Complete hypersurfaces in Euclidean spaces with vanishing r-mean curvatures

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Abstract

Hypersurfaces in euclidean spaces with vanishing r-mean curvatures are natural generalizations of minimal hypersurfaces. When they are complete and have finite total curvature, we prove that their topological structure is similar to that of minimal hypersurfaces. We apply this result to prove a theorem on stability, a gap theorem on the total curvature, and a non-existence theorem for one-ended hypersurfaces with even dimensions.

1 Introduction

Let $x\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface of the Euclidean space $\mathbb{R}^{n+1}$. Consider the elementary symmetric functions S_r, $r = 0, 1, \ldots, n$, of the principal curvatures $k_1, \ldots, k_n$ of x:

$$S_0 = 1, \quad S_r = \sum_{i_1<\cdots<i_r} k_{i_1}\ldots k_{i_r}, \quad i_1, \ldots, i_r = 1, \ldots, n,$$

and their associated r-mean curvatures H_r given by

$$H_r = \binom{n}{r}^{-1} S_r.$$

Hypersurfaces with $H_r = 0$ generalize minimal hypersurfaces ($H_1 = 0$). The relation is even deeper, since minimal hypersurfaces are critical points of the functional $A_0 = \int_M H_0\, dM$ for compactly supported variations of M whereas hypersurfaces with $H_{r+1} = 0$ are critical points of the functional $A_r = \int_M H_r\, dM$ again for compactly supported variations [12]. A breakthrough in the study of such hypersurfaces was made when Hounie and Leite [8] proved that the equation $H_{r+1} = 0$, $r \neq 0, n-1$, is elliptic provided that $H_n \neq 0$ everywhere (actually they proved that a weaker condition suffices but that

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is the one we want to use). In the case $r = 0$, no such condition is necessary, since the equation of a minimal hypersurface is automatically elliptic.

From now on, we assume that $M^n = M$ is orientable and fix an orientation on M. Let $g\colon M \to S_1^n \subset \mathbb{R}^{n+1}$ be the Gauss map in the given orientation, where S_1^n is the unit n-sphere. Recall that the linear operator $A\colon T_pM \to T_pM$, $p \in M$, associated to the second fundamental form is given by

$$\langle A(X), Y\rangle = -\langle \overline{\nabla}_X N, Y\rangle, \quad X, Y \in T_pM,$$

where $\overline{\nabla}$ is the covariant derivative of the ambient space and N is the unit normal vector in the given orientation. The map $A = -dg$ is self-adjoint and its eigenvalues are the principal curvatures $k_1, k_2, \dots, k_n$.

Assume now that the immersion is complete. We will say that the total curvature of the immersion is finite if $\int_M |A|^n \, dM < \infty$. Here $|A| = (\sum_i k_i^2)^{1/2}$.

Let $x\colon M^n \to \mathbb{R}^{n+1}$ be a complete minimal hypersurface with finite total curvature. Then M is (equivalent to) a compact manifold $\overline{M}$ minus finitely many points and the Gauss map extends to the punctures. This was proved by Osserman [11] for $n = 2$ (the equivalence here is conformal and the Gauss map extends to a (anti) holomorphic map $\overline{g}\colon \overline{M}^2 \to S_1^2$). For an arbitrary n, this was proved by Anderson [1] (here the equivalence is a diffeomorphism and the Gauss map extends smoothly).

In this paper, we want to show that a similar topological structure holds for the case $H_r = 0$, $r = 1, \dots, n-1$, $n \geq 3$. For various reasons (including ellipticity, but not only for that), we want the additional condition $H_n \neq 0$ to hold everywhere. Furthermore we also want $H_j \geq 0$, $j = 1, \dots, r-1$. More precisely, we prove

THEOREM 1.1. *Let $x\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be an orientable complete hypersurface with $H_r = 0$, $r = 1, \dots, n-1$, $H_j \geq 0$, $j = 1, \dots, r-1$, and $H_n \neq 0$ everywhere. Assume that the total curvature of the hypersurface is finite in the sense that $\int_M |A|^n \, dM < \infty$. Then:*

i) *M is diffeomorphic to a compact manifold $\overline{M}$ minus a finite number of points $q_1, \dots, q_k$.*

ii) *The Gauss map $g\colon M^n \to S_1^n$ extends continuously to the punctures.*

iii) *The extended Gauss map $\overline{g}\colon \overline{M} - S_1^n$ is a homeomorphism.*

We want to observe that the condition $H_j \geq 0$, $j = 1, \dots, r-1$ in Theorem 1.1 is automatically satisfied for the minimal case ($H_1 = 0$) and for the case of zero scalar curvature ($H_2 = 0$). For $H_1 = 0$, this is obvious, and for $H_2 = 0$, see Remark 4.1 in Section 4.

It follows that Theorem 1.1 applies to minimal hypersurfaces ($n \geq 3$) with $H_n \neq 0$ everywhere and finite total curvature. We do not know, however, whether the extension $\overline{g}$ can be made differentiable in our case.

We further remark that the condition $H_j \geq 0$, $j = 1, \ldots, r-1$, appears naturally in the statement of the Maximum Principle for hypersurfaces with vanishing H_r as described by Hounie and Leite in [9] (Lemma 1.2 and Theorem 1.3).

We want to apply Theorem 1.1 to a question on stability of complete hypersurfaces with $H_r = 0$. In [2], a definition of stability (see Section 4) was given for such hypersurfaces and the following conjecture was proposed: *There exists no complete, orientable, stable hypersurface $x\colon M^3 \to \mathbb{R}^4$ with $H_2 = 0$ and $H_3 \neq 0$ everywhere*. Here we show that with additional conditions a more general conjecture is true. Namely, we prove

THEOREM 1.2. *There exists no complete orientable, stable hypersurface $x\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, with $H_{n-1} = 0$, $H_j \geq 0$, $j = 1, \ldots, n-2$, $H_n \neq 0$ everywhere and finite total curvature.*

The paper is organized as follows. In Section 2, we discuss (Proposition 2.3) the rate of decay at infinity of the second fundamental form of a hypersurface in the hypothesis of Theorem 1.1. In Section 3, we show that each end of such a hypersurface has a unique "tangent plane at infinity" (see the definition before Lemma 3.4) and prove Theorem 1.1. In Section 4, we prove a lemma that yields domains in the unit n-sphere with n as first eigenvalue of the Laplacian (Lemma 4.3) and use it together with Theorem 1.1 to prove Theorem 1.2. In Section 5, we present further applications of Theorem 1.1 and discuss some related questions.

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2 The rate of decay of the second fundamental form

In the rest of this paper, we will be using the following notation for an immersion $x\colon M^n \to \mathbb{R}^{n+1}$:

$\rho =$ intrinsic distance in M.
$d =$ distance in $\mathbb{R}^{n+1}$; $0 =$origin of $\mathbb{R}^{n+1}$.
$D_p(R) = \{x \in M;\ \rho(x,p) < R\}$; $L_p(R) = \partial D_p(R)$.
$D_p(R,S) = \{x \in M;\ R < \rho(x,p) < S\}$.
$B(R) = \{x \in \mathbb{R}^{n+1};\ d(x,0) < R\}$; $S(R) = \partial B(R)$.
$A(R,S) = \{x \in \mathbb{R}^{n+1};\ R < d(x,0) < S\}$.

The following proposition will be used repeatedly in this and the next section.

PROPOSITION 2.1. (The C^2-Compactness Theorem). *Let $D \subset \mathbb{R}^{n+1}$ be a bounded domain with smooth boundary ∂D. Let (M_i) be a sequence of connected hypersurfaces in $\mathbb{R}^{n+1}$ with $(H_r)_i = 0$, $(H_n)_i \neq 0$ and $(H_j)_i \geq 0$, for $j = 1, \ldots, r-1$, everywhere, and such that $\partial M_i \cap D = \emptyset$. Assume that there exists a constant $C > 0$ such that $\sup_{x \in M_i} |A_i(x)|^2 < C$ and that there exists a sequence of points (x_i), $x_i \in M_i$, with a limit point $x_0 \in D$. Then a subsequence of (M_i) converges C^2 to a hypersurface $M_\infty \subset D$ and $(H_r)_\infty = 0$.*

PROOF. From the uniform bound of the curvature $|A_i|^2$, we conclude the existence of a number $\delta > 0$ such that near each $p_i \in M_i$, M_i can be graphed by a function $f_i^{p_i}$ over a disk $U_\delta(p_i) \subset T_{p_i}M$, of radius δ and center p_i in the tangent space $T_{p_i}M_i$, and that such functions have a uniform C^1 bound (independent of p_i and i). Notice that the normal curvature of the graph of $f_i^{p_i}$ at a point q in the direction v, $v \in T_q(M_i)$, $|v| = 1$, is given by

$$k_v^2 = [(f_i^{p_i})'']^2/(1 + [(f_i^{p_i})']^2)^3.$$

Here the primes denote derivatives at $t = 0$ of the restriction $f_i^{p_i}(c(t))$, and $c(t)$ is a curve on the graph parametrized by arc length with $c(0) = q$, $c'(0) = v$. Since $[(f_i^{p_i})']^2$ and k_v^2 are uniformly bounded, we obtain a uniform bound for the second derivative in any direction. By a standard procedure (see e.g. [5] p. 280), this implies a uniform C^2-bound on $f_i^{p_i}$.

Now, consider the sequence (x_i) with a limit point x_0, and let τ_i be the translation that takes x_i to x_0. The unit normals of $\tau_i(M_i)$ at x_0 have a convergent subsequence, hence a subsequence of the tangent planes $T_{x_0}(\tau_i M_i)$ converges to a plane π containing x_0. For i large, the parts of M_i that were graphs over $D_\delta(x_i)$ are now graphs over $D_{\delta/2}(x_0) \subset \pi$; we will denote the corresponding functions by $g_i^{x_0}$.

We need the following lemma.

LEMMA 2.2. *A subsequence of $(g_i^{x_0})$ converges C^2 to a function $g_\infty^{x_0}$.*

PROOF. By the bounds on the derivatives that we have obtained, the functions $g_i^{x_0}$ and their first and second derivatives are uniformly bounded, say, $|g_i^{x_0}|_{2;D_{\delta/2}(x_0)} < C_1$.

For each i, consider the parallel hypersurface of the graph of $g_i^{x_0}$ obtained by displacing by t this graph along its normal. For t small, we obtain a graph over $D_{\delta/4}(x_0) \subset \pi$ whose r-mean curvature is easily computed to be

$$(H_r(t))_i = \frac{1}{\binom{n}{r}} \left[\frac{t\binom{r+1}{r}(S_{r+1})_i + t^2\binom{r+2}{r}(S_{r+2})_i + \cdots + t^{n-r}\binom{n}{r}(S_n)_i}{\prod_{j=1}^{n}(1 + t(k_j)_i)} \right],$$

where we have used that $S_r = 0$. Let us denote the function corresponding to the parallel graph by $g_i^{x_0}(t)$. Since $(H_n)_i \neq 0$, we have that $(S_{r+1})_i \neq 0$ ([9], Lemma 1.2), and by taking t small enough (positive or negative as the case may be), we obtain that $(H_r(t))_i > 0$.

By construction, for small t, the function $g_i^{x_0}(t)$ and its derivatives are close to $g_i^{x_0}$ and its derivatives. So given i, there exists t_i such that

$$|g_i^{x_0}(t_i) - g_i^{x_0}|_{2;\Omega} < 1/i,$$

where $\Omega = D_{\delta/4}(x_0)$. Consider the sequence $(g_i^{x_0}(t_i))$. Since $g_i^{x_0}$ is uniformly C^2-bounded, so is $(g_i^{x_0}(t_i))$, say

$$|g_i^{x_0}(t_i)|_{2;\Omega} < C_2.$$

Now set $\psi_i = (H_r(t_i))_i$. Then, for each i, $g_i^{x_0}(t_i)$ satisfies the following fully nonlinear elliptic equation of second order:

$$H_r^{1/k} = \psi_i^{1/k}.$$

Set Γ = connected component in $\mathbb{R}^n$, containing $a = (1, \ldots, 1)$, of the set of principal curvatures vectors $(k_1, \ldots, k_n)$ for which $H_r > 0$. The conditions $(H_r)_i = 0$ and $(H_j)_i \geq 0$, $j = 1, \ldots, r-1$, guarantee that $((k_1)_i, \ldots, (k_n)_i) \in \partial\Gamma$ (see [9], Lemma 1.2; notice that the notation for Γ in [9] is C). Thus, since $(H_r(t_i))_i > 0$, we have that $((k_1(t_i))_i, \ldots, (k_n(t_i))_i) \in \Gamma$. Because $H_r^{1/r}$ is concave in Γ (see [15], Example 2), we can apply the known Hölder estimates for fully nonlinear elliptic equations ([7], Theorem 17.14, p. 461) to obtain

$$[D^2 g_i^{x_0}(t_i)]_{\alpha;\Omega'} < K,$$

where $\Omega' \subset B_{\delta/4}(x_0)$, $x_0 \in \Omega'$, and K is a constant that depends essentially on Ω' and C_2. It follows that $g_i^{x_0}(t_i)$ and its first and second derivative are uniformly bounded and equicontinuous. By the Theorem of Arzelá-Ascoli, a subsequence of $(g_i^{x_0}(t_i))$ converges C^2 to a function $g_\infty^{x_0}$. By construction, the corresponding subsequence $(g_i^{x_0})$ converges C^2 to $g_\infty^{x_0}$ and this completes the proof of the lemma. $\square$

Notice that we have obtained a subsequence of (M_i) with the property that those parts of M_i that are graphs around the points x_i, converge to a hypersurface passing through x_0. We will express this fact by saying that (M_i) *has a subsequence that converges locally at* x_0.

To complete the proof of Proposition 2.1, we need a covering argument that runs as follows.

Let L be the set of all limit points of sequences of the form (p_i), where $p_i \in M_i$, and let N be a connected component of L. Let $q_1, q_2, \ldots$ be a sequence of points in N that is dense in N. Let (q_1^i), $q_1^i \in M_i$, be a sequence that converges to q_1. As we did before, we can obtain a subsequence (M_i^1) of (M_i) that converges locally at q_1. From this sequence, we can extract a subsequence (M_i^2) that converges locally at q_1 and q_2. By induction, we can find sequences (M_i^n) that converge locally to $\bigcup_i q_i$, $i = 1, \ldots, n$. By using the Cantor diagonal process, we obtain a sequence $M_1^1, M_2^2, \ldots$ that converges C^2 to a hypersurface M_∞ that contains N. Clearly M_∞ satisfies $H_r = 0$. This completes the proof of Proposition 2.2. $\square$

Remark. Lemma 2.2 in the proof of the C^2-compactness theorem is the only reason why we need the condition $H_j \geq 0$, $j = 1, \ldots, r$ in Theorem 1.1. This comes from the fact that, in contrast to the minimal case, where the elliptic partial differential equation involved is quasi-linear, the case $H_r = 0$ involves a fully nonlinear equation on f of the type $F(Df, D^2 f) = 0$. For such equations, the available theorems that yields Hölder $C^{2+\alpha}$ bounds for f in terms of C^2 bounds require the function $F(Df, D^2 f)$ to be concave in the second argument. Since H_r is not a concave function, we approximated our graphs by using the function $(H_r)^{1/r}$ which is concave in Γ. Thus we must require our curvature vectors to be in $\partial\Gamma$, and this is equivalent to requiring $H_j \geq 0$, $j = 1, \ldots, r-1$.

The proof of the following Proposition is similar to that of [1], Proposition 2.2; for completeness, we present it here. We assume that the immersion passes through $0 \in \mathbb{R}^{n+1}$.

PROPOSITION 2.3. *Let $x\colon M^n \to \mathbb{R}^{n+1}$ be a complete immersion with $H_r = 0$, $H_j \geq 0$, $j = 1, \ldots, r-1$, and $H_n \neq 0$ everywhere. Assume that x has finite total curvature. Then, given $\varepsilon > 0$ there exists $R_0 > 0$ such that, for $R > R_0$,*

$$R^2 \sup_{x \in M - D_0(R)} |A|^2(x) < \varepsilon.$$

The proof of Proposition 2.3 depends on Lemmas 2.4 and 2.5 below. For these lemmas we use the following notation. We denote by $h\colon X^n \to \mathbb{R}^{n+1}$ an immersion into $\mathbb{R}^{n+1}$ of an n-manifold $X^n = X$ with boundary ∂X and such that $H_r = 0$, $H_j \geq 0$, $j = 1, \ldots, r-1$, $H_n \neq 0$ everywhere and that there exists a point $x \in X$ with $D_x(1) \cap \partial X = \emptyset$.

LEMMA 2.4. *There exists $\delta > 0$ such that for any $h\colon X^n \to \mathbb{R}^{n+1}$ as above, if*

$$\int_{D_x(1)} |A_h|^n \, dX < \delta,$$

then

$$\sup_{t \in [0,1]} \left[t^2 \sup_{D_x(1-t)} |A_h|^2 \right] \leq 4.$$

Here A_h is the linear map associated to the second fundamental of h.

PROOF. Suppose the lemma is false. Then there exist a sequence $h_i\colon X_i \to \mathbb{R}^{n+1}$ and points $x_i \in X_i$ with $D_{x_i}(1) \cap \partial X_i = \emptyset$, $(H_r)_i = 0$, $H_j \geq 0$, $j = 1, \ldots, r-1$, and $(H_n)_i \neq 0$, such that

$$\int_{D_{x_i}(1)} |A_i|^n \, dX_i \to 0$$

but

$$\sup_{t \in [0,1]} \left[t^2 \sup_{D_{x_i}(1-t)} |A_i|^2 \right] > 4,$$

for all i, where $A_i = A_{h_i}$.

Choose $t_i \in [0,1]$ so that

$$t_i^2 \sup_{D_{x_i}(1-t_i)} |A_i|^2 = \sup_{t \in [0,1]} \left[t^2 \sup_{D_{x_i}(1-t)} |A_i|^2 \right]$$

and choose $y_i \in D_{x_i}(1 - t_i)$ so that

$$|A_i|^2(y_i) = \sup_{D_{x_i}(1-t_i)} |A_i|^2.$$

By using that $D_{y_i}(t_i/2) \subset D_{x_i}(1-(t_i/2))$ and the choice of t_i, we obtain

$$\sup_{D_{y_i}(t_i/2)} |A_i|^2 \le \sup_{D_{x_i}(1-(t_i/2))} |A_i|^2 \le \frac{t_i^2}{t_i^2/4} \sup_{D_{x_i}(1-t_i)} |A_i|^2,$$

hence, by the choice of y_i, we have

$$\sup_{D_{y_i}(t_i/2)} |A_i|^2 \le 4|A_i|^2(y_i). \tag{2.1}$$

We now rescale the metric defining $d\tilde{s}_i^2 = |A_i|^2(y_i)ds_i^2$, that is, $d\tilde{s}_i^2$ is the metric on X_i induced by $\tilde{h}_i = d_i \circ h$, where d_i is the dilation of $\mathbb{R}^{n+1}$ about $h_i(y_i)$ by the factor $|A_i|(y_i)$ (by translation, we may assume that $h_i(y_i) = 0$). The symbol $\sim$ will indicate quantities measured with respect to the new metric $d\tilde{s}_i^2$.

By assumption, $|A_i|^2(y_i) > 4/t_i^2$. Thus

$$\tilde{D}_{y_i}(1) = D_{y_i}([|A_i|(y_i)]^{-1}) \subset D_{y_i}(t_i/2) \subset D_{x_i}(1-t_i/2) \subset D_{x_i}(1).$$

It follows that $\tilde{D}_{y_i}(1) \cap \partial X_i = \emptyset$. Now, use (2.1) and the fact that

$$|\tilde{A}_i|(p) = [|A_i|(y_i)]^{-1}|A_i|(p)$$

to obtain

$$\sup_{\tilde{D}_{y_i}(1)} |\tilde{A}_i|^2 \le 4.$$

Therefore, the sequence $\tilde{h}_i = \tilde{D}_{y_i}(1) \to \mathbb{R}^{n+1}$, $\tilde{h}_i(y_i) = 0$, is a sequence of immersions with $(H_r)_i = 0$, $H_j \ge 0$, $j = 1, \dots, r-1$, $(H_n)_i \ne 0$, and uniformly bounded second fundamental form. By the C^2-Compactness Theorem, a subsequence converges C^2 to an immersion

$$\tilde{h}_\infty \colon \tilde{D}_{y_\infty}(1) \to \mathbb{R}^{n+1}.$$

Furthermore, since

$$\int_{\tilde{D}_{y_i}(1)} |\tilde{A}_i|^n \, d\tilde{X}_i \le \int_{D_{x_i}(1)} |A_i|^n \, dX_i \to 0,$$

$\tilde{h}_\infty(\tilde{D}_{y_\infty}(1))$ is contained in an n-plane of $\mathbb{R}^{n+1}$. But $|\tilde{A}_i|(y_i) = 1$, for all i, hence $|\tilde{A}_\infty|(y_\infty) = 1$. This is a contradiction, and completes the proof of Lemma 2.4. □

Lemma 2.5. *Given $\varepsilon_1 > 0$, there exists $\delta > 0$, such that, for any $h \colon X^n \to \mathbb{R}^{n+1}$ as above, if*

$$\int_{D_x(1)} |A_h|^n \, dX < \delta,$$

then

$$\sup_{D_x(1/2)} |A_h|^2 < \varepsilon_1.$$

Proof. Assume the lemma is false. Then there exists a sequence of immersions $h_i\colon X_i \to \mathbb{R}^{n+1}$ with $(H_r)_i = 0$, $(H_j)_i \geq 0$, $j = 1, \ldots, r-1$, $(H_n)_i \neq 0$, and $D_{x_i}(1) \cap \partial X_i = \emptyset$ (by translation, we can assume $h_i(x_i) = 0$) such that

$$\int_{D_{x_i}(1)} |A_i|^n \, dX_i \to 0 \tag{2.2}$$

but

$$\sup_{D_{x_i}(1/2)} |A_i|^2 \geq C^2, \tag{2.3}$$

for some constant C.

By Lemma 2.4 (with $t = 1/2$), we have, for i sufficiently large,

$$\sup_{D_{x_i}(1/2)} |A_i|^2 \leq 16.$$

By the C^2-Compactness Theorem and (2.2), a subsequence of (h_i) converges smoothly to a domain in an n-plane in $\mathbb{R}^{n+1}$. This is a contradiction to (2.3) and proves Lemma 2.5. $\square$

Proof of Proposition 2.3. We first rescale the immersion x to $\tilde{x} = d_{2/R} \circ x$, where $d_{2/R}$ is the dilation by the factor $2/R$. Thus the metric induced by $\tilde{x}$ in M is $d\tilde{s}^2 = (4/R^2)ds^2$, where ds^2 is the metric induced by x. We will denote the quantities measured relative to the new metric by the superscript $\sim$. Notice that the second fundamental form $\tilde{A}$ satisfies $|\tilde{A}|^2 = \frac{R^2}{4}|A|^2$.

Therefore, Proposition 2.3 will be established once we prove that given $\varepsilon > 0$ there exists R_0 such that, for $R > R_0$,

$$\sup_{x \in M - \tilde{D}_0(2)} |\tilde{A}|^2 < \varepsilon/4.$$

Given the above ε, set $\varepsilon_1 = \varepsilon/4$ and let $\delta > 0$ be given by Lemma 2.5. Since M has finite total curvature, there exists R_0 such that, for $R > R_0$,

$$\delta > \int_{D_0(R/2,\infty)} |A|^n \, dM = \int_{\tilde{D}_0(1,\infty)} |\tilde{A}|^n \, d\tilde{M}.$$

Now, take $x \in M - \tilde{D}_0(2)$. Then $\tilde{D}_x(1) \subset \tilde{D}_0(1,\infty)$. By Lemma 2.5, the above inequality implies that

$$\sup_{\tilde{D}_x(1/2)} |\tilde{A}|^2 < \varepsilon_1 = \varepsilon/4,$$

hence

$$\sup_{x \in M - \tilde{D}_0(2)} |\tilde{A}|^2 < \varepsilon/4.$$

This completes the proof of Proposition 2.3. $\square$

3 Proof of Theorem 1.1

The proof of Theorem 1.1 depends on a series of lemmas to be presented now. In this section, $x\colon M^n \to \mathbb{R}^{n+1}$ will always denote a complete hypersurface with $H_r = 0$, $H_j \geq 0$, $j = 1, \dots, r-1$, $H_n \neq 0$ everywhere, and with finite total curvature.

The following lemma is similar to Lemma 2.4 in Anderson [1]. For completeness, we include its proof.

Lemma 3.1. *Let $x\colon M^n \to \mathbb{R}^{n+1}$ be as above. Assume that $x(M^n)$ passes through the origin 0 of $\mathbb{R}^{n+1}$, and let $r(p) = d(p, 0)$, where $p \in M$ and d is the distance in $\mathbb{R}^{n+1}$. Then x is proper and the gradient ∇r of r in M satisfies*

$$\lim_{r \to \infty} |\nabla r| = 1.$$

In particular, there exists r_0 such that if $r > r_0$, $\nabla r \neq 0$, i.e., the function r has no critical points outside the ball $B(r_0)$.

Proof. Let $\gamma(s)$ be any minimizing geodesic issuing from $0 \in M$ (a ray) parametrized by the arc length, and set $T = \gamma'(s)$. Let

$$X = (1/2)\overline{\nabla} r^2 = r \overline{\nabla} r,$$

be the position vector field, where $\overline{\nabla} r$ is the gradient of r in $\mathbb{R}^{n+1}$. Since $\overline{\nabla}_T X = T$, we have

$$T\langle X, T\rangle = \langle \overline{\nabla}_T X, T\rangle + \langle X, \overline{\nabla}_T T\rangle = 1 + \langle X, \overline{\nabla}_T T\rangle.$$

Since γ is a geodesic in M, the tangent component of $\overline{\nabla}_T T$ vanishes and

$$\overline{\nabla}_T T = \langle \overline{\nabla}_T T, N\rangle N = -\langle \overline{\nabla}_T N, T\rangle N = \langle A(T), T\rangle N.$$

It follows, by Cauchy-Schwary inequality, that

$$|\langle X, \overline{\nabla}_T T\rangle| \leq |X||A(T)||T| \leq |X||A|,$$

hence

$$T\langle X, T\rangle \geq 1 - |X||A|.$$

By using Proposition 2.3 with $\varepsilon = 1/n^2$, and the fact that $r = |X(s)| \leq s$, we obtain

$$T\langle X, T\rangle(s) \geq 1 - \frac{1}{n}, \tag{3.1}$$

for all $s > R_0$, where R_0 is given by Proposition 2.3. Integration of (3.1) from R_0 to s gives

$$\langle X, T\rangle(s) \geq (1 - \frac{1}{n})(s - R_0) + \langle X, T\rangle(R_0). \tag{3.2}$$

Since

$$\langle X, T\rangle(s) = \langle r\overline{\nabla}r, T\rangle(s) \le r|\nabla r|(s),$$

we have

$$|\nabla r|(s) \ge \frac{\langle X, T\rangle(s)}{s} \ge (1 - \frac{1}{n})(\frac{s - R_0}{s}) + \frac{\langle X, T\rangle(R_0)}{s}. \tag{3.3}$$

Because $r(s) = |X(s)| \ge \langle X, T\rangle(s)$, we see from (3.2) that r goes to infinity with s; thus M is properly immersed. Furthermore, by taking the limit in (3.3) as $s \to \infty$, we obtain that $\lim_{s\to\infty} |\nabla r| \ge 1 - (1/n)$. Since n is arbitrary and $|\nabla r| \le 1$, we obtain that $\lim_{r\to\infty} |\nabla r| = 1$. This concludes the proof of Lemma 3.1. $\square$

Now, let r_0 be chosen so that r has no critical points in $W = x(M) - (B(r_0) \cap x(M))$. By Morse Theory, W is diffeomorphic to $[x(M) \cap S(r_0)] \times [0, \infty]$. Let V be a connected component of $x^{-1}(W)$, to be called an *end* of M. Since $B(r_0) \cap x(M)$ is compact, M has only a finite number of ends. In what follows, we identify V and $x(V)$.

Let $r > r_0$ and set

$$\Sigma_r = \frac{1}{r}[V \cap S(r)] \subset S(1),$$
$$V_r = \frac{1}{r}[V \cap B(r)] \subset B(1).$$

Denote by A_r the second fundamental form of V_r. Then

$$|A_r|^2(x) = r^2|A|^2(rx).$$

Lemma 3.2. *For $r > r_0$, $V \cap B(r)$ is connected.*

Proof. Notice that $V = S \times [0, \infty)$ where S is a connected component S_{α_i} of $M \cap S(r_0)$. Consider the trajectories of ∇r in V. By the choice of r_0, these trajectories cover V simply (i.e., for each point of V there passes one and only one such trajectory).

Assume that $V \cap B(r)$ has two connected components, V_1 and V_2. Since $(V_1 \cup V_2) \cap S(r_0) = S_{\alpha_i}$ is connected, either $V_1 \cap S(r_0)$ or $V_2 \cap S(R_0)$ is empty. Assume it is V_2.

Let $p \in V_2$. Since the trajectories of ∇r cover V simply, there exists a trajectory $\varphi(t)$ with $\varphi(0) \in V_1 \cap S(r_0)$ and $\varphi(t_2) = p$. Thus, there exist $t_0, t_1 \in [0, t_2]$, such that a trajectory of ∇r satisfies $|\varphi(t_0)| = |\varphi(t_1)| = r$. We claim that this implies the existence of a critical point at some point of $\varphi(t)$.

Indeed, let $f(t) = r(\varphi(t), 0)$, where r is the distance in $\mathbb{R}^{n+1}$ restricted to M. Then $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function with $f(t_0) = f(t_1)$. Thus, there exists $\bar{t} \in [t_0, t_1]$ with $f'(\bar{t})$. But

$$f'(t) = dr\left(\frac{d\varphi}{dt}\right) = dr(\nabla r) = \langle \nabla r, \nabla r\rangle.$$

Therefore,

$$0 = f'(\bar{t}) = |\nabla r(\bar{t})|^2$$

and this proves our claim.

Thus we have reached a contradiction and this proves the lemma.

$\square$

LEMMA 3.3. *Let $0 < \delta < 1$ be given and fix a ring $A(\delta, 1) \subset B(1)$. Then, given $\varepsilon > 0$, there exists r_1 such that, for all $r > r_1$ and all $x \in A(\delta, 1)$, we have*

$$|A_r|^2(x) < \varepsilon.$$

PROOF. By Proposition 2.3, there exists r_0 such that for $r > r_0$

$$r^2 \sup_{x \in L_0(r)} |A|^2(x) < \delta^2 \varepsilon. \tag{3.4}$$

Take $r_1 = r_0/\delta$. Then, for $r > r_1$ and $x \in A(\delta, 1)$,

$$r|x| > r\delta > r_0.$$

Thus, by (3.4),

$$r^2|x|^2 \left[\sup_{y \in L_0(r|x|)} |A|^2(y) \right] < \delta^2 \varepsilon. \tag{3.5}$$

Now, by using (3.5), we obtain that

$$|A_r|^2(x) = r^2|A|^2(rx) < r^2 \left[\sup_{y \in L_0(r|x|)} |A|^2(y) \right] < \frac{\delta^2 \varepsilon}{|x|^2} < \varepsilon,$$

and this proves Lemma 3.3. $\square$

By Lemma 3.3, we see that $|A_r|^2 \to 0$ uniformly in the ring $A(\delta, 1)$. It follows from this and the fact that V_r is connected that we can apply the C^2-Compactness Theorem and conclude that a subsequence V_{r_i} of V_r converges to a piece of an n-plane π in $A(\delta, 1)$. Since δ is arbitrary, V_{r_i} converges to π in $B(1) - \{0\}$. Furthermore, since $\lim_{r \to \infty} |\nabla r| = 1$, π passes through the origin. Thus, Σ_{r_i} converges to an equatorial sphere $\Sigma_\infty = S^{n-1}(1) \subset S^n(1)$, possibly with multiplicity $m \geq 1$. By a covering argument, it is easily seen that $m = 1$, since S^{n-1} is simply-connected. Thus V is embedded.

The n-plane π spanned by Σ_∞ is called the *tangent plane at infinity of the end V associated to the sequence* $\{r_i\}$. A crucial point in the proof of Theorem 1.1 is to show that this plane does not depend on the sequence $\{r_i\}$.

LEMMA 3.4. *Each end V of M has a unique tangent plane at infinity.*

PROOF. Suppose that $\{s_i\}$ and $\{r_i\}$, $i = 1, \ldots, \infty$, are sequences of real numbers and that π_1 and π_2 are distinct tangent planes at infinity associated to $\{s_i\}$ and $\{r_i\}$, respectively. We can assume that the sequences satisfy

$$s_1 < r_1 < s_2 < r_2 < \cdots < s_i < r_i < \ldots.$$

Let K be the closure of $B(3/4) - B(1/4)$ and let N_1 be the normal to π_1, obtained as the limit of the normals to

$$K \cap \left\{ \frac{1}{s_i} V \right\} = \frac{1}{s_i}(V \cap s_i K).$$

Similarly, let N_2 be the normal to π_2 obtained as the limit of the normals to $K \cap \{(1/r_i)V\}$.

Now let U_1 and U_2 be neighborhoods in $S^n(1)$ of N_1 and N_2, respectively, such that $U_1 \cap U_2 = \emptyset$. Thus, there exists an index i_0 such that, for $i > i_0$, the normals to $K_i^1 = (s_i K) \cap V$ are in U_1 and the normals to $K_i^2 = (r_i K) \cap V$ are in U_2. If $K_i^1 \cap K_i^2 \neq \emptyset$, for some $i > i_0$, this contradicts the fact that $U_1 \cap U_2 = \emptyset$, and the lemma is proved.

Thus we may assume that, for all $i > i_0$, $K_i^1 \cap K_i^2 = \emptyset$. In this case, we have $(1/4)r_i > (3/4)s_i$; here, and in what follows, we always assume $i > i_0$. Set

$$W_i = V \cap (B(\frac{1}{4}r_i) - B(\frac{3}{4}s_i)).$$

Since $H_n \neq 0$ everywhere, the Gauss map g is a local diffeomorphism, hence $g(\partial W_i) \supset \partial(g(W_i))$. Since

$$g(S(\frac{1}{4}r_i) \cap V) \subset U_2,$$
$$g(S(\frac{3}{4}s_i) \cap V) \subset U_1,$$

we have $g(\partial W_i) \subset U_1 \cup U_2$. Thus

$$\partial(g(W_i)) \subset g(\partial W_i) \subset U_1 \cup U_2,$$

and, since $g(W_i)$ has nonvoid intersection with both U_1 and U_2, we obtain

$$g(W_i) \supset S^n(1) - \{U_1 \cup U_2\}. \tag{3.6}$$

On the other hand, because

$$\left(\Sigma k_i^2\right)^n > C k_1^2 \dots k_n^2,$$

for a constant $C = C(n)$, we have that

$$|H_n| < \frac{1}{\sqrt{C}} |A|^n.$$

Furthermore, since the total curvature is finite,

$$\int_{W_i} |A|^n \, dM \to 0, \quad i \to \infty.$$

Therefore, since

$$\text{Area } g(W_i) \leq \int_{W_i} |H_n| \, dM < (\frac{1}{\sqrt{C}}) \int_{W_i} |A|^n \, dM,$$

we have that Area $g(W_i) \to 0$. This a contradiction to (3.6), and completes the proof of Lemma 3.4. □

We can now prove our main Theorem.

Proof of Theorem 1.1. We prove (i) and (ii) together. Following Anderson [1], we consider each end V_i and apply to V_i the inversion $I\colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}$, $I(x) = x/|x|^2$. Thus $I(V_i) \subset B(1) - \{0\}$ and, by Lemma 3.4, for any sequence $q_n \in I(V_i)$ that converges to the origin 0, the normals of q_n converge to a unique normal $p_i \in S^n(1)$ (namely, to the normal of the plane at infinity of V_i). It follows that each V_i can be compacfied with a point q_i together with a tangent plane at q_i. Doing this for each V_i, we obtain a compact C^1-manifold $\overline{M}$ such that $\overline{M} - \{q_1, \dots, q_k\}$ is diffeomorphic to M, and a continuous extension $\overline{g}\colon \overline{M} \to S^n(1)$ of g by setting $\overline{g}(q_i) = p_i$. This proves (i) and (ii).

We prove (iii). Let $\{r_i^{n_i}\} \in M$, $i = 1, \dots, k$ and n_i running in a set I_i, be the inverse image of p_i by $g\colon M^n \to S^n(1)$. Since g is a local diffeomorphism, either the set $\{r_i^{n_i}\}$ is finite or the image set $\{g(r_i^{n_i})\}$ has p_i as the unique accumulation point. Clearly the map

$$g\colon M - \bigcup_{i=1}^{k} \{r_i^{n_i}\} \to S^n(1) - \{p_1, \dots, p_k\}$$

is proper and its Jacobian never vanishes. In this situation, it is known that the map is surjective and a covering map ([17], Corollary 1). Since $n \geq 3$, $S^n(1) - \{p_1, \dots, p_k\}$ is simply-connected. Thus g is a global diffeomorphism.

To complete the proof of (iii), we must show that: a) if $\overline{g}(q_i) = \overline{g}(r_i^{n_i})$, then $q_i = r_i^{n_i}$ (this shows that the sets $\{r_i^{n_i}\}$ are actually empty), and b) if $\overline{g}(q_i) = \overline{g}(q_j)$, then $q_i = q_j$. Let us prove (a). Suppose that the contrary is true, that is, there exist q_i and r_i^s with $\overline{g}(q_i) = \overline{g}(r_i^s) = p_i$ and $q_i \neq r_i^s$. Let $W \subset S^n(1)$ be a neighborhood of p_i. By continuity, there exist disjoint neighborhoods U_i of q_i and U_s of r_i^s such that $\overline{g}(U_i) \subset W$, $\overline{g}(U_s) \subset W$. Choose $p \in \overline{g}(U_i) \cap \overline{g}(U_s)$, $p \neq p_i$. Then, there exist $t_i \in U_i$ and $t_s \in U_s$ such that $g(t_i) = g(t_s) = p$. But this contradicts the fact that g is a diffeomorphism. This proves (a).

The proof of (b) is similar. This completes the proof of (iii) and of Theorem 1.1 $\square$

4 Proof of Theorem 1.2

Before going into the proof of Theorem 1.2, we need to fix some notation and to recall some facts on stability. Further details can be found in [12], [13], [3] and [2].

Let $x\colon M^n \to \mathbb{R}^{n+1}$ be an orientable hypersurface with $H_{r+1} = 0$ and $H_n \neq 0$ everywhere. A *regular domain* $D \subset M$ is a domain with compact closure and piecewise smooth boundary. We say that D is *stable* if either $A_r''(0) > 0$ for all variations with compact support in D or $A_r''(0) < 0$, for all such variations. Details can be found in [2], Remark 2.6. If for some variation with compact support in D we have $A_r''(0) > 0$, while for some other such variation, we have $A_r''(0) < 0$, we say that D is *unstable*.

Following [12], we define a linear map P_r of T_pM by

$$P_0 = I, \quad P_r = S_r I - AP_{r-1},$$

where I is the identity matrix and A is the linear map defined in the Introduction. Next, we define a second order linear operator L_r by

$$L_r f = \operatorname{div}(P_r \nabla f), \tag{4.1}$$

where ∇f is the gradient of f. We then write the Jacobi equation of the variational problem that defines the hypersurfaces with $H_{r+1} = 0$:

$$T_r f = L_r f - (r+2) S_{r+2} f. \tag{4.2}$$

The Jacobi equation (4.2) is the linearization of the equation $H_{r+1} = 0$. As we mentioned in the Introduction, $H_n \neq 0$ everywhere is a sufficient condition for (4.2) to be elliptic. By (4.1), this is equivalent to the fact that P_r has all its eigenvalues of the same sign. We denote by $\theta_i(r)$ the eigenvalues of $\sqrt{P_r}A$ when P_r is positive definite, and the eigenvalues of $\sqrt{-P_r}A$ when P_r is negative definite. We will assume for convenience that P_r is positive definite, leaving the details of the other case to the reader.

Remark 4.1. We can now justify the statement made in the Introduction that the condition $H_1 \geq 0$ is not necessary for the case $H_2 = 0$ and $H_n \neq 0$. Indeed, since the operator L_1 defined by (4.1) is elliptic, P_1 has all its eigenvalues with the same sign. It is known that (see, e.g., [3], Lemma 2.1(b))

$$\text{Trace } P_r = (n-r) S_r.$$

It follows that H_1 has a fixed sign which, by a change of orientation, can be made positive, as we claimed.

In the present case, we can rewrite the Jacobi operator T_r as ([2], p. 8)

$$T_r = L_r + ||\sqrt{P_r}A||^2,$$

where $||\sqrt{P_r}A||^2 = \sum_i \theta_i^2(r)$.

Under the condition $H_{n-1} = 0$, it can be shown that ([2], p. 4)

$$(\theta_i(n-2))^2 = -S_n. \tag{4.3}$$

Finally, we define the Morse index form I_r of our variational problem as

$$I_r(f,g) = -\int_M f T_r(g)\, dM.$$

For future reference, we need to quote some of the results of [2] as Theorem A and B below.

Theorem A (Theorem 1.1 of [2]). *Let $x\colon M^n \to \mathbb{R}^{n+1}$ be an orientable immersion with $H_{n-1} = 0$ and $H_n \neq 0$ everywhere. Let $D \subset M$ be a regular domain and assume that the* *first eigenvalue of $g(D) \subset S_1^n$ for the spherical Laplacian satisfies $\lambda_1(g(D)) > n$. Then D is stable.*

THEOREM B (Corollary 1.6 of [2]). *Let $x\colon M^n \to \mathbb{R}^{n+1}$ be as in Theorem A and $D \subset M$ be a regular domain. Assume that the Gauss map g restricted to $\overline{D}$ is a covering map onto $g(\overline{D})$. If the first eigenvalue of $g(D)$ for the spherical Laplacian satisfies $\lambda_1(g(D)) < n$, then D is unstable.*

We also need a lemma which is proved in [2] (Lemma 2.7) and that will be quoted here as Lemma A. We use $C_0^\infty(D)$ to denote the space of differentiable functions that vanish in the boundary ∂D of a regular domain D, and $C_c^\infty(D)$ to denote those differentiable functions that have a (compact) support in D.

LEMMA A ([2]; see also [16]). *The following statements are equivalent:*

i) *$\exists\ f \in C_c^\infty(D)$ such that $I_r(f,f) \leq 0$.*

ii) *$\exists\ f \in C_c^\infty(D)$ such that $I_r(f,f) < 0$.*

iii) *$\exists\ f \in C_0^\infty(D)$ such that $I_r(f,f) < 0$.*

We still need a definition. We say that the boundary ∂D of a regular domain D is *a first conjugate boundary* if there exists a Jacobi field that vanishes in ∂D and there exists no Jabobi field that vanishes in (the open set) D.

We observe that every domain properly contained in D is stable and every domain that contains D properly is unstable. In fact, if $D' \subsetneq D$ is not stable, there exists $f \in C_c^\infty(D')$ such that $I_r(f,f) \leq 0$. By Lemma A, there exists $f \in C_0^\infty(D')$ such that $I_r(f,f) < 0$. By the Morse Index theorem, there exists $D'' \subset D'$ and a Jacobi field vanishing in $\partial D''$. This is a contradiction and proves the first part of the statement.

To prove the second part of the statement, let $D'' \supsetneq D$. Since ∂D is a conjugate boundary, by the Morse Index Theorem there exists $f \in C_0^\infty(D'')$ with $I_r(f,f) < 0$. By Lemma A, there exists $f \in C_c^\infty(D'')$ with $I_r(f,f) < 0$, hence D'' is unstable.

REMARK 4.2. Although we have no need of it, it is not hard to show that the two-part statement that we just proved is an equivalent definition of a first conjugate boundary.

The proof of Theorem 1.2 will depend on Lemmas 4.3, 4.4 and 4.6 below.

LEMMA 4.3. *Let $x\colon M^n \to \mathbb{R}^{n+1}$ be a hypersurface with $H_{n-1} = 0$ and $H_n \neq 0$ everywhere. Assume that its Gauss map $g\colon M^n \to S^n$ is injective. Let $D \subset M$ be a regular domain such that ∂D is a first conjugate boundary. Then:*

a) *The first eigenvalue $\lambda_1(g(D))$ for the spherical Laplacian satisfies $\lambda_1(g(D)) = n$.*

b) *Let $f\colon g(D) \to \mathbb{R}$ be the first eigenfunction of $g(D)$. Then $u = f \circ g$ satisfies the Jacobi equation, $u > 0$ in D and $u = 0$ in ∂D.*

Proof. We will prove (a). Indeed, $\lambda_1(g(D))$ is not smaller than n. Otherwise, we could find a domain $D' \subset D$ such that $\lambda_1(g(D')) < n$. Thus D' is unstable by Theorem B and this contradicts the fact that every domain contained in D is stable (since ∂D is a first conjugate boundary). Also, it cannot occur that $\lambda_1(g(D)) > n$. Otherwise, we could find a domain $D'' \supset D$ such that $\lambda_1(g(D'')) > n$. By Theorem A, D'' is stable, and this is a contradiction. Thus $\lambda_1(g(D)) = n$ and this proves (a).

We now prove (b).

Since the Gauss map is injective $\partial(g(D)) = g(\partial D)$, and then

$$\tilde{\Delta} u + nu = 0, \quad u > 0 \text{ in } D, \quad u = 0 \text{ in } \partial D,$$

where $\tilde{\Delta}$ is the Laplacian in the pullback metric $\langle\langle\ ,\ \rangle\rangle$ on M by g (we recall that $H_n \neq 0$). By Stokes theorem,

$$0 = \int_D (||\tilde{\nabla} u||^2 - nu^2) dS = \int_D (||\tilde{\nabla} u||^2 - nu^2)|S_n| dM, \tag{4.4}$$

where dM and $dS = |S_n| dM$ are the volume elements of the induced metric and the pullback metric, respectively.

For notational simplicity, we write $(\theta_i(n-2))^2 = \theta_i^2$. Since, by (4.3), $\theta_i^2 = -S_n$, we have, assuming that P_r is positive definite,

$$\frac{1}{n}||\sqrt{P_r} A||^2 = \frac{1}{n}\sum_j \theta_j^2 = |S_n|.$$

Also, denoting by λ_i the eigenvalues of

$$\frac{\sqrt{P_r} A}{||\sqrt{P_r} A||},$$

we obtain that

$$\lambda_i^2 = \frac{\theta_i^2}{\sum_j \theta_j^2} = \frac{1}{n}.$$

Since, for any $X \in T_pM$,

$$||X||^2 = \left\| \left(\frac{\sqrt{P_r} A}{||\sqrt{P_r} A||} \right)^{-1} \right\|^2 \left\| \frac{\sqrt{P_r} A}{||\sqrt{P_r} A||} X \right\|^2,$$

we can write

$$||\tilde{\nabla} u||^2 = n \frac{||\sqrt{P_r} A \tilde{\nabla} u||^2}{||\sqrt{P_r} A||^2}.$$

Therefore, we have from (4.4),

$$\begin{aligned} 0 &= \int_D \left(\frac{n||\sqrt{P_r} A \tilde{\nabla} u||^2}{||\sqrt{P_r} A||^2} - nu^2 \right) \frac{1}{n} ||\sqrt{P_r} A||^2 \, dM \\ &= \int_D (||\sqrt{P_r} A \tilde{\nabla} u||^2 - ||\sqrt{P_r} A||^2 u^2) \, dM. \end{aligned}$$

By using that $\tilde{\nabla} = A^{-2}\nabla$ ([2], Lemma 2.9), that P_r commutes with A, and that $\langle\langle A^{-1}X, A^{-1}X\rangle\rangle = \langle X, X\rangle$, we have, by Stokes Theorem,

$$0 = \int_D (|\sqrt{P_r}\nabla u|^2 - ||\sqrt{P_r} A||^2 u^2) \, dM = I_r(u,u).$$

Now we use that ∂D is a first conjugate boundary. Thus for every $\varphi \in C_0^\infty(D)$, we have that $I_r(\varphi, \varphi) \geq 0$. Otherwise, there exists $g \in C_0^\infty(D)$ with $I_r(g,g) < 0$; by the Morse Index theorem, there exists a Jacobi field in $D' \subsetneq D$ vanishing in $\partial D'$, and this is a contradiction. Then, for any $v \in C_0^\infty(D)$, we obtain, for all $t \in \mathbb{R}$,

$$0 \leq I_r(u + tv, u + tv) = 2tI_r(u,v) + t^2 I_r(v,v),$$

hence $I_r(u,v) = 0$, and thus u satisfies the Jacobi equation. This proves (b) and completes the proof of Lemma 4.3. $\square$

Lemma 4.4. *Let $S_1^n \subset \mathbb{R}^{n+1}$ be the unit sphere of $\mathbb{R}^{n+1}$ and $p = (0, \ldots, 0, 1) \in S_1^n$. Then there exist a domain D, symmetric relative to the equator of S_1^n, and a function $f \colon S_1^n \to \mathbb{R}$ such that $\lambda_1(D) = n$ and f is the first eigenfunction of D. Furthermore, $\lim_{q \to \pm p} f(q) = -\infty$, $q \in S_1^n$; here $-p$ is the antipodal point to p.*

Proof. This is an application of 4.3 to rotation hypersurfaces. Let $\mathbb{R}^{n+1}$ have coordinates $x_1, \ldots, x_{n+1} = y$. Following [9] we let Ox_1 be the axis of rotation and let $y = h(x_1)$ be the equation of the generating curve C of the rotation hypersurface $x \colon M^n \to \mathbb{R}^{n+1}$ with $H_{n-1} = 0$. It is easily checked that $H_n \neq 0$ everywhere for such hypersurfaces and that the curve C is symmetric.

Now consider the domain $D \subset M$ bounded by the rotation of the points of contact of the tangent lines to C issued from the origin 0 of $\mathbb{R}^{n+1}$. It is known ([2], §3.8) that the support function $\langle x, N\rangle$ satisfies the Jacobi equation, is positive in D and vanishes in ∂D. Thus ∂D is a first conjugate boundary and, since the Gauss map of such rotation hypersurfaces is injective ([9], §2), Lemma 4.3 implies that the symmetric domain $g(D) \subset S_1^n$ satisfies $\lambda_1(g(D)) = n$. Furthermore, if f is the first eigenfunction of $g(D)$, then, again by Lemma 4.3, $u = f \circ g$ satisfies the Jacobi equation, $u > 0$ in D and $u = 0$ in ∂D. It follows that $u = \langle x, N\rangle$.

Since M behaves asymptotically like a parabola ([9], §2), we have that the support function, with a convenient choice of orientation, tends to $-\infty$, on both ends of M. Thus, f satisfies $\lim_{z \to \pm 1} f(z) = -\infty$, where $z = g \circ x_1$. $\square$

REMARK 4.5. If we know the explicit expresion of the generating curve C, we can write explictly the function f. For instance, in the case of a rotation hypersurface $x\colon M^3 \to \mathbb{R}^4$ with $H_2 = 0$, we know that the generating curve C is given by

$$y = 1 + \frac{x_1^2}{4}.$$

A simple computation shows that the support function transfered to S_1^n, i.e., $\langle x, N\rangle \circ g^{-1} = f$ is given by

$$f(z) = \frac{1-2z^2}{\sqrt{1-z^2}}, \quad z = g \circ x_1 = \frac{x_1}{\sqrt{4+(x_1)^2}}.$$

Since f is a radial function, one can easily check, by using the expression of the Laplacian for radial functions (see, for instance, Sakai [14], p. 263) that

$$\tilde{\Delta} f + 3f = 0,$$

as it should be.

Lemma 4.6 below follows an argument of do Carmo and Silveira [6].

LEMMA 4.6. *Given finitely many points $p_1, \ldots, p_k \in S_1^n$ there exists a domain $W \subset S_1^n$ that omits neighborhoods $U_i \subset S_1^n$ of p_i, $i = 1, \ldots, k$, and is such that $\lambda_1(W) = n$.*

PROOF. For each p_i, make a rotation of S_1^n so that $p_i = (0, \ldots, 0, 1)$ and let D_i and f_i the domain and the function given by Lemma 4.4. Set $h = \sum_i f_i$ and define W as a connected component of the set $\{p \in S_1^n;\ h \geq 0\}$.

We recall that a hemisphere H of S_1^n has eigenvalue n and that of all domains in S_1^n with the same area, the spherical cap has the smallest eigenvalue. Since $D_1 \cap D_2 \neq \emptyset$, the set $\{p \in S_1^n;\ f_1 + f_2 \geq 0\}$ is not empty. Thus a connected component D_{12} of $\{p \in S_1^n;\ f_1 + f_2 \geq 0\}$ has eigenvalue n with eigenfunction $f_1 + f_2$. By the above minimization property,

$$A(D_{12}) > A(H \subset S_1^n),$$

where $A(\quad)$ denotes the area of the enclosed domain. By the same token, $A(D_i) > A(H)$, $i = 1, \ldots, k$. Thus $D_{12} \cap D_3 = \emptyset$, and an induction shows that $A(W) > A(H)$. This shows that W is not empty. Clearly $\lambda_1(W) = n$, and h is the first eigenfunction of W. Since $\lim_{p \to p_i} f_i = -\infty$, W omit neighborhoods U_i of p_i, as we wished. $\square$

PROOF OF THEOREM 1.2 We assume the existence of an immersion $x\colon M^n \to \mathbb{R}^{n+1}$ as in Theorem 1.2. Since x has finite total curvature, Theorem 1.1 implies that, there exist a compact manifold $\overline{M}$ and points $q_1, \ldots, q_k \in \overline{M}$ such that M is diffeomorphic to $\overline{M} - \{q_1, \ldots, q_k\}$ and the Gauss map extends to a map $\overline{g}\colon \overline{M} \to S_1^n$. Set $p_i = \overline{g}(q_i)$, $i = 1, \ldots, k$. Let $W \subset S_1^n$ be the domain, given by Lemma 4.6, that omits neighborhoods U_i of p_i and is such that $\lambda_1(W) = n$. Let $W' \supsetneq W$ be a domain in S_1^n that still omits neighborhoods of p_i, and set $D = g^{-1}(W')$. Since g is bijective and $\lambda_1(W') < n$, we conclude, by Theorem B, that D is unstable. This contradicts the assumption and completes the proof. $\square$

Example. The following example shows that the hypothesis of stability in Theorem 1.2 cannot be dropped. As mentioned in [2] the hypersurface M in $\mathbb{R}^4$ generated by the rotation of the parabola $h(z) = 1 + z^2/4$ around the z-axis is a nonstable complete hypersurface with $H_2 = 0$ and $H_3 \neq 0$ everywhere. By using the orthogonal parametrization $x\colon M \to \mathbb{R}^4$,

$$x(z, \theta, \varphi) = (h\cos\theta\sin\varphi, h\sin\theta\sin\varphi, h\cos\theta, z),$$

we can easily compute that

$$\int_M |A|^3\, dM = \frac{27}{2}\pi^2.$$

Thus M has finite total curvature and this proves our claim.

5 Further applications and related questions

It is convenient to define $\mathcal{C}$ as the set of finite total curvature complete hypersurfaces $x\colon M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, with $H_r = 0$, $r = 1, \ldots, n-1$, $H_j \geq 0$, $j = 1, \ldots, r-1$, and $H_n \neq 0$ everywhere. It follows from Theorem 1.1 that there exists a universal lower bound for the total curvature in the set $\mathcal{C}$. More precisely, we prove

Theorem 5.1. (The Gap Theorem)

$$inf_{\mathcal{C}} \int_M |A|^n\, dM > \sqrt{24}\,\pi^2.$$

Proof. First, we easily compute that

$$|A|^{2n} > (n!)H_n^2.$$

Thus, since H_n is the determinant of the Gauss map $g\colon M^n \to S^{n-1}(1)$, we obtain

$$\int_M |A|^n\, dM > \sqrt{n!}\int_M |H_n|\, dM = \sqrt{n!} \text{ area of } g(M) \text{ with multiplicity.}$$

By Theorem 1.1, the extended map $\overline{g}\colon \overline{M} \to S^n(1)$ is a homeomorphism, hence

$$\text{area } g(M) = \text{ area } \overline{g}(\overline{M}) = \text{ area } S^n(1).$$

But the area σ_n of a unit sphere of $\mathbb{R}^{n+1}$ is given by ω_n/n, where ω_n is the volume of the corresponding ball given by

$$\omega_n = 2(\sqrt{\pi})^n/\Gamma(n/2);$$

here Γ is the gamma function. From this formula it is easily seen that σ_n increases with n and $\sigma_3 = 2\pi^2$.

It follows that, for all r and all n,

$$\int_M |A|^n\, dM > \sqrt{3!}\sigma_3 = \sqrt{24}\pi^2.$$

□

A natural question related to Theorem 5.1 is whether the infimum in $\mathcal{C}$ of the total curvature is reached for some hypersurface in $\mathcal{C}$.

A well known result about minimal surfaces in $\mathbb{R}^3$ with finite total curvature is that if it has only one embedded end, it is a plane. The embeddedness of the end is essential as shown by the example of the Enneper surface. Based on Theorem 1.1 and a recent result of Barbosa-Fuknoka-Mercuri [4], we will show that a similar result holds for the class $\mathcal{C}$, provided the dimension of the hypersurface is even.

THEOREM 5.2. *An immersion $x\colon M^n \to \mathbb{R}^{n+1}$ in the class $\mathcal{C}$, n even, has exatly two ends. In particular, there does not exist an immersion in $\mathcal{C}$ with a single end provided n is even.*

PROOF. The proof depends on the following considerations from differential topology (for details see [4]). By using a vector field v in $\mathbb{R}^{n+1}$, $|v| = 1$, such that v does not agree with the normal at any of the ends V_i of M, $i = 1, \ldots, k$, and projecting v orthogonally onto the tangent plane of $x(M)$, one obtains a smooth vector field ξ with singularities at the critical points of the height function $h = \langle x, v\rangle$. By compactifiction, ξ can be thought as a smooth vector field on $\overline{M} - \{q_1, \ldots, q_k\}$, with further singularities at the points q_i.

By using Hopf's theorem that the Euler characteristic $X(\overline{M})$ of $\overline{M}$ is equal to the sum of the indices of the vector field ξ, the following expression is obtained in [4] for a general situation which includes the elements of the class $\mathcal{C}$ as examples: if n is even,

$$X(\overline{M}) = \sum_{i=1}^{k}(1 + I(q_i)) + 2m\sigma.$$

Here $I(q_i)$ is the multiplicity of the end V_i (since $n \geq 3$, $I(q_i) = 1$ in our case), σ is ± 1 depending on the sign of H_n and m is the degree of the Gauss map g. From Theorem 1.1, g is a homeomorphism. Thus, $m = 1$ and, since n is even, $X(\overline{M}) = 2$. It follows that for even dimensional hypersurfaces in $\mathcal{C}$, we have

$$2 = 2k + 2\sigma$$

where k is the number of ends. Thus $k = 2$ and $\sigma = -1$, and the theorem follows. □

Before closing this Section, we want to mention a question related bo Theorems 1.1 and 5.2. *Is an immersion in $\mathcal{C}$ with exactly two ends a rotation hypersurface*?

This is related to a well known paper [15] of R. Schoen. In [15], a notion of *regular at infinity* for a complete minimal immersion M^n in $\mathbb{R}^{n+1}$ was introduced and the following facts were proved: 1) Let a minimal immersion M^n in $\mathbb{R}^{n+1}$, $n \geq 3$, have the property that for some compact K, $M - K$ is a union $M_1, \ldots, M_k$, where each M_i is a graph of bounded slope over the exterior of a bounded region in a hyperplane P_i. Then M is regular at infinity. 2) A complete minimal immersion M^n in $\mathbb{R}^{n+1}$, $n \geq 3$, with exactly two ends and regular at infinity is a rotation hypersurface.

Schoen's result (1) and Theorem 1.1 (or Theorem 3.2 of Anderson [1]) imply a minimal hypersurface in $\mathcal{C}$ is regular at infinity. By (2), if it has two ends, is a rotation hypersurface. To extend this result to the entire class $\mathcal{C}$, we must have an appropriate definition of regular at infinity.

Recent work of Hounie-Leite suggests that such a definition in $\mathcal{C}$ should be as follows. Let $x\colon M^n \to \mathbb{R}^{n+1}$ be an immersion in $\mathcal{C}$ with $H_r = 0$. The immersion is *regular at infinity* if for some compact K, $M - K$ consists of k connected components $M_1, \dots, M_k$ and the asymptotic behaviour of each M_i is the same as the asymptotic behaviour of an end of the (unique) complete rotational hypersurface with $H_r = 0$ (for details see [10]). Using this definition, we could ask if the elements in $\mathcal{C}$ are regular at infinity and then try to prove that regular at infinity plus two ends imply that the immersion is a rotation hypersurface. We mention that for $r = 2$, this last part was proved by Hounie-Leite in [10].

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