IMPLICIT LAGRANGIAN, *IL*

The **nonlinear complementarity problem** (see [3, 14]) is to find a point $x \in \mathbf{R}^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad \langle x, F(x)\rangle = 0, \tag{1}$$

where $F : \mathbf{R}^n \to \mathbf{R}^n$, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbf{R}^n$. A popular approach for solving the nonlinear complementarity problem (NCP) is to construct a *merit function* f such that solutions of NCP are related in a certain way to the optimal set of the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in C. \end{array}$$

Of practical interest is the case when the set C has simple structure and smoothness properties of F and dimensionality n of the variables space are preserved. There is a number of ways to reformulate the NCP as an equivalent optimization problem (for a survey, see [7]).

Unconstrained implicit Lagrangian. The first smooth unconstrained merit function was proposed by O.L. Mangasarian and M.V. Solodov [11]. This function is commonly referred to as the implicit Lagrangian; it has the following form :

$$\begin{aligned} M_\alpha(x) \;\; = \;\; & \langle x, F(x)\rangle \\ & + \frac{1}{2\alpha}\left(\|(x - \alpha F(x))_+\|^2 - \|x\|^2\right) \\ & + \frac{1}{2\alpha}\left(\|(F(x) - \alpha x)_+\|^2 - \|F(x)\|^2\right) \end{aligned}$$

where $\alpha > 1$ is a parameter and $(\cdot)_+$ denotes the orthogonal projection map onto the nonnegative orthant $\mathbf{R}^n_+$, i.e. the i-th component of the vector $(z)_+$ is $\max\{0, z_i\}$. It turns out that $M_\alpha(x)$ is nonnegative on $\mathbf{R}^n$ provided $\alpha > 1$, and is zero if and only if x is a solution of the NCP. If F is differentiable on $\mathbf{R}^n$, then so is $M_\alpha(\cdot)$ and its gradient vanishes at all solutions of NCP for $\alpha > 1$. Hence, one can attempt to solve the NCP by means of solving the smooth unconstrained optimization problem

$$\begin{array}{ll} \text{minimize} & M_\alpha(x) \\ \text{subject to} & x \in \mathbf{R}^n. \end{array} \tag{2}$$

The implicit Lagrangian owes its name to the way the function was first derived in [11]. Consider the constrained minimization problem (MP)

$$\begin{array}{ll} \text{minimize} & \langle x, F(x)\rangle \\ \text{subject to} & x \geq 0, \; F(x) \geq 0 \end{array}$$

which is related to the NCP (1) in the sense that its global minima of zero coincide with the solutions of NCP. Because of the special structure of the MP (the objective function is the inner product of the functions defining constraints), for every feasible $\bar{x}$ such that $\langle \bar{x}, F(\bar{x})\rangle = 0$ it can be observed that $\bar{x}$ plays the role of the *Lagrange multiplier* [2] for the constraint $F(x) \geq 0$, while $F(\bar{x})$ plays a similar role for the constraint $x \geq 0$. Keeping in mind this observation, consider the **augmented Lagrangian** [1] for the above MP :

$$\begin{aligned} L_\alpha(x, u, v) \;\; = \;\; & \langle x, F(x)\rangle \\ & + \frac{1}{2\alpha}\left(\|(-\alpha F(x) + u)_+\|^2 - \|u\|^2\right) \\ & + \frac{1}{2\alpha}\left(\|{-\alpha x} + v)_+\|^2 - \|v\|^2\right), \end{aligned}$$

where $u \in \mathbf{R}^n$ and $v \in \mathbf{R}^n$ are Lagrange multipliers corresponding to the constraints $F(x) \geq 0$ and $x \geq 0$ respectively. Since it is known a priori that at any solution $\bar{x}$ of MP (and NCP) one could take $\bar{u} = \bar{x}$ and $\bar{v} = F(\bar{x})$, it is intuitively reasonable to "solve" for multipliers u and v in terms of the original variables. Replacing u by x and v by $F(x)$ in the augmented Lagrangian, one obtains the implicit Lagrangian function $M_\alpha(x)$.

The parameter α must be strictly bigger than one, because it can be checked that $M_1(x) = 0$ for all $x \in \mathbf{R}^n$. Another interesting property is that the partial derivative of the implicit Lagrangian with respect to α is also nonnegative

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nonlinear complementarity problem
merit function
O.L. Mangasarian
M.V. Solodov
Lagrange multiplier
augmented Lagrangian

for all x, and is zero if and only if x is a solution of the NCP [10]. However, a merit function based on this derivative is nonsmooth.

Restricted implicit Lagrangian. When the implicit Lagrangian is restricted to the nonnegative orthant $\mathbf{R}^n_+$, where nonnegativity of x is explicitly enforced, the last two terms in the expression for $M_\alpha(x)$ can be dropped. Thus the restricted implicit Lagrangian is obtained :

$$\begin{aligned} N_\alpha(x) &= \langle x, F(x)\rangle \\ &\quad + \frac{1}{2\alpha}\left(\|(x-\alpha F(x))_+\|^2 - \|x\|^2\right), \end{aligned}$$

where $\alpha > 0$. In this form, the function was introduced in [11]. It is also equivalent to the **regularized gap function** proposed by `M. Fukushima` [6] in the more general context of **variational inequality** problems.

The restricted implicit Lagrangian is nonnegative for all $x \in \mathbf{R}^n_+$ provided the parameter α is positive, and its zeroes coincide with solutions of the NCP. It also inherits the differentiability of F. Thus the NCP can be solved via the bound constrained optimization problem

$$\begin{array}{ll} \text{minimize} & N_\alpha(x) \\ \text{subject to} & x \geq 0. \end{array} \tag{3}$$

Note that since it is known a priori that every solution of the NCP is nonnegative, one may also consider a bound constrained problem with the function $M_\alpha(x)$. However, for the constrained reformulations the function $N_\alpha(x)$ is probably preferable because it is somewhat simpler.

Regularity conditions. It should be emphasized that only global solutions of optimization problems (2) and (3) are solutions of the underlying NCP (1). On the other hand, standard iterative methods are guaranteed to find stationary points rather than global minima of optimization problems. It is therefore important to derive conditions which ensure that these stationary points are also solutions of NCP. One such condition is **convexity**. However, the implicit Lagrangian is known to be convex only in the case of *strongly monotone* affine F, and provided parameter α is large enough [17].Clearly, this is very restrictive. Thus other regularity conditions were investigated.

For the unconstrained problem (2), the first sufficient condition was given by `N. Yamashita` and `M. Fukushima` [22]. They established that if the Jacobian $\nabla F(\bar{x})$ is positive definite at a stationary point $\bar{x}$ of (2), then $\bar{x}$ solves the NCP. This result was later extended in [8] to the case when $\nabla F(\bar{x})$ is a *P-matrix*. Finally, `F. Facchinei` and `C. Kanzow` [5] obtained a certain regularity condition which is both necessary and sufficient for a stationary point point of the unconstrained implicit Lagrangian to be a solution of NCP (it is similar to the condition stated below for the restricted case).

For constrained problem (3), `M. Fukushima` [6] first showed the equivalence of stationary points to NCP solutions under the positive definiteness assumption on the Jacobian of F. A regularity condition which is both necessary and sufficient, was given by `M.V. Solodov` [19] : a point $x \in \mathbf{R}^n_+$ is said to be regular if $\nabla F(x)^\top$ reverses the sign of no nonzero vector $z \in \mathbf{R}^n$ satisfying

$$z_P > 0, \quad z_C = 0, \quad z_N < 0, \tag{4}$$

where

$$\begin{aligned} C &:= \{i \mid x_i \geq 0,\ F_i(x) \geq 0,\ x_i F_i(x) = 0\}, \\ P &:= \{i \mid x_i > 0,\ F_i(x) > 0\}, \\ N &:= \{i \mid x_i \geq 0,\ F_i(x) < 0\}. \end{aligned}$$

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regularized gap function
`M. Fukushima`
variational inequality
convexity
strongly monotone
`N. Yamashita`
`M. Fukushima`
P-matrix
`F. Facchinei`
`C. Kanzow`
`M. Fukushima`
`M.V. Solodov`

Recall [4] that the matrix $\nabla F(x)^\top$ is said to reverse the sign of a vector $z \in \mathbf{R}^n$ if

$$z_i[\nabla F(x)^\top z]_i \leq 0 \quad \forall i \in \{1, \ldots, n\}. \qquad (5)$$

Therefore a point $x \in \mathbf{R}^n_+$ is regular if the only vector $z \in \Re^n$ satisfying both (4) and (5) is the zero vector.

A stationary point of (3) solves the NCP if and only if it is regular in the sense of the given definition.

Derivative-free descent methods. When F is differentiable, so are the functions $M_\alpha(x)$ and $N_\alpha(x)$. Therefore, any standard optimization algorithm which makes use of first-order derivatives can be applied to problems (2) and (3). However, taking advantage of the underlying structure one can also devise special descent algorithms which do not use derivatives of F. This can be especially useful in cases when derivatives are not readily available or are expensive to compute.

In [22], it was shown that when F is *strongly monotone* and continuously differentiable, then the direction

$$\begin{aligned} d(x) &= (\beta - \alpha)\,(x - (x - \alpha F(x))_+) \\ &\quad +(1 - \alpha\beta)\,(F(x) - (F(x) - \alpha x)_+) \end{aligned}$$

is a descent direction for $M_\alpha(\cdot)$ at $x \in \mathbf{R}^n$, provided $\beta > 0$ is choisen appropriately. A descent method based on this direction with appropriate *linesearch*, converges globally to the unique solution of the NCP [22]. In [12], it was established that the *rate of convergence* is actually at least linear.

For the restricted implicit Lagrangian, a descent method with the direction

$$d(x) = (x - \alpha F(x))_+ - x$$

was proposed in [6]. The algorithm was proven to be convergent for the strongly monotone NCP (no rate of convergence has been established however). In [24], by using adaptive parameter α, this method was further extended to monotone (not necessarily strongly monotone) and Lipschitz continuous (not necessarily differentiable) functions.

Error bounds. The implicit Lagrangian also appears useful for providing bounds on the distance from a given point to the solution set of the NCP. In particular, if F is affine then there exists a constant $c > 0$ such that (see [10])

$$\mathrm{dist}(x, S) \leq c M_\alpha(x)^{1/2}$$

for all x close to S, where S denotes the solution set. This inequality is called a local **error bound**. `X.-D. Luo` and `P. Tseng` proved that, in the affine case, this bound is global (i.e. it holds for all $x \in \mathbf{R}^n$) if and only if the associated matrix is of the class R_0.

For the nonlinear case, `C. Kanzow` and `M. Fukushima` [9] showed that $M_\alpha(x)^{1/2}$ provides a global error bound if F is a *uniform P-function* which is Lipschitz continuous.

In the context of error bounds, the following relation established in [10] is useful : for all $x \in \mathbf{R}^n$,

$$\alpha^{-1}(\alpha - 1)\,\|r(x)\|^2 \leq M_\alpha(x) \leq (\alpha - 1)\,\|r(x)\|^2$$

where

$$r(x) = x - (x - F(x))_+$$

is the **natural residual** [13]. Therefore the implicit Lagrangian $M_\alpha(x)$ provides a local/global error bound if and only if so does the natural residual $r(x)$.

For the restricted implicit Lagrangian, one only has the following relation :

$$2\alpha N_\alpha(x) \geq \|r(x)\|^2 .$$

Thus, in principle, $N_\alpha(x)$ may provide a bound in cases when the natural residual does not.

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strongly monotone
linesearch
rate of convergence
error bound
`X.-D. Luo`
`P. Tseng`
`C. Kanzow`
`M. Fukushima`
uniform P-function
natural residual

For a general discussion of error bounds see [15].

Extensions. The implicit Lagrangian can be extended to the context of generalized complementarity problems and variational inequality problems via its relation with the **regularized gap function**. As observed by `J.-M. Peng` and `Y.X. Yuan` [18], the function $M_\alpha(x)$ can be represented as a difference of two regularized gap functions with parameters $1/\alpha$ and α. Since the regularized gap function can also be defined for variational inequalities, one might consider a similar expression in this more general context. `J.-M. Peng` [16] established the equivalence of the variational inequality problem to unconstrained minimization of a difference of regularized gap functions. This result was further extended by `N. Yamashita`, `K. Taji` and `M. Fukushima` [23] who obtained similar results for differences of regularized gap functions whose parameters are not necessarily the inverse of each other.

For a unified treatment of extensions of the implicit Lagrangian and the regularized gap function for the generalized complementarity problems see [21].

Yet another context where the implicit Lagrangian can be used is optimization reformulation of the extended linear complementarity problem [20].

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regularized gap function
J.-M. Peng
Y.X. Yuan
J.-M. Peng
N. Yamashita
K. Taji
M. Fukushima

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