A FORMULA WITH SOME APLICATIONS TO THE THEORY OF LYAPUNOV EXPONENTS

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Abstract. We prove an elementary formula about the average expansion of certain products of matrices. This permits us to quickly reobtain an inequality by M. Herman and a theorem by Dedieu and Shub, both concerning Lyapunov exponents. Indeed, we show that equality holds in Herman's result.

1. Introduction

A major problem in smooth ergodic theory is to determine whether a given measure-preserving diffeomorphism has non-zero Lyapunov exponents. This problem is also of interest in the more general setting of linear cocycles. However, it is difficult to show the existence of non-zero exponents without strong conditions like uniform hyperbolicity.

In [He], Herman devised a method to minorate the upper Lyapunov exponent of some cocycles and constructed the first examples of non-uniformly hyperbolic two-dimensional systems with a positive exponent. Such examples are very delicate: it is shown in [Bo] that the exponent of non-hyperbolic cocycles drops to zero with an arbitrarily small C^0-perturbation of the cocycle.

One of the methods of Herman allows to estimate the average of the upper Lyapunov exponent of systems in a special parametrized family. While each individual system may be unstable, this average estimate is robust. Using Herman's estimate, Knill proved in [Kn] that among bounded measurable $\mathrm{SL}(2,\mathbb{R})$-cocycles those with a positive exponent are dense.

This idea – to consider systems included in some suitable family and to show that global properties of the family imply good properties for many individual elements – is also present in the recent paper [DS]. This reasoning has been conjectured to work in more generality in [BPSW].

We will consider the following situation: take matrices $A_1, \dots, A_n$ in $\mathrm{SL}(2,\mathbb{R})$. Of course, the norm $\|A_n \cdots A_1\|$ can be much smaller than $\prod \|A_j\|$. Now we put those matrices inside a circle parametrized family: $A_{j,\theta} = A_j R_\theta$ (we indicate by R_θ a rotation of angle θ). Instead of looking at the norms, we will deal with the related quantity

$$N(A) = \log\left(\frac{\|A\| + \|A\|^{-1}}{2}\right).$$

In this note we will prove:

$$\frac{1}{2\pi}\int_0^{2\pi} N(A_{n,\theta} \cdots A_{1,\theta})\, d\theta = \sum_{j=1}^{n} N(A_j).$$

In particular, if the $\|A_j\|$ are large then $\|A_{n,\theta} \cdots A_{1,\theta}\|$ is of the order of $\prod \|A_j\|$ for most values of θ.

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The formula allows us to conclude that the mentioned bound of [He] is sharp and also to reobtain a theorem of [DS].

2. The formula

Notation. Given a real or complex matrix A, we denote:

$$\|A\| = \sup_{v\neq 0} \frac{\|Av\|}{\|v\|} \quad \text{where } \|\cdot\| \text{ is the euclidean norm.}$$

Also, we denote by $\rho(A)$ the spectral radius, that is, the maximum absolute value of the eigenvalues of A. We have $\rho(A) \leq \|A\|$. We will indicate by $\mathrm{SL}(2,\mathbb{R})$ the group of real two-by-two matrices with unit determinant and $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$. We define:

$$N(A) = \log\left(\frac{\|A\| + \|A\|^{-1}}{2}\right) \quad \text{for } A \in \mathrm{SL}(2,\mathbb{R}).$$

We define some special matrices in $\mathrm{SL}(2,\mathbb{R})$:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{for } \theta \in \mathbb{R}.$$

$$H_c = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \quad \text{for } c \geq 1.$$

Finally, we indicate by $\mathbb{D}$ the open unit disk in $\mathbb{C}$ and by $\mathbb{S}^1$ its boundary.

Our main formula is:

Theorem 1. *Let* $A_1, \ldots, A_n \in \mathrm{SL}(2,\mathbb{R})$. *Then*

$$\frac{1}{2\pi}\int_0^{2\pi} N(A_n R_\theta \cdots A_1 R_\theta)\, d\theta = \sum_{j=1}^{n} N(A_j).$$

Actually, Theorem 1 is a corollary of the formula below:

Theorem 2. *Let* $A_1, \ldots, A_n \in \mathrm{SL}(2,\mathbb{R})$. *Then*

$$\frac{1}{2\pi}\int_0^{2\pi} \log\rho\left(A_n R_\theta \cdots A_1 R_\theta\right) d\theta = \sum_{j=1}^{n} N(A_j).$$

Notice that $\log\|A\| - \log 2 < N(A) \leq \log\|A\|$. Let's give an interpretation of the quantity $N(A)$ through the following proposition:

Proposition 3. *Let* $A \in \mathrm{SL}(2,\mathbb{R})$. *Then*

$$N(A) = \frac{1}{2\pi}\int_0^{2\pi} \log\|A(\cos\theta, \sin\theta)\|\, d\theta.$$

Therefore the number $N(A)$ can be viewed as the "average rate of expansion" of the matrix $A \in \mathrm{SL}(2,\mathbb{R})$.

Proof. By the so-called singular value theorem, one can find numbers $\alpha, \beta \in [0, 2\pi]$ and $c \geq 1$ such that $A = R_\beta H_c R_\alpha$. Moreover, $\|A\| = c$. So we may suppose that $A = H_c$, and we have to prove

$$\frac{1}{2\pi}\int_0^{2\pi} \log\sqrt{c^2\cos^2\theta + c^{-2}\sin^2\theta}\, d\theta = \log\left(\frac{c + c^{-1}}{2}\right).$$

First we calculate

$$F(b) = \int_0^{\pi} \log(b^2\cos^2\theta + \sin^2\theta)\, d\theta.$$

We have

$$F'(b) = 2b \int_0^{\pi} \frac{d\theta}{b^2 + \tan^2\theta} = 2b \int_{-\infty}^{+\infty} \frac{dx}{(b^2 + x^2)(1 + x^2)} = \frac{2\pi}{b+1}.$$

(The last integral can be calculated by residues). The solution of this differential equation with initial condition $F(1) = 0$ is $F(b) = 2\pi \log \frac{b+1}{2}$. Therefore

$$\int_0^{2\pi} \log \sqrt{c^2 \cos^2\theta + c^{-2}\sin^2\theta}\, d\theta = -2\pi \log c + F(c^2) = 2\pi \log \frac{c + c^{-1}}{2}.$$

□

The corollary of Theorem 1 below is based on a idea from [Kn] and justifies the assertion made in the introduction:

Corollary 4. *Let* $A_1, \dots, A_n \in SL(2, \mathbb{R})$ $a > 0$ *and*

$$E = \left\{ \theta \in [0, 2\pi];\ \frac{1}{n} \|A_n R_\theta \cdots A_1 R_\theta\| > -a + \frac{1}{n} \sum \log \|A_j\| \right\}.$$

Let ν *denote the normalized Lebesgue measure in the circle. Then* $\nu(E) \geq 1 - \frac{\log 2}{a}$.

Proof. Let $f(\theta) = \frac{1}{n} N(A_n R_\theta \cdots A_1 R_\theta)$ and $M = \frac{1}{n} \sum N(A_j)$. Let $F = \{\theta;\ f(\theta) > M - b\}$, where $b = a - \log 2$. It is easy to see that $F \subset E$. Since $0 \leq f(\theta) \leq M + \log 2$, we have

$$M = \int f\, d\nu \leq (M - b)(1 - \nu(F)) + (M + \log 2)\nu(F).$$

This gives

$$\nu(E) \geq \nu(F) \geq \frac{b}{b + \log 2} = 1 - \frac{\log 2}{a}.$$

□

3. Proof of Theorem 2

The proof is based on complexification methods from [He].

By continuity, we only have to prove the theorem for a dense set of matrices A_i. So we can make the following assumption:

$$B_\theta = A_n R_\theta \cdots A_1 R_\theta \neq \pm I \quad \text{for all } \theta.$$

Define the following complex matrices:

$$S_z = \begin{pmatrix} \frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2i} \\ \frac{z-z^{-1}}{2i} & \frac{z+z^{-1}}{2} \end{pmatrix} \quad \text{for } z \in \mathbb{C}^*,$$

$$T_z = \begin{pmatrix} \frac{z^2+1}{2} & -\frac{z^2-1}{2i} \\ \frac{z^2-1}{2i} & \frac{z^2+1}{2} \end{pmatrix} \quad \text{for } z \in \mathbb{C}.$$

We have $T_z = z S_z$ and $S_{e^{i\theta}} = R_\theta$.

Given $A_1, \dots, A_n \in SL(2, \mathbb{R})$, we define

$$C_z = \prod_{j=1}^{n} A_j T_z = A_n T_z \cdots A_1 T_z \quad \text{for } z \in \mathbb{C}.$$

Lemma 5. *There are holomorphic functions* $\lambda_1, \lambda_2 : \mathbb{D} \to \mathbb{C}$, *which extend continuously to* $\overline{\mathbb{D}}$, *such that* $\{\lambda_1(z), \lambda_2(z)\}$ *are the eigenvalues of* C_z *and* $|\lambda_2(z)| < |\lambda_1(z)|$ *for every* $z \in \mathbb{D}$.

Lemma 5 implies that $\log \rho(C_z)$ is an harmonic function in the disk $\mathbb{D}$ which extends continuously to the boundary. Moreover,

$$z = e^{i\theta} \quad \Rightarrow \quad C_z = z^n \prod_{j=1}^{n} A_j R_\theta \quad \Rightarrow \quad \rho(C_z) = \rho(\prod_{j=1}^{n} A_j R_\theta).$$

Therefore

$$\frac{1}{2\pi}\int_0^{2\pi} \log \rho(\prod_{j=1}^{n} A_j R_\theta)\, d\theta = \log \rho(C_0).$$

So the proof of Theorem 2 will be complete once we prove Lemma 5 and the lemma below:

Lemma 6. *The eigenvalues of* C_0 *are zero and* $\prod_{j=1}^{n} \frac{\|A_j\|+\|A_j\|^{-1}}{2}$.

3.1. **Proof of Lemma 5.** It is enough to show that that the eigenvalues of C_z have different norm for all $z \in \mathbb{D}$. First we obtain the following criteria for identity of their norms:

Lemma 7. *Let* $C \in M(2,\mathbb{C})$ *with* $\det C \neq 0$ *and let* λ_1, λ_2 *be the eigenvalues of* C. *Then* $|\lambda_1| = |\lambda_2|$ *if and only if*

$$\frac{(\operatorname{tr} C)^2}{4 \det C} \in [0,1].$$

Proof. Let $t = \frac{\lambda_1}{\lambda_2}$. We have $u = \frac{(\operatorname{tr} C)^2}{4\det C} = \frac{1}{4}(t + t^{-1} + 2)$. Then $|\lambda_1| = |\lambda_2| \Leftrightarrow |t| = 1 \Leftrightarrow \frac{t+t^{-1}}{2} \in [-1,1] \Leftrightarrow u \in [0,1]$. $\square$

We have $\det T_z = z^2$ and so $\det C_z = z^{2n}$. Therefore, C_z has eigenvalues with equal modulus if and only if $\frac{(\operatorname{tr} C_z)^2}{4z^{2n}} \in [0,1]$, that is, if and only if $Q(z) = \frac{\operatorname{tr} C_z}{2z^n} \in [-1,1]$. So to prove lemma 5 we must prove that if $z \in \mathbb{D}$ then $Q(z) \notin [-1,1]$. The idea is that since $Q(z)$ is a rational map of degree at most $2n$, this can be checked by showing that the unit circle 'exhausts' all preimages of $[-1,1]$. This we will do with a topological argument.

Let $S = Q^{-1}([-1,1])$.

Lemma 8. *If* $S \cap \mathbb{S}^1$ *has at least* $2n$ *connected components then* S *is the union of* $2n$ *subintervals of* $\mathbb{S}^1$.

Proof. Notice that $\operatorname{tr} C_z$ is a polynomial of degree at most $2n$, so $Q(z)$ and $Q'(z)$ are rational maps of degree at most $2n$. Since $Q(\mathbb{S}^1) \subset \mathbb{R}$ we know that there is at least one 0 of $Q'(z)$ in each component of $\mathbb{S}^1 \setminus S$. In particular there are no zeros of $Q'(z)$ in S, which implies that each connected component of $S \cap \mathbb{S}^1$ is mapped diffeomorphically onto $[-1,1]$. $\square$

Define the following sets:

$$\begin{aligned} X &= PSL(2,\mathbb{R}) \setminus \{I\}, \\ Y &= \{A \in X;\ |\operatorname{tr} A| \le 2\}. \qquad\qquad (|\operatorname{tr}| \text{ is well-defined in } PSL(2,\mathbb{R})) \end{aligned}$$

Lemma 9. *There exists a continuous function* $F : X \to \mathbb{S}^1$ *such that* $F^{-1}(\{1\}) = Y$ *and that the induced homeomorphism* $F_\# : \pi_1(X) \to \pi_1(\mathbb{S}^1)$ *is an isomorphism.*

Proof. Let $A \in PSL(2,\mathbb{R})$. If $A \in Y$ then we define $F(A) = 1$. Otherwise A has two eigendirections $\pm v$ and $\pm w$, where $v, w \in \mathbb{S}^1 \subset \mathbb{C}$, with associated eigenvalues λ and λ^{-1}, where $|\lambda| > 1$. We then define $F(A)$ as v^2/w^2. It is easy to see that F is continuous at every $A \neq I$.

We have $\pi_1(X) = \mathbb{Z}$ (it is equal to $\pi_1(\mathrm{PSL}(2,\mathbb{R}))$, since $\mathrm{PSL}(2,\mathbb{R})$ is a three-dimensional manifold) and so it is enough to exhibit a closed path γ generating $\pi_1(X)$ such that $F \circ \gamma$ has degree one.

Let $\gamma : \mathbb{S}^1 \to X$ be defined as $e^{i\theta} \mapsto R_{\theta/2}M$, where $M \in X$ is symmetric. Using the identities $F(A^T) = \overline{F(A)}$ and $F(R_\theta^{-1} A R_\theta) = F(A)$, we easily see that $F \circ \gamma$ commutes with conjugacy ($F \circ \gamma(\overline{z}) = \overline{F \circ \gamma(z)}$). Furthermore, $e^{i\theta} = 1$ is the only value such that $\gamma(e^{i\theta})$ is symmetric, which is equivalent to $F(\gamma(e^{i\theta})) = -1$. This implies that $F \circ \gamma$ has degree one. □

Let $B_\theta = \prod_{j=1}^n A_j R_\theta$. We have $Q(e^{i\theta}) = \frac{1}{2} \operatorname{tr} B_\theta$. Notice that $e^{i\theta} \in S$ if and only if B_θ is an elliptic or parabolic matrix in $\mathrm{SL}(2,\mathbb{R})$.

Lemma 10. *$S \cap \mathbb{S}^1$ has at least $2n$ connected components.*

Proof. Let $g : \mathbb{S}^1 \to \mathrm{PSL}(2,\mathbb{R})$ be defined by $e^{i\theta} \mapsto R_{\theta/2}$. This is clearly a generator of the fundamental group of $\mathrm{PSL}(2,\mathbb{R})$.

Notice that $e^{i\theta} \mapsto B_\theta$ can also be seen as a path in $\mathrm{PSL}(2,\mathbb{R})$ and it follows from the definition that it is homotopic to g^{2n} (by the fact that $\mathrm{PSL}(2,\mathbb{R})$ is a group). Now we use the assumption made at the beginning: for all θ, $B_\theta \neq \pm I$.

In this case, the path $F \circ B_\theta$ has degree $2n$ and therefore the preimage of 1 has at least $2n$ connected components. This set coincides with $S \cap \mathbb{S}^1$. □

Lemmas 8 and 10 imply that $S \subset \mathbb{S}^1$, and Lemma 5 is proved.

3.2. Proof of Lemma 6.

We will list some facts to be used:

(1) One can find numbers $\alpha_j, \beta_j \in [0, 2\pi]$, $c_j \geq 1$ such that $A_j = R_{\beta_j} H_{c_j} R_{\alpha_j}$ for each j. Moreover, $\|A_j\| = c_j$.
(2) $A, B \in \mathrm{SL}(2,\mathbb{R}) \quad \Rightarrow \quad \rho(AB) = \rho(BA)$.
(3) For every $\theta \in \mathbb{R}$, $R_\theta T_0 = T_0 R_\theta = e^{-i\theta} T_0$.

(1) is the singular value theorem. For (2), notice that the spectral radius depends only on the trace, which has the property $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. For (3) we use that $S_z S_w = S_{zw}$. This implies $T_z T_w = T_{zw}$ and

$$R_\theta T_0 = S_{e^{i\theta}} T_0 = e^{-i\theta} T_{e^{i\theta}} T_0 = e^{-i\theta} T_0.$$

Using (1), (2) and (3) we obtain

$$\rho(C_0) = \rho(\prod_{j=1}^n A_j T_0) = \rho(\prod_{j=1}^n T_0 H_{c_j}).$$

Each matrix $T_0 H_{c_j}$ has an eigenvector $(i, 1)$ with corresponding eigenvalue $\frac{c_j + c_j^{-1}}{2}$. Therefore $\prod_{j=1}^n T_0 H_{c_j}$ has an eigenvalue

$$\prod_j (c_j + c_j^{-1})/2,$$

while C_0 is not invertible. This proves Lemma 6.

4. Proof of Theorem 1

Let $B_\theta = A_n R_\theta \cdots A_1 R_\theta$. Then, fixing θ we have, by Theorem 2,

$$\frac{1}{2\pi} \int_0^{2\pi} \log \rho(B_\theta R_{\theta'}) d\theta' = N(B_\theta).$$

On the other hand, fixing θ' we have, again by Theorem 2,

$$\frac{1}{2\pi}\int_0^{2\pi}\log\rho(B_\theta R_{\theta'})d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log\rho\left(A_n R_\theta\cdots(A_1 R_{\theta'})R_\theta\right)d\theta$$
$$= N(A_1 R_{\theta'}) + \sum_{j=2}^{n} N(A_j) = \sum_{j=1}^{n} N(A_j).$$

Then

$$\frac{1}{2\pi}\int_0^{2\pi} N(B_\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2\pi}\int_0^{2\pi}\log\rho(B_\theta R_{\theta'})\,d\theta d\theta' = \sum_{j=1}^{n} N(A_j).$$

This proves Theorem 1.

Remark. Inversely, Theorem 2 could be quickly deduced from Theorem 1, using

$$\log\rho(A) = \lim_{n\to\infty}\frac{1}{n}\log\|A^n\| = \lim_{n\to\infty}\frac{N(A^n)}{n}.$$

5. Herman's inequality reobtained

Let (X,μ) be a probability space and $T: X\to X$ an ergodic transformation. Let $A: X\to SL(2,\mathbb{R})$ be a measurable function satisfying the integrability condition

$$\int\log\|A\|d\mu < \infty.$$

We denote for $x\in X$ and $n\in\mathbb{N}$,

$$A^n(x) = A(T^{n-1}x)\cdots A(x)\,.$$

The function A is called a *linear cocycle*. In these conditions, there exists (see [FK] or [Le]) a number $\lambda^+(A)\geq 0$, called the *upper Lyapunov exponent*, such that

$$\lambda^+(A) = \lim_{n\to+\infty}\frac{1}{n}\log\|A^n(x)\| \quad \text{for } \mu\text{-a.e. } x\in X.$$

For $\theta\in\mathbb{R}$, we define a cocycle AR_θ by $(AR_\theta)(x) = A(x)R_\theta$. Clearly, $\theta\mapsto\lambda^+(AR_\theta)$ is a measurable function.

We now state Herman's inequality:

Theorem 11 ([He], §6.2, see also [Kn])**.** *If T, μ and A are as above then*

$$\frac{1}{2\pi}\int_0^{2\pi}\lambda^+(AR_\theta)d\theta \geq \int_X\log\left(\frac{\|A(x)\|+\|A(x)\|^{-1}}{2}\right)d\mu(x)\,.$$

Remark. Herman's inequality was stated in a different (but equivalent) way, involving the Iwasawa decomposition.

We will reobtain Theorem 11 and also show that equality holds.

Theorem 12. *If T, μ and A are as above then*

$$\frac{1}{2\pi}\int_0^{2\pi}\lambda^+(AR_\theta)d\theta = \int_X\log\left(\frac{\|A(x)\|+\|A(x)\|^{-1}}{2}\right)d\mu(x)\,.$$

Proof. Recall that $N(A)\leq\log\|A\| < \log 2 + N(A)$. By Theorem 1,

$$\sum_{j=0}^{n-1} N(A(T^j(x))) \leq \frac{1}{2\pi}\int_0^{2\pi}\log\|(AR_\theta)^n(x)\|\,d\theta \leq \log 2 + \sum_{j=0}^{n-1} N(A(T^j(x))).$$

Therefore, by Birkhoff's theorem,

$$\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{n}\log\|(AR_\theta)^n(x)\|\,d\theta = \int N(A(x))\,d\mu(x) \quad \text{for a.e. } x.$$

To finish the proof we must check that Dominated Convergence applies. We have

$$0 \le \frac{1}{n} \log \|(AR_\theta)^n(x)\| \le \frac{1}{n} \sum_{j=0}^{n-1} \log \|A(T^j x)\| = f_n(x).$$

$\{f_n\}$ is the sequence of Birkhoff means of the function $\log \|A\| \in L^1(\mu)$. In particular, $\{f_n(x)\}$ is bounded for a.e. x. □

Example 13. Consider the cocycle ([He], §4.1) where $T : \mathbb{S}^1 \to \mathbb{S}^1$ is an (uniquely ergodic) irrational rotation, $A : \mathbb{S}^1 \to \mathrm{SL}(2,\mathbb{R})$ is given by $A(e^{it}) = H_c R_t$ and $c \ge 1$ is fixed. We have $(AR_\theta)^n(z) = A^n(e^{i\theta} z)$ and therefore $\lambda^+(A) = \lambda^+(AR_\theta)$ for all θ. It follows from Theorem 12 that $\lambda^+(A) = \log\left(\frac{c+c^{-1}}{2}\right)$.

6. A theorem by Dedieu and Shub reobtained

We will use Proposition 3 and Theorem 2 (in the case $n = 1$) to give another proof of the following theorem by Dedieu and Shub:

Theorem 14 ([DS]). *Let μ be a probability measure in $\mathrm{SL}(2,\mathbb{R})$ such that $\int \log \|A\|\, d\mu(A)$ is finite. Suppose that μ is invariant by rotations, that is, $R_\theta^* \mu = \mu$ for all θ. Let $A_1, A_2, \ldots \in \mathrm{SL}(2,\mathbb{R})$ be independent random matrices with law μ and consider the associated upper Lyapunov exponent:*

$$\lambda^+ = \lim_{n\to+\infty} \frac{1}{n} \log \|A_n \cdots A_1\| \quad \textit{(w.p. 1)}.$$

Then

$$\lambda^+ = \int_{\mathrm{SL}(2,\mathbb{R})} \log \rho(A)\, d\mu(A).$$

Proof.

$$\begin{aligned}
\lambda^+ &= \int \int_0^{2\pi} \log \|Ae^{i\theta}\| \frac{d\theta}{2\pi}\, d\mu(A) && \text{(by Furstenberg's formula, see [Le])}\\
&= \int N(A) d\mu(A) && \text{(by Proposition 3)}\\
&= \int \int_0^{2\pi} \log \rho(AR_\theta) \frac{d\theta}{2\pi}\, d\mu(A) && \text{(by Theorem 2 with } n = 1)\\
&= \int_0^{2\pi} \int \log \rho(AR_\theta)\, d\mu(A) \frac{d\theta}{2\pi} && \text{(since } \log \rho(AR_\theta) \le \log \|A\| \in L^1)\\
&= \int \log \rho(A)\, d\mu(A) && \text{(since } \mu \text{ is invariant by rotations).}
\end{aligned}$$

□

7. Growth of the spectral radius

Let X, μ, T and A be as in section 5. In view of our results, it is somewhat natural to ask about the behaviour the spectral radius of the matrix $A^n(x)$ when $n \to \infty$. This question was already raised in [Co]. We have the following result:

Theorem 15. *Suppose T is invertible. Then for μ-a.e. $x \in X$,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \rho(A^n(x)) = \lambda^+(A).$$

Before giving the proof, we point out that in general the limit of $\frac{1}{n}\log\rho(A^n(x))$ does not exist. Moreover, the relation

$$\limsup_{n\to\infty}\frac{1}{n}\int\log\rho(A^n)\,d\mu=\lambda^+(A).$$

is in general *false*, as is shown by the following:

Example 16. Let $X=\{0,1\}^{\mathbb{Z}}$, μ be the $(1/2,1/2)$-Bernoulli measure and let $T:X\to X$ be the left shift. We define the following cocycle $A:X\to \mathrm{SL}(2,\mathbb{R})$:

$$A((x_i)_{i\in\mathbb{Z}})=\begin{cases}H_2 & \text{if } x_0=1,\\ I & \text{if } (x_{-1},x_0,x_1)=(0,0,0) \text{ or } (1,0,1),\\ R_{\pi/2} & \text{if } (x_{-1},x_0,x_1)=(1,0,0) \text{ or } (0,0,1).\end{cases}$$

It follows that

- $\lambda^+(A)=\frac{\log 2}{2}$,
- there exists a set $C\subset X$ of positive measure (the cilinder $(x_{-1},x_0,x_1)=(1,0,0)$) such that if $n\geq 0$ and $T^n(x)\in C$ then $\rho(A^{n+1}(x))=1$.

Proof of Theorem 15. We may regard the problem as being posed in $\mathrm{PSL}(2,\mathbb{R})$ instead of $\mathrm{SL}(2,\mathbb{R})$. Suppose that $\lambda^+(A)>0$ (otherwise there is nothing to prove). Consider (see [Le]) the Oseledets splitting $\mathbb{R}^2=E^+(x)\oplus E^-(x)$, defined for a.e. $x\in X$, where E^+ (resp. E^-) is associated to the exponent $\lambda^+(A)$ (resp. $-\lambda^+(A)$). By Oseledets' theorem,

$$\lim_{n\to\infty}\frac{1}{n}\log\sin\measuredangle\left(E^+(T^nx),E^-(T^nx)\right)=0.$$

For each x, take $B(x)\in\mathrm{PSL}(2,\mathbb{R})$ that sends the direction $\mathbb{R}(1,0)$ (resp. $\mathbb{R}(0,1)$) to the direction $E^+(x)$ (resp. $E^-(x)$). This defines a.e. a measurable function $B:X\to\mathrm{PSL}(2,\mathbb{R})$ such that

$$\lim_{n\to\infty}\frac{1}{n}\log\|B(T^nx)\|=0\quad\text{for a.e. } x.$$

We claim that

$$\liminf_{n\to\infty}\|B(x)B(T^nx)^{-1}-I\|=0\quad\text{for a.e. } x.$$

To prove it, let $\varepsilon>0$. Consider a countable cover of $\mathrm{PSL}(2,\mathbb{R})$ by open sets

$$U_j=\{M\in\mathrm{PSL}(2,\mathbb{R});\ \|M-M_j\|<\delta_j\},\quad\text{where } 2\delta_j(\|M_j\|+\delta_j)<\varepsilon.$$

Define $V_j=B^{-1}(V_j)\subset X$ and

$$\tilde V_j=\{x\in V_j;\ T^n(x)\in V_j \text{ for infinitely many } n\in\mathbb{N}\}.$$

By Poincaré's recurrence theorem, $\mu(\tilde V_j)=\mu(V_j)$. If $x\in\tilde V_j$ then, for infinitely many $n\in\mathbb{N}$, we have

$$\|B(x)B(T^nx)^{-1}-I\|\leq\|B(x)-B(T^nx)\|\cdot\|B(T^nx)\|<2\delta_j(\|M_j\|+\delta_j)<\varepsilon.$$

Therefore $\liminf\|B(x)B(T^nx)^{-1}-I\|\leq\varepsilon$ for every x in the full measure set $\bigcup\tilde V_j$. This proves the claim.

To prove the Theorem it's enough (since $\rho+\rho^{-1}=\max\{|\operatorname{tr}|,2\}$) to show that

$$\limsup_{n\to\infty}\frac{1}{n}\log|\operatorname{tr}A^n(x)|=0.$$

By construction, the matrix $H(x)=B(Tx)^{-1}A(x)B(x)$ is diagonal. We have $A^n(x)=B(T^nx)^{-1}H^n(x)B(x)$, in particular, $\lim\frac{1}{n}\log\|H^n(x)\|=\lambda^+(A)$. Write $B(x)B(T^nx)^{-1}=(b_{ij}(n,x))_{i,j=1,2}$. For a.e. x, we know that there are infinitely many $n\in\mathbb{N}$ such that

$$|b_{11}(n,x)-1|,\ |b_{22}(n,x)-1|<\frac{1}{2}.$$

Then

$$\begin{aligned}
|\operatorname{tr} A^n(x)| &= \left|\operatorname{tr}\left(B(x)B(T^n x)^{-1}D^n(x)\right)\right| \\
&= \left|\left[b_{11}(n,x)\cdot\|H^n(x)\| - b_{22}(n,x)\cdot\|H^n(x)\|^{-1}\right]\right| \\
&> \frac{1}{2}\|H^n(x)\| - \frac{3}{2}\|H^n(x)\|^{-1}.
\end{aligned}$$

The result follows. □

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