Color Representation: Theory and Techniques

Jonas Gomes and Luiz Velho

IMPA – Instituto de Matemática Pura e Aplicada
Estrada Dona Castorina, 110
22460-320 Rio de Janeiro, RJ, Brazil
{jonas | lvelho}@impa.br

Abstract

Representation models are very important in Applied Computational Mathematics (Requicha, 1980), (Gomes & Velho, 1992).

In this work we introduce a model for color space representation. Using this model we construct a mathematical framework that enables us to develop systematically the classical results in the area of colorimetry without the need to use Grassmann Laws as a basic set of axioms for the theory.

1 Introduction

The classical theory of colorimetry deals with finite dimensional color spaces. More specifically, trichromatic spaces are studied and the starting point are the Grassmann Laws (Pratt, 1978). These laws were stablished by H. Grassmann, (Grassmann, 1854), based on perceptual color experiments. The laws can be used as a set of postulates to get an axiomatic development of the theory of trichromatic spaces.

The extension of trichromatic space theory to n-dimensional color spaces is rather straightforward. But we should mention that trichromatic color theory is based on Grassmann Laws, and since these laws come from color perceptual experiments the extension to higher dimensions is a bit cumbersome. We will introduce a simple mathematical model that enable us to develop the theory of finite dimensional color spaces on a mathematical basis.

The organization of the paper is as follows: in section 2 we define the color space as an infinite dimensional vector space of functions; in section 3 we introduce a mathematical model to represent the color space as a finite dimensional vector space; in section 4 we give an overview of the basic colorimetric concepts from the point of view of color representation theory. Finally, on Section 5 we conclude making some remarks on possible future research topics related to this work.

2 Spectral Color Space

We will denote by $[\lambda_a, \lambda_b]$ the visible interval of the electromagnetic spectrum. A color which is the visual stimulus produced by an electromagnetic wave, is characterized by the spectral distribution of some radiometric physical unit (in general radiant power is used). Therefore, the color space can be defined to be an appropriate function vector space, $\mathbf{E}$, defined on a subset of the real line (e.g. the visible interval $[\lambda_a, \lambda_b]$ of the spectrum). Not all functions on $\mathbf{E}$ represent real colors because this space might contain unbounded or negative functions. We can conveniently consider different function spaces containing the spectral distributions to be a mathematical model of the color space. A natural way is to define $\mathbf{E}$ as the space of square integrable functions on the real numbers $\mathbf{R}$

$$\mathbf{E} = \{f : \mathbf{R} \to \mathbf{R} \ ; |f|^2 \ \text{ is Lebesgue integrable}\}. \tag{1}$$

Another possible choice is to define the spectral color space $\mathbf{E}$ as the vector space of bounded functions on the visible interval $[\lambda_a, \lambda_b]$ of the spectrum. In some problems it might be convenient to extend the spectral color space to include distributions.

The choice of a convenient function space as the mathematical model for the color space $\mathbf{E}$ has an influence on the metric that can be defined to compute the distance between two colors of the space. This is very important, particularly when we are concerned with approximation properties on the color space. We will not deal with these problems in this work, therefore we will use only the vector space structure of the spectral color space $\mathbf{E}$.

3 Color Space Representations

In order to represent color in the computer we must get finite dimensional representations of the spectral color space $\mathbf{E}$ defined on the previous section. The standard way to introduce these spaces in the literature is referring to color combination experiments and introducing Grassmann Laws. In fact an immediate consequence of these laws is the vector space nature of the trichromatic color space. In this section we will introduce a mathematical model for finite dimensional representations of the color space without resorting to the use of color experiment results.

Definition 1. Let $T : \mathbf{E} \to \mathcal{V}^n$ be a linear transformation from the spectral color space $\mathbf{E}$ to some n-dimensional vector space $\mathcal{V}^n$. If the image $T(\mathbf{E})$ is the whole space $\mathcal{V}^n$, we say that the triple $(T, \mathbf{E}, \mathcal{V}^n)$ is a *finite dimensional color space representation* of dimension n. The map T is called *representation map*, and the vector space V is called *representation space*.

When we choose some basis of the vector space $\mathcal{V}^n$, a coordinate system is defined on V and it becomes naturally isomorphic to the Euclidean space $\mathbf{R}^n$. We prefer to use $\mathcal{V}^n$ instead of $\mathbf{R}^n$ in the above definition, because of the different color coordinate systems used in colorimetry.

In Definition 1, $\mathcal{V}^n$ is a representation model of the spectral color space in the sense defined in (Requicha, 1980) [see also (Gomes & Velho, 1992)]. We will give two examples below in order to show that the above definition has an intrinsic physical nature.

Example 1. (Color Point Sampling) Consider the spectral color space $\mathbf{E}$ of the bounded functions on the visible interval $[\lambda_a, \lambda_b]$ of the spectrum. If λ_1, λ_2, ..., λ_n are n points on the visible interval $[\lambda_a, \lambda_b]$ of the spectrum, we can define a linear map T, in definition 1, by

$$T(C(\lambda)) = (C(\lambda_1), C(\lambda_2), \ldots, C(\lambda_n)). \tag{2}$$

T defines an n dimensional representation of the spectral color space.

Example 2. (Color Physical System) In a color physical system there are n sensors, $s_1, s_2, \ldots, s_n$, and each sensor outputs a signal when stimulated by electromagnetic waves in some interval I of the spectrum. Mathematically, each sensor s_i has an associated *spectral response curve*, $S_i(\lambda)$ so that the signal produced by s_i when stimulated by an electromagnetic wave with spectral power $C(\lambda)$ is

$$C_i = \int_I S_i(\lambda) C(\lambda) d\lambda, \tag{3}$$

Note that (2) is a special case of (3), if we consider the spectral response curve of the sensor s_i to be the Dirac delta distribution $\delta(\lambda - \lambda_i)$.

Since the integral is a linear operator, it is immediate that the map $T : \mathbf{E} \to \mathcal{V}^n$, defined by

$$T(C) = (C_1, C_2, \ldots, C_n), \tag{4}$$

C_i as in (3), is a linear map. Therefore T defines an n-dimensional representation of the spectral color space. This finite dimensional color space is called the *color space of the physical system*. The triple $(T, \mathbf{E}, \mathcal{V}^n)$ is called the *natural representation* of the color space of the physical system.

The eye is an example of a physical system as described in example 2 above. According to the classical Young-Helmholtz trichromatic theory, there are three types of molecules in the eye that are sensitive to electromagnetic waves. The spectral response curves of these "biological light sensors" were studied and measured by Konig and Brodhum (Konig and Brodhum,1889). For more details the reader should consult (Pratt, 1978) or (Wysecki & Stiles, 1983). This 3-dimensional color space representation is called *trichromatic color space*.

The linear transformation T from definition 1 defines a relation, $\equiv$, between the colors $C(\lambda)$ of the space $\mathbf{E}$: if $C_1(\lambda), C_2(\lambda) \in \mathbf{E}$, then $C_1(\lambda) \equiv C_2(\lambda)$ if $T(C_1(\lambda)) = T(C_2(\lambda))$. It is trivial to verify that $\equiv$ is an equivalence relation on $\mathbf{E}$:

- (1) $C_1(\lambda) \equiv C_1(\lambda)$;
- (2) If $C_1(\lambda) \equiv C_2(\lambda)$ then $C_2(\lambda) \equiv C_1(\lambda)$;
- (3) If $C_1(\lambda) \equiv C_2(\lambda)$ and $C_2(\lambda) \equiv C_3(\lambda)$, then $C_1(\lambda) \equiv C_3(\lambda)$.

In analogy with the study of trichromatic color space, we will call the relation $\equiv$ *methamerism* (Wyszecki & Styles, 1982). Two colors $C_1(\lambda)$ and $C_2(\lambda)$ in the space $\mathbf{E}$ are *methameric* if $C_1(\lambda) \equiv C_2(\lambda)$. We can view the finite dimensional color space representation $\mathcal{V}^n$ as the quotient space $\mathbf{E}/\equiv$ of the spectral color space by the methamerism relation.

4 Basic Colorimetric Concepts

In this section we will give a brief overview of all the basic colorimetric concepts with respect to the color space representation defined on the previous section.

4.1 Color Coordinate Systems

Suppose that $(T, \mathbf{E}, \mathcal{V}^n)$ is the natural representation of the color space associated to a color physical system (Example 2). For any color $C \in \mathcal{V}^n$, we have

$$C = \sum_k \beta_k P_k, \tag{5}$$

where P_i, $i = 1, \ldots, n$ are n linearly independent color vectors on the color space of the system. The basis $\{P_1, \ldots, P_n\}$ of color vectors is called a set of *primary sources* of the color space $\mathcal{V}^n$. The coefficients β_k, $k = 1, 2, \ldots, n$ are the n-dimensional coordinates of the color $C(\lambda)$ in the basis of primaries $\{P_1, \ldots, P_n\}$, and represent that color on the color space of the physical system. For $n = 3$ they are usually called *trichromatic coordinates.*

An important problem consists in obtaining $\mathcal{V}^n$ as a color space representation for the coordinate system defined by the basis $\{P_i\}$: we must find a linear map $\overline{T}\colon \mathbf{E} \to \mathcal{V}^n$ from some spectral color space $\mathbf{E}$, onto the finite dimensional color space $\mathcal{V}^n$, such that

$$T(C(\lambda)) = C = \sum_{k=1}^{n} \beta_k P_k,$$

for $C(\lambda) \in \mathbf{E}$.

Denote by $P_k(\lambda)$ the spectral distribution in $\mathbf{E}$ such that $T(P_k(\lambda)) = P_k$ for some representation of the color space $\mathcal{V}^n$.

If we define

$$\overline{C}(\lambda) = \sum_{k=1}^{n} \beta_k P_k(\lambda), \tag{6}$$

we can not guarantee that $C(\lambda) = \overline{C}(\lambda)$. However

$$\begin{aligned} T(\overline{C}(\lambda)) &= T\left(\sum_k \beta_k P_k(\lambda)\right) \\ &= \sum_k \beta_k T(P_k(\lambda)) \\ &= \sum_k \beta_k P_k \\ &= C = T(C(\lambda)). \end{aligned} \tag{7}$$

That is,

$$\overline{C}(\lambda) \equiv C(\lambda). \tag{8}$$

Since $T(C(\lambda)) \in \mathcal{V}^n$, we have

$$T(C(\lambda)) = (T_1(C(\lambda)), \ldots, T_n(C(\lambda))),$$

where, using (3) and (8), each coordinate function T_i is given by

$$\begin{aligned} T_i(C(\lambda)) &= \int_I C(\lambda)S_i(\lambda)d\lambda \\ &= \int_I \overline{C}(\lambda)S_i(\lambda)d\lambda \\ &= \int_I \left(\sum_{k=1}^{n} \beta_k P_k(\lambda)\right) S_i(\lambda)d\lambda \\ &= \sum_{k=1}^{n} \beta_k \int_I P_k(\lambda)S_i(\lambda)d\lambda \end{aligned} \tag{9}$$

If we denote by a_{ik} the signal produced by the sensor s_i when excited by the primary spectral distribution $P_k(\lambda)$, we have

$$a_{ik} = \int_I S_i(\lambda)P_k(\lambda)d\lambda, \tag{10}$$

and from (9) we obtain

$$\sum_{k=1}^{n} \beta_k a_{ik} = \int C(\lambda)S_i(\lambda)d\lambda. \tag{11}$$

Given the color spectral distribution $C(\lambda)$, from the knowledge of the primary spectral distribution $P_k(\lambda)$ and the response spectral curves $S_i(\lambda)$, we are able to compute the coordinates β_k from (10) and (11).

In practice, the coordinates β_k are normalized against a standard reference color W: we compute the coordinates $(w_1, \ldots, w_n)$ of W according to the procedure above, and the normalized coordinates of $C(\lambda)$ on the basis of primaries P_k, $k = 1, \ldots, n$, are defined by

$$C_k = \frac{\beta_k}{w_k}. \tag{12}$$

Take $C(\lambda) = E(\lambda)$ to be a constant spectral distribution in $\mathbf{E}$ (E is called the *equal energy white*). At a specific value λ the normalized components in the basis P_k will be denoted by $p_K(\lambda)$. From (12) we get

$$p_k(\lambda) = \frac{\beta_k}{w_k} \tag{13}$$

The functions $p_k(\lambda)$, $k = 1, \ldots, n$, are called *color matching functions* associated to the basis of primary colors $\{P_k\}$, $k = 1, \ldots, n$.

In order to compute $p_k(\lambda)$ at some specific wavelength λ, we take $C(\lambda)$ to be the Dirac distribution δ with center at λ. From equation (11), we obtain

$$\sum_{k=1}^{n} \beta_k a_{ik} = S_i(\lambda), \tag{14}$$

and from (13), we get

$$\sum_{k=1}^{n} w_k a_{ik} p_k(\lambda) = S_i(\lambda). \tag{15}$$

The color matching functions are of great importance in computing the coordinates of a color with spectral distribution $C(\lambda)$ in the color space representation with basis $P_1, \ldots, P_n$. This is the content of the Proposition below.

Proposition 1. *If $p_i(\lambda)$ are the color matching functions associated with a base $\{P_i\}$ of primary colors in a physical system, then the normalized coordinates, C_k, of a color $C(\lambda) \in \mathbf{E}$ on the basis $\{P_i\}$ are given by*

$$C_k = \int_I C(\lambda) p_k(\lambda) d\lambda. \quad k = 1, \ldots, n \tag{16}$$

Proof: Multiplying both sides of (15) by $C(\lambda)$, and integrating, we obtain

$$\sum_{k=1}^{n} w_k a_{ik} \int C(\lambda) p_k(\lambda) d\lambda = \int C(\lambda) S_i(\lambda) d\lambda. \tag{17}$$

From (11) we have

$$\sum_{k=1}^{n} w_k a_{ik} \int C(\lambda) p_k(\lambda) d\lambda = \sum_{k=1}^{n} \beta_k a_{ik}, \tag{18}$$

that is,

$$\sum_{k=1}^{n} a_{ik} \left[w_k \int C(\lambda) p_k(\lambda) d\lambda - \beta_k \right] = 0. \tag{19}$$

Since the matrix (a_{ik}) is non-singular (this is a required characteristic of the physical system) we have from (12) and (19)

$$\int_I C(\lambda) p_k(\lambda) d\lambda = \frac{\beta_k}{w_k} = C_k.$$

This proves the proposition.

Said differently, Proposition 1 defines a linear representation map $\overline{T}\colon \mathbf{E} \to \mathcal{V}^n$ from the spectral color space $\mathbf{E}$ to the color coordinate system defined on $\mathcal{V}^n$ by the primary basis $\{P_1, \ldots, P_n\}$. $\overline{T}$ is defined by

$$\overline{T}(C(\lambda)) = (C_1, \ldots, C_n),$$

where each C_k is computed using (16). Everything happens as though to each primary color P_k there exists an associated "virtual sensor" s_k of the physical system, and the color matching functions $p_k(\lambda)$ are the spectral response curves associated to these sensors (see Example 2).

We will finish this section with some computational remarks. If we know the spectral response curves $S_i(\lambda)$, and the spectral distribution $P_k(\lambda)$ of the primary colors of some basis $\{P_i\}$ of the color representation space $\mathcal{V}^n$, then from (10), (11) and (12) it is possible to compute the normalized coordinates C_k of a color with spectral distribution $C(\lambda)$ on the given basis. If we do not have the knowledge of the spectral response curves $S_i(\lambda)$, then it is possible to compute C_k using the color matching functions, $p_k(\lambda)$, associated to the basis.

In the physical system of the eye, the spectral response curves are very hard to be measured accurately (Pratt, 1978). The color matching functions on the other hand are obtained from perceptual color combination experiments and are tabulated (Wyszecki & Stiles, 1983).

4.2 Color Solid

A color in the spectral color space $\mathbf{E}$ is called a *real color* if it is visible in a perceptual experiment. Mathematically a function in $\mathbf{E}$ represents a real color if it is non-negative. The set of real colors in $\mathbf{E}$ is called *spectral color solid*, and it will be denoted by $\mathbf{SE}$.

A subset S of a vector space V is a *cone* if it is invariant under multiplication by a positive real number t. The following proposition is immediate

Proposition 2 *The spectral color solid* $\mathbf{SE}$ *is a cone of* $\mathbf{E}$.

If $T\colon \mathbf{E} \to \mathcal{V}^n$ is a color space representation, the set of color vectors in $\mathcal{V}^n$ that correspond to real spectral colors in $\mathbf{E}$ is called by *color solid*, and will be denoted by $\mathbf{S}$, that is $\mathbf{S} = T(\mathbf{SE})$. Since a cone is invariant under linear maps, the proposition below is also immediate

Proposition 3 *The color solid* $\mathbf{S}$ *is a cone of* $\mathcal{V}^n$.

A *pure spectral color* of wavelength λ is a color whose spectral distribution in $\mathbf{E}$ is null everywhere, except at λ. The proposition below is also very easy to prove

Proposition 4 *The pure spectral colors are on the boundary of the color solid.*

4.3 Luminance and Chrominance

Given a physical system such that the spectral response curve of the sensor s_i, is $S_i(\lambda)$, $i = 1, \ldots, n$, we define the *average spectral response curve* by

$$V(\lambda) = \sum_{i=1}^{n} a_i S_i(\lambda), \quad a_i \in \mathbf{R}, \quad a_i > 0 \tag{20}$$

where the constants a_i depend on the characteristics of the system.

For the thricromatic physical system of the eye, the curve $V(\lambda)$ is called *relative luminous efficiency function* and its values, obtained experimentally, are tabulated (Wyszecki & Styles, 1983). In what follows we will suppose that the function $V(\lambda)$ has compact support. The support of the relative luminous efficiency of the eye is the visible interval $[\lambda_a, \lambda_b]$ of the spectrum.

Definition 3. If $C(\lambda)$ is the spectral radiance distribution of a color C in the space $\mathbf{E}$, its *luminance*, with respect to a physical system with average spectral response curve $V(\lambda)$, is defined by

$$L(C(\lambda)) = K \int_{\mathbf{R}} C(\lambda) V(\lambda) d\lambda, \tag{21}$$

where K is a constant that depends on the unit system used.

The luminance defines a linear functional $L : \mathbf{E} \to \mathbf{R}$. If $C_1(\lambda)$ and $C_2(\lambda)$ are methameric spectral curves in $\mathbf{E}$, i.e. $T(C_1(\lambda)) = T(C_2(\lambda))$, then from (9) and (20), we obtain:

$$\begin{aligned}
L(C_1) &= K \int_{-\infty}^{+\infty} C_1(\lambda) V(\lambda) d\lambda \\
&= K \int_{-\infty}^{+\infty} C_1(\lambda) \left[\sum_{i=1}^{n} a_i S_i(\lambda) \right] d\lambda \\
&= K \sum_{i=1}^{n} \int_{-\infty}^{+\infty} C_1(\lambda) a_i S_i(\lambda) d\lambda \\
&= K \sum_{i=1}^{n} \int_{-\infty}^{+\infty} C_2(\lambda) a_i S_i(\lambda) d\lambda \\
&= K \int_{-\infty}^{+\infty} C_2(\lambda) \left[\sum_{i=1}^{n} a_i S_i(\lambda) \right] d\lambda
\end{aligned}$$

$$
\begin{aligned}
&= K \int_{-\infty}^{+\infty} C_2(\lambda) V(\lambda) d\lambda \\
&= L(C_2) \qquad (22)
\end{aligned}
$$

The above computation shows that the Luminance functional $L\colon \mathbf{E} \to \mathbf{R}$ induces naturally a linear functional $\overline{L}\colon \mathcal{V}^n = (\mathbf{E}/\equiv) \to \mathbf{R}$, in the color space of the physical system. As a consequence, $\overline{L}(T(C(\lambda))) = L(C(\lambda))$. The functional $\overline{L}$ is called the *luminance functional* of the physical system.

The kernel C_0 of the luminance functional $\overline{L}\colon \mathcal{V}^n \to \mathbf{R}$ is a $(n-1)$-dimensional subspace. Each affine hyperplane of the family $C_v = C_0 + v$, $v \in \mathcal{V}^n$, contains colors of constant luminance, and is therefore called a *chrominance hyperplane.* If $v \in \mathcal{V}^n$ is a vector, $v \notin C_0$, we will denote by $\langle v \rangle$ the subspace generated by v. There is a decomposition of the space $\mathcal{V}^n = C_0 \oplus \langle v \rangle$. Therefore each color vector $w \in \mathcal{V}^n$ can be written uniquely in the form $w = w_C + w_L$, $w_C \in C_0$ and $w_L \in \langle v \rangle$. The component w_C contains the chrominance information, and w_L the luminance information of the color vector w. The above decomposition is called a *chrominance-luminance decomposition* of the color space.

The chrominance-luminance decomposition is the starting point to define several color coordinate systems of great importance in colorimetry and its applications, such as the CIE-XYZ standard system and the NTSC system used in the television industry.

If the luminance of each of the sensors in a color physical system is non-zero, then the luminance subspace is transversal to the hyperplane $\Pi\colon x_1 + \cdots + x_n = 1$ of the color representation space. This result can be used to normalize the chrominance information of the color space as follows: for each real color vector C, there exists a positive real number t, such that $tC \in \Pi$. If $C = (C_1, \ldots, C_n)$ then

$$t = \frac{1}{C_1 + \cdots + C_n}. \qquad (23)$$

Geometrically, we are radially projecting the color solid C on the hyperplane Π. The resulting subset on the Π contains all of the chrominance information of the real colors, and it is called *chromaticity diagram.* It follows from Proposition 3 that it is a convex subset of Π (because it is a radial projection of a cone). For thricromatic color spaces, its horseshoe shape is well known. The coordinates of the projection of C on Π are computed by

$$c_i = \frac{C_i}{C_1 + \cdots + C_n}, \qquad (24)$$

and are called *chromaticity coordinates* of the color C.

4.4 Projective Model

If C is a color vector in some color space representation $\mathcal{V}^n$, and t is a non-zero real number, the vector tC has the same chrominance (same chromaticity coordinates) as C. This shows that it is possible to identify naturally the *chrominance space* as the set of all 1-dimensional subspaces of $\mathcal{V}^n$, excluding the origin. This is the real projective space of dimension $n-1$.

This simple remark is of great help when studying color space transformations: the natural transformation relating chromaticity coordinates in different color coordinate systems are projective transformations. This fact is not used explicitly in the literature and the explanation of chromaticity coordinate transformation is rather cumbersome.

4.5 Grassmann Laws

Grassmann Laws are used as the set of axioms in the development of colorimetry theory. These laws were established after perceptual experiments with color in the 19th century (Grassmann, 1854). Some of Grassmann's laws have an intrinsic perceptual nature (e.g. *components of a mixture of colored lights cannot be resolved by the human eye*). Other laws are of a more mathematical nature and can be shown to be valid in any finite dimensional color space representation. This assertion holds, for instance, in relation to Grassmann's fourth law: *The luminance of a mixture is equal to the sum of the luminance of its components.* This can be easily proved with the results from Section 4.3.

5 Conclusion and Research Topics

We introduced a mathematical framework to define a representation of the spectral color space as a finite dimensional vector space. This framework makes it possible to develop systematically the theory of colorimetry for n-dimensional color spaces, without resorting to results from perceptual color experiments (Grassmann's Laws). The basic colorimetric concepts are given in this framework.

It is interesting to establish conditions under which a color space representation is in fact an approximation of the color space as the dimension gets higher. In the case of the trichromatic color space of the eye, different metrics in the spectral color space can be investigated in order to study its relationship with the uniform Riemmanian metric of the finite dimensional color space representation.

6 References

Gomes, J. & Velho, Luiz (1992): *Abstraction Paradigms for Computational Mathematics.* IMPA.

Grassmann, H. (1854): *On the theory of compound colors.* Phil. Mag. Ser. 4, 7, 254-264.

Konig, A. & Brodhun, E. (1889): *Experimentelle Untersuchungen uber die Psychophysische Fundamental in Bezug auf den Gesichtssinn.* Zweite Mittlg. S. B. Preuss Akademic der Wissenschaften, 641.

Pratt, W. K. (1978): *Digital Image Processing* . Wiley, New York.

Requicha, A. (1980): *Representations for Rigid Solids: Theory, Methods, and Systems*, Computing Surveys, Vol. 12, N 4.

Wyszecki G. & Stiles W. S. (1982): *Color Science, Concepts and Methods.* Second edition, Wiley, New York.