A general Lagrangian approach for non-concave moral hazard problems

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Abstract

We establish a general Lagrangian for the moral hazard problem which generalizes the well known first order approach (FOA). It requires that besides the multiplier of the first order condition, there exist multipliers for the second order condition and for the binding actions of the incentive compatibility constraint. Some examples show that our approach can be useful to treat the finite and infinite state space cases. One of the examples is solved by the second order approach. We also compare our Lagrangian with Mirrlees'.

JEL classification: D82

Keywords: Moral hazard; Lagrangian; first order approach.

1. Introduction

The main tool in the literature to treat the moral hazard problem[1] is the first order approach (FOA) technique. Several authors have contributed in this direction, in which J.A. Mirrlees was the pioneer (see Mirrlees (1975) and (1986) and Rogerson (1985)). However, Mirrlees showed that in some cases this technique can not be applied and Rogerson gave some sufficient conditions for it: the density of the output resulting from the agents' action to have monotone likelihood ratio property (MLRP) and the convexity of the distribution function property (CDFP). The first condition has the fairly natural interpretation of more effort, more output, and also serves to imply (when the FOA is valid) that the agents payment is increasing in the observed output (see Milgrom (1981)). However, the second condition is by no means as easy to accept: most of the distributions commonly occurring in statistics (and economics) do not have the CDFP. Jewitt (1988) provides conditions which justify the FOA in the multi-statistic case and where the CDFP is replaced by conditions that are valid for problems with more than one variable (see also Sinclair-Desgagné (1994)). In these cases, the agent's utility function in the optimal contract is concave on the action variable and consequently the FOA follows.

In this paper, we obtain the general Lagrange multipliers for this problem which includes the FOA as a particular case. The Lagrangian can be described as follows: besides the multipliers relative to the individual rationality (IR) constraint and the first order condition of the incentive compatibility (IC) constraint, there exist a multiplier for the second order condition and a multiplier for the action variables where the IC constraint is binding. Moreover, there is a multiplier for the derivative of the agent's utility function with respect to action in the boundary.

Mirrlees (1975) and (1986) present a Lagrangian that is different from ours: he considers a finite dimensional state space and multi-dimensional action set; he does not consider the second order condition of the IC constraint; the boundary conditions do not appear in his Lagrangian (he works with an open action set); and finally, besides the multipliers of the binding actions of the IC constraint, there exist Lagrange multipliers for their first order conditions. This implies that the number of the variables (multipliers, state variables and action) is greater than the number of equations (the first order conditions of the Lagrangian and the complementary slackness Kuhn and Tucker's conditions) in a finite dimensional state space problem; in our case, these numbers are the same. As Mirrlees has himself pointed out, to follow his approach one has to know in advance the critical manifold. This is not the case in our paper: we determine the critical point jointly with the Lagrange multipliers. The reason is because Mirrlees uses a local Kuhn and Tucker Theorem in a finite dimensional state space.[2]

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[1] For a survey in this topic see Dutta and Radner (1994) and Rees (1987).

[2] He needs to put the binding equations of the IC constraints and their first order conditions as the constraints that replace the IC constraints in order to satisfy the regularity conditions of the Kuhn and

Grossman and Hart (1983) established a Lagrangian for finite state space and finite number of action and used it to approximate the solution when there are an infinite number of actions. For each action, we consider (as Grossman and Hart (1983)) the incentive scheme which minimizes the (expected) cost of inducing the agent to choose that action. Under the assumption that the agent's preferences over income lotteries are independent of the action he takes, we have that this cost minimization problem is a fairly straightforward convex programming problem. The assumption that the agent's preferences over income lotteries are independent of the action is a strong one. Yet it has been used in most of the applications of moral hazard problems. Special cases of this assumption occur when the agent's utility function is additively or multiplicatively separable in action and reward. However, we also obtain a local Kuhn and Tucker Theorem including the action variable without assuming this special type of agent's utility function (in this case we do not have necessarily a concave programming problem).

We provide four examples where the FOA is not valid: In the first one we use the global Kuhn and Tucker Theorem for a given action to solve the cost minimization problem in the state variable and we conclude that in this case the information of the second order condition (besides the first order one) is sufficient to characterize the optimal solution. This is an infinite dimensional example in the state variables, where the Lagrangian approach of Mirrlees (1975) and (1986) is not applicable. The expected utility function is constant in the action variable at the optimal contract, i.e., the agent is indifferent to all feasible actions at the optimal contract and the second order approach works. This example demonstrates the possibility raised by Mirrlees (1986) (see the last paragraph of page 1208). He said that, if there exist infinite states of nature, it is possible to have an optimal contract such that the agents expected utility function is constant on part of the set of feasible actions. The second example follows the same idea, but the number of states is finite and the optimal solution is such that the agent's expected utility presents two maximal points as a function of the action. This example shows the importance of the Lagrange multipliers of the binding actions of the IC constraint.

In the third example, we run the first order conditions derived from the local Kuhn and Tucker Theorem in the two variables (the action and the state variables) in a Mathematica program[3] and we see that the information of multiple maximal actions is important for the characterization of the optimal solution, i.e., the Lagrange multipliers have a non trivial component in a binding action of the IC constraint. This example is standard: there exist two states (the low and high output states), a convex cost function of action (the disutility of the action for the agent) and an additively separable form of the agent's utility function; the princi-

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Tucker Theorem. In our case, we use the Kuhn and Tucker Theorem for infinite dimensional spaces and our regularity conditions are quite different from Mirrlees'.

[3] We establish the first order equations and the Kuhn and Tucker's conditions of the Lagrangian of the problem and implement these as a system of nonlinear equations with the equal number of variables and equations and use an algorithm in Mathematica to solve the problem.

pal is risk neutral and the agent is risk averse. The optimal contract is monotone in output. However, the agent's expected utility function is not a concave function of the action variable and there exist two optimal feasible actions for the principal: the lower and the higher one; the optimal action from the viewpoint of the principal is the higher one.

The Mirrlees' counter example is reexamined under our approach: we can provide Lagrange multiplier for the optimal solution of that example. Some classical results in the literature valid under the FOA can be extended in our framework.

The paper is organized as follows. In section 2, we present a motivating example for the study of a general Lagrangian approach to the principal-agent problem. In section 3, we present the moral hazard model and the results. Section 4 gives the final conclusions and extensions. Finally, the appendix provides the proofs of the theorems.

2. An example

Suppose that the action set is $A = [0, 0.9]$. There are two states of nature: 1 (the low return state) and 2 (the high return state) with the following returns to the principal: $\pi_1 = 1$ and $\pi_2 = 5$. The cost of the action for the agent is $c(a) = a^2$, $a \in A$. The subjective beliefs of the principal and the agent about the state of nature are represented by a probability distribution conditioning in the action: $p_1(a) = 1 - a^3$ and $p_2(a) = a^3$, for all $a \in A$. The principal and the agent's preferences with respect to the monetary return are represented by $u(x) = x$ and $v(x) = \sqrt{x}$, respectively.

The principal offers a contract, i.e., a payment schedule given the return (or the state of nature): a monetary transfer x_i is paid for the agent if π_i occurs $(i = 1, 2)$.

The expected utilities for the principal and the agent are:

$$U(x, a) = p_1(a)u(\pi_1 - x_1) + p_2(a)u(\pi_2 - x_2)$$

and

$$V(x, a) = p_1(a)v(x_1) + p_2(a)v(x_2) - c(a)$$

respectively, where $x = (x_1, x_2)$.

Assume that the reservation utility of the agent is zero. If the principal observes the action, he can enforce the action as a part of the contract ("the first best problem"). However, we assume that the principal can not observe the agent's action, therefore he should induce the agent to take the action he would relatively prefer ("the second best problem").

Hence, the principal-agent problem is

$$\text{(I)} \qquad \begin{aligned} &\max_{\substack{x \in C \\ a \in A}} U(x, a) \\ &\text{s.t.} \quad V(x, a) \geq V(x, \hat{a}), \forall\, \hat{a} \in A \\ &\qquad\quad V(x, a) \geq 0 \end{aligned}$$

The FOA is the substitution (when it is valid) of the IC constraint by its first order condition (the derivative of V with respect to the action equals to zero). This is not possible in general because when the expected utility is not a concave function of the action (calculated at the optimal contract), the first order condition is not sufficient to characterize the optimal action in the IC constraint. We need more information: the second order condition, the multiple maxima and the behavior of $V(x,\cdot)$ in the boundary of the action set. Our Lagrangian captures all these aspects.

This example illustrates this case. We claim that the optimal contract is $x^* = (0, 1.23457)$ and the optimal action is $a^* = 0.9$. If this is true, the expected utility of the agent at x^*, $V(x^*,\cdot)$ admits exactly two maximal actions: the boundary actions 0 and 0.9 where $V(x^*,\cdot)$ is null. We also claim that these properties are sufficient to characterize the optimum. If this is the case, the problem becomes:

$$\text{(II)} \qquad \begin{aligned} &\max_{\substack{x\in C\\ a\in A}} U(x,a) \\ &\text{s.t.} \quad V(x,a) \geq V(x,0) \\ &\qquad\quad V(x,a) \geq 0 \end{aligned}$$

Thus, the Lagrangian can be written as

$$L(x,a) = U(x,a) + \lambda_1 V(x,a) + \mu_1(V(x,a) - V(x,0))$$

The first order conditions of the Lagrangian are

$$\frac{u'(\pi_i - x_i)}{v'(x_i)} = \lambda_1 + \mu_1\left(1 - \frac{p_i(0)}{p_i(a)}\right), \; i = 1, 2$$

$$\sum_{i=1}^{2}(u(\pi_i - x_i) + (\lambda_1 + \mu_1)v(x_i))p_i'(a) - (\lambda_1 + \mu_1)c'(a) = 0$$

The first order conditions are satisfied if and only if $x_1^* = 0$, $x_2^* = 1.23457$, $a^* = 0.9$, $\lambda_1 = 1.62$ and $\mu_1 = 0.602222$. Moreover, x_1^*, x_2^* and a^* are the unique critical points of the Lagrangian for the given values of λ_1 and μ_1 and the Lagrangian tends to $-\infty$ uniformly in a when the norm of (x_1, x_2) tends to ∞. Therefore, given a pair of feasible contract and action (x, a) for the problem (I), it is also feasible for the problem (II) and we have

$$U(x^*, a^*) = L(x^*, a^*) \geq L(x, a) \geq U(x, a)$$

Hence, (x^*, a^*) is the solution of the principal-agent problem. Observe that we only use the multiple maxima information (the FOA is not valid here).

The principal welfare is 3.016 and the graphic of the agent's utility function at the optimal contract as a function of action is:

Figure 1

In the next section, we present a general Lagrangian for the moral hazard model, even when there exist an infinite number of states of nature.

3. The main result

3.1 The Model

Let A be a non-degenerated compact interval in $\Re$ representing all the possible available actions to the agent. The space of states of nature will be represented by a non-empty set Ω and $\mathcal{A}$ will be a σ-algebra of events on Ω. Let $u: \Re \to \Re$ be the principal's utility function defined over the monetary outcomes which is concave and differentiable, $v: I \times A \to \Re$ is the agent's utility function defined over the monetary payoffs and actions, where I is an open interval in $\Re$.

First, we will assume that $v(x,a) = S(a) + M(a)\overline{v}(x)$, for all $(x,a) \in I \times A$, where $S, M: A \to \Re$ are functions in $C^2(A)$ and $\overline{v}: I \to \Re$ is a concave, increasing and differentiable function, i.e., the agents utility function is a von Newmann-Morgenstern utility function separable in actions and outcomes. This type of agents utility function was used by Grossman and Hart (1983).

The subjective beliefs of the principal and the agent about the state of nature subject to the agent's action are the function $p: \mathcal{A} \times A \to [0,1]$ such that $p(\cdot \mid a)$ is a probability measure on $(\Omega, \mathcal{A})$, for each $a \in A$.[4]

We shall assume that the set of all possible contracts is a convex set[5] C of a linear subspace L of the real vector space of all real measurable functions on $(\Omega, \mathcal{A})$. Let $\pi \in L$ be the principal's monetary outcome which is observed by both the principal and the agent.

For each action $a \in A$ and contract $x \in C$, the principal and the agent expected utilities are

$$U(x,a) = \int_\Omega u(\pi(w) - x(w))dp(w \mid a)$$

[4] We can also assume different subjective beliefs for the principal and for the agent.

[5] We assume that $x(w) \in I, \forall w \in \Omega, \forall x \in C$.

and

$$V(x,a) = \int_\Omega v(x(w),a)dp(w \mid a)$$
$$= S(a) + M(a)\int_\Omega \overline{v}(x(w))dp(w \mid a),$$

respectively.

We assume that $U, V : C \times A \to \Re$ are well defined, i.e., the integrals above exist and $a \in A \mapsto \int_\Omega \overline{v}(x(w))dp(w \mid a)$ is an element of $C^2(A)$ for each $x \in C$.

The principal-agent's problem is

$$\text{(P)} \qquad \begin{aligned} &\max_{\substack{x\in C\\ a\in A}} U(x,a) \\ &\text{s.t.} \quad V(x,a) \geq V(x,\hat{a}), \forall\, \hat{a} \in A \\ &\qquad\quad V(x,a) \geq \widetilde{V} \end{aligned}$$

where $\tilde{V}$ is the minimum level of utility for the agent.

The first constraint of the problem is a consequence of the non-observability of the action by the principal. It is known as the incentive compatibility (IC) constraint. The second constraint is determined by market forces or bargaining power. This is called the individual rationality (IR) constraint.

We separate the above problem into two parts. In the first one we fix an action $a^* \in A$ and solve the problem:

$$\text{(P')} \qquad \begin{aligned} W(a^*) = &\max_{x\in C} U(x,a^*) \\ &\text{s.t.} \quad V(x,a^*) \geq V(x,a), \forall\, a \in A \\ &\qquad\quad V(x,a^*) \geq \widetilde{V} \end{aligned}$$

In the second one we solve:

$$\text{(P")} \qquad \max_{a^*\in A} W(a^*)$$

We are not going to discuss the existence of solution for problems (P), (P') and (P") (for a reference see Grossman and Hart (1983) and Page (1987)). Our aim is to characterize the solution of problem (P') writing its Lagrangian for a fixed $a^* \in A$. By a simple change of variable, we can suppose that $\bar{v}(x) = x$, for all $x \in I$. Without loss of generality, we can assume that $\tilde{V} = 0$.

3.2 Mathematical Framework

Let A be a compact non degenerated interval on $\Re$ of the form $[\underline{a}, \overline{a}]$. We denote by $C^n(A)$ the space of n times continuously differentiable real functions defined on A, $n \geq 0$, with the topology of the uniform convergence until the derivative of order n, i.e., given $f \in C^n(A)$, we define the norm of f by $||f||_n = \max_{0 \leq k \leq n} ||f^{(k)}||_s$, where $||f||_s = \sup_{a \in A} |f(a)|$, $f^{(k)}$ is the derivative of order k of f and $f^{(0)} = f$. We have that $(C^n(A), || \cdot ||_n)$ is a Banach space.

Now fix $a^* \in A$ and define $F = \{f \in C^n(A); f^{(k)}(a^*) = 0, 0 \leq k < n\}$. It should be clear that F is a closed vector subspace of $C^n(A)$. Let Λ be the set of all functions in F such that a^* is a global minimum which is equivalent to the set of all non-negative functions in F. Thus, Λ is a positive cone on F. We also define the following concepts: $\mathcal{M}(A)$ is the set of all finite measures defined on the Borel sets of A and $\mathcal{M}_+(A)$ is the non-negative elements of $\mathcal{M}(A)$; F^* is the topological dual of F with respect to the norm topology and F^*_+ is the set of positive elements with respect to the positive cone Λ (i.e., $\lambda \in F^*_+$ if and only if $\lambda \in F^*$ and $\lambda(f) \geq 0$, $\forall\, f \in \Lambda$); $C_+(A)$ (respectively $C_{++}(A)$) is the set of the non-negative (respectively positive) continuous functions defined on A.

Let

$$\begin{array}{rcl} D\colon C^n(A) & \to & C^{n-1}(A) \\ f & \mapsto & f' \end{array} \quad \text{and} \quad \begin{array}{rcl} I\colon C^n(A) & \to & C^{n+1}(A) \\ f & \mapsto & \int_{a^*}^{\cdot} f(a)\, da \end{array}$$

be the differential and integral operator, respectively. The following lemma characterizes the subspace F and it is easy to prove:

Lemma 3.1.

(i) If $n = 2$, $D^2\colon F \to C(A)$ is a continuous linear isomorphism and $C_+(A) \subset D^2(\Lambda)$.

(ii) If $n = 1$, $D\colon F \to C(A)$ is a continuous linear isomorphism and if $a^* = \underline{a}$ (respectively $a^* = \overline{a}$), then $C_+(A) \subset D(\Lambda)$ (respectively $-C_+(A) \subset D(\Lambda)$).

Lemma 3.1 implies also that $\text{int}(\Lambda) \neq \phi$ since $C_{++}(A)$ is an open set of $C(A)$. We can be more precise: If $n = 2$, every function in F with strict global minimum at a^* and positive second derivative at a^* is an interior point of Λ. If $n = 1$ and $a^* = \underline{a}$ (respectively $a^* = \overline{a}$), every function in F with strict global minimum at a^* and positive (respectively negative) first derivative at a^* is an interior point of Λ. The reciprocal is also true in both cases.

By Lemma 3.1, we have that the topological dual of F, F^*, is isomorphic to the topological dual of $C(A)$, $\mathcal{M}(A)$. Therefore, we can characterize F^* using the following lemma (the proof is given in the appendix:[6])

Lemma 3.2. (The positive dual of F).

(i) If $n = 2$, for each $\lambda \in F^*_+$ there exist a non-negative measure μ on the Borel sets of A, $\alpha_i \geq 0$, $i = 0, 1, 2$ such that for all $f \in F$

$$\lambda(f) = \int_A f d\mu + \alpha_0 f''(a^*) + \alpha_1 f'(\underline{a}) - \alpha_2 f'(\overline{a})$$

[6] Since we don't know a proof of Lemma 3.2 in the Functional Analysis literature, we will give our proof.

(ii) If $n = 1$ and $a^* = \underline{a}$ (respectively $a^* = \overline{a}$), for each $\lambda \in F_+^*$ there exist a non-negative measure μ on the Borel sets of A and $\alpha_0 \geq 0$ such that for all $f \in F$

$$\lambda(f) = \int_A f d\mu \underset{(-)}{+} \alpha_0 f'(a^*)$$

From the proof of Lemma 3.2 (i), if the support of μ is a subset of $[\underline{a}+\epsilon, \overline{a}]$ (respectively $[\underline{a}, \overline{a}-\epsilon]$) for some $\epsilon > 0$, then $\alpha_1 = 0$ (respectively $\alpha_2 = 0$), because, if this is not the case, the right hand side of the definition of λ will not be an element of F_+^*. Moreover, the measure μ is finite on the complement of every open interval around a^*. However, it can be an unbounded measure. For instance, the measure that has density proportional to $\frac{1}{(a-a^*)^2}$ belongs to F_+^*. This measure is a Lèvy measure because it defines a continuous functional on F but it can be unbounded at a^*. We have the following:

Corollary. (The dual of $C^2(A)$ and $C^1(A)$).

(i) If $\lambda \in C^2(A)^*$, then there exist $\lambda_i \in \Re$, $1 \leq i \leq 5$, and a Lèvy measure with sign μ on the Borel sets of A such that

$$\lambda(f) = \int_A (f - f(a^*))d\mu + \lambda_1 f(a^*) + \lambda_2 f'(a^*) + \lambda_3 f''(a^*) + \lambda_4 f'(\underline{a}) + \lambda_5 f'(\overline{a})$$

for all $f \in C^2(A)$.

(ii) If $\lambda \in C^1(A)^*$, then there exist $\lambda_i \in \Re$, $i = 1, 2$, and a Lèvy measure with sign μ on the Borel sets of A such that

$$\lambda(f) = \int (f - f(a^*))d\mu + \lambda_1 f(a^*) + \lambda_2 f'(a^*)$$

for all $f \in C^2(A)$, where $a^* = \underline{a}$ or $a^* = \overline{a}$.

Proof. Since the map

$$C^2(A) \to F \times \Re^2$$
$$f \to (f - f(a^*), f(a^*), f'(a^*))$$

is an isomorphism and $F^* = F_+^* - F_+^*$, the result follows from the previous lemma.

The proof of (ii) is analagous. □

3.3 The Lagrangian Approach

Let L be a vector space, $C \subset L$ be a convex subset and $a^* \in A$. Suppose that $U, V : C \times A \to \Re$ are functions such that

Assumption A1. $U(\cdot, a^*)\colon C \to \Re$ *is a concave Gateaux differentiable functional in the set of internal points of* C, $\dot{C}$,[7] *such that the differential* $\delta U(x, a^*; \cdot)\colon L \to \Re$ *is a linear map, for all* $x \in \dot{C}$.

Assumption A2. $V(\cdot, a)\colon C \to \Re$ *is an affine map*[8] *for all* $a \in A$ *and* $V(x, \cdot)\colon A \to \Re$ *is a function in* $C^2(A)$, *for all* $x \in C$.

Because of the assumptions made in the beginning of this section, A1 and A2 are easily satisfied. Moreover, since $V(\cdot, a)$ is an affine map, for all $a \in A$, $V(\cdot, a)$, $V_a(\cdot, a)$ and $V_{aa}(\cdot, a)$ are Gateaux differentiable in $\dot{C}$, where V_a and V_{aa} are the first and the second derivative of V with respect to a, respectively (the same is valid for U). Moreover, $\delta V(x, a; \cdot) = \varphi(\cdot, a)$, $\delta V_a(x, a; \cdot) = \varphi_a(\cdot, a)$ and $\delta V_{aa}(x, a; \cdot) = \varphi_{aa}(\cdot, a)$, for all $x \in \dot{C}$ and for all $a \in A$, where $\varphi(h, a) = 1/\alpha[V(x + \alpha h, a) - V(x, a)]$, for all $x \in \dot{C}$, for all $h \in L$ and $\alpha > 0$ such that $x + \alpha h \in C$.

The following assumption is necessary:

Assumption A3. *There exists a solution* x^* *of (P') which belongs to* $\dot{C}$.

Before we establish our result we will list two more conditions which will be assumed as hypotheses. Each corresponds to a Slater condition of (P').

Condition C1. *There exists* $x_0 \in C$ *such that* $V(x_0, a^*) - V(x_0, \cdot) \in \mathrm{int}(\Lambda)$, $V(x_0, a^*) > 0$ *and* $\delta V_a(x, a^*; \cdot) \neq 0$ *for some* $x \in \dot{C}$.

Condition C2. *There exists* $x_0 \in C$ *such that* $V(x_0, a^*) - V(x_0, \cdot) \in \mathrm{int}(\Lambda)$ *and* $\delta V(x, a^*; \cdot)$ *and* $\delta V_a(x, a^*; \cdot)$ *are linearly independent functionals for some* $x \in \dot{C}$.[9]

By the remark following Lemma 3.1, the first part of C1 or C2 is equivalent to find x_0 such that a^* is the unique minimum point of $V(x_0, a^*) - V(x_0, \cdot)$ and $V_{aa}(x_0, a^*) < 0$. The following theorems are the characterization of the optimal solution of the moral hazard problem. They are an easy consequence of the Kuhn and Tucker Theorem and Lemma 3.2.

Theorem 3.0. *Assume that A1-A3 and C1 or C2 hold. Then there exist* $\lambda_i \in \Re$, $i = 1, 2$, *and* $\mu \in \mathcal{M}_+(A)$ *such that* x^* *maximizes:*

[7] $\dot{C} = \{x \in C; \forall\, h \in L, \exists\, \alpha > 0$ such that $x + \alpha h \in C\}$. The symbol δ represents the Gateaux differential with respect to x.

[8] In the sense that $V(\alpha x_1 + (1-\alpha)x_2, a) = \alpha V(x_1, a) + (1-\alpha)V(x_2, a)$, $\forall\, x_i \in C$, $i = 1, 2$, $\forall\, \alpha \in [0, 1]$, $\forall\, a \in A$.

[9] And consequently, it is valid for all $x \in \dot{C}$.

(i) if $a^* \in (\underline{a}, \overline{a})$,

$$L(x,a^*) = U(x,a^*) + \lambda_1 V(x,a^*) + \lambda_2 V_a(x,a^*) - \int_A V_{aa}(x,a)d\mu(a)$$

and $V_a(x^*,a^*) = 0$, $\int_A V_{aa}(x^*,a)d\mu(a) = 0$, $\lambda_1 V(x^*,a^*) = 0$.

(ii) if $a^* = \underline{a}$ *(respectively* $a^* = \overline{a}$*),*

$$L(x,a^*) = U(x,a^*) + \lambda_1 V(x,a^*) \underset{(+)}{-} \int_A V_a(x,a)d\mu(a)$$

and $V_a(x^*,a^*) \leq 0$ *(respectively* $V_a(x^*,a^*) \geq 0$*),* $\int_A V_a(x^*,a)d\mu(a) = 0$, $\lambda_1 V(x^*,a^*) = 0$[10] *and C1 implies* $\lambda_1 \geq 0$.

The proof is given in the appendix.

Theorem 3.1. Assume that A1-A3 and C1 or C2 hold. Then there exist $\lambda_i \in \Re$, $i = 1,2$, *and* μ *a positive measure on the Borel sets of* A *such that* x^* *maximizes:*

(i) if $a^* \in (\underline{a}, \overline{a})$,

$$\begin{aligned} L(x,a^*) =& U(x,a^*) + \lambda_1 V(x,a^*) + \lambda_2 V_a(x,a^*) \\ & - \alpha_0 V_{aa}(x,a^*) - \alpha_1 V_a(x,\underline{a}) + \alpha_2 V_a(x,\overline{a}) \\ & + \int_A (V(x,a^*) - V(x,a))d\mu(a) \end{aligned}$$

and $V_a(x^*,a^*) = 0$, $\alpha_0 V_{aa}(x^*,a^*) = 0$, $\alpha_1 V_a(x^*,\underline{a}) = 0$, $\alpha_2 V_a(x^*,\overline{a}) = 0$, $\int_A (V(x^*,a^*) - V(x^*,a))d\mu(a) = 0$, $\lambda_1 V(x^*,a^*) = 0$.

(ii) if $a^* = \underline{a}$ *(respectively* $a^* = \overline{a}$*),*

$$\begin{aligned} L(x,a^*) =& U(x,a^*) + \lambda_1 V(x,a^*) \underset{(+)}{-} \alpha_0 V_a(x,a^*) \\ & + \int_A (V(x,a^*) - V(x,a))d\mu(a) \end{aligned}$$

and $V_a(x^*,a^*) \leq 0$ *(respectively* $V_a(x^*,a^*) \geq 0$*),* $\alpha_0 V_a(x^*,a^*) = 0$, $\int_A (V(x^*,a^*) - V(x^*,a))d\mu(a) = 0$, $\lambda_1 V(x^*,a^*) = 0$[11]

The proof is given in the appendix.

Theorem 3.0 can be easily generalized to the multidimensional parameter case. However, it is not possible to extend Theorem 3.1 to the multidimensional case

[10] In this case, the conditions on $\delta V_a(x,a^*;\cdot)$ in conditions C1 and C2 are not necessary.

[11] See the last footnote.

using our method since we are not able to extend Lemma 3.2. (Even though we think that, in the multidimensional case, a similar characterization is possible.)

Corollary. Under the assumptions of Theorem 3.1, if x^* is such that $V(x^*, \cdot)$ has a finite number of maximal points: $a_1, ..., a_K$, the Lagrangian becomes
(i) if $a^* \in (\underline{a}, \overline{a})$

$$\begin{aligned} L(x, a^*) = U(x, a^*) &+ \lambda_1 V(x, a^*) + \lambda_2 V_a(x, a^*) \\ &- \alpha_0 V_{aa}(x, a^*) - \alpha_1 V_a(x, \underline{a}) + \alpha_2 V_a(x, \overline{a}) \\ &+ \sum_{k=1}^{K} \mu_k (V(x, a^*) - V(x, a_k)) \end{aligned}$$

(ii) if $a^* = \underline{a}$ (respectively $a^* = \overline{a}$)

$$\begin{aligned} L(x, a^*) = U(x, a^*) + \lambda_1 V(x, a^*) \underset{(+)}{-} \alpha_0 V_a(x, a^*) \\ + \sum_{k=1}^{K} \mu_k (V(x, a^*) - V(x, a_k)) \end{aligned}$$

Proof. We only have to observe that $\int_A (V(x^*, a^*) - V(x^*, a))\, d\mu(a) = 0$ implies that the measure μ is $\sum_{k=1}^{K} \mu_k \delta_{a_k}$, where δ_{a_k} is the Dirac measure concentrated at a_k. □

Remark 1. If L is the space of all real measurable functions on $(\Omega, \mathcal{A})$ and U is an expected utility function with an increasing kernel function u, then λ_1 is always non-negative. Moreover, from the remark following Lemma 3.2, if $\underline{a} \notin \{a_1, ..., a_K\}$ (respectively $\overline{a} \notin \{a_1, ..., a_K\}$), $\alpha_1 = 0$ (respectively $\alpha_2 = 0$).

Remark 2. Theorem 3.1 shows that, under the Slater condition (C1 or C2), the problem (P) has a Lagrangian.

Mirrlees and Roberts (1980) have the following result:
For almost all[12] C^∞ function V the number of distinct maxima in a variable is less than or equal to $n + 2$ for all $x \in \dot{C}$, and the dimension of the surface $\{(x, a) \in \dot{C} \times \dot{A};\ V_a(x, a) = 0\}$ corresponding to x with r distinct maxima is less than or equal to $n + 1 - r$, where n is the dimension of L (in the case of finite dimensional vector space).

Therefore, the remark above can be used for almost all functions V. However, our example 1 shows that this is not always the case.

[12] The set of such functions contains a countable intersection of dense sets in the Whitney or strong topology.

Now we are going to characterize the problem (P) in the variables x and a together. We give a local characterization of the optimal for the problem (P) (since this is not a concave problem in general).

Assume that there exists a solution $(x^*, a^*) \in \dot{C} \times A$ of the problem (P) and $U(\cdot, \cdot)$ is Gateaux differentiable in (x, a) with linear differential. Label *A1'* the assumption A1 without the concavity hypothesis. Suppose that V is a generic function (not necessarily affine in x) such that

Assumption A2'.[13] *δV_a is a continuous function along each direction of $L \times \Re$ and $V(x, \cdot)\colon A \to \Re$ is a function in $C^2(A)$, for all $x \in C$.*

The correspondent Slater condition is

Condition C3. There exists $h^0 \in L \times \Re$ such that $V(x^*, a^*) + \delta_{h^0} V(x^*, a^*) > 0$ and $V(x^*, a^*) - V(x^*, \cdot) + \delta_{h^0}(V(x^*, a^*) - V(x^*, \cdot)) \in int(\Lambda)$. If $a^* \in (\underline{a}, \overline{a})$, $V_{aa}(x^*, a^*) < 0$ and $\delta_{h^0} V_a(x^*, a^*) = 0$. If $a^* = \underline{a}$ (respectively $a^* = \overline{a}$), $V_a(x^*, a^*) + \delta_{h^0} V_a(x^*, a^*) \underset{(>)}{<} 0$.

Under these assumptions we have

Theorem 3.2. Assume that A1', A2' and C3 hold. Then there exists a Lagrangian function L (as in Theorem 3.1) such that

$$L_a(x^*, a^*) \begin{cases} \leq 0, & a^* = \underline{a} \\ = 0, & a^* \in (\underline{a}, \bar{a}) \\ \geq 0, & a^* = \bar{a} \end{cases}$$

and[14] *$L_x(x^*, a^*) = 0$, with the Kuhn and Tucker's conditions.*

The proof is given in the appendix.

Remark. Observe that if $a^* \in (\underline{a}, \overline{a})$ and $V_{aa}(x^*, a^*) < 0$, then the complementary slackness Kuhn and Tucker condition $\alpha_0 V_{aa}(x^*, a^*) = 0$ of Theorem 3.1 implies that $\alpha_0 = 0$ for Theorem 3.2.

The remarks following Theorem 3.1 can be also applied in the case of Theorem 3.2.

3.4 Related Literature

In the existing literature, we have two conditions that guarantee the so called FOA: the monotone likelihood ratio property (MLRP) and the convexity of the

[13] Now δ represents the Gateaux differential in the variables x and a together.

[14] L_x represents the Gateaux differential with respect to the variable x.

distribution function property (CDFP). MRLP has the fairly natural interpretation that a (first best) costly action increases the probability of high outcome than a less costly one. CDFP is not as easy to accept, most of the distributions commonly occurring in statistics do not have this property. For instance, suppose that output is subject to a simple additive disturbance $\tilde{\epsilon}$ with distribution F, and effort is measured in output terms. The realized output is given by $x = a + \epsilon$, and $\tilde{\epsilon}$ has distribution function $F(x - a)$ which is convex in effort if and only if ϵ has an increasing density.

Rogerson (1985) shows that, in the presence of MRLP and CDFP, the agent's utility function is a concave function of the action at the optimal contract and therefore the first order condition of the IC constraint is sufficient to characterize IC. From the complementary slackness Kuhn and Tucker conditions of Theorem 3.2, the FOA is also obtained in our framework. Mirrlees (1975) and (1986) has an example in which the FOA is not valid which we are going to reexamine. We present two more examples in which the FOA can not be applied.

The multipliers of the Theorem 3.1 and Theorem 3.2 have a quite simple meaning: λ_1 represents the IR constraint multiplier; λ_2 represents the multiplier of the first order condition of the IC constraint; α_0 is associated with the second order condition of the IC constraint in the case of Theorem 3.1 (i) and with the first order condition in the case of Theorem 3.1 (ii); α_i $(i = 1, 2)$ is the multiplier associated to the effect of the first derivative of the agent's expected utility function with respect to the action in the boundary of A, $\underline{a}$ and $\overline{a}$, whether $\underline{a}$ and $\overline{a}$ belong to the set of the binding actions of the IC constraint. Finally, the measure μ represents the multiplier which captures the information of multiple maxima because, by the complementary slackness Kuhn and Tucker conditions, μ has support on the set of the binding actions of the IC constraint. Therefore, besides the usual terms in the FOA Lagrangian, in the general case we have to consider the second order effect, the binding actions of the IC constraint and the behavior of the first derivative of $V(x^*, \cdot)$ on the boundary of A.

Mirrlees (1975) and (1986) also has a Lagrangian approach with the following differences from ours: he deals with a finite dimensional state space and multiple action variables and we treat the general state space case with an one-dimensional action variable (even though we have a Lagrangian form for multiple dimensional case. See the theorem in the appendix). In the Mirrlees' Lagrangian the multiplier of the second order condition is absent because he assumes that all critical action are non-degenerated[15] (i.e., V_{aa} is non-degenerated for all critical points, i.e., $V_a = 0$) and the multipliers of the first derivative at the boundary is also absent because Mirrlees assumes that the action set is open.

However, the main difference is that Mirrlees' Lagrangian has multipliers for the first order condition of the agent's utility function at each binding action. This implies that the number of variables is greater than the number of equations

[15] However, we assume that the optimal action, when it is in the interior of action set, is non-degenerated (see condition C3).

for the system generated by the first order condition of the Lagrangian and the complementary slackness Kuhn and Tucker conditions. In our case the number of the variables is the same of the equations.

Observe that the first order conditions of the binding actions of the IC constraint are actually redundant as constraints because if the action is binding in the IC constraint, then it automatically satisfies the first order condition. Mirrlees used these conditions as constraints (additionally with the equations of the binding IC constraints) for regularity reasons of the Kuhn and Tucker Theorem (see the introduction) and this is why the match of equations and variables does not occur in his case. In our case we do not need to use these conditions as constraints for the Lagrangian, but we use them as additional equations. The result is that we obtain the match of the number of equations and variables.[16]

And finally, he does not cover our Theorem 3.1 (ii) and Theorem 3.2 when the optimal action is in the boundary of the action interval. Mirrlees (1975) and (1986) also observe that when the state space is infinite dimensional, it would be possible to have the agent indifferent with respect to a continuous set of action at the optimum. Our example 1 below illustrates this possibility: the agent is indifferent to all the actions in the feasible domain.

In what follows, we give four examples where the FOA is not valid. The second example is the example of section 2. In the third example, we use a Mathematica program to compute optimal solutions using the first order conditions of the Lagrangian and the complementary slackness Kuhn and Tucker conditions of Theorem 3.2 (ii). The Mirrlees' counter example is analyzed in the example 4.

3.5 Examples

Example 1.

Let

- $A = [0,1]$; $I = \Re$; $\Omega = \{0,1,2,...\}$; $\mathcal{A} = \mathcal{P}(\Omega)$ (set of the parts of Ω);
- $u(x) = -\frac{x^2}{2}$, $x \in \Re$; $v(x) = x$, $x \in I$;
- $S(a) = a^2 - 2a$; $M = 1$, $a \in A$; $p(n|a) = \frac{a^n e^{-a}}{n!}$, $n \in \Omega$, $a \in A$ (Poisson distribution with parameter a); $\pi\colon \Omega \to \Re$ such that $\pi(n) = n$, for all $n \in \Omega$;
- $L = C = \{x\colon \Omega \to \Re;\ \sum_{n \geq 0} \frac{x_n^2}{n!} < \infty\}$.

Thus,

$$U(x,a) = -\sum_{n=0}^{\infty} \frac{(n - x_n)^2}{2} p(n|a)$$

[16] Even in the case when we there exist the multipliers for the derivative of the agent's expected utility function with respect to the action in the boundary actions, because these points are fixed.

and

$$V(x,a) = \sum_{n=0}^{\infty} x_n p(n|a) \, + \, a^2 - 2a$$

where $x\colon \Omega \to \Re$ is an element of C, $a \in A$ and $x_n = x(n)$.

Fix $a^* \in (0,1)$. By Theorem 3.1, the associated problem (P') has the following Lagrangian:

$$L(x,a^*) = U(x,a^*) + \lambda_1 \, V(x,a^*) + \lambda_2 \, V_a(x,a^*) - \alpha_0 V_{aa}(x,a^*)$$

where $\lambda_1 \geq 0$, $\lambda_2 \in \Re$ and $\alpha_0 \geq 0$. (Assume, for a while, that the Lagrange multipliers α_1, α_2 and the measure μ of Theorem 3.1 are null.)

If $x^* \in C$ is a solution of the problem then $(x^*, \lambda_1, \lambda_2, \alpha_0)$ is a saddle point of the Lagrangian, i.e.,

(I') $$L_{x_n}(x^*,a^*) = 0, \ \forall n = 0,1,2,..$$

(II') $$V_a(x^*,a^*) = 0$$

(III') $$\lambda_1 V(x^*,a^*) = 0$$

(IV') $$\alpha_0 V_{aa}(x^*,a^*) = 0$$

where L_{x_n} represents the derivative of the Lagrangian with respect to the variable x_n.

Therefore, we can rewrite (I') to (IV') as

(I) $$x_n^* \, - \, n = \lambda_1 \, + \, \lambda_2 \frac{n - a^*}{a^*} \, - \, \alpha_0 \frac{n^2 - (1+2a^*)n + {a^*}^2}{{a^*}^2},$$

for all $n \in \Omega$

(II) $$\sum_{n \geq 0} (x_{n+1}^* - x_n^*) p(n|a^*) + 2a^* - 2 = 0$$

(III) $$\lambda_1 (\sum_{n \geq 0} x_n^* p(n|a^*) \, + \, {a^*}^2 - 2a^*) = 0, \ \lambda_1 \geq 0$$

(IV) $$\alpha_0 (\sum_{n \geq 0} (x_{n+2}^* - 2x_{n+1}^* + x_n^*) p(n|a^*) \, + 2) = 0, \ \alpha_0 \geq 0.$$

Making the calculations, we have the following results for the Lagrange multipliers:

$$\lambda_1 = a^* - {a^*}^2, \ \lambda_2 = a^*(1 - 2a^*) = a^* - 2{a^*}^2 \text{ and } \alpha_0 = {a^*}^2.$$

Hence, the optimal solution is

$$x_n^* = 3n - n^2, \ \ n = 0, 1, 2, ...$$

Observe that $\lambda_1, \lambda_2, \alpha_0$ and x^* satisfy the optimality conditions. Therefore, x^* is the solution of our problem. Observe also that x^* is independent of a^*. It is straightforward to conclude that the agent is indifferent to all actions at the optimal contract, i.e., $V(x^*, \cdot) = 0$. Since $\alpha_0 = {a^*}^2 > 0$ we can say that the second order approach is valid here.

Moreover, MLRP is satisfied but CDFP is not. It is easy to see that the solution x^{**} obtained by the FOA is

$$x_n^{**} = {a^*}^2 + 2(1 - a^*)n$$

and

$$V(x^{**}, a) = (a - a^*)^2$$

which does not satisfy the IC constraint: Indeed, a^* minimizes $V(x^{**}, \cdot)$.

Example 2.

Let $\Omega = \{1, 2, 3\}$; $A = [1, 5]$; $M(a) = a(a^3 + a^2 - a + 1)$; $S(a) = -(\frac{a^4}{2} + 10a^2 + 16)$; $p(a) = \frac{1}{a^3+a^2-a+1}(a^3, a^2 - a, 1)$; $u(x) = x$; $v(x) = \sqrt{2x}$.

Suppose that the principal would like to induce effort 2, i.e., he would like to solve problem (P') for $a^* = 2$. Proceeding as in the example 1 above we have the following results:

$$\begin{cases} x^* = (\frac{1}{32}, \frac{9}{2}, 288) \\ \lambda_1 = 1.189 \\ \lambda_2 = -0.0535 \\ \mu = 0.0714 \ \delta_4 \\ \alpha_i = 0, \ i = 0, 1, 2 \end{cases}$$

Observe that the FOA is not valid because μ is non-null (it is a multiple of the Dirac measure at the 4). The figure below is the agent's expected utility at the optimal contract as a function of the action:

Figure 2

Observe that at the optimal contract the agent is indifferent between two efforts (2 and 4), however from the principal's point of view the better effort is the low one.

Example 3. (Example of the section 2)

To solve this example we constructed a simple algorithm in Mathematica to compute critical points of the Lagrangian with the complementary slackness Kuhn and Tucker conditions of Theorem 3.2.[17] Taking the Lagrangian of Theorem 3.2 when $a^* = \overline{a}$ (and consequently the Lagrangian of Theorem 3.1 (ii)), we have the following results:

Optimal action: $\{ \; a^* = 0.9$

Optimal contract: $\begin{cases} x_1^* = 0 \\ x_2^* = 1.23457 \end{cases}$

Lagrange multipliers of Theorem 3.2 (ii): $\begin{cases} \lambda_1 = 1.62 \\ \alpha_0 = 0 \\ \mu = 0.602222 \; \delta_0 \end{cases}$

Example 4. (Mirrlees' counter example).

In Mirrlees (1975) and (1986), there is an important example showing that the FOA is not valid in general. Under the light of what we said in this paper, how can we solve such example?

The example is: Let

$$U(x,a) = -(a-1)^2 - (x-2)^2$$

[17] Using the Kuhn and Tucker's conditions and the first order conditions of the Lagrangian and separating the items (i) and (ii) of Theorem 3.2, we have 20 possible nonlinear systems. We calculate the critical points of these systems (when they exist) and select the one that maximizes the principal's expected utility function.

$$V(x,a) = xe^{-(a+1)^2} + e^{-(a-1)^2}$$

where $x \in \Re$ and $a \in \Re$.[18] Suppose that there is no uncertainty and the reservation utility is $\tilde{V} = -\infty$ (i.e., there is no IR constraint).

In Mirrlees (1975) and (1986), the solution is shown to be $x^* = 1$ and $a^* = 0.957$ and the first order approach can not be applied. However, $V(x, \cdot)$ has just one (regular) maximum, for all $x \neq 1$. Therefore, by our previous analysis, if $x^* \neq 1$ were the solution, then the FOA would be valid. As long as this is not true, the unique viable solution is $x^* = 1$.

Observe that $V(1, \cdot)$ has two maximal points: -0.957 and 0.957 and it is also easy to find the Lagrange multipliers such that $(1, 0.957)$ is the unique critical point of

$$L(x,a) = U(x,a) + \lambda_1 V_a(x,a) + \mu_1(V(x,a) - V(x,a_1))$$

with $V_a(x,a) = V_a(x,a_1) = 0$ and $V(x,a) = V(x,a_1)$ (where a_1 can be shown to be -0.957).

The main feature of this example is its robustness: one can make a small perturbation of the principal's and the agent's utility function and obtain the same type of solution. This means that, even though the FOA is very frequent in the literature, one can not neglet situations where FOA is not valid.

Some results in the literature, valid under FOA, can be easily generalized under our framework. We give three applications:

(1) Holmstrom (1979) showed that $\lambda_2 \geq 0$ under FOA. However, our examples 1 and 2 show that λ_2 can be negative in some cases.

(2) Using Theorem 3.2, we can easily prove that if the MLRP holds and the optimal action is the supreme of the binding action set of the IC constraint, then the optimal contract should be monotone.

(3) Let s be a random variable representing a signal. We can show that the return π is sufficient for (π, s) with respect to $a \in A$ if and only if the distribution of π and s given a $f(\pi, s|a)$ is multiplicatively separable in s and a, i.e.,

$$f(\pi, s|a) = g(\pi|a)h(\pi, s).$$

We say that s is informative about $a \in A$ whenever x is not sufficient for (π, s) with respect to $a \in A$. Holmstrom (1979) and Shavell (1979) showed that there exists a new contract using s that strictly Pareto dominates the optimal contract without using s if and only if s is informative about $a \in A$. We can easily extend this result under our assumptions (see the proof in the references and use our Lagrangian approach to do the extension).

4. Conclusions

[18] Indeed, we can restrict our attention to a compact interval in $\Re$ without loss of generality.

In this paper we studied a general characterization of optimal solutions for the moral hazard problem when the set of parameters is a compact interval in the real line. We obtained a Lagrangian for the infinite dimensional state space. In the finite state space case, our approach matches the numbers of equations and variables of the first order conditions of the Lagrangian which does not happen in the case of Mirrlees (1975) and (1986).

Acknowledgements.

A first version of this paper was presented at the XVII Meeting of the Brazilian Econometric Society and it was part of the H. Moreira's Ph.D. thesis. Financial support from CNPq of Brazil is acknowledged by the authors.

Appendix

Since Λ strictly contains a set that is isomorphic to $C_+(A)$ under the isomorphism described at Lemma 3.1, F^*_+ is equivalent to a proper subset of $\mathcal{M}_+(A)$. We are going to describe such subset in a more specific way in the following lemmas:

Lemma 1.

(i) If $n = 2$, for each $\lambda \in F^*$ there exists $\mu \in \mathcal{M}(A)$ such that $\lambda(f) = \int_A f'' d\mu$, for all $f \in F$, and conversely. Moreover, if $\lambda \in F^*_+$ then $\mu \in \mathcal{M}_+(A)$.

(ii) If $n = 1$ and $a^* = \underline{a}$ (respectively $a^* = \overline{a}$), for each $\lambda \in F^*$ there exists $\mu \in \mathcal{M}(A)$ such that $\lambda(f) = \int_A f' d\mu$, for all $f \in F$, and conversely. Moreover, if $\lambda \in F^*_+$ then $\mu \in \mathcal{M}_+(A)$ (respectively $-\mu \in \mathcal{M}_+(A)$).

Proof. It is an immediate consequence of Lemma 3.1 and the fact that the topological dual of $C(A)$ is $\mathcal{M}(A)$. □

Lemma 2.

(i) If $n = 2$, for each $\lambda \in F^*_+$ there exists a non-decreasing function $\xi\colon A \to \Re_+$ such that $\xi\big|_{[a^*,\overline{a}]}$ is concave, $\xi\big|_{[\underline{a},a^*]}$ is convex and $\lambda(f) = \int_A f''\xi'\, da + f''(a^*)[\xi(a^*) - \xi(a^*_-)]$,[19] for all $f \in F$, where $\xi'\colon A \to \Re_+$ is an integrable function on A equal to the derivative of ξ almost everywhere.

(ii) If $n = 1$ and $a^* = \underline{a}$ (respectively $a^* = \overline{a}$), for each $\lambda \in F^*_+$ there is a non-decreasing concave (respectively convex) function $\xi\colon A \to \Re_+$ such that $\lambda(f) = \int_A f'\xi' da + f'(\underline{a})\xi(\underline{a})$ (respectively, $\lambda(f) = -\int_A f'\xi' da - f'(\overline{a})[\xi(\overline{a}) - \xi_-(\overline{a})]$), for all $f \in F$, where $\xi'\colon A \to \Re_+$ is an integrable function on A equal to the derivative of ξ almost everywhere.

Proof. (i) Given $\lambda \in F^*_+$, by Lemma 1 there exists $\mu \in \mathcal{M}_+(A)$ such that $\lambda(f) = \int_A f'' d\mu$, $\forall\, f \in F$. Since μ is a finite non-negative measure, the set $D = \{a \in$

[19] $\xi(a_-) = \lim_{t \to a^-} \xi(t)$.

$A; \mu(\{a\}) \neq 0\}$ is an enumerable set. Let $b \in A$ and $0 < \eta < \epsilon$ such that $a^* \leq b - \epsilon$ and $b, b-\epsilon,\ b+\epsilon \notin D$. Let $\varphi\colon \Re \to [-1,1]$ a C^∞ function such that

$$\varphi(a) = \begin{cases} 0 & \text{if} \quad a \leq b-\epsilon \text{ or } a \geq b+\epsilon \\ 1 & \text{if} \quad b-\epsilon+\delta \leq a \leq b-\delta \\ -1 & \text{if} \quad b+\delta \leq a \leq b+\epsilon-\delta \end{cases}$$

and $a \to \varphi(a-b)$ is an odd function, where $\delta = (\epsilon - \eta)/2$. Define $f = I^2(\varphi)$. We have that $f \in \Lambda$.

Then $0 \leq \lambda(f) = \int_{b-\epsilon}^{b+\epsilon} \varphi(a) d\mu(a)$. Making $\delta \to 0$, we have that $\mu([b-\epsilon, b]) \geq \mu([b, b+\epsilon])$. Analogously, if $b \in A$ is such that $b+\epsilon \leq a^*$ and $b, b-\epsilon, b+\epsilon \notin D$ then $\mu([b, b+\epsilon]) \geq \mu([b-\epsilon, b])$.

Define $\xi\colon A \to \Re_+$ by $\xi(a) = \mu([\underline{a}, a])$, $\forall\, a \in A$. Then ξ is a non-decreasing right continuous function. Given $x, y \in [a^*, \overline{a}] - D$ such that $x < y$ and $z = \frac{x+y}{2} \notin D$,

$$\xi(z) - \xi(x) = \mu([x,z]) \geq \mu([z,y]) = \xi(y) - \xi(z)$$
$$\Rightarrow \xi\left(\frac{x+y}{2}\right) \geq \frac{1}{2}\xi(x) + \frac{1}{2}\xi(y).$$

Moreover, if $b \in (x, y)$, the right continuity of ξ implies

$$\xi\left(\frac{x+b}{2}\right) = \lim_{y \downarrow b} \xi\left(\frac{x+y}{2}\right)$$
$$\geq \lim_{y \downarrow b} \frac{\xi(x)+\xi(y)}{2} = \frac{\xi(x)+\xi(b)}{2}.$$

Therefore, taking the limit in x, we have $\xi(b_-) \geq \frac{\xi(b_-)+\xi(b)}{2}$. Since ξ is non-decreasing, $\xi(b_-) = \xi(b)$. Thus ξ is continuous at b and $D \cap [a^*, \overline{a}] \subset \{a^*\}$. This implies that ξ is a concave function on $[a^*, \overline{a}]$. Analogously, ξ is a convex function on $[\underline{a}, a^*]$. Therefore, ξ is absolutely continuous on every compact subset of $(\underline{a}, a^*) \cup (a^*, \overline{a})$. By Rudin (1974), there exists $\xi'\colon A \to \Re_+$ an integrable function such that $\xi(y) - \xi(x) = \int_x^y \xi'(a)\, da$, for any $x, y \in (\underline{a}, a^*)$ or $x, y \in (a^*, \overline{a})$, and this completes the proof of (i).

The proof of (ii) is analogous. $\square$

Proof of Lemma 3.2. (i) Let $\xi'\colon A \to \Re_+$ be as in Lemma 2 (i). We can assume that ξ' is right continuous, non-decreasing on $[\underline{a}, a^*)$ and non-increasing on $(a^*, \overline{a}]$. Then ξ'' is equal, in the distributional sense on $\Re$, to a measure defined on the Borel sets of A non-negative on $[\underline{a}, a^*)$ and non-positive on $(a^*, \overline{a}]$ (see Rudin (1974), Theorem 8.6). Taking φ' in the place of φ, $b \in A - \{a^*\}$ in the proof of Lemma 2 (i) and integrating by parts in the distributional sense (see Rudin (1991) for the definitions):

$$\int_A \varphi \xi'\, da = -\int_A I(\varphi) \xi''\, da$$

we conclude by a similar argument of the proof of Lemma 2 that $\xi'\big|_{[\underline{a},a^*)}$ and $\xi'\big|_{(a^*,\overline{a}]}$ are convex functions. Therefore, ξ'' can be identified with a right continuous function non-decreasing and non-negative on $[\underline{a}, a^*)$ and non-decreasing and non-positive on $(a^*, \overline{a}]$, almost everywhere. Analogously, in the distributional sense the derivative of ξ'' can be identified with a non-negative measure μ defined on the Borel sets of A (use Theorem 8.6 of Rudin (1974) and the fact that ξ'' is non-decreasing on $[\underline{a}, a^*)$ and on $(a^*, \overline{a}]$, almost everywhere).

Since ξ is convex on $[\underline{a}, a^*)$ and concave on $(a^*, \overline{a}]$, we have the following inequalities

$$\xi(a) - \xi(a^*_-) \leq \xi'(a)(a - a^*) \leq 0, \text{ when } a < a^*,$$

$$\xi(a) - \xi(a^*) \geq \xi'(a)(a - a^*) \geq 0, \text{ when } a > a^*,$$

respectively. Thus, $\lim_{a \to a^*} \xi'(a) f'(a) = \lim_{a \to a^*} (a - a^*)\xi'(a)\frac{f'(a)}{a - a^*} = f''(a^*) \lim_{a \to a^*} (a - a^*)\xi'(a) = 0$, for all $f \in F$. Integrating by parts we get[20]

$$\begin{aligned}\int_A f''\xi'\, da &= f'\xi'\big|_{\underline{a}}^{\overline{a}} - \int_A f'\xi''\, da \\ &= f'(\overline{a})\xi'(\overline{a}) - f'(\underline{a})\xi'(\underline{a}) + \int_A f d\mu, \ \forall f \in F.\end{aligned}$$

Using Lemma 2 (i) we conclude the proof.

The proof of (ii) is analogous. $\square$

Proof of Theorem 3.0. We will only prove (i) because the proof of (ii) is analogous. We have to consider two cases:

(a) Assume C1. Define $D = \{x \in C; V_a(x, a^*) = 0\}$. By A2, D is a convex set. Then the following problem is equivalent to (P'):

$$\begin{aligned}&\max_{x \in D} U(x, a^*) \\ &\text{s.t.} \quad V(x, a^*) - V(x, \cdot\,) \in \Lambda \\ &\qquad\;\; V(x, a^*) \geq 0.\end{aligned}$$

By the Slater condition C1, Kuhn and Tucker Theorem (see Luenberger (1969), Theorem 1, sec 8.3) and Lemma 1, there exist $\lambda_1 \geq 0$ and $\mu \in \mathcal{M}_+(A)$ such that x^* maximizes

$$L_1(x, a^*) = U(x, a^*) + \lambda_1 V(x, a^*) - \int_A V_{aa}(x, a) d\mu(a)$$

in D such that $\lambda_1 V(x^*, a^*) = 0$ and $\int_A V_{aa}(x^*, a) d\mu(a) = 0$.

[20] The term $-f\xi''\big|_{\underline{a}}^{\overline{a}}$ is considered as a part of $\int_A f d\mu$.

To complete the proof we only have to guarantee the existence of Lagrange multipliers for the problem

$$\max_{x\in C} L_1(x,a^*)$$
$$\text{s.t.}\quad V_a(x,a^*)=0.$$

Under assumptions A1-A3 and C1, this is the case, i.e., there exists $\lambda_2 \in \Re$ such that x^* is a critical point of $L(x,a^*)$. Since $L(\,\cdot\,,a^*)$ is a concave functional, x^* is a global maximum of $L(x,a^*)$.

(b) Assume C2. If $V(x^*,a^*)=0$, the proof is analogous to the proof of (a). Suppose that $V(x^*,a^*)>0$. Then $x^*\in\dot{D}$ (because $V(\,\cdot\,,a^*)$ is an affine map) which implies that (P') is equivalent to the problem:

$$\max_{x\in D} U(x,a^*)$$
$$\text{s.t.}\quad V(x,a^*)-V(x,\,\cdot\,)\in\Lambda$$

The rest of the proof is similar to (a) with $\lambda_1=0$. □

Proof of Theorem 3.1. Theorem 3.1 is a corollary of Theorem 3.0 and Lemma 3.2. □

Proof of Theorem 3.2. First suppose that $a^*\in(\underline{a},\overline{a})$. Define $f,g_1,g_2\colon L\times\Re\to\Re$ such that

$$f(h)=\delta_h U(x^*,a^*)$$
$$g_1(h)=V(x^*,a^*)\,+\,\delta_h V(x^*,a^*)$$
$$g_2(h)=\delta_h V_a(x^*,a^*)$$

and $g_3\colon L\times\Re\to C^2(A)$ such that

$$g_3(h)=V(x^*,a^*)\,-\,V(x^*,\,\cdot\,)\,+\,\delta_h(V(x^*,a^*)\,-\,V(x^*,\,\cdot\,))$$

It is easy to see that f,g_1,g_2,g_3 are affine functions.

It suffices to prove that the problem

$$\max_{h\in L\times\Re} f(h)$$
$$\text{s.t.}\quad g_1(h)\geq 0$$
$$g_2(h)=0$$
$$g_3(h)\in\Lambda$$

has a Lagrangian and $0\in L\times\Re$ is its solution. Condition C3 guarantees the Slater condition for this problem. Therefore, we only have to show that 0 is its solution. Take $h\in L\times\Re$ satisfying the constraints of the problem above. Define for each $\lambda\in[0,1]$, $h_\lambda=\lambda h^0+(1-\lambda)h$. Thus, $g_1(h_\lambda)>0$, $g_2(h_\lambda)=0$ and $g_3(h_\lambda)\in\mathrm{int}(\Lambda)$, for all $\lambda \in (0,1]$, by C3 and the fact that g_1, g_2, g_3 are affine functionals. Fix $\lambda \in (0,1]$. For $\epsilon > 0$ (sufficiently small), define $\varphi\colon (-\epsilon,\epsilon) \times (\underline{a}, \overline{a}) \to \Re$ such that

$$\varphi(t,a) = V_a(x^* + th^1_\lambda, a + th^2_\lambda).$$

We have that φ is C^1, $\varphi(0,a^*) = 0$ and $\varphi_a(0,a^*) < 0$ (by C3). By the Implicit Function Theorem and C3 we can put a as a C^1 function of t, $a(t)$, in a neighborhood of $(0,a^*)$ such that $a(0) = a^*$, $\varphi(t,a(t)) = 0$ is satisfied for all t near 0 and $a'(0) = 0$ (since $g_2(h_\lambda) = 0$). The Taylor formula implies that

$$\begin{aligned} V(x^* + th^1_\lambda, a(t) + th^2_\lambda) &= V(x^*, a^*) + t\delta_{h_\lambda} V(x^*, a^*) + r(t) \\ &= tg_1(h_\lambda) + (1-t)g_1(0) + r(t), \end{aligned}$$

where $\lim\limits_{t\to 0} \frac{r(t)}{t} = 0$.

Then

$$\frac{V(x^* + th^1_\lambda, a(t) + th^2_\lambda)}{t} = g_1(h_\lambda) + (\frac{1}{t} - 1)g_1(0) + \frac{r(t)}{t}.$$

Since $g_1(h_\lambda) > 0$, $g_1(0) \geq 0$, for $t > 0$ sufficiently small, $V(x^* + th^1_\lambda, a(t) + th^2_\lambda) > 0$. Moreover, $V_a(x^* + th^1_\lambda, a(t) + th^2_\lambda) = 0$ and $V_{aa}(x^* + th^1_\lambda, a(t) + th^2_\lambda) < 0$, for t sufficiently small, i.e., $a(t) + th^2_\lambda$ is a strict local maximum of $V(x^* + th^1_\lambda, \cdot\,)$.

Analogously,

$$\frac{V(x^* + th^1_\lambda, a(t) + th^2_\lambda) - V(x^* + th^1_\lambda, a)}{t} = g_3(h_\lambda)(a) + (\frac{1}{t} - 1)g_3(0)(a) + \frac{r(t,a)}{t}$$

where $\lim\limits_{t\to 0} \frac{r(t,a)}{t} = 0$, for each $a \in A$.

However, $a(t) + th^2_\lambda$ is a strict local maximum of $V(x^* + th^1_\lambda, \cdot\,)$, $g_3(h_\lambda) \in \mathrm{int}(\Lambda)$ and $g_3(0) \in \Lambda$, which implies that $V(x^* + th^1_\lambda, a(t) + th^2_\lambda) - V(x^* + th^1_\lambda, \cdot\,) \in \mathrm{int}(\Lambda)$.[21]

Since (x^*, a^*) is the solution of the moral hazard problem and $(x^* + th^1_\lambda, a(t) + th^2_\lambda)$ is feasible for that problem, $U(x^* + th^1_\lambda, a(t) + th^2_\lambda) \leq U(x^*, a^*)$, for all $t > 0$ sufficiently small.

Therefore, $\delta_{h_\lambda} U(x^*, a^*) \leq 0$, since $a'(0) = 0$. Making $\lambda \to 0$, $\delta_h U(x^*, a^*) \leq 0$.

If $a^* = \underline{a}$ (respectively $a^* = \overline{a}$), the proof is analogous. □

[21] By the compactness of the complement of an open interval around a^*.

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Figure 1

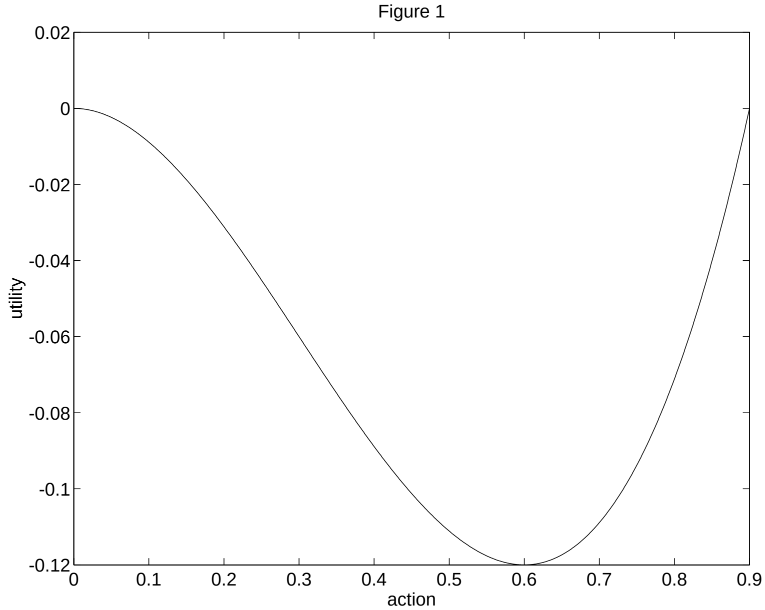

Figure 2