ATTRACTORS OF GENERIC DIFFEOMORPHISMS ARE PERSISTENT

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Abstract. We prove that given a compact n-dimensional boundaryless manifold M, $n \geq 2$, there exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that if Λ is an Ω-isolated and transitive set of $f \in \mathcal{R}$, then Λ admits a continuation in a generic neighborhood of f; such sets are called *almost robustly transitive* or *generically transitive* sets. Furthermore, if Λ is a transitive attractor of f, then the continuation of Λ is also an attractor.

This implies that Ω-isolated transitive sets of generic diffeomorphisms always admit weakly hyperbolic dominated splittings; in particular, given any surface diffeomorphism f in a residual subset of $\mathrm{Diff}^1(M^2)$, then every Ω-isolated transitive set of f (such as a transitive attractor) is hyperbolic. We also show that, generically, Ω-isolated transitive sets are either hyperbolic or approached by heterodimensional cycles, a type of homoclinic bifurcation.

1. Statement of the Results

Throughout this paper, M denotes a compact boundaryless manifold of dimension $n \geq 2$ and $\mathrm{Diff}^1(M)$ is the space of C^1-diffeomorphisms on M with the usual topology.

The Spectral Decomposition Theorem states that the nonwandering set of any hyperbolic (i.e., Axiom A) diffeomorphism f admits a partition into a finite number of transitive pairwise disjoint sets called *basic sets*. If f also has no cycles, then it is Ω-stable, and each of its basic sets persists in a C^1-neighborhood of f.

There are several ways of defining nonhyperbolic generalizations of the concept of basic set, such as *robustly transitive sets*, as in [BDP]. The weakest such generalization is to simply require that a set be Ω-isolated and transitive:

Definition 1. A compact invariant set Λ of $f \in \mathrm{Diff}^1(M)$ is *transitive* if there is a point $q \in \Lambda$ such that the future f-orbit of q is dense in Λ. This is equivalent to the following condition: given any two nonempty subsets U and V of Λ, there is a natural number $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$.

Λ is *Ω-isolated* if there exists an open neighborhood V of Λ such that

$$\Omega(f) \cap V = \Lambda.$$

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Λ is *isolated* or *locally maximal* if there exists an open neighborhood V of Λ such that

$$\Lambda = \bigcap_{k \in \mathbb{Z}} f^k(V).$$

Our first result deals with the persistence of Ω-isolated transitive sets of "typical" (i.e., generic) diffeomorphisms:

Theorem A. *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that if Λ is an Ω-isolated transitive set of $f \in \mathcal{R}$ then there exist a neighborhood V of Λ and a neighborhood $\mathcal{W}$ of f in $\mathrm{Diff}^1(M)$ such that if $g \in \mathcal{W} \cap \mathcal{R}$ then*

$$\Omega(g) \cap V \equiv \Lambda_g,$$

where Λ_g is a compact transitive set of g.

Moreover, if $\{g_k\} \subset (\mathcal{W} \cap \mathcal{R})$ converges to f in the C^1 topology, then Λ_{g_k} converges to Λ in the Hausdorff topology.

The proof of Theorem A relies on generic properties of *homoclinic classes*, defined in Section 2 below.

Transitive sets that persist in a generic neighborhood as in Theorem A above were introduced in [A] in the study of the "basic sets" of nonhyperbolic diffeomorphisms that admit spectral decompositions (i.e., their nonwandering sets admit partitions into finitely many pairwise disjoint transitive sets). To be precise, a transitive compact set Λ of a diffeomorphism $f \in \mathrm{Diff}^1(M)$ is *almost robustly transitive* or *generically transitive* if there is a neighborhood V of Λ and a neighborhood $\mathcal{W}$ of f in $\mathrm{Diff}^1(M)$ such that if $g \in \mathcal{W} \cap \mathcal{R}$ then $\Omega(g) \cap V \equiv \Lambda_g$, where Λ_g is a compact transitive set of g. Hence Theorem A essentially says that generically any Ω-isolated transitive compact set is generically transitive.

Generic transitivity is essentially a weak form of robust transitivity. In fact, there are no known examples of generically transitive sets which are not robustly transitive. This naturally raises the following question:

Question. *Does generic transitivity imply robust transitivity?*

We now define a weak type of hyperbolicity, first introduced by [BDP]:

Definition 2. Let Λ be a compact invariant set of $f \in \mathrm{Diff}^1(M)$. Let E, F, and F be Df-invariant subbundles of $T_\Lambda M$ such that $T_\Lambda M = E \oplus C \oplus F$.

Then $E \oplus C \oplus F$ is a *dominated splitting* over Λ if $(C \oplus F)$ dominates E and F dominates $(E \oplus C)$ in the usual sense. The subbundle E is *uniformly volume contracting* if there exists some $k \in \mathbb{N}$ such that $|det(Df^k|_E)| < 1$ at all points of Λ; the subbundle F is *uniformly volume expanding* if it is uniformly volume contracting for f^{-1}.

Λ is said to be *volume partially hyperbolic* if it admits a dominated splitting $T_\Lambda M = E \oplus C \oplus F$, with E and F nonempty, such that E is uniformly volume contracting and F is uniformly volume expanding. Note that if E or F is one-dimensional, then it is hyperbolic (i.e., uniformly contracting/expanding).

Finally, Λ is *partially hyperbolic* if it admits a dominated splitting $T_\Lambda M = E \oplus F$, where E is uniformly contracting or F is uniformly expanding.

Theorem 4 of [BDP] states that robustly transitive sets are volume partially hyperbolic. The next result is a generic version of that theorem: it says that, generically, any generically transitive set is volume partially hyperbolic.

Proposition A. *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that if Λ is a generically transitive set of $f \in \mathcal{R}$, then Λ is volume partially hyperbolic (or else the orbit of a periodic sink/source).*

The proof of Proposition A is an adaptation of the proof of Theorem 4 of [BDP]; it is set out in the last section of this paper.

Combining Theorem A with Proposition A we immediately obtain the following result:

Corollary A. *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M^n)$ such that if Λ is an Ω-isolated transitive set of $f \in \mathcal{R}$ then Λ is volume partially hyperbolic (or else the orbit of a periodic sink/source).*

In particular, if $n = 3$ then every Ω-isolated transitive set of f is partially hyperbolic, and if $n = 2$ then every Ω-isolated transitive set of f is a hyperbolic set.[1]

Thus, generically any nonhyperbolic basic set (here meaning any Ω-isolated transitive set) admits a volume hyperbolic dominated splitting. In the case of surface diffeomorphisms, they are hyperbolic, and therefore robustly transitive. This (essentially) means that generically transitive sets of surface diffeomorphisms are robustly transitive, and thus the Question above has an affirmative answer in the context of two-dimensional diffeomorphisms.

The result in [AP] is an analogue for three-dimensional flows of the two-dimensional statement in Corollary A.

An important class of transitive Ω-isolated sets are topological attractors, defined as follows:

Definition 3. An *attractor* of $f \in \mathrm{Diff}^1(M)$ is an invariant compact transitive set $\Lambda \subset M$ such that there is a neighborhood U of Λ (which we call an *attracting neighborhood* of Λ) with

$$cl(f(U)) \subset U \text{ and } \cap_{n\in\mathbb{N}} f^n(U) = \Lambda$$

It is easily seen that attractors are always Ω-isolated. The simplest attractors are periodic sinks. Given any Axiom A diffeomorphism with no cycles, at least one of its basic sets is an attractor in the sense above; there are also many examples of nonhyperbolic attractors (the best-known of which are the Hénon and the Lorenz attractors). Note that Corollary A implies

[1] C. Morales has pointed out that in the 2- and 3-dimensional cases this result may be obtained more directly, without the need for generic transitivity, by respectively using arguments from Mañé [M] and Diaz-Pujals-Ures [DPU].

that C^1-generically in dimension 2 there exist no "Hénon-type" attractors. (Hénon attractors are two-dimensional attractors in our sense, but are non-hyperbolic.)

We also obtain a stronger version (for attractors) of Theorem A; it says that generic attractors are not only persistent, but also persistent *relative to some uniform attracting neighborhood.*

Theorem B. *There exists a residual subset $\mathcal{R}$ of* $\mathrm{Diff}^1(M)$ *such that if Λ is an attractor of $f \in \mathcal{R}$ then there exist an attracting neighborhood V of Λ and a neighborhood $\mathcal{W}$ of f in* $\mathrm{Diff}^1(M)$ *such that if $g \in \mathcal{W} \cap \mathcal{R}$ then*

$$\Lambda_g \equiv \cap_{n \in \mathbb{N}} g^n(V)$$

is an attractor (with V an attracting neighborhood).

Moreover, if $\{g_k\} \subset (\mathcal{W} \cap \mathcal{R})$ converges to f in the C^1 topology, then Λ_{g_k} converges to Λ in the Hausdorff topology.

Finally, we characterize hyperbolicity for Ω-isolated transitive sets of generic diffeomorphisms in terms of approximation by homoclinic bifurcations. Let us first define the type of homoclinic bifurcation we are dealing with:

Definition 4. A diffeomorphism $f \in \mathrm{Diff}^1(M)$ exhibits a *heterodimensional cycle* if there are periodic saddles P and Q of different stability indices such that

$$W^s(P) \cap W^u(Q) \neq \emptyset \text{ and } W^u(P) \cap W^s(Q) \neq \emptyset.$$

Generic transitivity allows us to obtain

Theorem C. *There exists a residual subset $\mathcal{R}$ of* $\mathrm{Diff}^1(M)$ *such that if Λ is an Ω-isolated transitive set of $f \in \mathcal{R}$ then either (1) or (2) holds:*

1) *Lambda is a hyperbolic set of f*
2) *Λ is nonhyperbolic and f is C^1-approached by diffeomorphisms which exhibit heterodimensional cycles*

Theorem C above is a "setwise" version of the following (global) conjecture by Palis [P]:

Conjecture (Palis). *There exists a dense subset $\mathcal{D}$ of* $\mathrm{Diff}^k(M)$ *such that if $f \in \mathcal{D}$, then f either is hyperbolic or exhibits homoclinic tangencies or heterodimensional cycles.*

This paper is organized as follows: in Section 2 we list some definitions and results that are used in the remainder of the paper; in Section 3 we prove Theorem A using the results of Section 2; in Section 4 we deduce Theorem B from theorem A; in Section 5 we show how to adapt the proof of Theorem 4 of [BDP] in order to obtain Proposition A; and in Section 6 we use Theorem A to prove Theorem C.

2. Technical Preliminaries

We now state some definitions and results to be used later.

Definition 5. Let $f \in \mathrm{Diff}^1(M)$, p a periodic hyperbolic point of f. The *homoclinic class* of f relative to p is given by

$$H(p,f) = cl[W^s(p) \pitchfork W^u(p)],$$

where $\pitchfork$ denotes points of transverse intersection of the invariant manifolds.

$H(p,f)$ is a transitive compact f-invariant subset of $\Omega(f)$. $H(p,f)$ is not necessarily hyperbolic, but when f is Axiom A, its basic sets are hyperbolic homoclinic classes. In the absence of ambiguity, we may write $H(p)$ for $H(p,f)$.

Note that by definition the homoclinic class of a periodic sink/source coincides with its orbit.

Theorem 1. [Pu] *There exists a residual subset $\mathcal{R}_1 \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}_1$, then $\Omega(f) = cl(Per(f))$ and $Per(f) = Per_h(f)$, where $Per_h(f)$ denotes the set of hyperbolic periodic points of f.*

Theorem 2. [CMP] *There exists a residual subset $\mathcal{R}_2 \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}_2$ and $H(p,f)$ is a homoclinic class of f, then $H(p,f)$ has no 1-cycle (i.e., $W^s(H(p,f)) \cap W^u(H(p,f)) = H(p,f)$).*

Theorem 3 below is an immediate consequence of Theorem B of [BD] combined with Theorem 1 above.

Theorem 3. *There exists a residual subset $\mathcal{R}_3 \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}_3$ then, given any Ω-isolated transitive set Λ of f, we have $\Lambda = H(p,f)$ for some periodic point $p \in \Lambda$.*

Definition 6. Let p be a periodic saddle of $f \in \mathrm{Diff}^1(M)$. Then $H(p)$ *varies C^1-continuously at f* if given any sequence $g_j \to f$ in the C^1 topology, we have $H(p^{g_j}, g_j) \to H(p,f)$ in the Hausdorff topology, where p^g is the continuation of p relative to g.

The next result is an immediate consequence of the lower semi-continuity of homoclinic classes:

Theorem 4. *There exists a residual subset $\mathcal{R}_4 \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}_4$ and p is a saddle of f, then $H(p)$ varies C^1-continuously at f.*

The following lemma is a well-known fact from general topology (see for instance [Ku]):

Topological Lemma. *Let X be a Baire topological space and $\Gamma : X \to \mathbb{N}$ a lower semicontinuous map. Then there exists a residual subset N of X such that $\Gamma|_N$ is locally constant at each point of N.*

3. Proof of Theorem A

Given two periodic points p, q of $f \in \mathrm{Diff}^1(M)$, we say that the homoclinic classes af p and q are *distinct* if they do not coincide as sets; that is, if $H(p,f) \neq H(q,f)$.

Now let $\{U_k\}_{k\in\mathbb{N}}$ be a (countable) base of open subsets of M. For each $k \in \mathbb{N}$, define $\Gamma_k : \mathrm{Diff}^1(M) \to (\mathbb{N} \cup \{\infty\})$ by setting

$$\Gamma_k(g) \equiv \mathrm{card}\{\text{distinct homoclinic classes of } g \text{ that intersect } U_k\}$$

That is, Γ_k is the map that "counts" the number of homoclinic classes that intersect the open set U_k.

We will need to use the following "local" version of Proposition 2 of [A]:

Proposition 1. *There exists a residual subset $\mathcal{R}_5 \subset \mathrm{Diff}^1(M)$ such that, for all $k \in \mathbb{N}$, if $f \in \mathcal{R}_5$ is such that $\Gamma_k(f) \in \mathbb{N}$, then there exists a neighborhood $\mathcal{V}$ of f in $\mathcal{R}_5$ such that if $g \in \mathcal{V}$ then $\Gamma_k(g) \in \mathbb{N}$.*

The proof of Proposition 1 is almost identical to that of Proposition 2 of [A], so we will not write it out in detail. Only two modifications to the proof of Proposition 2 of [A] are needed: first, replace "homoclinic classes of g" by "homoclinic classes of g that intersect U_k", thereby obtaining for each $k \in \mathbb{N}$ a residual subset $\mathcal{S}_k$ of $\mathrm{Diff}^1(M)$; second, intersect the $\mathcal{S}_k$'s in order to obtain $\mathcal{R}_5$.

Lemma 1. *There exists a residual subset $\mathcal{R}'$ of $\mathrm{Diff}^1(M)$ such that given any $k \in \mathbb{N}$ and $g \in \mathcal{R}'$ with $\Gamma_k(g)$ finite, then Γ_k is constant in a neighborhood $\mathcal{A}_k$ of g in $\mathcal{R}'$.*

Proof. By Theorem 4, there exists a residual subset $\mathcal{R}_4$ of $\mathrm{Diff}^1(M)$ where all the homoclinic classes vary continuously. Let $\mathcal{R}^* \equiv \mathcal{R}_4 \cap \mathcal{R}_5$, where $\mathcal{R}_5$ is as in Proposition 1 above. Then $\mathcal{R}^*$ is a residual subset of $\mathrm{Diff}^1(M)$ such that, for all $k \in \mathbb{N}$, if $f \in \mathcal{R}^*$ only has finitely many distinct homoclinic classes that intersect U_k, then there is a neighborhood $\mathcal{V}$ of f in R^* such that every $g \in \mathcal{V}$ only has finitely many distinct homoclinic classes that intersect U_k.

Set $\mathcal{R}^k{}_{fin} \equiv \{g \in \mathcal{R}^* : \Gamma_k(g) \in \mathbb{N}\}$ and $\mathcal{R}^k_\infty \equiv \mathcal{R}^* \setminus \mathcal{R}^k{}_{fin}$. Then for each $k \in \mathbb{N}$ we have that $\mathcal{R}^* = \mathcal{R}^k{}_{fin} \cup \mathcal{R}^k_\infty$, where the union is disjoint and $\mathcal{R}^k{}_{fin}$ is open in $\mathcal{R}^*$.

Given $k \in \mathbb{N}$ and $g \in \mathcal{R}^k{}_{fin}$, then for all $g' \in \mathrm{Diff}^1(M)$ sufficiently near g, we have by their continuity that all the continuations of the "original" distinct homoclinic classes of g that intersect U_k will still intersect U_k and be pairwise distinct. That is, there is a neighborhood $\mathcal{A}'$ of g in $\mathcal{R}^k{}_{fin}$ such that $\Gamma_k(g')$ is greater than or equal to $\Gamma_k(g)$ for all $g' \in \mathcal{A}'$.

Hence, Γ_k is lower-semicontinuous on $\mathcal{R}^k{}_{fin}$. Therefore, by the Topological Lemma, there exists a residual subset $\mathcal{S}_k$ of $\mathcal{R}^k{}_{fin}$ such that Γ_k is locally constant when restricted to $\mathcal{S}_k$. Setting $\mathcal{R}^k \equiv \mathcal{S}_k \cup \mathcal{R}^k_\infty$ we clearly have that $\mathcal{R}^k$ is residual in $\mathrm{Diff}^1(M)$, and moreover $\Gamma_k|_{\mathcal{R}^k}$ is constant in a neighborhood of any $g \in \mathcal{R}^k$ with $\Gamma_k(g)$ finite.

Setting $\mathcal{R}' \equiv \cap_{k\in\mathbb{N}}\mathcal{R}^k$ we obtain a residual subset of $\mathrm{Diff}^1(M)$ with the desired properties. □

We can now prove Theorem A:

Proof of Theorem A.
Let $\mathcal{R}$ be the residual subset of $\mathrm{Diff}^1(M)$ given by

$$\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3 \cap \mathcal{R}_4 \cap \mathcal{R}',$$

where $\mathcal{R}_i$ is as in Theorem i, with $i \in \{1, 2, 3, 4\}$, and $\mathcal{R}'$ is as in Lemma 1. Take $f \in \mathcal{R}$ and Λ an Ω-isolated transitive set of f, with U an open neighborhood of Λ such that $U \cap \Omega(f) = \Lambda$. By Theorem 3, we have $\Lambda = H(p, f)$ for some periodic $p \in \Lambda$.

Take now a smaller neighborhood V of Λ such that $cl(V) \subset U$. Then we have that

$$\Lambda = \Omega(f) \cap cl(V).$$

Consider now the open base $\{U_k\}$ of M defined above. U being open, we have

$$U = \cup_{i\in\mathbb{N}} U_{k_i}$$

for some subsequence $\{U_{k_i}\}$ of $\{U_k\}$. Since $\cup_{i\in\mathbb{N}} U_{k_i}$ contains $cl(V)$, by compacity there exists a finite number of such sets U_{k_i} whose union covers $cl(V)$. We will rename these sets $U_1, \dots, U_s$ in order to simplify the notation.

Since $f \in \mathcal{R}$, it follows by Lemma 1 that for each U_i, $i \in \{1, \dots, s\}$, there is a neighborhood $\mathcal{A}_i$ of f in R such that Γ_i is constant in $\mathcal{A}_i$. But note that U only intersects one homoclinic class (namely, $H(p, f)$), so we have either $\Gamma_i \equiv 1$ or $\Gamma_i \equiv 0$ in each neighborhood $\mathcal{A}_i$ of f.

Let $\mathcal{W} \equiv \mathcal{A}_1 \cap \dots \cap \mathcal{A}_s$. Then each $\Gamma_1, \dots, \Gamma_s$ is constant (and equal to 1 or 0) when restricted to $\mathcal{W}$. It is easily seen that this means that $U_1 \cup \dots \cup U_s$ only intersects one homoclinic class (the continuation $H(p^g, g)$ of $H(p, f)$) for each $g \in \mathcal{W}$. By Theorem 1, it follows that

$$(U_1 \cup \dots \cup U_s) \cap \Omega(g) = H(p^g, g)$$

for each $g \in \mathcal{W}$. Therefore

$$V \cap \Omega(g) = H(p^g, g)$$

for each $g \in \mathcal{W}$, as desired. □

4. Proof of Theorem B

We now prove Theorem B:

Proof of Theorem B. Consider the residual set $\mathcal{R}$ given by Theorem A, and let Λ be an attractor of $f \in \mathcal{R}$ with U an attracting neighborhood. By Theorem 3 and Lemma 2 above we have $U \cap \Omega(f) = \Lambda = H(p, f)$ for some $p \in \mathrm{Per}(f)$.

Applying Theorem A to Λ, we obtain a neighborhood V of Λ in M and a neighborhood $\mathcal{W}^*$ of f in $\mathrm{Diff}^1(M)$ such that if $g \in \mathcal{R} \cap \mathcal{W}^*$ then

$$V \cap \Omega(g) = \Lambda_g \equiv H(p^g, g).$$

Taking V sufficiently small, we can and do assume that $\mathrm{cl}(V) \cap \Omega(g) = \Lambda_g$.

Since $\Lambda \subset V \subset U$, we have

$$\Lambda \subset \cap_{k\in\mathbb{N}} f^k(V) \subset \cap_{k\in\mathbb{N}} f^k(U) = \Lambda,$$

and therefore $\cap_{k\in\mathbb{N}} f^k(V) = \Lambda$. By Lemma 2.9 of [S], we may assume that $f(cl(V)) \subset V$ by shrinking V if necessary. We conclude that V is an attracting neighborhood of Λ for f.

We must now show that V is also an attracting neighborhood for Λ_g, given any $g \in \mathcal{W}^* \cap \mathcal{R}$. The condition $cl(f(V)) \subset V$ is open: it holds for all g C^1-near f. We therefore need only prove that

$$\cap_{k\in\mathbb{N}} g^k(V) = H(p^g, g) \equiv \Lambda_g.$$

for all $g \in \mathcal{W}^*$.

Taking $\mathcal{W}^*$ sufficiently small, we guarantee by the continuity of homoclinic classes that $H(p^g, g) \subset V$ for all $g \in \mathcal{W}^*$. We therefore have

$$\cap_{k\in\mathbb{N}} g^k(V) \supset H(p^g, g) \equiv \Lambda_g$$

for all $g \in \mathcal{W}^*$, and need only prove the other inclusion.

Assume by contradiction that there is some $x \in \cap_{k\in\mathbb{N}} g^k(V) \setminus H(p^g, g)$. Then the future g-orbit $\theta_+(x)$ of x is contained in V since $g(V) \subset V$, and the past g-orbit $\theta_-(x)$ of x is contained in V since $x \in \cap_{k\in\mathbb{N}} g^k(V)$. Now, because the future and past g-orbits of every point of M converge to the nonwandering set of g, and $\mathrm{cl}(V) \cap \Omega(g) = \Lambda_g \equiv H(p^g, g)$, we have (taking convergent subsequences if necessary)

$$lim_{k\to+\infty} g^k(x) \in \mathrm{cl}(V) \cap \Omega(g) = \Lambda_g \text{ and } lim_{k\to-\infty} g^k(x) \in \mathrm{cl}(V) \cap \Omega(g) = \Lambda_g.$$

But this means that $\Lambda_g = H(p^g, g)$ has a 1-cycle, contradicting Theorem 2. Hence we have that $\cap_{k\in\mathbb{N}} g^k(V) = H(p^g, g) \equiv \Lambda_g$, and Theorem B is proven. □

5. Proof of Proposition A

This section consists of the proof of Proposition A. For the sake of brevity, we only point out the necessary modifications to the proof of Theorem 4 of [BDP], freely using notation and concepts from that paper.

Let us first recall that [CMP] proved that, generically, Ω-isolated transitive sets are isolated (i.e., locally maximal) homoclinic classes. Combining this result with Theorem A above and Proposition 6.2 of [BDP] we easily obtain the following "generic version" of Proposition 6.2 of [BDP]:

Proposition 6.2'. *There is some residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that if Λ is an Ω-isolated transitive set of $f \in \mathcal{R}$ that admits some dominated splitting $E \oplus F$, $E \prec F$, then there is some neighborhood U of Λ and some neighborhood $\mathcal{W}$ of f such that:*

(a) *if $g \in \mathcal{R} \cap \mathcal{W}$ then Λ_g admits a dominated continuation $E_g \oplus F_g$ of the original splitting; and*
(b) *either (i) $Df|E$ uniformly contracts volume or (ii) there are arbitrarily small C^1-perturbations g of f such that there is some periodic hyperbolic point q of g in U such that Dg expands volume on $E_g(q)$.*

We may now prove Proposition A by closely following the proof of Theorem 4 of [BDP]. Essentially, the only difficulty here is adapting the original proof, which deals with robustly transitive sets, to the case of generically transitive sets.

Proof of Proposition A. Let Λ be a non-trivial generically transitive set of $f \in \mathrm{Diff}^1(M)$. That is , there are a neighborhood U of Λ in M, a neighborhood $\mathcal{W}$ of f in $\mathrm{Diff}^1(M)$, and a residual subset $\mathcal{R}$ of $\mathcal{W}$ such that if $g \in \mathcal{R}$, then

$$\Lambda_g \equiv \Omega(g) \cap U$$

is a compact transitive continuation of Λ (indeed, there is some periodic saddle p such that $\Lambda_g = H(p^g, g)$ for all $g \in \mathcal{R}$). Since intersections of residual sets are residual, we may by Corollary 0.3 of [BDP] assume that all such continuations Λ_g of Λ admit dominated splittings (as does Λ itself). Finally, we assume also that $\mathcal{R}$ satisfies the properties of Proposition 6.2' above.

Let N be the dimension of M and $K > 0$ a strict upper bound for the norms of Df and Df^{-1}. Assume that $\mathcal{W}$ is an open ball in $\mathrm{Diff}^1(M)$, centered on f, with diameter small enough so that K is an upper bound for all $g \in \mathcal{W}$.

Let $F_1 \oplus F_2 \oplus \cdots \oplus F_k$ be the finest dominated splitting of Df over Λ. Let $E \equiv F_1$ and $F \equiv F_2 \oplus \cdots \oplus F_k$ and fix L so that the splitting $E \oplus F$ is $2L$-dominated. Let $l > 0$ be the constant associated to $K, N, 2L$, and $\epsilon \equiv \frac{\mathrm{diam}(\mathcal{W})}{2}$ as in Lemma 6.1 of [BDP].

Assume by contradiction that $Df|E$ does not contract volume uniformly. Then by Proposition 6.2' there are $g \in \mathcal{W}$ and some periodic hyperbolic saddle $q \in U$ such that Dg expands the volume in $E_g(q)$. By generic transitivity and the persistence of periodic hyperbolic points, we may assume after a perturbation that $g \in \mathcal{R}$ and therefore that $q \in \Lambda_g = H(p^g, g)$.

Consider the set Σ of periodic saddles in $\Lambda_g (= H(q, g))$ that are homoclinically related to q. Then Σ is dense in Λ_g and g induces the periodic linear system $(\Sigma, g, T_\Sigma M, Dg)$, which has dimension N, is bounded by K, and has transitions. Moreover, for g close to f, $E(g) \oplus F(g)$ is an L-dominated splitting for this system and the bundle $E(g)$ does not admit any l-dominated splitting (by Lemma 1.4 of [BDP]). Applying Lemma 6.1 of [BDP] we obtain an ϵ-perturbation B of $(\Sigma, g, T_\Sigma M, Dg)$ and some periodic point $s \in \Sigma$ such that all the eigenvalues of $M_B(s)$ have modulus greater than one. Then using Franks' Lemma we obtain some perturbation $g' \in \mathcal{W}$ of g such that s is a periodic hyperbolic source of g' in $\Omega(g') \cap U$. Again using generic transitivity and the persistence of periodic hyperbolic points, we may assume after a small perturbation that $g' \in \mathcal{R}$ and that therefore the source s belongs to the (non-trivial and transitive) set $\Lambda_{g'}$, a contradiction. We have thus proved that $Df|E$ uniformly contracts volume.

Applying the same argument to f^{-1} and to the splitting $E' \oplus F'$, where $E' \equiv F_1 \oplus \cdots \oplus F_{k-1}$ and $F' \equiv F_k$, we obtain that $Df|F$ uniformly expands volume, as desired. □

6. Proof of Theorem C

Given Λ a compact invariant set of $f \in \mathrm{Diff}^1(M)$, let $\mathrm{Per}_f(\Lambda)$ denote the set of periodic points of f that belong to Λ.

Let U be an open subset of M. Then we set $\Im^1(\mathcal{U}) \equiv \{f \in \mathrm{Diff}^1(M)$: there exists an open neighborhood $\mathcal{W}$ of f in $\mathrm{Diff}^1(M)$ such that if $g \in \mathcal{W}$, then all periodic points of g in U are hyperbolic $\}$. Hayashi proved in [H1] that the set $\Im^1(M)$ concides with the set of Ω-stable diffeomorphisms. It is easy to see that the arguments in his proof also yield the following "local" version of that result:

Theorem 5. [H1] *Let Λ be a transitive compact Ω-isolated set of $f \in \mathrm{Diff}^1(M)$ such that $Per_f(\Lambda)$ is dense in Λ and such that there is a neighborhood U of Λ such that $f \in \Im^1(U)$. Then Λ is a hyperbolic set of f.*

We will also need the following version of Hayashi's Connecting Lemma [H2]:

Connecting Lemma. [WX] *Consider $f \in \mathrm{Diff}^1(M)$ with hyperbolic periodic points p and q. Assume that there are sequences of points (x_j) and natural numbers (k_j) such that*

$$(x_j) \to p^s \in W^s(p) \setminus \{p\} \text{ and } f^{k_j}(x_j) \to q^u \in W^u(q) \setminus \{q\}$$

Then given any $\varepsilon > 0$ there exists an ε-perturbation g of f such that $W^s(p^g) \cap W^u(q^g) \neq \emptyset$.

We now prove a lemma, whose proof follows that of Lemma 3.3 of Diaz-Pujals-Ures [DPU], and then use it to prove Theorem C.

Lemma 2. *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that if p and q are periodic saddles of $f \in \mathcal{R}$ such that $ind(q) \neq ind(p)$ and such that there is some transitive set Λ that contains both p and q, then for every $\varepsilon > 0$ there is some $g \in \mathrm{Diff}^1(M)$ which is ε-C^1-close to f and which exhibits a heterodimensional cycle.*

Proof. Let $\mathcal{R}$ be the residual as in Theorem B of [BD2] and $f \in \mathcal{R}$, p, q, and Λ be as in the statement. Then p and q are *persistently connected*; that is, there is a neighborhood $\mathcal{W}$ of f in $\mathrm{Diff}^1(M)$ and a dense subset $\mathcal{D}$ of $\mathcal{W}$ such that given $g \in \mathcal{D}$ then the continuations p^g and q^g of p and q belong to some common transitive set.

By hypothesis q and p have different indices. Assume $ind(p) > ind(q)$; the other case is identical if we switch the roles of p and q. We first want to create an intersection between $W^s(p)$ and $W^u(q)$.

Let $\varepsilon > 0$. By transitivity, there exists some point x in Λ whose future orbit is dense in Λ. This means that there are sequences of increasing natural numbers (m_j) and (d_j) with $d_j > m_j$ such that

$$f^{m_j}(x) \to p \text{ and } f^{d_j}(x) \to q.$$

By considering fundamental domains D^s of p in ${W^s}_{loc}(p)$ and D^u of q in ${W^u}_{loc}(q)$, we can suppose (after changing the values of the sequences (m_j) and (d_j)) that we have instead

$$f^{m_j}(x) \to p^s \in W^s(p) \setminus \{p\} \text{ and } f^{d_j}(x) \to q^u \in W^u(q) \setminus \{q\}.$$

By setting $x_j = f^{m_j}(x)$ and $k_j = d_j - m_j$ we can now apply the Connecting Lemma to obtain an $\varepsilon/4$-perturbation g of f with $W^s(p^g) \cap W^u(q^g) \neq \emptyset$.

Since

$$\begin{aligned} dim(W^s(p)) + dim(W^u(q)) &= ind(p) + (n - ind(q)) \\ &> ind(q) + n - ind(q) = n, \end{aligned}$$

where n is the dimension of M, it follows that after another $\varepsilon/4$-perturbation we obtain $g' \in \mathcal{W}$ such that the intersection $W^s(p^{g'}) \cap W^u(q^{g'})$ is transversal at some point, and therefore persistent. Now, making another $\varepsilon/4$-perturbation if necessary, we can assume that g' belongs to $\mathcal{D}$, and therefore that $q^{g'}$ and $p^{g'}$ belong to some common transitive set.

Finally, to obtain an intersection between $W^u(p)$ and $W^s(q)$ after an $\varepsilon/4$-perturbation f' of g', we use the same argument as above. The end result is an ε-perturbation of f that has a heterodimensional cycle between the continuations of p and q, as desired. $\square$

Proof of Theorem C. Let $\mathcal{R}$ be the residual subset of $\mathrm{Diff}^1(M)$ obtained by intersecting the residual sets given by Theorem A and by Lemma 2, and let $f \in \mathcal{R}$ and Λ be as in the statement of Theorem C. Then Theorem A applies, so take V and $\mathcal{W}$ as in the statement of Theorem A. Choose now some hyperbolic periodic point $p \in \Lambda$ with stability index $ind(p) = s$; shrinking $\mathcal{W}$ if necessary, we can assume that the continuation p^g of p persists and belongs to V for all $g \in \mathcal{W}$.

Assume that Λ is nonhyperbolic. Then by Theorem 5 we have that f does not belong to $\Im^1(V)$. Choose some $\varepsilon > 0$. To prove the theorem we must create a heterodimensional cycle after an ε-perturbation of f.

Since $f \notin \Im^1(V)$, it follows that there exists some g in $\mathcal{W}$, $\varepsilon/4$-close to f, with some nonhyperbolic periodic point $q \in V$. Since q is nonhyperbolic (i.e., it has some eigenvalue with norm 1), we can $\varepsilon/4$-perturb f along the orbit of q so as to obtain some $f' \in \mathcal{W}$ with a periodic hyperbolic point q' such that $ind(q') \neq s$ and whose f'-orbit coincides with the f-orbit of q; thus we have $q' \in V$.

Clearly q' may not be a sink/source, because otherwise we would contradict the generic transitivity of Λ given by Theorem A. Therefore q' is a hyperbolic periodic saddle whose index differs from that of the continuation $p^{f'}$ of p. The saddle q' is hyperbolic and therefore persistent, so after another $\varepsilon/4$-perturbation we obtain $g' \in \mathcal{R}$ such that the continuation $q'^{g'}$ of q' still belongs to V. By Theorem A, we have that both $q'^{g'}$ and $p^{g'}$ belong to $\Lambda_{g'} = V \cap \Omega(g')$, which is a transitive set. So by Lemma 2 we can create a heterodimensional cycle after an $\varepsilon/4$ perturbation of g'.

So after four succesive $\varepsilon/4$-perturbations we create a heterodimensional cycle, as desired. $\square$

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