# On Deformation of Foliations with a Center in the Projective Space

Hossein Movasati[1]

**Abstract**

Let $\mathcal{F}$ be a foliation in $\mathbb{C}P(2)$ with a first integral of the type $\frac{F^p}{G^q}$, where $F$ and $G$ are two polynomials on an affine coordinate $\mathbb{C}^2$, $\frac{deg(F)}{deg(G)}=\frac{q}{p}$ and $g.c.d.(p,q)=1$. Let $z$ be a nondegenerate critical point of $\frac{F^p}{G^q}$, which is a center singularity of $\mathcal{F}$, and $\mathcal{F}_t$ be a deformation of $\mathcal{F}$ in the space of foliations of degree $deg(\mathcal{F})$ such that its unique deformed singularity $z_t$ near $z$ persists in being center. We will prove that the foliation $\mathcal{F}_t$ has a first integral of the same type of $\mathcal{F}$. To prove this fact we will introduce the notion of generalized integrating factor. We will also encounter a certain class of relatively exact 1-forms modulo the foliation $\mathcal{F}$. Finally we will give a lower bound for the maximum number of limit cycles of real polynomial differential equations of a fixed degree in $\mathbb{R}^2$.

## 1 Introduction and Basic Results

### 1.1 Foliations in the Projective Space

Let us consider a polynomial differential equation in $\mathbb{C}^2$:

$$\omega = P(x,y)dy - Q(x,y)dx = 0 \tag{1}$$

where $P$ and $Q$ are relatively prime polynomials. Such an equation defines a foliation $\mathcal{F}$ in $\mathbb{C}^2$ with a finite number of singularities $\{P=0\}\cap\{Q=0\}$. In other words, the foliation $\mathcal{F}$ is given by the 1-form $\omega = Pdy - Qdx$. By $\mathcal{F}(\omega)$ we will mean this statement.

The foliation $\mathcal{F}(\omega)$ extends to $\mathbb{C}P(2)$ as follows: The projective space $\mathbb{C}P(2)$ has three affine coordinates $(x,y)$, $(u,v)$, $(u',v')$ related by the transition functions:

$$\phi(u,v) = (\frac{1}{u},\frac{v}{u}) = (x,y) \text{ and } \phi'(u',v') = (\frac{v'}{u'},\frac{1}{u'}) = (x,y)$$

Let $k$ be the smallest positive integer number such that

$$u^k\phi^*(\omega)(u,v) = \tilde{P}(u,v)dv - \tilde{Q}(u,v)du = 0 \tag{2}$$

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is a polynomial equation in the coordinate $(u,v) \in \mathbb{C}^2$. The foliation $\mathcal{F}(\omega)$ extends to the coordinates $(u,v) \in \mathbb{C}^2$ by the equation (2). In the same way the foliation $\mathcal{F}(\omega)$ extends to the coordinate $(u',v') \in \mathbb{C}^2$. The foliation $\mathcal{F}(\omega)$ may have singularities in the line at infinity $L_\infty = \mathbb{C}P(2)\backslash\mathbb{C}^2$. We denote by $sing(\mathcal{F})$ the set of singularities of $\mathcal{F}$ in $\mathbb{C}P(2)$.

**Definition 1.1.1** Let $\omega = Pdy - Qdx$ be a polynomial 1-form and

$$P = P_0 + \cdots + P_{d+1},\ Q = Q_0 + \cdots + Q_{d+1},\ Q_{d+1} \not\equiv 0 \text{ or } P_{d+1} \not\equiv 0$$

be the homogeneous decomposition of $P$ and $Q$. Define the degree of $\omega$ as follows:

$$deg(\omega) = \begin{cases} d & \text{if } P_{d+1}y - Q_{d+1}x \equiv 0 \\ d+1 & \text{if } P_{d+1}y - Q_{d+1}x \not\equiv 0 \end{cases}$$

Throughout the text, we will see that why this definition is reasonable. For example the reader can check that the number $k$ appeared in the equation (2) is $deg(\omega)+2$.

It is a well-known fact that each holomorphic singular foliation in $\mathbb{C}P(2)$ is given by a unique polynomial differential equation considered in (1) , up to multiplication by a constant (see [13]). Therefore, we can define the degree of the foliation $\mathcal{F}(\omega)$ as the degree of the 1-form $\omega$. It is a good exercise to the reader to show that $deg(\mathcal{F})$ is the number of tangency points of a generic line in $\mathbb{C}P(2)$ with the foliation $\mathcal{F}$ (see [13]). Denote by $\mathcal{F}(2,d)$ the set of degree d foliations in $\mathbb{C}P(2)$. It follows immediately that $\mathcal{F}(2,d)$ is a Zariski open subset of the projectivization of the vector space

$$\{\omega \mid \omega \text{ is a polynomial 1-form of degree } \leq d\}$$

Let $\mathcal{F}(Pdy - Qdx)$ be a germ of foliation at $(\mathbb{C}^2, 0)$ which has nondegenerate singularity at 0 i.e., the map

$$\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$$

$$\phi(x,y) = (P(x,y), Q(x,y))$$

satisfies $det(D\phi(0)) \neq 0$. Let also $\mathcal{F}_t(P_t dy - Q_t dx)$, $t \in (\mathbb{C}, 0)$ be a holomorphic deformation of $\mathcal{F}$. Applying the implicit function theorem to the map

$$(x,y,t) \to (P_t(x,y), Q_t(x,y))$$

at the point $(0,0,0)$, we conclude that there is a unique singularity $p_t$ of $\mathcal{F}_t$ near 0 and moreover, $p_t$ as a function with variable $t$ is holomorphic.

The reader is referred to [13] for more informations about holomorphic foliations in the projective spaces.

## 1.2 Integrable Foliations

In this section we will introduce some of simple foliations and singularities.

**Definition 1.2.1** Let $\mathcal{F}$ be a germ of singular foliation at $(\mathbb{C}^2, 0)$. We say that 0 is a center singularity of $\mathcal{F}$ or simply a center of $\mathcal{F}$, if there exists a germ of holomorphic function $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ which has nondegenerate critical point at 0, and the leaves of $\mathcal{F}$ are given by $f = const.$. The point 0 is also called the Morse singularity of $f$.

Morse lemma in the complex case implies that there exists a local coordinate system $(x, y)$ in $(\mathbb{C}^2, 0)$ with $x(0) = 0,\ y(0) = 0$ and such that $f(x, y) = x^2 + y^2$. Near the center the leaves of $\mathcal{F}$ are homeomorphic to the cylinder, therefore each leaf has a nontrivial closed cycle that will be called the Lefschetz vanishing cycle (see Figure 1 in Section 2). In this new coordinate the foliation is given by

$$\mathcal{F} : d(x^2 + y^2)/2 = xdx + ydy = 0$$

In the sequel by $\mathcal{P}_d$ we will refer to the set of polynomials of maximum degree $d$ on $\mathbb{C}^2$. Let $(F, G) \in \mathcal{P}_{a+1} \times \mathcal{P}_{b+1}$, $\frac{a+1}{b+1} = \frac{q}{p}$ and $g.c.d.(p, q) = 1$. The foliation $\mathcal{F} = \mathcal{F}(pGdF - qFdG)$ has the first integral:

$$f : \mathbb{C}P(2) \backslash (\{F = 0\} \cap \{G = 0\}) \to \overline{\mathbb{C}}$$

$$f(x, y) = \frac{F(x, y)^p}{G(x, y)^q}$$

i.e., the leaves of the foliation $\mathcal{F}$ are contained in the level surfaces of $f$. $\mathcal{F}$ has two distinct type of singularities:

- The isolated critical points of $f$, $C(\mathcal{F}) = \{p_1, p_2, \dots, p_r\}$. We denote by $c_1, \dots, c_r$ the corresponding critical values of $f$. If $p_i$ is a nondegenerate critical point of $\mathcal{F}$ then $p_i$ is a center of $\mathcal{F}$;

- The dicritical singularities of $\mathcal{F}$, $R(\mathcal{F}) = \{F = 0\} \cap \{G = 0\} = \{q_1, q_2, \dots, q_s\}$.

Let $q_i$ be a dicritical singularity of $\mathcal{F}$ in an affine coordinate $\mathbb{C}^2$ and let $\{F = 0\}$ intersect $\{G = 0\}$ transversally at $q_i$. After the change of coordinate

$$\phi : (\mathbb{C}^2, q_i) \to (\mathbb{C}^2, 0)$$

$$(x, y) \to (F, G) = (\tilde{x}, \tilde{y})$$

in a neighborhood of $q_i$, the foliation $\mathcal{F}$ in this coordinate is given by the equation:

$$p\tilde{y}d\tilde{x} - q\tilde{x}d\tilde{y} = 0 \tag{3}$$

Throughout the text by a dicritical singularity we will mean a singularity of the form (3). In a dicritical singularity $q_i$ after a finite number of blow ups at the singularity, the leaves separate from each other (see [3] and [13] ).

**Definition 1.2.2** We denote by $\mathcal{I}(a,b)$ the set of all integrable foliations $\mathcal{F}(pGdF-qFdG)$, where $(F,G)\in\mathcal{P}_{a+1}\times\mathcal{P}_{b+1}$ and we denote by $\mathcal{I}_m(a,b)$ the set of $\mathcal{F}(pGdF-qFdG)\in\mathcal{I}(a,b)$ such that all singularities of $\mathcal{F}$ are center or dicritical and the image of the centers under $\frac{F^p}{G^q}$, 0 and $\infty$ are distinct points in $\overline{\mathbb{C}}$.

**Proposition 1.2.1** *For every $\mathcal{F}\in\mathcal{I}_m(a,b)$ we have:*

$$s=\#R(\mathcal{F})=(a+1).(b+1),\ \ deg(\mathcal{F})=a+b,\ \ r=\#C(\mathcal{F})=a^2+b^2+a.b$$

**Proof:** According to the definition of dicritical singularities, $\{F=0\}$ intersects $\{G=0\}$ transversally and so $s=\#R(\mathcal{F})=(a+1).(b+1)$.
The Baum-Bott index of a center is zero and of a dicritical is $\frac{(p+q)^2}{pq}$. If $p$ is a dicritical or a center singularity then $\mu(p)=1$, where $\mu(p)$ is the Milnor number of the singularity $p$. Let $\mathcal{F}\in\mathcal{F}(2,d)$ then we have the following formulas:

$$\sum_{p\in sin(\mathcal{F})} BB(p)=(d+2)^2$$

$$\sum_{p\in sin(\mathcal{F})} \mu(p)=d^2+d+1$$

About the Milnor number and Baum-Bott indices and the above formulas see [13]. Let $a+1=d'.q$ and $b+1=d'.p$, where $d'=g.c.d.(a+1,b+1)$. For the foliation $\mathcal{F}\in\mathcal{I}_m(a,b)$ the first formula implies that

$$\frac{(p+q)^2}{pq}.s=(p+q)^2d'^2=(d+2)^2\Rightarrow d=a+b$$

Also, the second formula implies that $r=d^2+d+1-s=(a+b)^2+(a+b)+1-(a+1)(b+1)=a^2+b^2+a.b.\diamond$

**Proposition 1.2.2** *$\mathcal{I}_m(a,b)$ is an open dense set in $\mathcal{I}(a,b)$.*

The proof can be found in the appendix of [17]. The transversality theorem in differentiable topology or algebraic geometry is used in the proof of this proposition.

**Remark 1** It is easy to see that $\mathcal{F}\in\mathcal{I}_m(a,b)$ is a rational fibration (a fibration whose generic fiber has genus zero) if and only if $a+b\le 2$. In fact only in the case $a+b\le 2$ the fibration $\mathcal{F}$ has reducible critical fibers. In the case $a=b=1$ Theorem 1.3.3 is not true and hence Theorem 1.3.1 is not true also. In this case it is easy to find counterexamples for Theorem 1.3.1 by performing deformations inside the logarithmic foliations. From now on we will assume that $a+b>2$.

## 1.3 Main Theorem

Centers appear in two areas of mathematics: In Algebraic Geometry, first in the famous article of S. Lefschetz [12], and in Differential Equations, first (as the author know) in the article of H. Dulac [8]. To study the topology of an algebraic variety $X$, Lefschetz uses a pencil of hyperplanes in generic position with respect to $X$ and obtains an integrable foliation with dicritical and center singularities (the notations used in this article). The Centers in his work are due to the hyperplanes which are not in generic position with respect to the variety $X$. For our purpose, we have used some results in this area of mathematics, especially the results obtained in [11] and [5], in the proof of Theorem 1.3.3.
Also, Dulac in his article [8] has classified all polynomial differential equations of degree two in $\mathbb{C}^2$ with a center singularity at origin and the line at infinity invariant. After him, D. Cerveau and A. Lins Neto in [6] completed this classification with the new definition of degree, which we have used in this article. Such classifications show how the existence of a center in a polynomial differential equation can affect the whole equation. Yu.S. Ilyashenko in [10] looks to the existence of a center in polynomial equations in another way. He shows that the persistence of a center after deformation of a generic Hamiltonian equation implies that the deformed equation is also Hamiltonian. In this article we are concerned with the last type of problem related with centers and we follow Ilyashenko's program: Dividing the main theorem 1.3.1 into three theorems 1.3.2, 1.3.3, 1.3.4.

Let $\mathcal{F}_t \in \mathcal{F}(2,d)$, $t \in (\mathbb{C},0)$ be a holomorphic deformation of the foliation $\mathcal{F} = \mathcal{F}_0$ which has at least one center for all $t \in (\mathbb{C},0)$. If we know the structure of $\mathcal{F}$, for example the type of singularities, holonomy groups of some leaves and etc., then what can be said about the deformed foliation $\mathcal{F}_t$?
The foliation $\mathcal{F}$ has a finite number of singularities therefore, there exists a center $p \in sing(\mathcal{F})$ and a sequence of centers $p_{t_i} \in sing(\mathcal{F}_{t_i})$ such that $p_{t_i} \to p$ when $t_i \to 0$. The deformed holonomy $h_t(x)$ (see Section 3), related to the vanishing cycle around $p$, is the identity for $t = t_i$. Since $h_t$ is holomorphic in $t$, $h_t$ is the identity for all $t$. Later in corollary 1.4.1, we will see that for all $t \in (\mathbb{C},0)$, $p_t$ is a center singularity of $\mathcal{F}_t$.

Given an integrable foliation $\mathcal{F}(pGdF - qFdG) \in \mathcal{I}(a,b)$. Let $p$ be one of the center singularities of $\mathcal{F}$ and let $\mathcal{F}_t$ be a holomorphic deformation of $\mathcal{F}$ in $\mathcal{F}(2,d)$, where $d = a+b$, such that its unique singularity $p_t$ near $p$ is still a center. Our principal theorem in this article is the following:

**Theorem 1.3.1** *In the above situation, there exists an open dense subset $U$ of $\mathcal{I}(a,b)$ such that for all $\mathcal{F}(pGdF - qFdG) \in U$, $\mathcal{F}_t$ is also an integrable foliation. More precisely, there exist polynomials $F_t$ and $G_t$ such that $\mathcal{F}_t = \mathcal{F}(pG_tdF_t - qF_tdG_t)$, where $F_t$ and $G_t$ are holomorphic in $t$ and $F_0 = F$ and $G_0 = G$.*

This theorem also says that the persistence of one center implies the persistence of all other center and dicritical singularities.

The idea of the proof of Theorem 1.3.1 lies in the theorems 1.3.2, 1.3.3, 1.3.4 which we state below. First we consider the local situation of the theorem.

Given a germ of center singularity $\mathcal{F}(xdx+ydy)$. Let $\mathcal{F}_t(xdx+ydy+t\omega_1+h.o.t.)$ be a deformation of $\mathcal{F}$ such that its unique deformed singularity $p_t$ near 0 is still a center. The 1-form $\omega_1$ is called "tangent vector" of the deformation $\mathcal{F}_t$.

**Theorem 1.3.2** *In the above situation we have:*

$$\int_\delta \omega_1 = 0$$

*for all Lefschetz vanishing cycles $\delta$ in the leaves around the center* 0.

In Section 3, we will introduce the concept of multivalued integrating factor and this will give us a more general result.

**Definition 1.3.1** Let $\mathcal{F} \in \mathcal{F}(2,d)$ and let $\delta$ and $\delta'$ be two closed cycles contained in two leaves of $\mathcal{F}$. We say that the cycle $\delta$ is $\mathcal{F}$-equivalent with $\delta'$, if there is a continuous sequence of closed cycles $\delta_t$ , $0 \leq t \leq 1$ such that

- $\delta_t$ is a cycle in some leaf of $\mathcal{F}$;
- $\delta_0 = \delta$ and $\delta_1 = \delta'$.

A closed cycle $\delta$ is called a vanishing cycle if it is $\mathcal{F}$-equivalent with a zero cycle, which is a point $p$, in $(\mathbb{C}P(2)\backslash sing(\mathcal{F})) \cup \{p\}$, and is called a Lefschetz vanishing cycle if it is $\mathcal{F}$-equivalent with the singularity $p$ of a center in $(\mathbb{C}P(2)\backslash sing(\mathcal{F}))\cup\{p\}$. The Novikov's definition of vanishing cycles can be obtained by this definition.

Note that here we consider a singularity as a leaf and therefore the cycles introduced around the centers satisfy this definition.

**Remark 2** Let $\delta$ be a closed cycle with initial and end point $p$ in a leaf $L$ of a foliation $\mathcal{F}$, and $\Sigma$ be a transverse section at $p$ to $\mathcal{F}$. Suppose that the holonomy $h$ of $\delta$ in a transversal section $\Sigma$ is trivial. In this case we can lift the cycle $\delta$ to a closed cycle $\delta_{z,h(z)} \subset L_z$, where $z$ is a point in $\Sigma$ and $L_z$ is the leaf through $z$. Define the holomorphic function

$$\tilde{h}(z) = \int_{\delta_{z,h(z)}} \omega_1$$

where $\omega_1$ is a meromorphic 1-form on $\mathbb{C}P(2)$. By definition, it can be seen that $\delta_{z,h(z)}$ is $\mathcal{F}$-equivalent with $\delta$. The reader can verify easily that if $\tilde{h} \equiv 0$ then for all cycles $\delta'$ $\mathcal{F}$-equivalent with $\delta$ in $\mathbb{C}P(2)\backslash(sing(\mathcal{F}) \cup (\omega_1)_\infty)$ we have also $\int_{\delta'} \omega_1 = 0$.

Keeping the notations used in the section 1.2, we have

**Theorem 1.3.3** *Let $\mathcal{F}(pGdF - qFdG) \in \mathcal{I}_m(a,b)$, $L$ be a regular leaf of $\mathcal{F}$ and $\delta$ be a Lefschetz vanishing cycle in $L$. Then the set of cycles in $L$ which are $\mathcal{F}$-equivalent with $\delta$ in $\mathbb{C}P(2)\backslash(\cup_{i=1}^r f^{-1}(c_i) \cup \{F=0\} \cup \{G=0\})$ generate the first homology group of $L$ in an arbitrary field.*

The proof ot Theorem 1.3.3 can be found in [17].

Let $p$ be the center of $\mathcal{F}$ in the theorem 1.3.1 which persists in being center after deformation of $\mathcal{F}$. Applying Theorem 1.3.2 to the deformation

$$\mathcal{F}_t : p\frac{dF}{F} - q\frac{dG}{G} + t\frac{\omega_1}{FG} + h.o.t.$$

we obtain that

(4) $$\int_\delta \frac{\omega_1}{FG} = 0$$

for all vanishing cycles $\delta$ in the leaves around $p$ and by theorem 1.3.3 and Remark 2, we conclude that the equality (4) holds for all closed cycles $\delta$ in the leaves of the foliation $\mathcal{F}$, where the integral is defined. The 1-form $\frac{\omega_1}{FG}$ is called a relatively exact 1-form modulo the foliation $\mathcal{F}(pGdF - qFdG)$.

**Theorem 1.3.4** *Let $\mathcal{F}(pGdF - qFdG) \in \mathcal{I}_m(a,b)$ and a polynomial 1-form $\omega_1$ have the same degree of $\mathcal{F}$. Let also $\frac{\omega_1}{FG}$ be relatively exact modulo $\mathcal{F}$. Then there is $(P,Q) \in \mathcal{P}_{a+1} \times \mathcal{P}_{b+1}$ such that $\omega_1$ has the form*

(5) $$\omega_1 = pGdP - qPdG + pQdF - qFdQ$$

In [18] we have classified all relatively exact meromorphic 1-forms modulo $\mathcal{F}(pGdF - qFdG)$ and as a corollary we have proved Theorem 1.3.4.

## 1.4 Space of Foliations with Centers

Let $\mathcal{M}(2,d)$ be the set of foliations of degree $d$ in $\mathbb{C}P(2)$ with at least one center. The following example gives us a huge number of these foliations:

**Example 1** Let $\tau : \mathbb{C}P(2) \to M$ be a holomorphic map between $\mathbb{C}P(2)$ and a complex compact manifold $M$ with $dim(M) \geq 3$. Let also $\mathcal{F}$ be a holomorphic codimension one singular foliation in $M$ (see [13]). We say that $\tau$ has a tangency point $a \in \mathbb{C}P(2)$ of order two with $\mathcal{F}$ if $\tau(a)$ is a regular point of the foliation $\mathcal{F}$ and, if $\mathcal{F}$ in a coordinate $(x,y) \in (\mathbb{C}^{n-1},0) \times (\mathbb{C},0)$ around $a$ is given by $y = const.$, then $a$ is a nondegenerate critical point of $y \circ \tau$. This says that the pullback foliation $\tau^*(\mathcal{F})$ has a center at the point $a$.

I have learned the statement and proof of the following proposition from A. Lins Neto.

**Proposition 1.4.1** *$\mathcal{M}(2,d)$ is an algebraic subset of $\mathcal{F}(2,d)$.*

**Proof:** Let $\mathcal{M}_0(2,d)$ be the set of all foliations in $\mathcal{M}(2,d)$ with a center at origin $(0,0) \in \mathbb{C}^2 \subset \mathbb{C}P(2)$ and with a local first integral of the type

$$f = xy + f_3 + f_4 + \cdots + f_n + h.o.t. \tag{6}$$

in a neighborhood of $(0,0)$. Let us prove that $\mathcal{M}_0(2,d)$ is an algebraic subset of $\mathcal{F}(2,d)$.
Let $\mathcal{F}(\omega) \in \mathcal{M}_0(2,d)$ and $\omega = \omega_1 + \omega_2 + \omega_3 + \ldots + \omega_{d+1}$ be the homogeneous decomposition of $\omega$, then in a neighborhood around $(0,0)$ in $\mathbb{C}^2$, we have

$$\omega \wedge df = 0 \Rightarrow (\omega_1 + \omega_2 + \omega_3 + \cdots + \omega_{d+1}) \wedge (d(xy) + df_3 + df_4 + \cdots) = 0$$

Putting the homogeneous parts of the above equation equal to zero, we obtain

$$\begin{cases} \omega_1 \wedge d(xy) = 0 \Rightarrow \omega_1 = k.d(xy), \ k \text{ is constant} \\ \omega_1 \wedge df_3 = -\omega_2 \wedge d(xy) \\ \ldots \\ \omega_1 \wedge df_n = -\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy) \\ \ldots \end{cases} \tag{7}$$

Dividing the 1-form $\omega$ by $k$, we can assume that $k = 1$. Let $\mathcal{P}_n$ denote the set of homogeneous polynomials of degree $n$. Define the operator :

$$S_n : \mathcal{P}_n \to (\mathcal{P}_n dx \wedge dy)$$

$$S_n(g) = \omega_1 \wedge d(g)$$

We have

$$S_{i+j}(x^i y^j) = d(xy) \wedge d(x^i y^j) = (xdy + ydx) \wedge (x^{i-1} y^{j-1}(jxdy + iydx))$$

$$= (j - i)x^i y^j dx \wedge dy$$

This implies that when $n$ is odd $S_n$ is bijective and so in (7), $f_n$ is uniquely defined by the terms $f_m's, \ m < n$, and when $n$ is even

$$Im(S_n) = (\mathcal{P}_n \backslash \{a(xy)^{\frac{n}{2}} \mid a \in \mathbb{C}\})dx \wedge dy.$$

Then, when n is even the existence of $f_n$ implies that the coefficient of $(xy)^{\frac{n}{2}}$ in

$$-\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy)$$

which is a polynomial, say $P_n$, with variables

$$\text{coefficients of } \omega_2 \ldots \omega_{n-1}, f_2, \ldots, f_{n-1}$$

is zero. Now an algebraic set

$$X : P_4 = 0 \ \& \ P_6 = 0 \ \& \ldots \& \ P_n = 0 \ \ldots$$

consists of all foliations $\mathcal{F}$ in $\mathcal{F}(2,d)$ which have a formal first integral of the type 6 at $(0,0)$. From theorem A in [15], it follows that $\mathcal{F}$ has a holomorphic first integral of the type (6). This implies that $\mathcal{M}_0(2,d)=X$ is algebraic. Note that by Hilbert zeroes theorem, a finite number of $P_i$'s defines $\mathcal{M}_0(2,d)$.
The set $\mathcal{M}(2,d)$ is obtained by the action of the group of automorphisms of $\mathbb{C}P(2)$ on $\mathcal{M}_0(2,d)$. Since this group is compact we conclude that $\mathcal{M}(2,d)$ is also algebraic. $\diamond$

Now consider $\mathcal{F}\in\mathcal{M}(2,d)$ and $p$ a center of $\mathcal{F}$. Let $\{\mathcal{F}_t\}_{t\in D}$ be an analytic deformation of $\mathcal{F}$ and $\{p_t\}_{t\in D}$ be the deformation of the singularity $p$, where $D=\{t \mid\mid t\mid<\epsilon\}$, $\mathcal{F}_0=\mathcal{F}$ and $p_0=p$.

**Corollary 1.4.1** *In the above situation, if there exists a sequence $t_n\to 0$ in $D$ such that $p_{t_n}$ is a center of $\mathcal{F}_{t_n}$, then for all $t\in D$, $p_t$ is a center of $\mathcal{F}_t$.*

**Proof:** By the implicit function theorem, we know that $p_t$ is a holomorphic function in $t$. Let $\Phi_t$ be a linear map which sends 0 to $p_t$. Then the holomorphic map

$$A: D\to\mathcal{F}(2,d)$$

$$A(t)=\Phi_t^*(\mathcal{F}_t)$$

intersects an algebraic set $\mathcal{M}_0(2,d)$ in all $t_i$. This implies that $Im(A)\subset\mathcal{M}_0(2,d)$ which means that 0 is a center of $\Phi_t^*(\mathcal{F}_t)$ or equivalently $p_t$ is a center of $\mathcal{F}_t$. $\diamond$

Let $X$ be an irreducible component of $\mathcal{M}(2,d)$. For any $\mathcal{F}\in X$ we say that $\mathcal{F}$ has $X$-property and for any $\mathcal{F}\in X\backslash\overline{(\mathcal{M}(2,d)\backslash X)}$ we say that $\mathcal{F}$ has generic $X$-property. Now the reader can prove the general form of the Theorem 1.3.1!

**Theorem 1.4.1** *Let $\mathcal{F}\in\mathcal{F}(2,d)$ have generic $X$-property and $p\in sing(\mathcal{F})$ be a center singularity. Let also $\mathcal{F}_t$, $t\in(\mathbb{C},0)$ be a deformation of $\mathcal{F}=\mathcal{F}_0$ such that the deformed singularity $p_t$, $t\in(\mathbb{C},0)$ is still a center for a sequence $t_n\to 0$. Then $\mathcal{F}_t$ has generic $X$-property for all $t\in(\mathbb{C},0)$.*

In fact this theorem will be interesting if we say explicitly what are generic $X$-properties, that is, what are irreducible components of $\mathcal{M}(2,d)$.

**Theorem 1.4.2** *$\mathcal{I}(a,b)$ is an irreducible component of $\mathcal{M}(2,d)$, where $d=a+b$.*

**Proof:** Since $\mathcal{I}(a,b)$ is parameterized by $\mathcal{P}_{a+1}\times\mathcal{P}_{b+1}$, $\mathcal{I}(a,b)$ is an irreducible variety. For any $\mathcal{F}(pGdF-qFdG)\in\mathcal{I}_m(a,b)$, we have seen in the theorems 1.3.2, 1.3.3 and 1.3.4 that

$$[\omega_1]\in T_{\mathcal{F}}\mathcal{M}(2,d)\Rightarrow$$

$$\omega_1=pGdP-qPdG+pFdQ-QdF,\ (P,Q)\in\mathcal{P}_{a+1}\times\mathcal{P}_{b+1}\Rightarrow$$

$$[\omega_1]\in T_{\mathcal{F}}\mathcal{I}(a,b)$$

this and the fact that $\mathcal{I}(a,b) \subset \mathcal{M}(2,d)$ imply that

$$T_{\mathcal{F}}\mathcal{M}(2,d) = T_{\mathcal{F}}\mathcal{I}(a,b),\ \forall \mathcal{F} \in \mathcal{I}_m(a,b)$$

Since $\mathcal{I}_m(a,b)$ is an open dense subset of $\mathcal{I}(a,b)$, we conclude that $\mathcal{I}(a,b)$ is an irreducible component of $\mathcal{M}(2,d)$. $\diamond$

**Proof of the theorem 1.3.1:** It is a direct consequence of Theorem 1.4.2.

## 1.5 Star Tangent Space

The idea of the following definition comes from the converse version of Theorem 1.3.2

**Definition 1.5.1** Let $X$ be an irreducible component of $\mathcal{M}(2,d)$, $\mathcal{F}(\omega_0) \in X$ and $p$ be a center of $\mathcal{F}$. There is a coordinate $(x,y)$ in a small neighborhood $U$ of $p$ such that in this coordinate $p=(0,0)$ and

$$\omega_0 = gd(\tilde{f}),\ \tilde{f} = x^2+y^2+h.o.t.,\ g(0) \neq 0$$

where $\tilde{f}$ and $g$ are holomorphic functions on $U$. Define $T^*_{\mathcal{F}}X$ as the set of all 1-forms $[\omega_1] \in T_{\mathcal{F}}\mathcal{F}(2,d)$ such that

$$\int_\delta \frac{\omega_1}{g} = 0$$

for all Lefschetz vanishing cycles in the leaves of $\mathcal{F}$ around $p$. By Theorem 1.3.2, we know that

$$T_{\mathcal{F}}X \subset T^*_{\mathcal{F}}X \tag{8}$$

$X$ is called a good irreducible component of $\mathcal{M}(2,d)$ if for a generic choice of $\mathcal{F} \in X$, the equality holds in (8).

The above definition does not depend on the choice of $g$. More precisely, if

$$\omega_0 = gd\tilde{f} = g'd\tilde{f}' \Rightarrow \frac{g}{g'}d\tilde{f} = d\tilde{f}' \Rightarrow d(\frac{g}{g'}) \wedge \omega_0 = 0$$

this means that $\frac{g}{g'}$ is constant on the leaves of $(\mathcal{F},p)$ and so

$$\int_\delta \frac{\omega_1}{g'} = \int_\delta \frac{g}{g'}\frac{\omega_1}{g} = \frac{g}{g'}\int_\delta \frac{\omega_1}{g'} = 0$$

which implies that the definition does not depend on $g$.

**Proposition 1.5.1** *$\mathcal{I}(a,b)$ is a good irreducible component of $\mathcal{M}(2,d)$.*

**Proof:** This is a direct consequence of Theorem 1.3.3, Remark 2 and Theorem 1.3.4.
**Question:** Is any other irreducible component of $M$ good?
Now the reader can go directly to Section 2 to see the applications of what we have done until now, in real polynomial differential equations.

## 1.6 Challenge

We have proved that

$$\mathcal{I}(0,d),\mathcal{I}(1,d-1),\mathcal{I}(2,d-2),\ldots,\mathcal{I}([\frac{d}{2}],d-[\frac{d}{2}])$$

are irreducible components of $\mathcal{M}(2,d)$ (Note that $\mathcal{I}(a,b)=\mathcal{I}(b,a)$). What are other irreducible components of $\mathcal{M}(2,d)$? For the cases $d=1,2$ the complete description of irreducible components of $\mathcal{M}(2,d)$ is done in [6]. If the reader has read Section 2, he finds that for any irreducible component of $\mathcal{M}(2,d)$, namely $X$, which has a real element $\mathcal{F}$ with a real center, we can construct a real polynomial equation of degree $d$ and with at least

$$dim(T_{\mathcal{F}}\mathcal{F}(2,d)/T^*_{\mathcal{F}}X)-1$$

limit cycles. Let us introduce an algebraic set in $\mathcal{F}(2,d)$ which seems to be an irreducible component of $\mathcal{M}(2,d)$.

**Logarithmic Foliation:** Let $\mathcal{F}$ be a foliation in $\mathbb{C}P(2)$ given by the polynomial 1-form

(9) $$\omega(f,\lambda)=\omega(f_1,\ldots,f_r,\lambda_1,\ldots,\lambda_r)=f_1\cdots f_r\sum_{i=1}^r\lambda_i\frac{df_i}{f_i}$$

where $f_i$'s are irreducible polynomials in $\mathbb{C}^2$, $\sum d_i\lambda_i=0$ and $d_i=deg(f_i)$. $\mathcal{F}$ is called a logarithmic foliation and it has the multivalued first integral $f=f_1^{\lambda_1}\cdots f_r^{\lambda_r}$ in $U=\mathbb{C}P(2)\backslash(\cup_{i=1}^r\{f_i=0\})$. By the same method used in Proposition 1.2.1, we can prove that generically, the degree of $\mathcal{F}$ is

$$d=\sum_{i=1}^r d_i-2$$

and has

$$d^2+d+1-\sum_{1\le i<j\le r}d_id_j$$

centers.

Let $\mathcal{L}(d_1,d_2,\ldots,d_r)$ be the set of all logarithmic foliations of the above type. Is $\mathcal{L}(d_1,d_2,\ldots,d_r)$ an (good) irreducible components of $\mathcal{M}(2,d)$, where $d=\sum_{i=1}^r d_i-2$? If we want to prove this fact with our methods, Theorem 1.3.3 is the first obstacle that we will encounter in this case. Note that if we succeed to do the present work for logarithmic foliations, then we get a better number of limit cycles which the coefficient of $d^2$ in this number is one.

I want to use this opportunity to express my thanks to my advisor Alcides Lins Neto, who introduced me with the article [10]. Discussions with him during this work have provided valuable information and new ideas. Also, thanks go to Cesar Camacho for his interest and support. My special thanks go to Paulo Sad whose comments made me to see the necessity of Remark 1. The author is also grateful to the exceptional scientific atmosphere in IMPA that made this work possible.

# 2 Irreducible Components of $\mathcal{M}(2,d)$ and Limit Cycles

In this section we will apply the following concept to each irreducible component $X$ of $\mathcal{M}(2,d)$:

*A Vanishing Cycle Generates A Limit Cycle*

to obtain real polynomial differential equations with

$$dim(T_{\mathcal{F}}\mathcal{F}(2,d)/T^*_{\mathcal{F}}X) - 1 \tag{10}$$

limit cycles.

## 2.1 Real Polynomial Differential Equations

Let us consider a polynomial differential equation in $\mathbb{R}^2$:

$$\omega = P(x,y)dy - Q(x,y)dx = 0 \tag{11}$$

where $P$ and $Q$ are relatively prime polynomials with real coefficients. Like as complex case, this equation defines a real singular foliation $\mathcal{F}_r$ in the real projective space $\mathbb{R}P(2)$. Now if we consider (11) as an equation with complex variables $x$ and $y$, we also obtain a holomorphic foliation $\mathcal{F}$ in $\mathbb{C}P(2)$ such that the intersection of a leaf of $\mathcal{F}$ with the real submanifold $\mathbb{R}P(2) \subset \mathbb{C}P(2)$ gives us several leaves of the real foliation $\mathcal{F}_r$. In another words, $\mathcal{F}$ is the complexification of $\mathcal{F}_r$. We have the same argument for the germs of real analytic functions $h_r : (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$. In the sequel, by a real 1-form, foliation, germ and etc., we mean those which are defined by real coefficients.

**Definition 2.1.1** Let $\delta_r$ be a closed orbit of $\mathcal{F}_r$, then $\delta_r$ is a closed path in some leaf, namely $L$, of $\mathcal{F}$. A cycle $\delta'$ in $L$ is called real if it is homotop to some $\delta_r$.
A cycle $\delta$ in the leaf $L$ of the foliation $\mathcal{F}$ is called a limit cycle if it has nontrivial holonomy. A similar definition can be made in the real case.

Note that the definition of limit cycle is more meaningful in the real case than in the complex case. In the real case, if the holonomy (Poincaré first return map) $h_r : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ is not trivial, then all the orbits which pass near enough $\delta$, accumulate in $\delta$. All theorems are stated for both cases real and complex, with the difference that in the real case, sometimes, we need more hypotheses.

**Lemma 2.1.1** *Let $h(t,z) : (\mathbb{C},0) \times (\mathbb{C},0) \to (\mathbb{C},0)$ be a germ of holomorphic function such that $h(0,.) \equiv Id$ and $\tilde{h} = \frac{\partial h}{\partial t}(0,.)$ has multiplicity $n$ at $z = 0$, then for any $t \in (\mathbb{C},0)$, $h_t(.) = h(t,.)$ has $n$ fixed point, counting with multiplicity, near $0$. Moreover, if $h$ is a real germ and $n$ is odd, then for all $t \in (\mathbb{R},0)$, $h_t$ has at least one real fixed point near $0$.*

**Proof:** By the hypothesis we can write

$$h(t,z)-z=t(\tilde{h}(z)+t\tilde{\tilde{h}}(z)+\dots)=tf(t,z)$$

where $f(0,z)=z^n+h.o.t.$. Applying Weierstrass Preparation Theorem to $f$, we conclude that there are germs of holomorphic functions in $(\mathbb{C},0)$, namely $b_1,\ b_2,\dots,b_n$, and $u$ in $(\mathbb{C}^2,0)$ such that

$$h(t,z)-z=tf(x,t)=tu(t,z)(z^n+b_1(t)z^{n-1}+\dots+b_n(t)) \tag{12}$$

$$u(0,0)\neq 0,\ b_1(0)=b_2(0)=\dots=b_n(0)=0$$

For any $t$ near 0, $h(t,z)-z=0$ has $n$ solution $z_1(t),z_2(t),\dots,z_n(t)$ which are fixed points of the function $h_t$.
Let $h$ be a real germ, then we can write

$$h(t,z)-z=ta(z)(z^n+tb(t,z))$$

where $\tilde{h}(z)=z^na(z)$, $a(0)\neq 0$ and $b$ is a germ of analytic function in $(\mathbb{R}^2,0)$. For small values of $t$ and big positive number $z$, $z^n+tb(t,z)$ is positive, and for big negative number $z$, $z^n+tb(t,z)$ is negative, and so by mean-value theorem there is a real number $z$ such that $z^n+tb(t,z)=0$.
Note that when $n$ is even we have the counterexample $h_t=z+tz^2+t^3$ with the fixed points $it$ and $-it$ which are not in the real line. $\diamond$

**Definition 2.1.2** Let $\delta$ be a closed path in the leaf $L$ of the foliation $\mathcal{F}$ with trivial holonomy. Let also $\mathcal{F}_t$ be a deformation of $\mathcal{F}$, $h_t$ be the related deformed holonomy along $\delta$ and $n=mul(\frac{\partial h_t}{\partial t}\mid_{t=0},z=0)$. According to the previous lemma, for $t$ near 0, there are $n$ closed paths in the leaves of $\mathcal{F}_t$, near the path $\delta$, with nontrivial holonomy. We say that $\delta$ generates $n$ limit cycles. Note that, according to the previous lemma, if $n$ is odd, $\mathcal{F}$ is a complexification of some real foliation and $\delta$ is a real cycle, then at least one of the generated cycles lies in the real plane $\mathbb{R}^2$.

**Complex and Real centers:** Let $\omega_0$ be a polynomial 1-form in $\mathbb{R}^2$ and $p\in\mathbb{R}^2$ be a singularity of $\mathcal{F}_r(\omega)$. Let also $\mathcal{F}_r$ has a nondegenerate first integral around $p$ i.e., there exist germs of analytic functions $f,g:(\mathbb{R}^2,p)\to\mathbb{R}$ such that $f$ has nondegenerate critical point at $p$ and

$$\omega_0=gdf$$

By Morse Lemma in the real case, there exists a coordinate $(x,y)$ around $p$ such that in this coordinate $p=(0,0)$ and $f$ is given by one of the following forms:

$$x^2-y^2,\ x^2+y^2$$

therefore, the real foliation $\mathcal{F}_r$ around the singularity $p$ is given by the 1-form

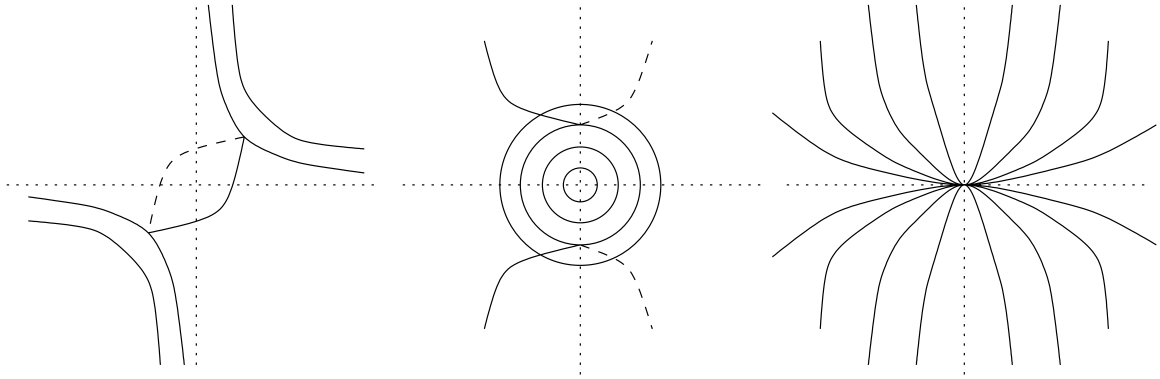


Figure 1: Singularities

- $xdx - ydy$, which is called a complex center;
- $xdx + ydy$, which is called a real center.

Using the linear isomorphism

$$(x, y) \to (x, iy)$$

we see that, in the complex domain the real and complex centers are of the same type. Note that the vanishing cycles near a real center lie in the real plane and the vanishing cycles near a complex center lie in the complex domain. (see Figure 1)

## 2.2 Main Theorem

Let $X$ be an irreducible component of $\mathcal{M}(2, d)$, $\mathcal{F}(\omega_0) \in X$ be a smooth point of $X \backslash (\mathcal{M}(2, d) \cap X)$ and $p$ be a center singularity of $\mathcal{F}$. Furthermore, if $\omega_0$ is a real 1-form assume that $\mathcal{F}_r(\omega_0)$ has a real center at $p \in \mathbb{R}^2$. The main theorem of this section is the following:

**Theorem 2.2.1** *There is a real polynomial differential equation (11) of degree d and with at least*

$$N = dim(T_{\mathcal{F}}\mathcal{F}(2, d)/T^*_{\mathcal{F}}X) - 1$$

*limit cycles in the real plane* $\mathbb{R}^2$*.*

We will obtain this equation by a small deformation of $\mathcal{F}(\omega_0)$, where $\omega_0$ is a real 1-form.

Choose a basis

$$[\tilde{\omega}_0], [\tilde{\omega}_1], \ldots, [\tilde{\omega}_N]$$

for the vector space

$$T_{\mathcal{F}}\mathcal{F}(2,d)/T^*_{\mathcal{F}}X$$

If $\omega_0$ is a real 1-form, then we can choose the $\tilde{\omega}_i$'s with real coefficients. Keeping the notations used in Definition 1.5.1, Let $D = \{z \mid\mid z \mid < \epsilon\}$ be a small disk contained in the $\tilde{f}(U)$, define

$$f : D \to \mathbb{C}^{N+1}$$

$$f(\check{b}) = (\int_{\delta(\check{b})} \frac{\tilde{\omega}_0}{g}, \int_{\delta(\check{b})} \frac{\tilde{\omega}_1}{g}, \dots, \int_{\delta(\check{b})} \frac{\tilde{\omega}_N}{g})$$

where $\delta(\check{b})$ denotes the vanishing cycle in the leaf $\tilde{f}^{-1}(\check{b})$ around $p$. If $\omega_0$ is a real 1-form, then $f$ induces the real analytic function

$$f_r = f \mid_{D_r} : D_r \to \mathbb{R}^{N+1}$$

where $D_r = \{z \in D \mid Im(z) = 0\}$. The fact that $\delta(\check{b})$ is a vanishing cycle, implies that $f(0) = 0$. The idea of the function $f$ comes from Ilyashenko's article [10] in deformation of Hamiltonian equations.

**Remark 3** Given a hyperplane $H(z_0, z_1, \dots, z_N) = \sum_i a_i z_i = 0$ in $\mathbb{C}^{N+1}$, which passes through 0. Define

$$\omega_H = \sum_{i=0}^{N} a_i \tilde{\omega}_i$$

If H is a real hyperplane then $\omega_H$ is also a real 1-form. Consider the deformation

$$\mathcal{F}_t : \omega_0 + t\omega_H$$

Let $f$ intersect $H$ at $f(\check{b}_0)$ and let $\Sigma$ be a transverse section to $\mathcal{F}(\omega_0)$ at some point of $\delta(\check{b}_0)$. Let also $h(t,z)$ be the deformed holonomy along $\delta(\check{b}_0)$ in $\Sigma$. By Theorem 3.3.1 in Section 3, we have

$$\tilde{h}(z) = \frac{\partial h}{\partial t} \mid_{t=0} = a(z) \int_{\delta_{z,h(z)}} \frac{\omega_H}{g}, \ \forall z \in \Sigma$$

where $\delta_{z,h(z)}$ is the lifting of $\delta$ to the point $z \in \Sigma$ and $a$ is a holomorphic function in $\Sigma$ with $a(0) \neq 0$. The function $j = \tilde{f} \mid_\Sigma$ is a local isomorphism between $\Sigma$ and a small neighborhood of $\check{b}_0$ and so we have

$$H \circ f(\check{b}) = \int_{\delta(\check{b})} \frac{\omega_H}{FG} = \frac{\tilde{h}(z)}{a(z)}$$

where $j(\check{b}) = z$. Knowing $a(0) \neq 0$, the above formula shows that the tangency multiplicity between $f$ and $H$ at $\check{b}_0$ is $mul(\tilde{h}, z = 0)$, therefore $f$ intersects $H$ transversally at $\check{b}_0$ if and only if $mul(\tilde{h}, z = 0) = 1$.

**Theorem 2.2.2** *Suppose that a hyperplane $H = \sum_i a_i z_i = 0$ intersects $f$ at $\check{b}_0, \check{b}_1, \ldots, \check{b}_s$. Then each vanishing cycle $\delta(\check{b}_i)$ generates at least one limit cycle in the deformation $\mathcal{F}_t(\omega_0 + t\omega_H)$. Moreover in the real case, if $H$ intersects $f_r$ in these points with odd tangency number, then we have the same assertion in this case.*

**Proof:** This theorem is direct consequence of Corollary **??**, Lemma 2.1.1 and the previous remark.

**Proposition 2.2.1** *Locally, the image of $f$ (resp. $f_r$) lies in no hyperplane in $\mathbb{C}^{N+1}$ (resp. $\mathbb{R}^{N+1}$).*

**Proof:** Let $V$ be an open subset in $D$ and let $f(V)$ be subset of the hyperplane $\sum_{i=0}^N a_i z_i = a$. This means that

$$\sum_{i=0}^{N} a_i \int_{\delta(\check{b})} \frac{\tilde{\omega}_i}{g} = \int_{\delta(\check{b})} \frac{\omega_H}{g} = a$$

for all $\check{b} \in V$. Since $\delta(\check{b})$ is a vanishing cycle we have $a = 0$. By the definition, we conclude that

$$\omega_H = \sum_{i=0}^{N} a_i \tilde{\omega}_i \in T^*_{\mathcal{F}} X$$

which contradicts the fact that, the $\tilde{\omega}_i$'s form a basis for the space $T_{\mathcal{F}}\mathcal{F}(2, d)\backslash T^*_{\mathcal{F}}X$. $\diamond$

In the situation of the previous proposition, we say that $f$ ($f_r$) is complete.

**Proposition 2.2.2** *Let $f$ be a complete holomorphic map from a complex manifold $S$ of dimension one to $\mathbb{C}^{N+1}$. Then generically for all $N$ points $x_1, x_2, \ldots, x_N$ in $S$, the hyperplane passing through $0, f(x_1), f(x_2), \ldots, f(x_N)$ intersects $f$ transversally in these points.*

We can also state this Proposition in the real case.

**Proof of Theorem 2.2.1:** Let $H = \sum_{i=0}^N a_i z_i = 0$ be a hyperplane at $\mathbb{R}^N$ which passes through $0, f(\check{b}_1), f(\check{b}_2), \ldots, f(\check{b}_N)$, where $\check{b}_1, \check{b}_2, \ldots, \check{b}_N \in D_r \backslash \{0\}$, and is transverse to $f_r$ in these points. The real foliation $\mathcal{F}_t(\omega_0 + t\omega_H)$ is desired.$\diamond$

## 2.3 Applications

Let us apply the obtained results to the good irreducible component $\mathcal{I}(a, b)$ of $\mathcal{M}(2, d)$, where $d = a + b$. We fix $d$ and look for the maximum of

$$N + 1 = codim_{\mathcal{F}(2,d)}\mathcal{I}(a, b) = dim(T_{\mathcal{F}}\mathcal{F}(2, d)/T^*_{\mathcal{F}}\mathcal{I}(a, b))$$

**Proposition 2.3.1** *The codimension of $\mathcal{I}(a, b)$ in $\mathcal{F}(2, d)$ is*

$$N + 1 = (d + 1)(d + 3) - \frac{1}{2}((a + 2)(a + 3) + (b + 2)(b + 3)) + \epsilon_{p,q} \tag{13}$$

*where $\epsilon_{p,q} = ([\frac{1}{p}] + [\frac{1}{q}] + 1)$ and $[.]$ denotes the integer part of a number.*

**Proof:** By definition any 1-form of degree $d$ can be written as:

$$\omega = \omega_0 + \omega_1 + \omega_2 + \cdots + \omega_d + G.(xdy - xdy)$$

where $\omega_i,\ i = 0, 1, \ldots, d$ is a homogeneous 1-form of degree $i$, and $G$ is a homogeneous polynomial of degree $d$, therefore

$$(14)\ dim(\mathcal{F}(2, d)) + 1 = 2(1 + 2 + 3 + \cdots + d + 1) + (d + 1) = (d + 1)(d + 3)$$

The parameterization

$$i : \mathcal{P}_{a+1} \times \mathcal{P}_{b+1} \to \mathcal{I}(a, b)$$
$$i(F, G) = pGdF - qFdG$$

has the kernel of dimension $\epsilon_{p,q}$ and so

$$dim(\mathcal{I}(a, b)) + 1 =$$

$$dim(\mathcal{P}_{a+1} \times \mathcal{P}_{b+1}) - \epsilon_{p,q} = \frac{1}{2}((a + 2)(a + 3) + (b + 2)(b + 3)) - \epsilon_{p,q} \quad (15)$$

The number $+1$ in the left side of these equalities are due to the projectivization of our spaces. The equalities (14) and (15) give us (13). $\diamond$

$N$ reaches to its maximal value by putting

$$\begin{cases} a = b = \frac{d}{2} & \text{if } d \text{ is even} \\ a = \frac{d-1}{2}, b = \frac{d+1}{2} & \text{if } d \text{ is odd} \end{cases}$$

Putting these numbers in (13), and by Theorem 2.2.1 we conclude that

**Corollary 2.3.1** *There is a real polynomial differential equation of degree $d$ and with at least*

$$\begin{cases} \frac{3}{4}(d^2 + 2d - 4/3) & \textit{if } d \textit{ is even} \\ \frac{3}{4}(d^2 + 2d - 13/3) & \textit{if } d \textit{ is odd} \end{cases} \quad (16)$$

*limit cycles.*

**Remark 4** In Theorem 2.2.1 we have obtained limit cycles in a small neighborhood of a center singularity. The following discussion allows a much free placement of limit cycles in the real plane.

Let $\mathcal{F}(pGdF - qFdG) \in \mathcal{I}_m(a, b)$ and let $c_1, c_2, \ldots, c_r$ be critical values due to the nondegenerate critical points. Fix $c_1$ and let

$$U = \overline{\mathbb{C}} \backslash \{c_2, \ldots, c_r, 0, \infty\}$$

Consider the universal covering $\tau : \check{U} \to U$ with base point $c_1$ i.e., any point $\check{b}^i$ of $\check{U}$ represents a homotopy class of a path in $U$ with initial point at $b$ and end point at $c_1$. The upperscript $i$ means the different choices of homotopy class of such path,

for fixed $b \in U$. When there is no confusion, we drop the upperscript $i$.
For any $\check{b} \in \tilde{U}$ denote by $\delta(\check{b})$ the Lefschetz vanishing cycle along the path $\check{b}$ in $L_b$ (see [17] or [11]). We know that this vanishing cycle is unique up to homotopy and change of sign.
Now consider $F$ and $G$ with real coefficients. The real foliation $\mathcal{F}_r(pGdF - qFdG)$ has three type of singularities: Real and Complex centers and dicritical singularities (see Figure 1). In a neighborhood of a dicritical singularity $\mathcal{F}$ is given by the 1-form

$$pydx - qxdy$$

Define

$$\check{U}_r = \{\check{b} \in \check{U} \mid b \in \bar{\mathbb{R}} \& \ \delta(\check{b}) \text{ is a real cycle } \}$$

$\check{U}_r$ is a real manifold of $\check{U}$ of dimension one. Note that the path $\check{b}$ may not lie in the $\bar{\mathbb{R}}$, but the vanishing cycle $\delta(\check{b})$ is a real cycle.
The reader can check easily that, the definition domain of the functions $f$ and $f_r$ can be extended to $\check{U}$ and $\check{U}_r$, respectively.
Now we can restate Theorem 2.2.1 and 2.2.2, Proposition 2.2.1 and 2.2.2 for the extended $f$ and $f_r$ and obtain a much freer placement of limit cycles because of the following fact:
Consider any other real Lefschetz vanishing cycle $\delta$ in the leaf $(\frac{F^p}{G^q})^{-1}(c)$. Corollary 3.1.2 in [17] implies that, there exists a path $\check{b} \in \check{U}_r$ such that $\delta = \delta(\check{b})$. This means that, fixing $N$ real vanishing cycles in the real plane, we can produce a limit cycle around each of them, where $N$ is the number in (13).

**Remark 5** Let us come back to the old definition of the degree of real polynomial differential equations. Define the degree of the real polynomial differential equation (11) to be $max\{deg(P), deg(Q)\}$. If an equation (11) is of degree $d$ then it is of degree $d-1$ or $d$ with the new definition of degree. With this definition we have a real polynomial differential equation of degree $d+1$ and with at least $N$ limit cycles, where $N$ is the number in (13).

# 3 Holonomy in Deformed Foliation

In this section we will work with a codimension one foliation $\mathcal{F}(\omega)$ in a complex manifold $M$ of dimension two, where $\omega$ is a meromorphic 1-form on the manifold $M$. By $zer(\omega)$ and $pol(\omega)$ we mean the zero and pole divisors of $\omega$, respectively. Also by $L_p$ we mean the leaf through $p \in M$ and by $\Sigma_p$ we mean a small transverse section to $\mathcal{F}$ at $p \in M$.

## 3.1 Integrating Form

For every point $p \in M$, there is a neighborhood $U$ and a meromorphic 1-form $\alpha_U$ defined in $U$ such that

$$d\omega = \alpha_U \wedge \omega \text{ in } U$$

For example, if $\mathcal{F}$ is given by a 1-form $Pdy - Qdx$ in a coordinate $(x, y)$ around $p$, then we can define $\alpha_U$ as follows:

$$\alpha_U = \frac{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}{P} dx$$

For every two such 1-forms $\alpha_U$ and $\alpha_V$ in open sets $U$ and $V$, respectively, in the open set $U \cap V$ we have:

$$d\omega = \alpha_U \wedge \omega = \alpha_V \wedge \omega \Rightarrow (\alpha_U - \alpha_V) \wedge \omega = 0 \Rightarrow$$

$$\alpha_U \mid_L = \alpha_V \mid_L \text{ for any leaf } L \text{ of } \mathcal{F}$$

Therefore the $\alpha_U$ 's coincide in the leaves of $\mathcal{F}$. They define a meromorphic 1-form on each leaf $L$, namely $\alpha_\omega(L)$. By $\alpha_\omega$ we mean the collection of all 1-Forms $\alpha_U$. The 1-form $\alpha_\omega$ can be considered as a multivalued meromorphic 1-form on $M$ which is one valued on the leaves of $\mathcal{F}$ and satisfies:

$$d\omega = \alpha_\omega \wedge \omega$$

The 1-form $\alpha_\omega$ is defined up to meromorphic 1-forms which induce the foliation $\mathcal{F}$ and it depends on $\mathcal{F}$ up to 1-forms of the type $\frac{df}{f}$, where f is a meromorphic function on $M$. More precisely, if two meromorphic 1-forms $\omega_1$ and $\omega_2$ induce the same foliation $\mathcal{F}$, then there is a meromorphic 1-form $f$ on $M$ such that $\omega_2 = f\omega_1$ and therefore:

$$d\omega_2 = d(f\omega_1) = df \wedge \omega_1 + f d\omega_1 = (\frac{df}{f} + \alpha_{\omega_1}) \wedge (\omega_2) \Rightarrow$$

$$\alpha_{\omega_2} = \alpha_{f\omega_1} = \alpha_{\omega_1} + \frac{df}{f} \tag{17}$$

Note that the above equalities are up to meromorphic 1-forms which induce the foliation $\mathcal{F}$. Put

$$pol(\alpha_\omega) = \overline{\cup_L pol(\alpha_\omega(L))}, \;\; zer(\alpha_\omega) = \overline{\cup_L zer(\alpha(L))}$$

$\alpha_\omega(L)$'s are one valued 1-forms and so, $pol(\alpha_\omega)$ and $zer(\alpha_\omega)$ are well-defined.

Let $p$ a point in a leaf $L$ of the foliation $\mathcal{F}(\omega)$ and let $(U, (x, y))$ be a foliation chart around $p$. By a foliation chart $(U, (x, y))$ around $p$ we mean a chart in which $p = (0, 0)$ and the foliation $\mathcal{F}$ is given by $y = const.$. In the coordinate $(x, y)$ we have

$$\omega = f(x, y)dy$$

and so

$$\alpha_\omega = \frac{\partial f / \partial x}{f} dx$$

Using this fact, the reader can verify that

$$pol(\alpha) \subset pol(\omega) \cup zer(\omega), \;\; zer(\alpha) \subset zer(d\omega)$$

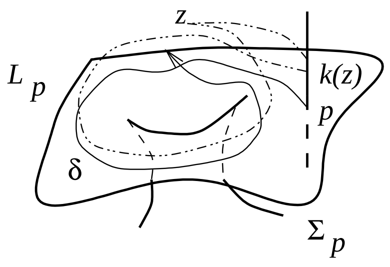


Figure 2: Holonomy

## 3.2 Integrating Factor

**Holonomy:** Let $\delta : [0,1] \to L$ be a path in a leaf $L$ of the foliation $\mathcal{F}$ with initial point $p$. Assume that $\delta$ has a finite number of self intersecting points. There is a neighborhood $U_\delta$ of the path $\delta$ such that for every $t \in [0,1]$ and $z \in U_\delta$ near $\delta(t)$, the lifting path $\delta_{k(z),z}$ of $\delta\mid_{[0,t]}$ in the leaf $L_z$ is well-defined. Roughly speaking, the path $\delta_{k(z),z}$, in the leaf $L_z$, connects $k(z) \in \Sigma_p$ to $z$ in the direction of the path $\delta\mid_{[0,t]}$. In Figure 2 we have shown that in the self intersecting points of $\delta$, depending on the choice of $t$, we can choose non-homotop $\delta_{k(z),z}$'s. These paths are depicted by dash-dot-dot lines.

**Remark 6** *Let $\tilde{U}_\delta$ be the set of all homotopy classes $[\delta_{k(z),z}]$ in an small neighborhood $U_\delta$ of $\delta$. The reader can easily verify that $\tilde{U}_\delta$ is a complex manifold and the natural map $\tau : \tilde{U}_\delta \to U_\delta$ may not be one to one near the self intersecting points of $\delta$ (see Figure 2). All functions, for example $k(z)$, that we define on the set $U_\delta$ are multivalued near such points and are one valued in $\tilde{U}_\delta$. For simplicity, we will work with $U_\delta$ instead of $\tilde{U}_\delta$.*

Let $q = \delta(t_1)$, $0 \le t_1 \le 1$, be a point of $\delta$ and $\Sigma_q$ be a small transverse section at $q$ to $\mathcal{F}$. For any point $z \in \Sigma_q$, the lifting $\delta_{k(z),z}$ of $\delta\mid_{[0,t_1]}$ defines the holomorphic function $k : \Sigma_q \to \Sigma_p$. The function

$$h = k^{-1} : \Sigma_p \to \Sigma_q$$

is called the holonomy of $\mathcal{F}$ along $\delta$ from $\Sigma_p$ to $\Sigma_q$. If $\delta$ is a closed path, $q = \delta(1)$ and $\Sigma_p = \Sigma_q$, we have the holomorphic germ

$$h : \Sigma_p \to \Sigma_p$$

$h$ is called the holonomy of $\mathcal{F}$ along $\delta$ in $\Sigma_p$.

**Integrating Factor:** Define the "integrating factor" $I_\omega$ as follows:

$$I_\omega : U_\delta \to \mathbb{C}$$

$$I_\omega(z) = exp(\int_{\delta_{k(z),z}} \alpha_\omega)$$

If $L_z \subset pol(\alpha)$ then $I_\omega$ has pole at $L_z \cap U_\delta$, in other cases the path $\delta_{k(z),z}$ is chosen so that it does not intersect $pol(\alpha_\omega)$.
The multivalued function $I_\omega$ is the generalization of the old concept of integrating factor. The foliation $\mathcal{F}(\omega)$ has an integrating factor $f$, where $f$ is a meromorphic function on $M$, if

$$d(\frac{\omega}{f}) = 0$$

If the foliation $\mathcal{F}(\omega)$ has an integrating factor $f$ then:

$$\alpha_\omega = \frac{df}{f},\ I_\omega(z) = \frac{f(z)}{f(k(z))}$$

Choosing an arbitrary branch of $k(z)$ in a small neighborhood of $z$, we can see that the function $f(k(z))$ is constant on the leaves of $\mathcal{F}$. By (17) and the definition of $I_\omega$, we can see that for every meromorphic function $f$ on $M$

$$I_{f\omega}(z) = \frac{f(z)}{f(k(z))} I_\omega(z)$$

In what follows, sometimes, we will use the notation $\int_{k(z)}^z$ instead of $\int_{\delta_{k(z),z}}$.

**Theorem 3.2.1** *The following statements are true:*
*1.* $d(\frac{\omega}{I_\omega}) = 0$ *in the definition domain* $U_\delta$ *of* $I_\omega$*;*
*2. Let* $a = \delta(t_1), b = \delta(t_2),\ 0 \leq t_1 \leq t_2 \leq 1,$ *and* $h : \Sigma_a \to \Sigma_b$ *be the holonomy along the path* $\delta' = \delta \mid_{[t_1,t_2]}$*, then*

$$h^*(\frac{\omega}{I_\omega} \mid_{\Sigma_b}) = \frac{\omega}{I_\omega} \mid_{\Sigma_a} \tag{18}$$

**Proof:** Let us prove the mentioned facts in a local coordinate. Let $q \in \delta$ and $(U, (x, y))$ be a foliation chart around $q$ such that in the coordinate the leaf $L_p$ is given by $y = 0$ and

$$\omega = f dy,\ \alpha_\omega = Adx + Bdy$$

Fix two points $c = (x_1, 0)$ and $d = (x_2, 0)$ in $\delta \cap U$. For any point $z$ in $U$, let $r$ be the intersection point of the leaf $L_z$ and $\Sigma_c$ (Figure 3). Define

$$g(z) = \int_{k(z)}^z \alpha_\omega$$

In this coordinate we have

$$g(z) = g(x, y) = \int_{k(z)}^r \alpha_\omega + \int_r^z \alpha_\omega = s(y) + \int_{x_1}^x A(\xi, y) d\xi$$

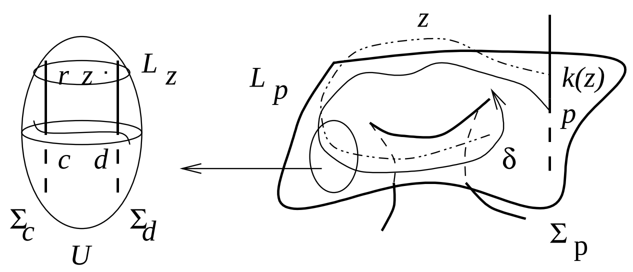


Figure 3: Holonomy

where $s(y)$ only depends on $y$ and so

$$dg = s'(y)dy + Adx + (\int_{x_1}^{x} \frac{dA}{dy})dy \Rightarrow dg \wedge \omega = dg \wedge f dy = \alpha_\omega \wedge \omega$$

By the definition, $I_\omega = e^g$ and so

$$d(\frac{\omega}{I_\omega}) = d(e^{-g}\omega) = e^{-g}(-dg \wedge \omega + d\omega) = e^{-g}(-\alpha_\omega \wedge \omega + d\omega) = 0$$

and here the proof of the first statement finishes.
In the coordinate $(x, y)$ we can write

$$\frac{\omega}{I_\omega} = Gdy$$

The first part of the lemma implies that $\frac{\partial G}{\partial x} = 0$ or equivalently $G = G(y)$ does not depend on the variable $x$.

$$h^*(\frac{\omega}{I_\omega} \mid_{\Sigma_d}) = h^*(Gdy \mid_{\Sigma_d})) = h^*(G \mid_{\Sigma_d})d(h^*(y \mid_{\Sigma_d})) = Gdy \mid_{\Sigma_c} = \frac{\omega}{I_\omega} \mid_{\Sigma_c} \Rightarrow$$

$$h^*(\frac{\omega}{I_\omega} \mid_{\Sigma_d}) = \frac{\omega}{I_\omega} \mid_{\Sigma_c} \tag{19}$$

Let $(U_i, (x_i, y_i)),\ i = 0, 1, 2, \ldots, n$ be foliation charts which cover the path $\delta \mid_{[t_1,t_2]}$. Let also $\Sigma_{c_i}$ and $\Sigma_{d_i}$ be small transverse sections at $c_i, d_i \in U_i$ to $\mathcal{F}$ such that

$$c_0 = p,\ c_i = d_{i-1},\ \Sigma_{c_i} = \Sigma_{d_{i-1}},\ i = 1, 2, \ldots, n-1,\ d_n = b$$

Now our affirmation is obtained by the combination of (19)'s in each chart $U_i$. $\diamond$

The next result is a consequence of the second part of the previous Lemma.

**Corollary 3.2.1** *Let $\delta$ be a closed path in a leaf $L$ of the foliation $\mathcal{F}$, $\Sigma$ be a transverse section at $p \in \delta$ to the foliation and $h : \Sigma \to \Sigma$ be the holonomy along $\delta$. If $\eta = \omega \mid_\Sigma$ then*

$$h^*(\eta) = e^{\int_{\delta z, h(z)} \alpha_\omega} \eta$$

**Proof:** lifting of the paths $\delta\mid_{[0,0]}$ and $\delta\mid_{[0,1]}$, gives us two branches of $I_\omega$ in $\Sigma$

$$I_\omega\mid_\Sigma\equiv 1,\ e^{\int_{\delta_{k(z),z}}\alpha_\omega}$$

we have

$$h^*(e^{\int_{\delta_{k(z),z}}\alpha_\omega})=e^{\int_{\delta_{z,h(z)}}\alpha_\omega}$$

putting these equalities in (18), our affirmation is proved. $\diamond$
The Poincaré formula $h'(0)=e^{\int_\delta\alpha}$ is a direct consequence of this fact. The ideas of Theorem 3.2.1 come from the author's first course in complex dynamical system with S. Shahshahani in Iran.

## 3.3 Integrating Factor and Deformation

Now consider a deformation

$$\mathcal{F}_t:\omega+t\omega_1+h.o.t.$$

of $\mathcal{F}(\omega)$. The 1-form $\omega_1$ is called the tangent vector of the deformation. We can consider $\mathcal{F}_t$'s as a codimension two foliation $\bar{\mathcal{F}}=\{\mathcal{F}_t\}_{t\in(\mathbb{C},0)}$ in $(\mathbb{C},0)\times M$.
For any point $a$ in the leaf $L_p$ and a transverse section $\Sigma_a$ to $\mathcal{F}$, we can consider $(\mathbb{C},0)\times\Sigma_a$ as a transverse section to $\bar{\mathcal{F}}$ at $(0,a)$. The deformed holonomy along $\delta$ is defined by

$$\mathcal{H}:(\mathbb{C},0)\times\Sigma_a\to(\mathbb{C},0)\times\Sigma_b$$

$$\mathcal{H}(t,z)=(t,\mathcal{H}_t(z))$$

where $a$ and $b$ are two points in the path $\delta$ and $\mathcal{H}_t:\Sigma_a\to\Sigma_b$ is the holonomy of the foliation $\mathcal{F}_t$ along the path $\delta$. With previous notations we have $\mathcal{H}_0=h$.
Now consider a coordinate system $(U_\delta,(x,y))$ in a neighborhood of $\delta$ such that in this coordinate for all $a=(x,y)$ in a small neighborhood of $\delta$, the set

$$\Sigma_x=\Sigma_a=\{b\in U_\delta\mid\pi_1(b)=x\}$$

is a transverse section to $\mathcal{F}$, where $\pi_1$ is the projection in the first coordinate. In this coordinate let

$$\omega=Pdy-Qdx$$

$$\omega_1=Ady-Bdx$$

**Remark 7** For example in the projective space of dimension 2, $\mathbb{C}P(2)$, fix an affine chart $(\mathbb{C}^2,(x,y))$ and let $\mathcal{F}$ be given by $\omega=Pdy-Qdx$ in this affine chart. Choosing a proper element in the homotopy class of $\delta$, we can assume that $\delta$ doesn't intersect $P=0$. This is a desired coordinates system.

Define the function $\mathcal{H}$ as follows:

$$\mathcal{H} : (\mathbb{C}, 0) \times \pi_1(U_\delta) \times \Sigma_p \to \mathbb{C}$$

For any $(t, x, \tilde{y}) \in (\mathbb{C}, 0) \times \pi_1(U_\delta) \times \Sigma_p$, $(x, \mathcal{H}(t, x, \tilde{y}))$ is the unique common point of the leaf of $\mathcal{F}_t$ through $\tilde{y} \in \Sigma_p$ and the transverse section $\Sigma_x$ in the direction of the path $\delta$. In another words the holonomy of the foliation $\mathcal{F}_t$ from $\Sigma_p$ to $\Sigma_x$ is given by

$$\tilde{y} \to (x, \mathcal{H}(t, x, \tilde{y}))$$

Define

$$\tilde{h}(x, \tilde{y}) = \frac{\partial \mathcal{H}}{\partial t} |_{t=0}$$

An another interesting formula which $I_\omega$ has a fundamental role in it, is stated in the following theorem.

**Theorem 3.3.1** *Keeping the above notations, we have*

$$\tilde{h}(x, \tilde{y}) = -\frac{I_\omega}{P}(z) \int_{\delta_{z,k(z)}} \frac{\omega_1}{I_\omega}$$

*where* $z = (x, y) = (x, h(x, \tilde{y}))$.

**Proof:** In the coordinate $(x, y)$, let $\mathcal{F}_t$ be given by

$$P_t dy - Q_t dx = 0$$

We have assumed that $P_t$ has not zero in some neighborhood $U_\delta$ of $\delta$ and so

$$\mathcal{R}(t, x, y) = \frac{Q_t}{P_t}(x, y), \;\; R = \mathcal{R} |_{t=0}$$

are holomorphic functions in $(\mathbb{C}, 0) \times U_\delta$ and $U_\delta$, respectively.
The foliation $\mathcal{F}$ is induced by the 1-form

$$\omega' = dy - Rdx$$

in the open set $U_\delta$. By definition of $\mathcal{H}$ we have

$$\frac{\partial \mathcal{H}}{\partial x} = \mathcal{R}(t, x, \mathcal{H}(t, x, \tilde{y}))$$

$$\Rightarrow \frac{\partial \tilde{h}}{\partial x} = \frac{\partial(\frac{\partial \mathcal{H}}{\partial t})}{\partial x} = \frac{\partial(\frac{\partial \mathcal{H}}{\partial x})}{\partial t} \Rightarrow$$

$$\frac{\partial \tilde{h}}{\partial x} = \frac{\partial \mathcal{R}}{\partial t}(t, x, \mathcal{H}(t, x, y)) |_{t=0} + \frac{\partial R}{\partial y}(x, h(x, y))\tilde{h}(x, y)$$

The solution of the above differential equation with initial condition $h(x_0, \tilde{y}) \equiv 0$ is

$$(20) \quad \tilde{h}(x,y) = e^{\int_{x_0}^{x} \frac{\partial R}{\partial y}(\xi, h(\xi,y))d\xi} \int_{x_0}^{x} \frac{\partial \mathcal{R}}{\partial t}(t,\xi,\mathcal{H}(t,\xi,y)) \mid_{t=0} e^{\int_{x_0}^{\xi} -\frac{\partial R}{\partial y}(s,h(s,y))ds} d\xi$$

where $x_0 = \pi_1(p)$. By $\int_{x_0}^{x}$ we mean the integral along the path $\delta' = \pi_1(\delta)$.
We have

$$\alpha_{\omega'} = \frac{\partial R}{\partial y} dx \Rightarrow$$

$$\tilde{h}(x,\tilde{y}) = I_{\omega'}(z) \int_{x_0}^{x} \frac{\partial \mathcal{R}}{\partial t}(t,\xi,\mathcal{H}(t,\xi,y)) \mid_{t=0} I_{\omega'}(t,\xi,\mathcal{H}(t,\xi,y)) d\xi$$

Now $\omega = P\omega'$ and so

$$\alpha_{\omega} = \alpha_{\omega'} + \frac{dP}{P} \Rightarrow$$

$$(21) \qquad I_{\omega}(z) = \frac{P(z)}{P(\tilde{y})} I_{\omega'}(z)$$

Also

$$\mathcal{R} = \frac{Q + tB + ...}{P + tA + ...} \Rightarrow \frac{\partial \mathcal{R}}{\partial t} \mid_{t=0} = \frac{BP - AQ}{P^2} \Rightarrow$$

$$\frac{\partial \mathcal{R}}{\partial t}(x,\mathcal{H}(t,x,\tilde{y}))dx \mid_{t=0} = (\frac{BP - AQ}{P^2})(x,h(x,\tilde{y}))dx = (\frac{B - A\frac{Q}{P}}{P})(x,h(x,\tilde{y}))dx$$

We have $\frac{\partial h}{\partial x} = \frac{Q}{P}(x,h(x,\tilde{y}))$ and so, lifting the above function to the lef $L_z$, it is equal

$$(22) \qquad \frac{Bdx - Ady}{P} \mid_{L_z} = -\frac{\omega_1}{P} \mid_{L_z}$$

Putting (21) and (22) in (20) we have

$$\tilde{h}(x,\tilde{y}) = \frac{P(\tilde{y})}{P(z)} I_{\omega}(z) \int_{x_0}^{x} \frac{(-\omega_1)}{P(\xi,h(\xi,\tilde{y}))} (\frac{P(\tilde{y})}{P(\xi,h(\xi,\tilde{y}))} I_{\omega}(\xi,h(\xi,\tilde{y})))^{-1}$$

$$= -\frac{I_{\omega}(z)}{P(z)} \int_{\delta_{z,k(z)}} \frac{\omega_1(\xi)}{I_{\omega}(\xi)}$$

$\diamond$

Theorem 1.3.2 is a direct consequence of the above theorem.

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Hossein Movasati
Instituto de Matemática Pura e Aplicada, IMPA
Estrada Dona Castorina, 110, 22460-320
Rio de Janeiro, RJ, Brazil
E-mail: hossein@impa.br