in $D_\theta(L_p^k)$, for some angle θ. Indeed, the filled-in Julia set is inside a sector supported on P_0^k. We follows the argument in [10]: Consider the interval J of monotonicity of f^{N_k} which contains the periodic point u in ∂P_0^k. We have $J \subset_\delta f^{N_k}(J)$. Let $g = f^{-N_k}$ be the inverse branch of f^{N_k} such that $g(D(f^{N_k}(J)))$ is deeply inside $D(J)$, for large k. The map g has exactly one fixed point u in $D(J)$ and $\lambda_1 \ge (f^{N_k})'(u) \ge \lambda_2 > 1$, where λ_1, λ_2 does not depend on k. We can linearize g inside $D(f^{N_k}(J))$. Do the pullback of $D_\theta(L_p^k)$ along the k-cycle a sufficient number of times (which does not depend on k) such that the pullback will be contained in $D_{\tilde{\theta}}(H)$, for some interval H such that the component of $H \setminus P_0^k$ which contains u is contained in $g^2(B_{\delta|P_0^k|}(u))$, for a small δ. In particular, since $K(R_p^k(f)) \subset D_{\tilde{\theta}}(H)$, the Julia set cuts the fundamental ring $B_{\delta|P_0^k|}(u) \setminus g(B_{\delta|P_0^k|}(u))$ at most a certain angle. Using that the linearization has small distortion on $B_{\delta|P_0^k|}(u)$, if δ is small, and the invariance of $K(R_p^k(f))$ by $R_p^k(f)$, we obtain the sector. By invariance of $K(R_p^k(f))$, we obtain the sector in other point of ∂P_0^k. $\square$

Remark 5.2. *If f satisfies the standard conditions, the complex extensions can be taken unbranched. In fact, there exists δ such that δ-$P_0^k \cap P(f) = P_0^k \cap P(f)$. By the control of geometry of the ring $V \setminus K(R_p^k)$ (see the figure 3), the ring $(V \setminus K(R_p^k(f))) \cap$*

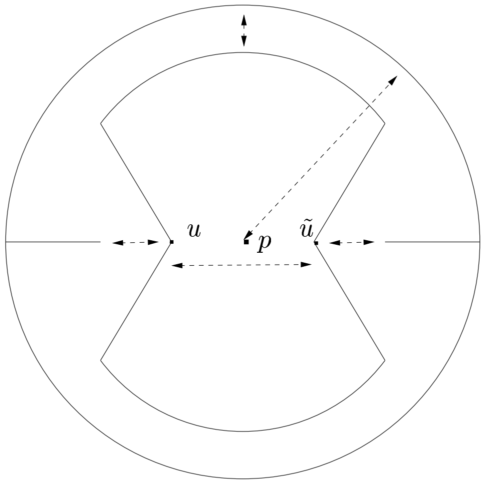

FIGURE 3. This ring is contained in $V \setminus K(R_p^k(f))$. The extern boundary is ∂V. All measures indicated in picture are commensurable. Moreover, the angles in u and $\tilde{u}$ are under control.

$\mathbb{C}_{\delta\text{-}P_0^k}$ has the modulus bounded below. Now it is easy to find an unbranched polynomial like extension such that the modulus is also bounded below.

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