# Iterative Algorithms for Equilibrium Problems

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**Abstract**

We consider equilibrium problems in the framework of the formulation proposed by Blum and Oettli, which includes variational inequalities, Nash equilibria in noncooperative games, and vector optimization problems, for instance, as particular cases. We show that such problems are particular instances of convex feasibility problems with infinitely many convex sets, but with additional structure, so that projection algorithms for convex feasibility can be modified in order to improve their convergence properties, mainly achieving global convergence without either compactness or coercivity assumptions. We present a sequential projections algorithm with an approximately most violated constraint control strategy, and two variants where exact orthogonal projections are replaced by approximate ones, using separating hypeplanes generated by subgradients. We include full convergence analyses of these algorithms.

**Keywords:** Equilibrium Problems, Convex Feasibility Problems, Proyection Methods.

## 1 Introduction

The problem of interest, which we call *Equilibrium Problem*, abbreviated EP, is defined as follows:

$$EP: \text{Find } \bar{x} \in K \text{ such that } f(\bar{x}, y) \geq 0 \text{ for all } y \in K,$$

where:

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1. $K$ is a nonempty closed and convex subset of $\mathbb{R}^n$,

2. $f : K \times K \to R$ is a continuous function satisfying:

   P1. $f(x, x) = 0$ for all $x \in K$.

   P2. $f(x, \cdot) : K \to \mathbb{R}$ is convex for all $x \in K$.

This problem was considered in the past, with slight variations in the assumptions, under various headings, for instance in [5], [1], [4], [3], [15], [17] and [12]. The purpose of most these works was to extend results concerning particular problems to more general settings. For example, [5] extended results on variational inequalities, corresponding to the case where $f(x, y) = \langle A(x), y - x\rangle$ and $A$ is a monotone operator, see Example (e) below. It has been pointed out in [4], [15] that this formulation includes, as particular cases, optimization problems, Nash equilibria problems, complementarity problems, fixed point problems, variational inequality problems and convex vector optimization problems. We show next how these six problems can be formulated as particular cases of EP. We follow [4] for Examples (a) thorough (e) and [15] for Example (f). For each example, we exhibit the appropriate set $K$ and function $f$. In all six cases, the solution sets of the corresponding problem coincides with the solution set of EP for the proposed $K$ and and $f$, $f$ is continuous and both P1 and P2 hold.

**Examples**

**a) Convex minimization problem:** Let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex, continuous and proper function. The convex minimization problem is defined as:

$$\text{Find } \; x \in K \text{ such that } \; h(x) \leq h(y) \text{ for all } y \in K.$$

Take $K := \{x \in \mathbb{R}^n : h(x) < +\infty\}$ and $f(x, y) := h(y) - h(x)$ for all $x$, $y \in K$.

**b) Fixed point problem:** Let $T : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be an upper semicontinuous point-to-set mapping such that $T(x) \cap K$ is a nonempty, convex, compact set for each $x \in K$. The fixed point problem is defined as:

$$\text{Find } \; x \in K \text{ such that } \; x \in T(x).$$

Take $f(x, y) := \max_{u \in T(x) \cap K} \langle x - u, y - x\rangle$ for all $x$, $y \in K$.

**c) Complementarity problem:** Let $K \subset \mathbb{R}^n$ be a closed convex cone and $K^+ := \{x \in \mathbb{R}^n : \langle x, y\rangle \geq 0 \text{ for all } y \in K\}$ its polar cone. Let

$T : K \to I\!R^n$ be a continuous operator. The complementarity problem is defined as:

$$\text{Find } \ x \in K \text{ such that } \ T(x) \in K^+, \ \ \langle T(x), x \rangle = 0.$$

Take $f(x, y) := \langle T(x), y - x \rangle$ for all $x$, $y \in K$.

**d) Nash equilibria problem in noncooperative games:** Let $I$ be a finite set (the set of players). For each $i \in I$, consider a nonempty closed convex subset $K_i$ of $I\!R^n$ (the strategy set of the $i-$th player). Let $X := \prod_{i \in I} Y_i$, $K := \prod_{i \in I} K_i$. For each $i \in I$ consider a continuous function $f_i : K \to I\!R$ (the loss function of $i-$th player, depending on the strategies of all players) which is convex in each one of its arguments. For each $x$, $y \in K$, we define $x(y_i)$ as $(x(y_i))_j := x_j$ for all $j \neq i$, and $(x(y_i))_i := y_i$. The Nash equilibria problem in noncooperative games is defined as:

$$\text{Find } \ x \in K, \text{ such that } \ f_i(x) \leq f_i(x(y_i)) \text{ for all } i \in I \text{ and for all } y \in K.$$

Take $f(x, y) := \sum_{i \in I} (f_i(x(y_i)) - f_i(x))$ for all $x$, $y \in K$.

**e) Variational Inequality Problem:** Let $K$ be a closed and convex subset of $I\!R^n$ and $T : K \to \mathcal{P}(I\!R^n)$ an upper semicontinuous point-to-set mapping, such that $T(x)$ is a compact set for all $x \in K$. The variational inequality problem is defined as:

$$\text{Find } \ x \in K, \ u \in T(x), \text{ such that } \ \langle u, y - x \rangle \geq 0 \text{ for all } y \in K.$$

Take $f(x, y) = \max_{u \in T(x)} \langle u, y - x \rangle$ for all $x$, $y \in K$.

**f) Vector Minimization Problem:** Let $C \subset I\!R^m$ be a closed convex cone with nonempty interior such that its polar cone ($C^+$) has nonempty interior too. Consider the partial order in $I\!R^m$ given by

$$x \preceq y \text{ if and only if } y - x \in C,$$

$$x \prec y \text{ if and only if } \ y - x \in \text{int}(C).$$

A function $F : K \subset I\!R^n \to I\!R^m$ is said to be $C$-convex if and only if $K$ is convex and $F(tx + (1-t)y) \preceq tF(x) + (1-t)F(y)$ for all $x$, $y \in K$ and for all $t \in (0, 1)$. Let $C \subset X = I\!R^m$ be a closed convex cone with nonempty interior, $C^+ = \{y \in I\!R^m \ / \ \langle x, y \rangle \geq 0 \text{ for all } x \in C\}$ and $F : K \to I\!R^m$ a $C$-convex and continuous function. The vector minimization problem is defined as:

$$\text{Find } \ \bar{x} \in K \text{ such that } F(x) \not\prec F(\bar{x}) \text{ for all } x \in K.$$

Take $f(x,y) := \max_{||z||=1,\ z\in C^+} \langle z, F(y) - F(x)\rangle$.

For $y \in K$, let $L_f(y) = \{x \in K : f(y,x) \leq 0\}$, which is a convex set by P2. We will show in the following section that the solution set of EP is contained in $Y_f := \cap_{y\in K} L_f(y)$. The problem of finding a point in $Y_f$ is a particular case of the Convex Feasibility Problem (CFP from now on), which in general consists of finding a point belonging to the intersection of a family of convex sets. The algorithms in this paper are devised for finding solutions of EP by solving the CFP given by $Y_f$. Observe that, since $K$ is a nonempty closed and convex set, the cardinality of $K$ is infinite, unless $K$ is a singleton, so that the corresponding instance of CFP, if nontrivial, involves infinitely many closed convex sets. Thus, we are interested in methods for CFP which are not restricted to a finite number of convex sets.

The most usual tools for solving convex feasibility problems are the so called *Projection Methods*. Recent reviews and comprehensive unifying approaches can be found in [8] and [2]. Projection methods can be classified in two cathegories, succesive and simultaneous, which we briefly describe next, for the instance of CFP given by $Y_f$.

Tipically, successive projection methods construct a sequence $\{x^m\} \subset I\!R^n$ from an arbitrary initial point $x^0$. Given $x^m$, a set $L_f(y^m)$ is chosen and a step is made in the direction of the orthogonal projection of $x^m$ onto $L_f(y^m)$, i.e. $x^{m+1} = x^k + \lambda_m(P_{L_f(y^m)}(x^m) - x^m)$, where, for $y \in K$, $P_{L_f(y)}$ denotes the orthogonal projection onto $L_f(y)$ and $\{\lambda_m\} \subset [\alpha, 2-\alpha]$ with $\alpha \in (0,1)$, is a sequence of exogenously given relaxation parameters. The way in which the sequence $\{y^m\}$ is chosen defines the so called *control strategy* and characterizes each method within the family of succesive projection methods. For a finite number of convex sets the standard control strategy is the cyclic one. Three control sequences adequate for the case of infinitely many convex sets are the following.

a) *Remotest set control.* This is obtained by determining $y^m \in K$ such that

$$||x^m - P_{L_f(y^m)}(x^m)|| = \max_{y\in K} ||x^m - P_{L_f(y)}(x^m)||. \tag{1}$$

b) *Approximately remotest set control.* Denote

$$\theta(x) = \sup_{y\in K} ||x - P_{L_f(y)}(x)|| \tag{2}$$

and choose $y^m \in K$ satisfying the following condition:

$$\lim_{m\to\infty} ||x^m - P_{L_f(y^m)}(x^m)|| = 0 \text{ implies } \lim_{m\to\infty} \theta(x^m) = 0. \tag{3}$$

c) *Most violated constraint control.* This control strategy measures the proximity of $x^m$ to a set in the family not through the Euclidean distance, but through the values of the function $f(\cdot, x^m)$, i.e. $y^m \in K$ is chosen so that

$$f(y^m, x^m) = \max_{y \in K} f(y, x^m). \tag{4}$$

Two realizations of succesive projection methods devised to deal with the case of infinitely many convex sets are the POCS algorithm in [13] and Algorithm A in [6].

Simultaneous projection methods construct a sequence $\{x^m\} \subset I\!R^n$, starting from an arbitrary $x^0 \in I\!R^n$, and computing, given $x^m$, the orthogonal projection onto all the convex sets $L_f(y)$ $(y \in K)$. Such projections must be somehow "averaged" in order to obtain the following iterate $x^{m+1}$. In the case of infinitely many convex sets, such averaging requires the introduction of a probability measure on the set $K$, so that finite averages can be replaced by integration of the projections with respect to this measure. As an example of a simultaneous projection method of this sort, we have Algorithm B in [6].

Of course, one can solve EP by applying to CFP any of the projection methods mentioned above. It happens to be the case that the structure of EP, and particularly the seemingly harmless asssumption P1, allows for substantial improvements over succesive projection methods devised to solve arbitrary convex feasibility problems (i.e., where the convex sets do not originate from an EP), e.g. over the POCS method of [13]. The improvement is related to the following feature of succesive projection methods, when applied to infinitely many convex sets: the three control strategies describe above, and all of them in general, require some sort of maximization in $y$ in order to choose the convex set onto which the $m$-th iterate is going to be projected. Existence of such maximizer, and/or efficient methods for finding it, demand some compactness assumption on $K$ or coercivity of $f(\cdot, y)$. We will show next that the EP structure allows for modifying succesive projection methods so that the maximization step in each iteration is limited to a compact set, but without imposing any compactness assumption on $K$ (nor coercivity on $f(\cdot, x)$). This can be done without affecting at all the convergence properties of the method, which fails to occur if similar modifications are attempted for the case of arbitrary convex feasibility problems, in which case cluster points of the generated sequences may be neither unique nor solutions of CFP. Algorithm 1, a procedure of this kind with a control strategy related to the most violated constraint one, is presented and analyzed in Section 3. In section 5 we comment on similar modifications of algorithms

with control strategies related to the remotest set control one.

We have found no similar improvements for the case of simultaneous projection methods, and thus we restrict our analysis to succesive ones. Algorithms such as Algorithm B in [6] can be effectively used for EP in suitable cases, but we discovered no way to use the EP structure so that we can add something significant to the existing convergence analysis of such methods.

Succesive projection methods for convex feasibility problems have been improved by using approximate projections instead of exact ones. More precisely, if a convex set $C$ is presented as $C = \{x \in I\!R^n : g(x) \leq 0\}$, where $g : I\!R^n \to I\!R$ is convex, then, for a point $x$ which does not belong to $C$, the orthogonal projection $P_C(x)$ is approximated by the orthogonal projection onto a hyperplane separating $x$ from $C$, namely $H = \{y \in I\!R^n : g(x) + \langle \xi, y - x \rangle = 0\}$, where $\xi$ is a nonnull subgradient of $g$ at $x$. The approximate projection $\tilde{P}_C(x)$ is then defined as $\tilde{P}_C(x) = x + [g(x)/||\xi||^2]\xi$, with $\xi \in \partial g(x)$, and it replaces the orthogonal projection in algorithms such as those mentioned above. The resulting algorithms are called *subgradient projection methods*; se e.g. [7], [14]. In our case, due to the presence of $K$, the convex sets of interest do not have naturally the required form. More precisely, if we assume that the domain of $f$ is $I\!R^n \times I\!R^n$ instead of $K \times K$ (which is frequently the case), and we call $Q_f(y) = \{x \in I\!R^n : f(y, x) \leq 0\}$, then we have that $L_f(y) = Q_f(y) \cap K$. In general, it should not be expected that $K$ could also be given as a level set of a single convex function, and so it seems natural to approximate with subgradients only the set $Q_f(y)$, performing thus the projection of $x^m$ onto the hyperplane separating $x^m$ from $Q_f(y^m)$ (using a subgradient $\xi^m$ of $f(y^m, \cdot)$ at $x^k$), followed by an orthogonal projection of the resulting point onto $K$, or alternatively, projecting $x^m$ both onto $K$ and onto the separating hyperplane, and taking then a convex combination of both projections. In the first case, we use in the algorithms $P_K\left(x^m - [\max\{0, f(y^m, x^m)\}/||\xi^m||^2]\xi^m\right)$ instead of $P_{L_f(y^m)}(x^m)$; in the second case we use $\mu_m\left(x^m - [\max\{0, f(y^m, x^m)\}/||\xi^m||^2]\xi^m\right) + (1 - \mu_m)P_K(x^m)$, with $\mu_m \in (0, 1)$. Subgradient projection methods adapted to EP are presented in Section 4.

## 2 Preliminaries

Consider EP as defined in Section 1 and let $L_f(y) := \{x \in K \ / \ f(y, x) \leq 0\}$. The following Convex Feasibility Problem:

$$\text{CFP} \quad \text{Find } \bar{x} \in K \text{ such that } \bar{x} \in \cap_{y \in K} L_f(y),$$

was introduced in [9] as an auxiliary problem for solving EP. Its relevance lies in the fact that the solution set of EP contains the solution set of CFP.

**Theorem 2.1.** *The solution set of CFP is a subset of the solution set of EP, with CFP and EP as defined above.*

**Proof**: Let $x$ be a solution of CFP and take any $y \in K$. Define $w_t := ty + (1-t)x$ for each $t \in (0,1)$. Thus, $0 = f(w_t, w_t) \leq tf(w_t, y) + (1-t)f(w_t, x)$ for each $t \in (0,1)$, but $f(w_t, x) \leq 0$ for each $t \in (0,1)$, because $x \in \cap_{y \in K} L_f(y)$ and $w_t \in K$. Then, $0 \leq tf(w_t, y)$ for each $t \in (0,1)$, so that $0 \leq f(w_t, y)$ for each $t \in (0,1)$. Taking limits as $t \to 0$ we conclude that $0 \leq f(x, y)$ by continuity of $f$. □

We remark that the converse of Theorem 2.1 does not hold. For example, consider $X = \mathbb{R}$, fix $a \in \mathbb{R}, a > 0$, take $K = [0, a]$ and define

$$f(x,y) := \begin{cases} 0 & \text{if } x = 0 \\ -\frac{y}{x} + 1 & \text{if } x > 0 \ . \end{cases}$$

Then the solution set of EP is $\{0, a\}$, but $\cap_{y \in K} L_f(y) = \cap_{y \in [0,a]} [y, a] = \{a\}$.

$f$ is said to be *pseudomonotone* when for all $x, y \in K$, if $f(x,y) \geq 0$ then $f(y,x) \leq 0$. The converse of Theorem 2.1 holds when $f$ is pseudomonotone, in which case both CFP and EP have the same solution set (Proposition 2.5 in [9]).

The following lemma gives a sufficient condition for existence of solutions of CFP.

**Lemma 2.2.** *([11], Lemma 1) Take $Y \subset \mathbb{R}^n$. For each $y \in Y$, consider a closed subset $C(y)$ of $\mathbb{R}^n$. If the following two conditions hold:*

*C1) the convex hull $co\{x_1, ...., x_p\}$ of any finite subset $\{x_1, .., x_p\}$ of $Y$ is contained in $\bigcup_{i=1}^{p} C(x_i)$,*

*C2) $C(x)$ is compact for at least some $x \in Y$,*

*then $\bigcap_{y \in Y} C(y) \neq \emptyset$*

We need in the sequel the following modification of Lemma 2.2, introduced in [9].

**Lemma 2.3.** *Take $Y \subset \mathbb{R}^n$. For each $y \in Y$, consider a closed subset $C(y)$ of $\mathbb{R}^n$. If the following two conditions hold:*

*C1) the convex hull* $co\{x_1, ...., x_p\}$ *of any finite subset* $\{x_1, .., x_p\}$ *of* $Y$ *is contained in* $\bigcup_{i=1}^{p} C(x_i)$,

*C2' The closure* $\overline{co}Y$ *of the convex hull of* $Y$ *is compact,*

*then* $\bigcap_{y\in Y} C(y) \neq \emptyset$.

**Proof**: For each $y \in Y$ we define $D(y) := \overline{co}Y \cap C(y)$. Consider any finite subset $\{ x_1, .., x_p \}$ of $Y$. We get from C1 that $co\{x_1, ...., x_p\} \subset \cup_{i=1}^{p} C(x_i)$ and that $co\{x_1, ...., x_p\} \subset \overline{co}Y$. It follows from these facts that

$$co\{x_1, ...., x_p\} \subset \overline{co}Y \cap \cup_{i=1}^{p} C(x_i),$$

implying

$$co\{x_1, ...., x_p\} \subset \cup_{i=1}^{p} \overline{co}Y \cap C(x_i) = \cup_{i=1}^{p} D(x_i).$$

On the other hand, $D(y)$ is compact for each $y \in Y$. Then, by Lemma 2.2, we have $\cap_{y\in Y} D(y) \neq \emptyset$, implying $\cap_{y\in Y} C(y) \neq \emptyset$. □

**Corollary 2.4.** *If* $f$ *is pseudomonotone and* $\{y^m\} \subset K$ *is a bounded sequence, then* $\cap_{m=1}^{\infty} L_f(y^m) \neq \emptyset$.

**Proof**: Let $Y = \{y^m\}$ and $C(y) = L_f(y)$ for $y \in Y$. We claim that C1 and C2' of Lemma 2.3 hold. C2' holds because $\overline{co}Y$ is compact when $Y$ is bounded. For C1, consider any finite subset $\{y^{m_1}, .., y^{m_p}\}$ of $Y$ and any $\bar{x} \in co\{y^{m_1}, .., y^{m_p}\}$, i.e. $\bar{x} = \sum_{i=1}^{p} \lambda_i y^{m_i}$ with $\sum_{i=1}^{p} \lambda_i = 1$ and $\lambda_i \geq 0$ $(1 \leq i \leq p)$. Suppose, by contradiction, that $\bar{x} \notin \cup_{i=1}^{p} L_f(y^{m_i})$. It follows that $f(y^{m_i}, \bar{x}) > 0$ $(1 \leq i \leq p)$. By pseudomonotonicity of $f$, we have that $f(\bar{x}, y^{m_i}) < 0$ $(1 \leq i \leq p)$, and therefore, using P1 and P2 with $x = \bar{x}$,

$$0 = f(\bar{x}, \bar{x}) = f(\bar{x}, \sum_{i=1}^{p} \lambda_i y^{m_i}) \leq \sum_{i=1}^{p} f(\bar{x}, y^{m_i}) < 0. \tag{5}$$

The contradiction in (5) establishes the claim. The result follows then from Lemma 2.3. □

# 3 An exact projection method for EP

In this section we present our first algorithm for solving EP, based upon the succesive projection scheme with the control strategy of the most violated constraint, described in Section 1. Each iteration consists basically of two

steps: an inexact maximization of a continuous function (of the form $f(\cdot, x)$ for some $x \in K$) over a compact set, and an orthogonal projection onto a set of the form $L_f(y)$ for some $y \in K$. The algorithm requires an exogenous constant $\alpha \in (0,1)$ and two exogenous sequences, one of relaxation parameters $\{\lambda_m\} \subset [\alpha, 2-\alpha]$, and one of precision parameters $\{\varepsilon_m\} \subset I\!R_+$, satisfying $\lim_{m\to\infty} \varepsilon_m = 0$, for the inexact maximization. We use the following notation: for $y \in K$, $L_f(y) := \{x \in K : f(y,x) \leq 0\}$ and for a closed convex subset $C$ of $I\!R^n$, $P_C : I\!R^n \to C$ denotes the orthogonal projection onto $C$. The algorithm generates two sequences $\{x^m\}, \{y^m\} \subset K$ in the following way.

**Algorithm 1.**
**Initial step:** Choose $x^0 \in K$ and $\rho_0 = ||x^0||$.
**Iterative step** Given $x^m$, $\rho_m$,

a) Define

$$K_m = \{x \in K : ||x|| \leq \rho_m + 1\}. \tag{6}$$

b) Find $y^m \in K_m$ satisfying

$$\max_{y \in K_m} f(y, x^m) \leq f(y^m, x^m) + \varepsilon_m. \tag{7}$$

c) Compute $x^{m+1}$ as

$$x^{m+1} = x^m + \lambda_m (P_{L_f(y^m)}(x^m) - x^m). \tag{8}$$

d) Update $\rho_m$ through

$$\rho_{m+1} = \max\{\rho_m, ||x^{m+1}||\}. \tag{9}$$

We start now the convergence analysis of Algorithm 1.

**Proposition 3.1.** *Algorithm 1 is well defined.*

**Proof**: By (6) and (9), $K_m \subset K_{m+1}$ for all $m$. Since $x^0 \in K$, we get that $x^0$ belongs to $K_0$ and thus $x^0 \in K_m$ for all $m$. Thus, all the $K_m$'s are nonempty and trivially compact, since $K$ is closed. It follows, in view of the continuity of $f$, that$f(\cdot, y^m)$ attains its maximum over $K_m$, so that there exists $y^m$ satisfying (7). By convexity of $f(y^m, \cdot)$, which results from P2, and convexity and closedness of $K$, the set $L_f(y^m)$ is closed and convex, so that $x^{m+1}$ is uniquely defined by (8). □

The following theorem summarizes the convergence properties of Algorithm 1.

**Theorem 3.2.** *Let $\{x^m\}$ and $\{y^m\}$ be the sequences generated by Algorithm 1.*

*i) If $\bigcap_{m=1}^{\infty} L_f(y^m)$ is nonempty, then $\{x^m\}$ converges to a solution of EP.*

*ii) If EP lacks solutions, then $\{x^m\}$ is not convergent (though it might be bounded).*

**Proof**: i) Take any $\bar{x} \in \cap_{m=1}^{\infty} L_f(y^m)$. Then by (8):

$$||x^{m+1} - \bar{x}||^2 = ||x^m - \bar{x}||^2 + \lambda_m^2 ||x^m - P_{L_f(y^m)}(x^m)||^2 +$$

$$2\lambda_m \langle x^m - \bar{x}, P_{L_f(y^m)}(x^m) - x^m \rangle = ||x^m - \bar{x}||^2 + \lambda_m^2 ||x^m - P_{L_f(y^m)}(x^m)||^2 +$$

$$-2\lambda_m ||x^m - P_{L_f(y^m)}(x^m)||^2 + 2\lambda_m \langle P_{L_f(y^m)}(x^m) - \bar{x}, P_{L_f(y^m)}(x^m) - x^m \rangle. \tag{10}$$

Since $\bar{x}$ belongs to $L_f(y^m)$, the last term is the rightmost expression in (10) is nonnegative, because of the following well known property of orthogonal projections: $\langle P_C(x) - y, P_C(x) - x \rangle \leq 0$ for any closed and convex set $C$, any $x \in I\!R^n$ and any $y \in C$. Thus, it follows from (10) that, for all $m \in I\!N$,

$$||x^{m+1} - \bar{x}||^2 \leq ||x^m - \bar{x}||^2 - \lambda_m(2 - \lambda_m)||x^m - P_{L_f(y^m)}||^2 \leq ||x^m - \bar{x}||^2, \tag{11}$$

using the fact that $\lambda_m(2 - \lambda_m) > 0$. We conclude from (11) that the sequence $\{||x^m - \bar{x}||\}$ in nonincreasing, and henceforth, being nonnegative, it is convergent. Rewriting (11) and using the conditions on $\{\lambda_m\}$ we get

$$\alpha(2 - \alpha)||x^m - P_{L_f(y^m)}(x^m)||^2 \leq \lambda_m(2 - \lambda_m)||x^m - P_{L_f(y^m)}(x^m)||^2 \leq$$

$$||x^m - \bar{x}||^2 - ||x^{m+1} - \bar{x}||^2. \tag{12}$$

From (12) and the convergence of $\{||x^m - \bar{x}||\}$ we obtain, since $\alpha(2-\alpha) > 0$,

$$\lim_{m \to \infty} (x^m - P_{L_f(y^m)}(x^m)) = 0. \tag{13}$$

The convergence of $\{||x^m - \bar{x}||\}$ also implies that the sequence $\{x^m\}$ is bounded. Take $\bar{\rho}$ such that $||x^m|| \leq \bar{\rho}$ for all $m \in I\!N$. It follows easily from (9) that $\rho_m = \max\{||x^0||, \ldots, ||x^m||\}$, so that $\rho_m \leq \bar{\rho}$ for all $m \in I\!N$. By

(6), $K_m$ is contained in the ball of center at 0 and radius $\bar{\rho}+1$, and then it follows from (7) that $||y^m|| \leq \bar{\rho}+1$, i.e. that $\{y^m\}$ is bounded. Let $x^*$ be a cluster point of $\{x^m\} \subset K$. Since $K$ is closed, $x^*$ belongs to $K$. Select a subsequence $\{x^{k_m}\}$ of $\{x^m\}$ such that $\lim_{m\to\infty} x^{k_m} = x^*$ and that $\{y^{k_m}\}$ converges, say to $y^*$. By (13), $\lim_{m\to\infty} P_{L_f(y^{k_m})}(x^{k_m}) = x^*$. It follows from the continuity of $f$ that

$$f(y^*, x^*) = \lim_{m\to\infty} (f(y^{k_m}, P_{L_f(y^{k_m})}(x^{k_m})) \leq 0, \tag{14}$$

where the inequality follows from the definition of $L_f(y^{k_m})$ and the fact that $P_{L_f(y^{k_m})}(x^{k_m})$ belongs to $L_f(y^{k_m})$.

Observe that (6) and (9) imply that $x^m$ belongs to $K_m$ for all $m \in I\!N$. Thus, using P1 and (7), for all $m \in I\!N$,

$$0 = f(x^m, x^m) \leq \max_{y\in K_m} f(y, x^m) \leq f(y^m, x^m) + \varepsilon_m. \tag{15}$$

Since $\lim_{m\to\infty} \varepsilon_m = 0$ and $f$ is continuous, it follows from (15) that

$$0 \leq \lim_{m\to\infty} (f(y^{k_m}, x^{k_m}) + \varepsilon_{k_m}) = f(y^*, x^*). \tag{16}$$

We conclude from (14) and (16) that

$$f(y^*, x^*) = 0. \tag{17}$$

Define $\rho^* = \sup\{\rho_m\} + 1/2$, which is finite because the $\rho_m$'s are bounded, as proved above. Let $C$ be the ball with center at 0 and radius $\rho^*$. Since, in view of (9), $||x^m|| \leq \rho_m \leq \rho^* - 1/2$ for all $m$, we conclude that $x^*$ belongs to the interior of $C$. Let now $C' = C \cap K$, and consider the equilibrium problem EP', with function $f$ and feasible set $C'$. We claim that $x^*$ solves EP'. Observe that $\{\rho_m\}$ is nondecreasing. Take $M \in I\!N$ such that $\rho_M \geq \sup\{\rho_m\} - 1/2 = \rho^* - 1$. Then, $\rho^* \leq \rho_m + 1$ for all $m \geq M$, and thus $C' \subset K_m$ for all $m \geq M$. Take any $z \in C'$. We have that $z \in K_m$ for all $m \geq M$, and therefore

$$f(z, x^m) \leq \max_{y\in K_m} f(y, x^m) \leq f(y^m, x^m) + \varepsilon_m \tag{18}$$

for all $m \geq M$, so that, using continuity of $f$, (18) and (17) we obtain

$$f(z, x^*) = \lim_{m\to\infty} f(z, x^{k_m}) \leq \lim_{m\to\infty} [f(y^{k_m}, x^{k_m}) + \varepsilon_{k_m}] = f(y^*, x^*) = 0. \tag{19}$$

By (19), since $x^* \in K$, $x^*$ belongs to $\cap_{z \in C'} L_f(z)$, so that the claim holds by Theorem 2.1, i.e. $f(x^*, y) \geq 0$ for all $y \in C'$. Since $f(x^*, x^*) = 0$ by P1, we conclude that $x^*$ solves the convex optimization problem min $f(x^*, y)$ subject to $y \in C'$ which, in view of the definition of $C'$, can be rewritten as min $g(y)$ subject to $y \in C$, where $g : I\!R^n \to I\!R$ is defined as $g(y) = f(x^*, y)$ if $y \in K$, $g(y) = +\infty$ if $y \notin K$. Since we proved above that $x^*$, a minimizer of $g$ over $C$, belongs in fact to the interior of $C$, it follows from the convexity of $g$ that $x^*$ is an unrestricted minimizer of $g$, i.e. that $g(x^*) \leq g(y)$ for all $y \in I\!R^n$, which in turn, in view of P1 and the definition of $g$, means that $0 = f(x^*, x^*) \leq f(x^*, y)$ for all $y \in K$, i.e. that $x^*$ solves EP. Let us look now at the sequence $\{||x^m - x^*||\}$. Since $x^* \in \bigcap_{z \in C'} L_f(z) \subset \bigcap_{m=1}^{\infty} L_f(y^m)$ because $\{y^m\} \subset C'$, this sequence is nonincreasing. It is certainly nonnegative, and it has a subsequence, namely $\{||x^{k_m} - x^*||\}$, which converges to 0. It follows that the whole sequence $\{||x^m - x^*||\}$ converges to 0, i.e. that $\lim_{m \to \infty} x^m = x^*$.

ii) Suppose that $\{x^m\}$ converges, say to some $x^* \in I\!R^n$. Note that the hypothesis that $\cap_{m=1}^{\infty} L_f(y^m) \neq \emptyset$ (missing in this item) was used in the proof of item (i) only to establish that $\{x^m\}$ is bounded and that $\lim_{m \to \infty}(x^{m+1} - x^m) = 0$. When $\{x^m\}$ converges, these two facts clearly occur, and the remainder of the argument in the proof of (i) shows that $x^*$ solves EP, contradicting the hypothesis of this item. □

**Corollary 3.3.** *If CFP has solutions then the sequence $\{x^m\}$ generated by Algorithm 1 converges to a solution of EP.*

**Proof**: If CFP has solutions then $\emptyset \neq \cap_{y \in K} L_f(y) \subset \cap_{m=1}^{\infty} L_f(y^m)$. The result follows from Theorem 3.2(i). □

We point out that, when $\cap_{m=1}^{\infty} L_f(y^m) = \emptyset$, which may happen, even in cases in which EP has solutions, the sequence $\{x^m\}$ generated by Algorithm 1 might diverge or oscilate among more than one cluster point. The following corollary shows that the second option cannot take place when $f$ is pseudomonotone, in which case boundedness of either $\{x^m\}$ or $\{y^m\}$ is enough to guarantee convergence to a solution of EP. In other words, when $f$ is pseudomonotone and EP lacks solutions, both $\{x^m\}$ and $\{y^m\}$ are unbounded. We remark that the function $f$ in Example (a) is pseudomonotone, and the same happens with Examples (c) and (e) when $T$ is pseudomonotone in the sense of [10] (and a fortiori, when $T$ is monotone), and with Example (b) when $T$ is nonexpansive.

**Corollary 3.4.** *Let $\{x^m\}$ and $\{y^m\}$ be the sequences generated by Algorithm 1. If $f$ is pseudomonotone and either $\{x^m\}$ or $\{y^m\}$ is bounded, then the sequence $\{x^m\}$ converges to a solution of EP.*

**Proof**: If $\{x^m\}$ is bounded, then $\{y^m\}$ is bounded, with the same argument as in the proof of Theorem 3.2(i). Thus we may assume that $\{y^m\}$ is bounded. By Corollary 2.4 $\cap_{m=1}^{\infty} L_f(y^m) \neq \emptyset$. The statement follows from then Theorem 3.2(i). □

We make now a few comments on Algorithm 1. First, we emphasize that the novelty of our algorithm lies in the introduction of the compact constraint sets $K_m$ used in the maximization of $f$ in the first argument (the relaxation parameters $\lambda_m$ and the approximate maximization with the $\varepsilon_m$'s are rather standard). As we have announced, the introduction of the $K_m$'s avoid all the limitations of other succesive projection algorithms for CFP in the case of infinitely many convex sets, without making either compactness assumptions on $K$ or coercivity assumptions on $f$, and, as such, represents a significant improvement, dependent on the additional structure of EP with respect to general CFP's. If we attempt the same procedure for a CFP of the form: find $\bar{x} \in \cap_{y \in K} L(y)$ with $L(y) = \{x : f(y,x) \leq 0\}$, where $f : K \times K \to I\!R$ is continuous, and convex in its second argument, but which does not satisfy P1, then the algorithm is well defined and we can establish, with the same argument as in the proof of Theorem 3.2, that both $\{x^m\}$ and $\{y^m\}$ are bounded, that $\lim_{m\to\infty}(x^m - x^{m+1}) = 0$, and that any cluster point $x^*$ of $\{x^m\}$ solves the convex feasibility problem of finding $\bar{x} \in \cap_{y \in C'} L(y)$, but we cannot show that $x^*$ belongs to $\cap_{y \in K} L(y)$, i.e. that it solves the problem of interest, because in the absence of P1 we cannot establish that $x^*$ solves a constrained convex optimization problem. The fact that $x^*$ solves such a problem, while at the same it belongs to the interior of the feasible set, is crucial, in the proof of Theorem 3.2, for showing that $x^*$ solves the full CFP, and thus the original problem too.

We also remark that the endogenous definition of the sequence $\rho_m$, depending upon the norms of the previous iterates $x^j$ $(1 \leq j \leq m)$, seems also to be esssential for the validity of Theorem 3.2. If we take instead an exogenous sequence $\{\rho_m\}$, then the sequences $\{x^m\}$ and $\{y^m\}$ are still well defined, and it is easy to prove that $\{x^m\}$ is bounded and that $\lim_{m\to\infty}(x^m - x^{m+1})$, but we get in trouble in connection with $\{y^m\}$: if we take $\rho_m$ such that $\lim_{m\to\infty} \rho_m = +\infty$ (e.g. $\rho_m = m$) then we cannot establish in general boundedness of $y^m$ and consequently optimality of the cluster points of $\{x^m\}$. We have been able to carry over the convergence analysis for these $\rho_m$'s only for the case in which $f$ is linear in $y$, in which case it follows from

P1 and P2 that, if $K$ has nonempty interior, then necessarily $f(x,y) = x^t Ay$, where $A \in I\!R^{n\times n}$ is skew symmetric. If, on the other hand, we take a sequence $\{\rho_m\}$ which is bounded "a priori", then $\{y^m\}$ is obviously bounded, but the cluster points of $\{x^m\}$ only solve an EP problem with a constraint set which in general is strictly contained in $K$, without solving the original EP, with constraint set $K$.

# 4 Subgradient projection methods for EP

In this section we modify Algorithm 1, allowing for approximate projections onto separating hyperplanes. We remind that for a convex function $g : I\!R^n \to I\!R \cup \{+\infty\}$, the subdifferential $\partial g(x)$ of $g$ at a point $x \in I\!R^n$ is defined as $\partial g(x) = \{u \in I\!R^n : \langle u, y - x\rangle \leq g(y) - g(x) \text{ for all } y \in I\!R^n\}$ and the domain $\mathrm{dom}(g)$ of $g$ as $\mathrm{dom}(g) = \{x \in I\!R^n : g(x) \in I\!R\}$. For each $x \in K$ define $f_x : I\!R^n \to I\!R \cup \{+\infty\}$ as $f_x(y) = f(x,y)$ if $y \in K$, $f_x(y) = +\infty$ if $y \notin K$, and $\Gamma_f : I\!R^n \to \mathcal{P}(I\!R^n)$ as $\Gamma_f(x) = \partial f_x(x)$.

We will need in this section the following assumption on $f$:

A) $\Gamma_f(x) \neq \emptyset$ for all $x \in K$ and $\Gamma_f$ is bounded on compact subsets of $K$ (i.e., if $C \subset K$ is compact, then $\cup_{x\in C}\Gamma_f(x)$ is bounded).

Later on, we will discuss conditions on $f$ under which (A) holds. Next we present two algorithms for solving EP, where the orthogonal projections of Algorithm 1 are replaced by approximate ones. As before, $P_K : I\!R^n \to K$ will denote the orthogonal projection onto $K$.

**Algorithm 2**

Algorithm 2 requires the same exogenous data as Algorithm 1 and it coincides with it, except that step (c) is replaced by:

c') If $0 \in \Gamma_f(y^m)$ then stop; otherwise, take $\xi^m \in \Gamma_f(y^m)$ and define

$$z^m = x^m - \lambda_m \left[\max\{0, f(y^m, x^m)\}/||\xi^m||^2\right] \xi^m, \tag{20}$$

$$x^{m+1} = z^m + \lambda_m[P_K(z^m) - z^m]. \tag{21}$$

**Algorithm 3**

Algorithm 3 requires, besides the exogenous data of Algorithm 1, a constant $\beta \in (0, 1/2)$ and a sequence $\{\mu_m\} \subset [\beta, 1-\beta]$. It also differs from Algorithm 1 only in step (c), which is replaced by:

c") If $0 \in \Gamma_f(y^m)$ then stop; otherwise, take $\xi^m \in \Gamma_f(y^m)$ and define

$$x^{m+1} = \mu_m[x^m - \lambda_m(\max\{0, f(y^m, x^m)\}/||\xi^m||^2)\xi^m] +$$

$$(1-\mu_m)[x_m + \lambda_m(P_K(x^m) - x^m)], \tag{22}$$

The following theorem summarizes the convergence properties of Algorithms 2 and 3.

**Theorem 4.1.** *Let $\{x^m\}$ be the sequence generated by either Algorithm 2 or Algorithm 3. Assume that $f$ satisfies assumption (A).*

*i) If the solution set of CFP is nonempty, then $\{x^m\}$ converges to a solution of EP.*

*ii) If EP lacks solutions, then $\{x^m\}$ is not convergent (though it might be bounded).*

**Proof**: We discuss first the case of finite termination, i.e. when $0 \in \Gamma_f(y^m)$. In such a case, $y^m$ minimizes $f_{y^m}(\cdot)$, and henceforth $f(y^m, \cdot)$ on $K$. Therefore, in view of P1, we have $0 = f(y^m, y^m) \leq f(y^m, z)$ for all $z \in K$, so that $y^m$ solves EP.

Now, take a solution $\bar{x}$ of CFP. We begin with Algorithm 2. Let $S_m = \{v \in I\!R^m : \langle \xi^m, v - x^m \rangle + f(y^m, x^m) \leq 0\}$ and $P_{S_m} : I\!R^m \to S_m$ the orthogonal projection onto $S_m$. It follows easily from (20) that $z^m = x^m + \lambda_m[P_{S_m}(x^m) - x^m]$. Observe that $\bar{x} \in S_m$, because, since $\bar{x}$ belongs to $L_f(y)$ for all $y \in K$ and $y^m \in K$ by step (b) of the algorithm, we have that $0 \geq f(y^m, \bar{x}) \geq f(y^m, x^m) + \langle \xi^m, \bar{x} - x^m \rangle$, using the fact that $\xi^m \in \Gamma_f(x^m)$. Then, with the same argument used to establish (10) in the proof of Theorem 3.2, we get

$$||z^m - \bar{x}||^2 \leq ||x^m - \bar{x}||^2 - \lambda_m(2-\lambda_m)||x^m - P_{S_m}(x^m)||^2 =$$

$$||x^m - \bar{x}||^2 - \lambda_m(2-\lambda_m)\left(\frac{\max\{0, f(y^m, x^m)\}}{||\xi^m||}\right)^2. \tag{23}$$

By the same token, using (21) and the fact that $\bar{x}$ belongs to $K$ because $L_f(y) \subset K$ for all $y \in K$, we get

$$||x^{m+1} - \bar{x}||^2 \leq ||z^m - \bar{x}||^2 - \lambda_m(2-\lambda_m)||z^m - P_K(z^m)||^2. \tag{24}$$

Combining (23) and (24), and uisng the fact that $\lambda_m(2-\lambda_m) > 0$, we get

$$||x^{m+1} - \bar{x}||^2 \leq ||x^m - \bar{x}||^2 -$$

$$\lambda_m(2-\lambda_m)\left[\left(\frac{\max\{0,f(y^m,x^m)\}}{||\xi^m||}\right)^2+||z^m-P_K(z^m)||^2\right]\leq||x^m-\bar{x}||^2. \tag{25}$$

As in the proof of Theorem 3.2, we get from (25) that $\{||x^m-\bar{x}||\}$ is nonincreasing, henceforth convergent, and that $\{x^m\}$ is bounded, implying that $\{\rho_m\}$ is bounded and consequently that $\{y^m\}$ is bounded. Also, (25) and the conditions on the sequence $\{\lambda_m\}$ imply that

$$\alpha(2-\alpha)\left[\left(\frac{\max\{0,f(y^m,x^m)\}}{||\xi^m||}\right)^2+||z^m-P_K(z^m)||^2\right]\leq$$

$$||x^m-\bar{x}||^2-||x^{m+1}-\bar{x}||^2. \tag{26}$$

Since $\{||x^m-\bar{x}||\}$ is convergent and $\alpha(2-\alpha)>0$, (26) implies that

$$\lim_{m\to\infty}||z^m-P_K(z^m)||=\lim_{m\to\infty}\frac{\max\{0,f(y^m,x^m)\}}{||\xi^m||}=0. \tag{27}$$

Since $\{x^m\}\subset K$ is bounded and $\xi^m\in\Gamma_f(x^m)$, we get from Assumption (A) that $\{\xi^m\}$ is bounded. Thus, we are able to conclude from (27) that $\lim_{m\to\infty}\max\{0,f(y^m,x^m)\}=0$, i.e. that $\limsup_{m\to\infty}f(y^m,x^m)\leq 0$, so that $f(y^*,x^*)\leq 0$ for any cluster point $(y^*,x^*)$ of $\{(y^m,x^m)\}$. The result follows using verbatim the argument in the proof of Theorem 3.2 after (14).

We consider now Algorithm 3. It follows from (22) that $x^{m+1}=\mu_m[x^m+\lambda_m(P_{S_m}(x^m)-x^m)]+(1-\mu_m)[x_m+\lambda_m(P_K(x^m)-x^m)]$. By convexity of $||\cdot||^2$, we get

$$||x^{m+1}-\bar{x}||^2\leq\mu_m||x^m+\lambda_m(P_{S_m}(x^m)-x^m)-\bar{x}||^2+$$

$$(1-\mu_m)||x^m+\lambda_m(P_K(x^m)-x^m)-\bar{x}||^2. \tag{28}$$

With the same argument used to establish (10) in the proof of Theorem 3.2, we get from (28), using the fact that $\lambda_m(2-\lambda_m)>0$,

$$||x^{m+1}-\bar{x}||^2\leq\mu_m\left[||x^m-\bar{x}||^2-\lambda_m(2-\lambda_m)||x^m-P_{S_m}(x^m)||^2\right]+$$

$$(1-\mu_m)\left[||x^m-\bar{x}||^2-\lambda_m(2-\lambda_m)||x^m-P_K(x^m)||^2\right]=||x^m-\bar{x}||^2-$$

$$\lambda_m(2-\lambda_m)\left[\mu_m\left(\frac{\max\{0,f(y^m,x^m)\}}{||\xi^m||}\right)^2+(1-\mu_m)||x^m-P_K(x^m)||^2\right]\leq$$

$$||x^m - \bar{x}||^2. \tag{29}$$

By (29), $\{||x^m - \bar{x}||\}$ is nonincreasing, hence convergent, so that $\{x^m\}$, and consequently $\{\rho_m\}$ and $\{y^m\}$, are bounded. Also, by (29) and the conditions on $\{\lambda_m\}$, $\{\mu_m\}$,

$$\alpha(2-\alpha)\beta\left[\left(\frac{\max\{0, f(y^m, x^m)\}}{||\xi^m||}\right)^2 + ||x^m - P_K(x^m)||^2\right] \leq$$

$$||x^m - \bar{x}||^2 - ||x^{m+1} - \bar{x}||^2. \tag{30}$$

Since $\{||x^m - \bar{x}||\}$ converges, and $\alpha(2-\alpha)\beta > 0$, it follows from (30) that $\lim_{m\to\infty} ||x^m - P_K(x^m)|| = 0$, so that all cluster points of $\{x^m\}$ belong to $K$, and that (27) holds also in this case. Then, the proof follows exactly as in the previous case. Item (ii) is proved with the argument in the proof of Theorem 3.2(ii). □

The same comments made about Algorithm 1, in the sense that its convergence properties depend upon the structure of EP, so that convergence to a solution is not guaranteed for convex feasibility problems which do not originite in an equilibrium problem, particularly when P1 does not hold, are still relevant for Algorithms 2 and 3.

Next we discuss conditions which guarantee the validity of assumption (A).

**Proposition 4.2.** *If $f : K \times K \to \mathbb{R}$ is the restriction to $K \times K$ of a continuous function $f : U \times U \to \mathbb{R}$, which satisfies P1 and P2 in $U \times U$, where $U$ is an open set containing $K$, and such that either*
*i) $f$ is of class $\mathcal{C}^1$ in $U \times U$, or*
*ii) $f$ is monotone (i.e. $f(x,y) + f(y,x) \leq 0$ for all $x, y \in U$),*
*then $f$ satisfies assumption (A).*

**Proof**: Under hypothesis (i), we know that $\nabla f : U \times U \to \mathbb{R}^n \times R^n$ is a continuous function. If we write $\nabla f(u,v) = (\nabla_x f(u,v), \nabla_y f(u,v))$ where $\nabla_y f(u,v)$ denotes the gradient of $f(u,\cdot)$ evaluated at $v \in U$, it is clear that $\nabla_y f$ is continuous on $U \times U$, so that $\gamma_f : U \to \mathbb{R}^n$, defined as $\gamma_f(x) = \nabla_y f(x,x)$, is also continuous. Observe that, for $x \in K \subset U$, $\Gamma_f(x)$ reduces to the singleton $\{\gamma_f(x)\}$. Since $\gamma_f$ is continuous on $U \supset K$, $K$ is closed and $U$ is open, the image through $\gamma_f$ of any bounded subset of $K$ is clearly bounded.

Now we look at hypothesis (ii). We claim that $\Gamma_f$ is monotone, i.e. that for all $x, y \in K$ and all $u \in \Gamma_f(x), v \in \Gamma_f(y)$ it holds that $\langle u - v, x - y\rangle \geq 0$. Observe that, since subdifferentials of convex functions are maximal monotone, the operator $T_y(x) = [\partial f(y, \cdot)](x)$ is maximal monotone for all $y \in K$, but $\Gamma_f(x) = [\partial f(x, \cdot)](x)$ is not a subdifferential, and thus a direct proof of monotonicity is indeed required, and the monotonicity of $f$ will be needed. Since $u \in \Gamma_f(x)$, $v \in \Gamma_f(y)$, using the definition of $\Gamma_f$ and P1 we have, for all $z \in K$,

$$\langle u, z - x\rangle \leq f(z, x) - f(x, x) = f(z, x), \tag{31}$$

$$\langle v, z - y\rangle \leq f(z, y) - f(y, y) = f(z, y). \tag{32}$$

Taking $z = y$ in (31), $z = x$ in (32), adding both equations, and using monotonicity of $f$,

$$\langle u - v, x - y\rangle \geq -f(y, x) - f(x, y) \geq 0, \tag{33}$$

establishing monotonicity of $\Gamma_f$. It has been proved in [16] that every monotone operator $T$ is locally bounded in the interior int $D(T)$ of its domain, meaning that for all $x \in$ int $D(T)$ there exists a neighborhood $V$ such that $\cup_{z \in V} T(z)$ is bounded. It follows, using a standard compactness argument, that $\cup_{z \in C} T(z)$ is bounded, whenever $C$ is a compact subset of int $D(T)$. The fact that $f$ can be extended to $U \times U$ keeping all its properties, means that $\Gamma_f$ can be extended to a monotone operator with domain $U$. In such a case, since $K \subset U$ and $U$ is open, every compact subset $C$ of $K$ is contained in int $D(\Gamma_f) = U$, so that $\cup_{z \in C} \Gamma_f(z)$ is bounded, because $\Gamma_f$ is monotone, and so $f$ satisfies (A). □

We remark that the function $f$ in Example (a) is monotone in the sense of hypothesis (ii) in Proposition 4.2, and the same happens with Examples (c) and (e) when $T$ is monotone, and with Example (b) when $T$ is nonexpansive.

# 5 An algorithm with an approximately remotest set control strategy for EP

If we compare the remotest set and the most violated constraint control strategies presented in Section 1, the latter seems generally preferable, because evaluating $f(y, x)$ looks simpler that computing the distance from $x$ to the level set $L_f(y)$ of $f(y, \cdot)$. In fact, the remotest set control has been

introduced for cases in which the convex sets are not presented as level sets of convex functions, but through other specifications (e.g. as the convex hull of certain points). Since, by definition of EP, the convex sets of instances of CFP which originate in EP are given as the level sets $L_f(y)$ of the convex functions $f(y,\cdot)$, algorithms for EP with the remotest set control strategy do not sem very attractive. Nevertheless, since there might be cases in which the evaluation of the distance to a level set of a function could be more convenient that the evaluation of the function itself, we state next an algorithm with an approximately remotest set control strategy, where the same sets $K_m$ as in Algorithms 1-3 are used, and which is convergent, when CFP has solutions, without compactness assumptions on $K$ or coercivity hypotheses on $f$. Again the convergence properties depend upon the EP structure and do not hold for convex feasibility properties which do not originate from EP. The convergence analysis, similar to those of Algorithms 1-3, is ommitted for the sake of brevity.

**Algorithm 4**

Algorithm 4 shares steps (a), (c) and (d) of Algorithm 1, but step (b) of Algorithm 1 is replaced by
b') Find $y^m \in K_m$ satisfying

$$\sup_{y \in K_m} ||x^m - P_{L_f(y)}(x^m)|| \leq ||x^m - P_{L_f(y^m)}(x^m)|| + \varepsilon_m.$$

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