Homoclinic tangencies and fractal invariants in arbitrary dimension *

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Abstract. For generic families of diffeomorphisms in arbitrary dimension unfolding a homoclinic tangency associated to a hyperbolic basic set, we prove that unifoprm hyperbolicity prevails in parameter space if and only if the Hausdorff dimension of the basic set is less than 1.

Tangences homoclines et invariants fractaux en dimension arbitraire

Resumé. Pour des familles génériques de difféomorphismes en dimension quelconque qui developent une tangence homocline associé a un ensemble basique hyperbolique, nous montrons que les dynamiques uniformément hyperboliques sont prévalentes dans l'espace des parametres si et seulement si la dimension de Hausdorff de l'ensemble basique est plus petite que 1.

Version française abrégée:

Les résultats sur les tangences homoclines des difféomorphismes des surfaces, obtenus dans le courant des trois dernières décennies par Newhouse, Palis, Takens, Yoccoz et Moreira dévoilèrent une relation profonde entre des invariants fractaux d'ensembles hyperboliques et les comportements dynamiques typiques, dans une famille paramétrée, après la bifurcation homocline. Voir [5, 6, 9, 3] et [7, chapitre 7] pour un panorama de la théorie.

Voici, en quelques mots, comment se manifeste cette relation : si la dimension de Hausdorff de l'ensemble hyperbolique associé à la tangence est plus petite que 1, les dynamiques hyperboliques sont prédominantes (densité de Lebesgue totale) dans l'espace des paramètres près de la tangence; si la dimension de Hausdorff est plus grande que 1, l'hyperbolicité n'est pas prédominante dans l'espace des paramètres, par contre l'union de l'hyperbolicité et des tangences persistantes l'est.

L'extension de ces résultats aux dimensions supérieures présente des difficultés importantes, liées au manque de régularité des feuilletages invariants des ensembles hyperboliques. Par example, on ne sait même pas si la dimension de Hausdorff locale est bien

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définie pour les ensembles hyperboliques dans les variétés de dimension plus grande que 2. Une partie de ces difficultés fut résolue par Palis et Viana [8], à l'aide d'une notion de différentiabilité à la Whitney, dans leur généralisation en dimension arbitraire du théorème de Newhouse [4] sur la coexistence d'une infinité de puits.

Le but de ce travail est de démontrer que, pourtant, le paradigme

$$\textit{l'hyperbolicité est prédominante} \Leftrightarrow \textit{la dimension de Hausdorff est plus petite que } 1$$

peut être étendu aux bifurcations homoclines en dimension arbitraire.

Plus précisément, nous considérons des familles à un paramètre de difféomorphismes $\varphi_\mu : M^n \to M^n$, $\mu \in (-1, 1)$ de classe C^2 qui développent une tangence homocline associée à un point périodique p_0 inclus dans un fer-à-cheval Λ_0 de φ_0. Nous supposons que φ_μ est hyperbolique (Axiome A) pour $\mu < 0$, ce qui implique que les valeurs propres les plus faibles, contractante et dilatante, de p_0 sont réelles.

Soient d_s et d_u les dimensions de Hausdorff de $W^s(p_0) \cap \Lambda_0$ et $W^u(p_0) \cap \Lambda_0$, respectivement. Nous montrons que, pour un sous-ensemble résiduel de difféomorphismes φ_0, les dimensions d_s et d_u dépendent continûment de φ_0 et on a :

i) Si $d_s + d_u < 1$ alors $H = \{\mu > 0 \mid \varphi_\mu \text{ est hyperbolique}\}$ a densité de Lebesgue totale en $\mu = 0$.

ii) Si $d_s + d_u > 1$ alors $T_s = \{\mu > 0 \mid \varphi_\mu$ présente des tangences homoclines persistantes associées à la continuation hyperbolique Λ_μ de $\Lambda_0\}$ est un ensemble ouvert avec densité de Lebesgue positive en $\mu = 0$.

La demonstration est basée sur une extension des techniques introduites par Newhouse-Palis [5], Palis-Takens [6] et Moreira-Yoccoz [2, 3] pour montrer les résultats correspondants en dimension 2, et par Palis-Viana [8] pour étudier les bifurcations homoclines de codimension 1 en dimension arbitraire.

Un des lemmes les plus importants dans ce travail est la construction (peut-être après une perturbation de φ_0) des feuilletages stable-fort et instable-fort de codimension 1 pour des sous-ensembles hyperboliques de Λ_0 avec presque les mêmes dimensions d_s et d_u. Ces feuilletages sont utilisées pour réduire l'étude des géometries des feuilletages stable et instable près de la tangence homocline initiale a une situation proche du cas bidimensionel.

Il est intéressant de remarquer que la dimension critique 1 est independante de la dimension n de l'espace ambiant. Donc, dans un certain sens, le deuxième cas devient relativement plus fréquent quand n croît.

1 Introduction

Over the last three decades, a number of results have been proved by Newhouse, Palis, Takens, Yoccoz and Moreira, unveiling a deep connection between fractal invariants of hyperbolic sets and the typical dynamics observed in the unfolding of a homoclinic tangency by a parametrized family of surface diffeomorphisms. See [5, 6, 9, 3] and [7, chapter 7] for an overview.

In brief terms, if the Hausdorff dimension of the hyperbolic set involved in the tangency is less than 1, then hyperbolicity prevails in parameter space (full Lebesgue density at the tangency parameter); if the Hausdorff dimension is larger than 1 then hyperbolicity alone is not prevalent, but the union of hyperbolicity and persistent tangencies is.

The extension of these results to higher dimension involves considerable difficulties, related to the lack of smoothness of invariant foliations of hyperbolic sets of codimension larger than 1. For instance, it is not even known whether the local Hausdorff dimension is well-defined for hyperbolic sets in high dimension manifolds. Part of these difficulties could be handled by Palis and Viana [8], by means of a notion of differentiability à la Whitney, in their generalization to arbitrary dimension of Newhouse's theorem [4] on coexistence of infinitely many sinks.

Our goal in this work is to prove that, nevertheless, the paradigm

$$\textit{hyperbolicity prevails} \Leftrightarrow \textit{the Hausdoff dimension is smaller than } 1$$

does extend to homoclinic bifurcations in any dimension.

More precisely we consider 1-parameter families of diffeomorphisms $\varphi_\mu : M^n \to M^n$, $\mu \in (-1, 1)$ of class C^2, unfolding a generic homoclinic tangency associated to a periodic point p_0 contained in a horseshoe Λ_0 of φ_0. We suppose that φ_μ is hyperbolic (Axiom A) for $\mu < 0$, which implies that the weakest contracting and weakest expanding eigenvalues of p_0 are real numbers.

Let d_s and d_u be the Hausdorff dimension of $W^s(p_0) \cap \Lambda_0$ and $W^u(p_0) \cap \Lambda_0$, respectively. We show that, for a residual subset of diffeomorphisms, d_s and d_u are continuous with respect to φ_0, and we have:

i) If $d_s + d_u < 1$ then $H = \{\mu > 0 \mid \varphi_\mu \text{ is hyperbolic}\}$ has full Lebesgue density at $\mu = 0$.

ii) If $d_s + d_u > 1$ then $T_s = \{\mu > 0 \mid \varphi_\mu$ presents persistent homoclinic tangencies associated to the hyperbolic continuation Λ_μ of $\Lambda_0\}$ is an open set with positive lower density at $\mu = 0$.

The proof is based on an extension of the techniques introduced by Newhouse-Palis [5], Palis-Takens [6], and Moreira-Yoccoz [2, 3] to prove the corresponding results in dimension 2, and by Palis-Viana [8] to investigate homoclinic bifurcations of codimension one in arbitrary dimension.

One of our main lemmas is the construction (perhaps after perturbation of φ_0) of strong-stable and strong-unstable foliations of codimension 1 for hyperbolic subsets of Λ_0 with almost the same dimensions d_s and d_u. These foliations are used to (essentially) reduce the study of the geometries of the stable and unstable foliations near the initial homoclinic tangency to the bidimensional case.

It is interesting to note that the separating dimension 1 is independent of the dimension n of the ambient space. Thus, in some sense, the second case above becomes more frequent when n increases.

2 Geometry of horseshoes : the upper dimension

Let Λ be a horseshoe for a diffeomorphism $\varphi : M \to M$, and P be some Markov partition of Λ. We call *vertical n-cylinder* any subset of Λ defined by prescribing the first n symbols in the backward itinerary with respect to P. Let $\mathcal{V}_n$ be the set of vertical n-cylinders. There is a dual notion of *horizontal n-cylinder*, where one considers forward itinerary.

Fix $\varepsilon > 0$ small. For each $V \in \mathcal{V}_n$ let $D_s(V) = \sup\{\operatorname{diam}(W^s_\varepsilon(x) \cap V) \mid x \in \Lambda \cap V\}$. We define $\widetilde{\lambda}_n$ by the relation

$$\sum_{V \in \mathcal{V}_n} D_s(V)^{\widetilde{\lambda}_n} = 1.$$

and we let the *upper stable dimension* be given by $\overline{d}_s(\Lambda) = \lim_{n\to\infty} \widetilde{\lambda}_n$. There is a dual notion of *upper unstable dimension* $\overline{d}_u(\Lambda)$, dealing with W^u_ε instead of W^s_ε.

It is not difficult to show that the limit always exists, and $\overline{d}_s$ is an upper-semicontinuous function of φ. Moreover,

$$\operatorname{HD}(W^s_\varepsilon(x) \cap \Lambda) \le \overline{d}_s(\Lambda) \quad \text{for each } x \in \Lambda.$$

The following result shows that the equality holds in most cases where $\overline{d}_s < 1$:

Theorem 1. *Suppose $\overline{d}_s(\Lambda) < 1$. Then there is a residual set of diffeomorphisms $\mathcal{R}$ in a neghbourhood of φ such that, for all $\psi \in \mathcal{R}$ and all $x \in \Lambda_\psi$,*

$$\operatorname{HD}(\Lambda_\psi \cap W^s_\varepsilon(x)) = \overline{d}_s(\Lambda_\psi).$$

In addition, $\mathcal{R}$ may be taken so that its elements are points of continuity of $\overline{d}_s$.

The idea of the proof is to consider k-parameter families of diffeomorphisms ψ, for some large k, and to show that for each $V \in \mathcal{V}_n$ we have $\operatorname{diam}(W^s_\varepsilon(x) \cap V) \ge D_s(V)^{1-\delta}$ for every $x \in \Lambda_\psi$, with large probability in parameter space. This gives that, for most parameters, most of the vertical cylinders have subexponential variation of their horizontal diameters. Using this fact, we obtain lower estimates for $\operatorname{HD}(W^s_\varepsilon(x) \cap \Lambda_\psi)$ that imply the conclusion of the theorem.

Now we can state our

Main Result. *There are open sets $\mathcal{U}$ and $\mathcal{V}$ of families of diffeomorphisms unfolding a homoclinic tangency as in the Introduction, such that $\mathcal{U} \cup \mathcal{V}$ is dense and*

i) For families in $\mathcal{U}$ we have $\overline{d}_s(\Lambda) + \overline{d}_u(\Lambda) < 1$ and $\lim_{\delta\to 0} m(H \cap [0,\delta])/\delta = 1$, where $H = \{\mu \mid \varphi_\mu$ is hyperbolic $\}$.

ii) For families in $\mathcal{V}$ we have $\overline{d}_s(\Lambda) + \overline{d}_v(\Lambda) > 1$ and $\liminf_{\delta\to 0} m\ (T_s \cap\ [0,\delta])/\delta > 0$, where $T_s = \{\mu \mid \varphi_\mu$ presents a persistent homoclinic tangency associated to $\Lambda_\mu\}$.

In the next sections we indicate main ideas of the proof of this theorem.

3 Prevalence of hyperbolicity

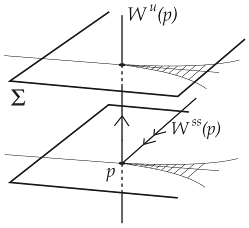
In this section we outline the proof of the first part of the main theorem: hyperbolicity prevails in parameter space when the sum of the stable and unstable upper dimensions is less than 1. The first main step is a reduction to the case when the horseshoe Λ does not intersect the strong-stable manifold, nor the strong-unstable manifold of the point p.

Theorem 2. *Assume $\overline{d}_s(\Lambda) < 1$ (respectively, $\overline{d}_u(\Lambda) < 1$). For an open and dense subset of maps in some neighborhood of φ we have*

$$\Lambda \cap W^{ss}(p) = \{p\} \quad (\textit{respectively, } \Lambda \cap W^{uu}(p) = \{p\}).$$

By continuity, $\overline{d}_s(\Lambda_\psi) < 1$ for any ψ in a whole neigborhood of φ. The density statement in the theorem is proved by deformation. We embed ψ in a one-parameter family of diffeomorphisms ψ_ν such that $\Lambda_\nu \cap W^s(p_{\psi_\nu})$ moves with respect to $W^{ss}(p_{\psi_\nu})$ when ν varies. The fact that the upper Hausdorff dimension is less than 1 ensures that $\Lambda_{\psi_\nu} \cap W^s(p_{\psi_\nu})$ intersects $W^{ss}(p_{\psi_\nu})$ only for a zero Lebesgue measure set of parameters ν. This implies density, and openness is clear from compactness of Λ_{ψ_ν} and of a fundamental domain of $W^{ss}(p_{\psi_\nu})$.

Theorem 2 has the important geometric consequence that $\Lambda \cap W^s(p)$ is contained in a cuspidal region around the weak-stable direction of p. Let K^s be the projection of $\Lambda \cap W^s(p)$ to this weak-stable direction along the strong-stable foliation of p. Then, for any $\delta > 0$ there exists $\varepsilon > 0$ such that each $x \in \Lambda \cap W^s(p)$ in the ε-neighborhood of p is within distance $\varepsilon\delta$ from some $y \in K_s$. We say that $\Lambda \cap W^s(p)$ is *well-represented* by K_s near p.



Then a similar fact is true replacing $W^s(p)$ by any other cross-section Σ to the unstable direction at any $q \in W^u(p)$: the intersection $W^u(p) \cap \Sigma$ is well-represented by K_s near q. Using also smoothness of the unstable leaves, together with Hölder continuity of their tangent bundle, we conclude that the intersection of $W^u(\Lambda)$ with a neighborhood of q is well-represented by the product of K_s by $T_qW^u(p)$ near q.

We consider q to be the point of homoclinic tangency. The previous paragraph, together with a dual statement for $W^s(\Lambda)$, mean that the stable and unstable foliations of Λ are well-represented by affine models

$$K_s \times T_qW^u(p) \quad \text{and} \quad K_u \times T_qW^s(p)$$

near q. By continuity of the invariant foliations this remains true when the tangency is unfolded along a one-parameter family φ_μ: if μ is close to zero the $W^s(\Lambda_\mu)$ and $W^u(\Lambda_\mu)$ are well-represented by translates

$$[K_s \times T_q W^u(p)] + \mu v_1 \quad \text{and} \quad [K_u \times T_q W^s(p)] + \mu v_2$$

for some vectors v_1, v_2. So, along the lines of [6] and [10], to ensure that the two foliations are transverse and, in fact, the angles at their intersection are bounded away from zero for most parameters, it suffices to prove a corresponding fact for these affine models:

Lemma 3. *Let K_1, K_2 be subsets of $\mathbb{R}^+$ with sum of limit capacities $c(K_1) + c(K_2) < 1$. Let V_1 and V_2 be subspaces of $\mathbb{R}^n$ with dim $V_1 = u - 1$, dim $V_2 = s - 1$, and $u + s = n$, and $w_1 \notin V_1$ and $w_2 \notin V_2$ be vectors such that the closed half-subspaces $\mathbb{R}^+ w_1 \oplus V_1$ and $\mathbb{R}^+ w_2 \oplus V_2$ intersect at the origin. Then most translates of $K_1 w_1 \oplus V_1$ and $K_2 w_2 \oplus V_2$ do not intersect: for any vectors v_1, and v_2 of $\mathbb{R}^n$ in general position, and for any $\delta > 0$*

$$d(K_1 w_1 \oplus V_1 + \mu v_1, K_2 w_2 \oplus V_2 + \mu v_2) > 3\varepsilon\delta$$

for a fraction of parameters in $[-\varepsilon, \varepsilon]$ that goes to 1 when $\varepsilon \to 0$.

4 Global strong-stable and strong-unstable foliations

In order to prove the second part of our main result, we consider compact subsets of our basic set that are invariant by some iterate of the dynamics at the first bifurcation parameter and have good geometric properties (perhaps after a small perturbation of the family). More precisely, in case ii) of the theorem we prove that, generically, there are disjoint subsets Λ_1 and Λ_2 of Λ, invariant by some iterate $\varphi_0^{n_0}$, with the following properties:

a) Λ_1 has a globally defined strong-stable foliation, invariant under $\varphi_0^{n_0}$.

b) Λ_2 has a globally defined strong-unstable foliation, invariant under $\varphi_0^{n_0}$.

c) $\overline{d}_s(\Lambda_1) < 1$, $\overline{d}_u(\Lambda_2) < 1$, and $\overline{d}_s(\Lambda_1) + \overline{d}_u(\Lambda_2) > 1$

d) For each $p_1 \in \Lambda_1$ and $p_2 \in \Lambda_2$, the projections $\pi_{ss} \mid (\Lambda_1 \cap W^s(p_1))$ along strong-stable leaves and $\pi_{uu} \mid (\Lambda_2 \cap W^u(p_2))$ are injective and have *intrinsically* $C^{1+\varepsilon}$ inverses, in the sense of [8].

e) Either Λ_1 contracts area in the weak (bidimensional) direction or Λ_2 expands area in that weak direction.

The main and most delicate points of the above statement are items a) and b). Let us give some ideas of their proof.

First, we show that, perhaps after a small perturbation, Λ has a periodic point whose weakest, contracting and expanding, eigenvalues are real. Next, we use the existence of this periodic point to show that a sizable proportion of the vertical (and of the horizontal) n-cylinders in an advanced stage $n \gg 1$ of the construction of Λ are exponentially squeezed along the corresponding direction.

Finally, we consider families of perturbations with a large number of parameters, whose effect is to rotate the rectangles of a Markov partition of Λ with enough independence, in order to show that, with large probability, there are invariant subsets of Λ with almost the same stable and unstable upper dimensions as Λ, and admitting invariant strong-stable (respectively, strong-unstable) cone fields.

5 Positive density of persistent tangencies

Conditions a) to d) of the previous section imply that the projection of $\Lambda_1 \cap W^s(p_1)$ onto a weak-stable (1-dimensional) manifold, along the strong-stable foliation, is a regular $C^{1+\varepsilon}$ Cantor set K^s, whose conjugation class does not depend on $p_1 \in \Lambda_1$ nor on the choice of the weak-stable leaf. The same holds for $\pi_{uu}(\Lambda_2 \cap W^u(p_2)) = K^u$.

We adapt arguments of [1] to show the next lemma. It is assumed that the eigenvalues of φ at the point p satisfy a generic non-resonance condition.

Lemma 4. *If there are $\lambda > 0$ and $t \in \mathbb{R}$ such that K^s intersects $\lambda K^u + t$ stably, then $(\varphi_\mu)_\mu$ persistently exhibits positive density of stable tangencies at $\mu = 0$.*

In order to get the stable intersection between K^s and $\lambda K^u + t$, we first prove that condition e) implies that either K^s or K^u is a C^2-regular Cantor set on the central leaf. This is important to show that, generically, K^s and K^u satisfy the Scale Recurrence Lemma, the main technical tool of [2, 3]. Afterwards, we consider again a family with a large number of parameters of local perturbations of the diffeomorphism φ_0 and show that for most parameters there are λ and t such that K^s stably intersects $\lambda K^u + t$. This final step is adapted from the perturbation arguments of [2, 3]. This allows us to to use the previous lemma and conclude the proof of part ii) of the main theorem.

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