Some results about a bidimensional version of the generalized BO

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Abstract

For the bidimensional version of the generalized Benjamin-Ono equation:

$$u_t - H^{(x)}u_{xy} + u^p u_y = 0,\ t \in \mathbb{R},\ (x,y) \in \mathbb{R}^2,$$

we use the method of parabolic regularization to prove local well-posedness in the spaces $H^s(\mathbb{R}^2)$, $s > 2$ and in the weighted spaces $\mathcal{F}_r^s = H^s(\mathbb{R}^2) \cap L^2\big((1+x^2+y^2)^r dxdy\big)$, $s > 2$, $r \in [0,1]$ and $\mathcal{F}_{1,k}^k = H^k(\mathbb{R}^2) \cap L^2\big((1+x^2+y^{2k})dxdy\big)$, $k \in \mathbb{N}$, $k \geq 3$. As in the case of BO there is lack of persistence for both the linear and nonlinear equations (for p odd) in $\mathcal{F}_2^s$. That leads to unique continuation principles in a natural way. By standard methods based on $L^p - L^q$ estimates of the associated group we obtain global well-posedness for small initial data

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and nonlinear scattering for $p \geq 3,\ s > 3$. Nonexistence of square integrable solitary waves of the form $u(x,y,t) = v(x, y-ct),\ c > 0,\ p \in \{1,2\}$ is obtained using the results about existence of solitary waves of the BO and variational methods.

1 Introduction

The Korteweg-de Vries (KdV) and Benjamin-Ono (BO) equations:

$$u_t + u_{xxx} + uu_x = 0, \qquad \text{(KdV)}$$

$$u_t - Hu_{xx} + uu_x = 0, \qquad \text{(BO)[1]}$$

are classical models in water waves propagation theory. KdV governs the surface wave propagation in a shallow channel of water (c.f.[23]) while BO arises in the study of long internal waves in deep stratified fluids (see [7], [28]).

Notice that they describe unidirectional motion. When transversal effects are also present, the resulting equations are bidimensional. The most accepted generalizations of KdV to two dimensions are the equation of Kadomtsev and Petviashvili and the equation of Zakharov and Kuznetsov (see [21], [40]). Various 2D versions analogous to KP and ZK, have also been derived for the BO case (c.f.[5], [30], [38]).

In [3] and [4] the authors present the following 2D version of the BO:

$$\left(u_t - H^{(x)}u_{xx} + uu_x\right)_x = \left(-H^{(x)}u_{xx} + uu_x\right)_y, \tag{1}$$

[1] Here H denotes the Hilbert transform, see Sect.2.

where we have written $H^{(x)}$ for the Hilbert transform with respect to the variable x. This equation is obtained after a suitable limiting process [2] from a compatibility condition of a 2D-version of the ILW's Lax pair. The asymmetries in both the linear and the nonlinear parts and the presence of a nonlocal operator, make its analysis interesting from the mathematical point of view. However, as far as we know it does not have physical interest.

Assuming enough regularity and decay at infinity, from Eqn.(1) we obtain:

$$u_t - H^{(x)}u_{xx} + uu_x = -H^{(x)}u_{xy} + uu_y. \tag{2}$$

The structure of this equation suggests the use of the change of variables $(x, y) \mapsto (x + y, -y)$. In the new variables it takes the form:

$$u_t - H^{(x)}u_{xy} + uu_y = 0. \tag{3}$$

We can consider higher order nonlinearities as:

$$u_t - H^{(x)}u_{xy} + u^p u_y = 0,\ p \in \mathbb{N}. \tag{4}$$

In the sense just explained, the above equation can be understood as a 2D version of the generalized BO (gBO) (see [15], [26], [16]):

$$u_t - Hu_{xx} + u^p u_x = 0,\ p \in \mathbb{N}. \tag{5}$$

Observe that Eqns.(4) and (5) share the same (formal) Hamiltonian:

$$\mathcal{H}(u) = \frac{1}{2}\int uH^{(x)}u_x - \frac{1}{(p+1)(p+2)}\int u^{p+2}. \tag{6}$$

A natural first question is about well-posedness. Well-posedness in Sobolev spaces $H^s(\mathbb{R}^n)$, $s > \frac{n}{2} + 1$ for the initial value problem (IVP)

[2] Based on the relation between BO and the intermediate long wave equation (ILW).

associated to nonlinear dispersive equations is now well understood. Kato's theory for quasilinear equations (see [22]) and the method of parabolic regularization (see [17], [31]) are the techniques most widely used. However, these methods do not exploit the dispersive structure of the equations. This has been done by various authors, (see [10], [25], [27]), who showed that larger spaces can be dealt with. On the other side, the results obtained are not so general in the sense that they depend strongly on the properties of the symbol of the linear part of the equation at hand. We will restrict ourselves to the second method mentioned and consider well-posedness in the spaces $H^s = H^s(\mathbb{R}^2),\ s > 2$.

The consideration of weighted spaces is natural when one wants to study the decay in space of the solutions. One may study polynomial decay by considering spaces of the type $\mathcal{F}^s_r(\mathbb{R}^n) = H^s(\mathbb{R}^n) \cap L^2((1+|x|^2)^r dx)$. Iorio ([17], [18]) proved the local well-posedness for BO in the spaces $\mathcal{F}^s_r(\mathbb{R}),\ s > 3/2,\ r \in \{0,1\}$ and global well-posedness in $\mathcal{F}^2_r(\mathbb{R}),\ r \in [0,2]$. For $s \geq 3$ and $r = 3$ it was shown that persistence requires further restrictions on the initial data and for $s = r = 4$ there is no persistence at all. These features imply unique continuation principles (see [11], [20], [39]). We will prove here local well-posedness for Eqn.(4) in $\mathcal{F}^s_r(\mathbb{R}^2),\ s > 2,\ r \in [0,1]$ and that for $r = 2$, persistence fails to occur for odd values of p. As a consequence we also obtain unique continuation principles.

Abdelouhab [1] proved a result on local well-posedness for a wide class of equations in the weighted Sobolev spaces $H^{s'}(\mathbb{R}^n) \cap H^s_r(\mathbb{R}^n),\ s' > \frac{n}{2} + 1$, where $H^s_r(\mathbb{R}^n) = [w^{r/2}(1-\Delta)^{s/2}]^{-1} L^2(\mathbb{R}^n),\ r \geq 0$ and s satisfies further restrictions. The local results obtained here for Eqn.(4) in $\mathcal{F}^s_r(\mathbb{R}^2),\ s >$

2, $r \in \{0,1\}$ follow from his conclusions. We will also consider a wider class of weights that includes nonisotropic ones (see Sect.4). We think this type of analysis illustrates better the relation between the smoothness of the symbol of the dispersive operator and the decay rate of the solutions. Results in this direction has been obtained for other equations by several authors (see, for example, [14], [22] and [36].)

For several nonlinear equations it has been proved that for higher order nonlinearities and small initial data, the solutions behave as the solutions of the associated linear equation (see [24] and [34], for example). We obtain here an explicit expression for the kernel of $e^{tH\partial^2_{xy}}$ (see Lemma 5.1) that permits us to obtain $L^p - L^q$ estimates analogous to those of the Schrödinger group $e^{it\Delta}$ in dimension two. By standard methods (see [24]), these estimates allow us to obtain global existence in H^s, $s > 3$ for initial data with small $L^1_1 \cap H^s$ norm and nonlinear scattering for such solutions of Eqn.(4).

The solitary waves, or waves of permanent form, are known to arise in several nonlinear models as a consequence of the combined effects of the linear and the nonlinear parts. The existence and uniqueness of solitary waves for the BO is proved in [6] (see also [37], where the gBO is considered). We use this result to prove nonexistence of square integrable solitary waves for Eqn.(4), $p = 1$. A similar conclusion follows for $p = 2$ by variational arguments.

The plan of this paper is the following. In Section 2 we present the basic notations and results that we will need. In Sections 3 and 4 we study the local well-posedness in Sobolev and weighted Sobolev spaces, respectively for Eqn.(4). In Section 5 we prove the global well-posedness of small initial data and nonlinear scattering for $p \geq 3$. Finally, Section 6 is devoted to the proof of nonexistence of square integrable solitary waves of the form $u(x, y, t) = v(x, y - ct)$.

2 Notation and preliminaries

Let $\Delta_n = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ denote the n-dimensional Laplacian and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions in $\mathbb{R}^n$, dual to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Define the (real) Lebesgue spaces by:

$$L_s^p(\mathbb{R}^n, \mathbb{R}) = J^{-s}L^p = \{f \in \mathcal{S}'(\mathbb{R}^n) : Imf = 0,\ J^s f \in L^p\},$$

where $s \in \mathbb{R},\ 1 \leq p \leq +\infty$ and $J^s = (1 - \Delta_n)^{s/2}$ is the Bessel potential of order $-s$. The norm of these spaces will be denoted by $\|f\|_{L_s^p} = |f|_{s,p}$. For $p = 2$ we will write $L_s^2 = H^s$ (the usual L^2-type Sobolev spaces) and $\|f\|_{H^s} = \|f\|_s$. In the case $s = 0$ the corresponding norm will be denoted as $\|\cdot\|$. Observe that the equality $L_0^p = L^p$ is trivial. The notation $\|f\|_{L^p} = |f|_p$ will also be used. We will need the following results:

Lemma 2.1 *Let $f \in \mathcal{S}(\mathbb{R}^n)$, $p_0,\ p_1 \in (1, +\infty)$ and $s_0,\ s_1 \in \mathbb{R}$. Then:*

$$|J^s f|_p \leq c|J^{s_0} f|_{p_0}^{\theta} |J^{s_1} f|_{p_1}^{1-\theta} \tag{7}$$

where

$$s = \theta s_0 + (1 - \theta)s_1$$

and

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1 - \theta}{p_1}.$$

Proof: See ([8], Chapter 6).□

The next two lemmas have been proved in [24].

Lemma 2.2 *If $s > 0$ and $1 < p < +\infty$, then*

$$|[J^s, M_f]g|_p \leq c(|\nabla f|_\infty |J^{s-1}g|_p + |J^s f|_p |g|_\infty) \tag{8}$$

where M_f denotes the operator of multiplication by f, and $f, g \in \mathcal{S}(\mathbb{R}^n)$.

Lemma 2.3 *With the assumptions of Lemma 2.2, $L^p_s \cap L^\infty$ is an algebra. Moreover,*

$$|fg|_{s,p} \leq c(|f|_\infty |g|_{s,p} + |f|_{s,p}|g|_\infty) \tag{9}$$

The following Sobolev type inequality will be useful. It is proved in ([31], Lemma 6):

Lemma 2.4 *For any $f \in H^{n/2+\epsilon}$, $\epsilon \in (0, 1/2]$ we have:*

$$|f|_\infty \leq c(n)\epsilon^{-1/2}\|f\|_{n/2+\epsilon}. \tag{10}$$

Recall that the Hilbert Transform, defined by

$$Hf(x) = p.v.\frac{1}{\pi}\int \frac{f(y)}{x-y}dy = (-isgn(.)\hat{f}(\cdot))\check{}(x),$$

is a unitary operator on $L^2(\mathbb{R})$, such that $H^2 = -I$ and that $H\partial_x = \partial_x H$ is a positive operator on $L^2(\mathbb{R})$ with domain $H^1(\mathbb{R})$. Here the symbols $\hat{}$, $\check{}$ denote the Fourier transform and its inverse, respectively. For the sake of clarity, when we deal with two spatial variables, we will denote $H^{(x)}$ the Hilbert transform with respect to the variable x.

Theorem 2.1 *(Calderón's Commutator Theorem) Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then for any $f \in \mathcal{S}(\mathbb{R})$ we have:*

$$\|[H, A]f'\| \leq C|A'|_\infty \|f\|. \tag{11}$$

Proof: See [12] or [13].□

Following Triebel [35] we introduce:

Definition 2.1 *A positive function $\rho \in C^\infty(\mathbb{R}^2; \mathbb{R})$ will be called an* ***admissible weight*** *if all the following conditions hold:*
a) $\rho(x,y) \geq 1$, $\forall (x,y) \in \mathbb{R}^2$;
b) $|\rho(x,y)| \to \infty$, as $|(x,y)| \to +\infty$;
c)$|\nabla \rho(x,y)| \leq c\rho^2(x,y)$, $\forall (x,y) \in \mathbb{R}^2$.

It is easy to see that the functions ρ defined by $\rho(x,y) = (1+x^2+y^2)^{r/\nu}$ and $\rho(x,y) = (1 + x^{2r_1} + y^{2r_2})^{1/\nu}$, with $\nu > 0,\ r > 0,\ r_j > 0,\ j = 1, 2$, are admissible weights.

In what follows ρ will always denote an admissible weight.

Consider:

$$H^s(\rho^\nu) = H^s \cap L^2(\rho^\nu),\ s \geq 0,\ \nu > 0$$

where $L^2(\rho^\nu)$ denotes the space of all real valued measurable functions such that:

$$\|f\|_{L^2(\rho^\nu)} = \left(\int f^2(x,y) \rho^\nu dx dy \right)^{1/2} < \infty.$$

They are Hilbert spaces when provided with the inner product:

$$(f, g)_{H^s(\rho^\nu)} = (f, g)_s + (f, g)_{L^2(\rho^\nu)}$$

and they possess the following interpolation property:

Theorem 2.2 *Let $s \geq 0$, $\nu \geq 2s$, $0 \leq j \leq s$, $j \in \mathbb{Z}$ and $\chi_j = \nu(1 - \frac{j}{s})$. Then there exists a positive number c such that for all $f \in H^s(\rho^\nu)$*

$$\sum_{|\alpha|=j} \int \rho^{\chi_j} |D^\alpha f(x,y)|^2 dxdy \leq c\|f\|^2_{H^s(\rho^\nu)}.$$

Proof: See [35], Chapter 3.□

For the particular weight $\rho^\nu(x,y) = (1 + x^2 + y^2)^r$, we will denote $H^s(\rho^\nu) = \mathcal{F}^s_r$, $\|\cdot\|_{H^s(\rho^\nu)} = \|\cdot\|_{s,r}$. Analogously for $\rho^\nu(x,y) = (1+x^{2r_1}+y^{2r_2})$ we set $H^s(\rho^\nu) = \mathcal{F}^s_{r_1,r_2}$. The following propositions are easy consequences of the preceding theorem.

Proposition 2.1 *Let $\mathbf{x} = (x,y)$, $\alpha, \beta \in (\mathbb{Z}^+)^2, |\alpha| + |\beta| \leq k$, $k \in \mathbb{N}$ and $f \in \mathcal{F}^k_k$. Then*

$$\|\mathbf{x}^\beta D^\alpha f\| \leq c(|\alpha|)\|f\|_{k,k}.$$

Proposition 2.2 *Let $\alpha \in (\mathbb{Z}^+)^2$, $\beta \in \mathbb{Z}^+$, $|\alpha| + \beta \leq k$, $k \in \mathbb{N}$ and $f \in \mathcal{F}^k_{1,k}$. Then*

$$\|y^\beta D^\alpha f\| \leq c(|\alpha|)\|f\|_{\mathcal{F}^k_{1,k}}.$$

For these spaces we also have:

Proposition 2.3 *Let $r_j > 0$, $j = 1,2$, $s > 1$, then $\mathcal{F}^s_{r_1,r_2}$ is a Banach algebra.*

Proof: Observe that by the Sobolev's Embedding Theorem:

$$\|fg\|_{\mathcal{F}^s_{r_1,r_2}} \leq c_s\|f\|_s\|g\|_{\mathcal{F}^s_{r_1,r_2}}.$$

for all $f, g \in \mathcal{F}^s_{r_1,r_2}$. Then the result follows since $\|f\|_s \leq \|f\|_{\mathcal{F}^s_{r_1,r_2}}$ obviously holds.□

For X, Y Banach spaces, we will denote $\mathcal{B}(X,Y)$ the space of bounded linear operators from X to Y. When $X=Y$ we will simply write $\mathcal{B}(X)$. In what follows different constants may be denoted by the same letter when its precise value is of no relevance. When necessary, dependence on other quantities will be indicated.

3 Local theory in Sobolev spaces

We adopt the notion of well-posedness of a Cauchy problem as in [22], that is: existence, uniqueness, persistence in the sense that the solution describes a curve in the space which contains the initial data and continuous dependence on the initial data.

In order to investigate the local well-posedness in H^s of the IVP associated to Eqn.(4), we will use the method of parabolic regularization as in [17] and [19]. For any $\mu \geq 0$, $p \in \mathbb{N}$ we consider the equation:

$$u_t - Hu_{xy} + u^p u_y = \mu \Delta u. \tag{12}$$

Using the properties of the solutions to these equations with $\mu > 0$ we will be able to obtain the solution for $\mu = 0$.

The linear problem associated to Eqn.(12) is:

$$\begin{aligned} u_t - Hu_{xy} &= \mu \Delta u & (13)\\ u(0) &= \phi & (14)\end{aligned}$$

where $\phi \in H^s,\ s \in \mathbb{R}$. Its (unique) solution $u^\mu(t) = E_\mu(t)\phi$, is given by:

$$E_\mu(t)\phi = (e^{t(i|\xi|\eta - \mu(\xi^2+\eta^2))}\hat{\phi})\check{} = e^{t(H\partial^2_{xy}+\mu\Delta)}\phi. \tag{15}$$

It is convenient to define:

$$G_\mu(t,\xi,\eta) = e^{t(i|\xi|\eta-\mu(\xi^2+\eta^2))}, \tag{16}$$

then $(E_\mu(t)\phi)\hat{} = G_\mu(t,\xi,\eta)\hat{\phi}$.

The following lemma provides us a very useful inequality. Its proof is straightforward.

Lemma 3.1 *For any $a > 0,\ b > 0$ and $r \geq 0$, we have:*

$$h(r) = r^a \exp\left(-br^2\right) \leq \left(\frac{\sqrt{2}}{2}\sqrt{\frac{a}{b}}\right)^a \exp\left(-\frac{a}{2}\right) \tag{17}$$

Some interesting properties of E_μ can now be derived.

Lemma 3.2 *Let $s \in \mathbb{R},\ \mu \geq 0$. Then:*

(a) E_μ defines a contraction semigroup in H^s that can be extended, when $\mu = 0$ to a strongly continuous unitary group.

(b) Let $\lambda \geq 0,\ \mu > 0$, then for all $t > 0$:

$$E_\mu(t) \in \mathcal{B}(H^s, H^{s+\lambda}) \tag{18}$$

and satisfies the estimate:

$$\|E_\mu(t)f\|_{s+\lambda} \leq K_\lambda(1+(2\mu t)^{-\lambda})^{1/2}\|f\|_s, \tag{19}$$

for any $f \in H^s$.

Proof: The result is completely analogous to Theorem 2.1 of [17]. It follows from Lemma 3.1 with the aid of the Fourier transform. □

Note that (b) expresses a regularizing property of the semigroup $E_\mu(t),\ \mu > 0$. Now we state our result on local well-posedness in H^s.

Theorem 3.1 *Given an initial condition* $\phi \in H^s$, $s > 2$, *there exists* $\tilde{T}^\mu = \tilde{T}(\mu, \|\phi\|_s)$ *such that the IVP for Eqn.(12) with* $\mu \geq 0$ *has a unique solution* u^μ *in the class:*

$$C\left([0, \tilde{T}^\mu]; H^s\right) \cap C^1\left([0, \tilde{T}^\mu]; H^{s-2}\right)$$

that satisfies

$$\|u^\mu(t)\|_s \leq \|\phi\|_s \exp\{c \int_0^t |u^\mu(\tau)|_\infty^{p-1} |\nabla u^\mu(\tau)|_\infty d\tau\}, \ \forall t \in [0, \tilde{T}^\mu]. \tag{20}$$

Moreover, the map $\phi \to u^\mu$ *is continuous from* H^s *to* $C\left([0, \tilde{T}^\mu]; H^s\right)$ *and if* $\phi \in H^{s'}$, $s' > s > 2$ *then the above results hold with* s' *in place of* s.

Proof: Suppose first that $\mu > 0$. Using the Lemma 3.2, one can find a $\tilde{T}^\mu > 0$ such that

$$Ff(t) = E_\mu(t)\phi - \int_0^t E_\mu(t-\tau) f^p f_y(\tau) d\tau \tag{21}$$

is a contraction in the complete metric space

$$\mathcal{X}_s(\tilde{T}^\mu) = \{f \in C([0, \tilde{T}^\mu]; H^s) : \sup_{t \in [0, \tilde{T}^\mu]} \|f(t) - E_\mu(t)\phi\|_s \leq \|\phi\|_s\}. \tag{22}$$

By Banach's Fixed Point Theorem there exists a unique solution in $\mathcal{X}_s(\tilde{T}^\mu)$. The uniqueness in $C([0, \tilde{T}^\mu]; H^s)$ can be derived estimating the H^s-norm of the difference of two solutions as in [19]. The restriction on the order of the spaces is involved in the procedure described above, since for the calculations we use that H^{s-1} is a Banach algebra (or that the elements of H^s are differentiable functions).

It is easy to see that the integral equation:

$$u^\mu(t) = E_\mu(t)\phi - \int_0^t E_\mu(t-\tau)(u^\mu)^p u_y^\mu(\tau) d\tau \tag{23}$$

is equivalent to the initial value problem associated to Eqn.(12), with $u^\mu(0) = \phi$. Then (18) and a bootstrapping argument allow to show that if $u^\mu \in C([0,\tilde{T}^\mu]; H^s)$ is a solution of Eqn.(12) with initial value ϕ, then $u^\mu \in C((0,\tilde{T}^\mu]; H^\infty)$ (see [17], [18] for details).

Fix now $t \in (0,\tilde{T}^\mu]$. We have that $\frac{d}{dt}u^\mu(t)$, $u^\mu(t)$ and J^s belong to H^∞, for any $s \in \mathbb{R}$. Then using Eqn.(12) we obtain:

$$\frac{1}{2}\frac{d}{dt}\|J^s u^\mu\|^2 + (J^s H u^\mu_{xy}, J^s u^\mu) + (J^s (u^\mu)^p u^\mu_y, J^s u^\mu) = \mu (J^s \Delta u^\mu, J^s u^\mu), \quad (24)$$

where the functions are evaluated at t. Then integration by parts, estimates (8)-(9) and the skew symmetry of $J^s H \partial^2_{xy}$ lead to:

$$\frac{d}{dt}\|u^\mu(t)\|_s^2 \le c|u^\mu(t)|_\infty^{p-1}|\nabla u^\mu(t)|_\infty \|u^\mu(t)\|_s^2, \quad t \in (0,\tilde{T}^\mu]. \qquad (25)$$

Gronwall's inequality yields (20).

Suppose now that $\phi \in H^{s'}$, $s' > s > 2$. Then (25) and the Sobolev's Embedding Theorem imply that

$$\frac{d}{dt}\|u^\mu(t)\|_{s'}^2 \le c_s \|u^\mu(t)\|_s^p \|u^\mu(t)\|_{s'}^2, \; t \in (0,T], \qquad (26)$$

where $T \in (0,\tilde{T}^\mu]$ is any time of existence of u^μ in $H^{s'}$. Then Gronwall's inequality and an extension argument imply the result.

To treat the case $\mu = 0$, one may use the strategy followed in [17] and [19](Chapter 6). That is, first one proves that the solutions u^μ, $\mu > 0$ can be extended to an interval of the form $[0,T_s)$, where $T_s = c\|\phi\|_s^{-p}$, for some $c = c(s,p)$. Then for any $\tilde{T}^0 \in (0,T_s)$ and any $\mu > 0$, it holds that $u^\mu \in C([0,\tilde{T}^0]; H^s)$. The solution for $\mu = 0$ is then obtained as the limit of this net in L^2 uniformly in time. This limit turns out to hold also weakly in H^s, uniformly in time. Then (20) and the last assertion in the theorem follow by the weak lower semicontinuity of the H^s-norm.

To prove the continuous dependence on the initial data, one must use the Bona-Smith approximations taking as auxiliary initial data, for example $\phi^\tau = (exp(-\tau(1+|\cdot|^2)^{s/2})\hat{\phi}(\cdot))\check{}$. For the details we refer [19]. □

Remark 3.1 *When $\mu = 0$ we can solve the equation for negative times. It follows that the interval of existence can be taken symmetric with respect to the origin. To see this observe that the solutions of Eqn.(12) are invariant under the transformation $(t,x,y) \mapsto (-t,x,-y)$.*

Remark 3.2 *Regarding global well-posedness, observe that since for any $\mu \geq 0$, $\tilde{T}^\mu$ can be chosen commonly for all ϕ in a bounded set, then Theorem 3.1 holds in any interval $[0,T]$, $T \in (0,T_s^\mu)$, where $[0,T_s^\mu)$ is the maximal interval of existence of u^μ in H^s and $T_s^\mu = +\infty$ or $\|u^\mu(t)\|_s \to \infty$ as $t \nearrow T_s^\mu$. Theorem 3.1 says that $T_s^\mu = T_{s'}^\mu$ if $\phi \in H^{s'}$, $s' > s > 2$. Moreover, we can prove the following extension principle (see [2], [31]):*

Proposition 3.1 *Under the assumptions of Theorem 3.1, if $\mu > 0$ then either $T_s^\mu = +\infty$ or $\|u^\mu(t)\|_2 \to +\infty$ as $t \nearrow T_s^\mu$.*

Proof: Fix $\mu > 0$, and suppose that $M_1 = \sup_{t\in(0,T_s^\mu)} \|u^\mu(t)\|_2 < \infty$ with $T_s^\mu < \infty$. Combining (7) and (25) with Lemma 2.4 one obtains that for any $\epsilon \in (0,1/2]$ such that $2+\epsilon < s$ and $t \in (0,T_s^\mu)$ we have,

$$\begin{aligned}
\frac{d}{dt}\|u^\mu(t)\|_s^2 &\leq \frac{c}{\epsilon^{1/2}}\|u^\mu\|_2^{p-1}\|u^\mu\|_{2+\epsilon}\|u^\mu\|_s^2 \\
&\leq \frac{c}{\epsilon^{1/2}}\|u^\mu\|_2^{p-\theta}\|u^\mu\|_s^{2+\theta} \\
&\leq \frac{c(1+M_1^p)}{\epsilon^{1/2}}\|u^\mu\|_s^{2+\theta}
\end{aligned}$$

where $\theta = \frac{\epsilon}{s-2}$. Then $\|u^\mu(t)\|_s \leq \rho^{1/2}(t),\ \forall t \in [0, \min\{T_\epsilon, T_s^\mu\})$ where ρ solves:

$$\begin{aligned} \frac{d}{dt}\rho(t) &= \frac{c(1+M_1^p)}{\epsilon^{1/2}}\rho^{1+\theta/2} \\ \rho(0) &= \|\phi\|_s^2 \end{aligned}$$

and $[0, T_\epsilon)$ is the maximal interval of existence of ρ. It is easy to see that

$$T_\epsilon = \frac{2(s-2)}{c(1+M_1^p)\|\phi\|_s^\theta \epsilon^{1/2}}.$$

Then one can clearly choose ϵ small enough such that $T_\epsilon > T_s^\mu$ and an extension argument yields a contradiction to the fact that $\|u_\mu(t)\|_s \to \infty$ as $t \nearrow T_s^\mu$. $\square$

4 Decay properties

In this section we study now the asymptotic behavior of the solutions of Eqn.(12), $\mu \geq 0$ as $|(x,y)| \to \infty$. The procedure that we employ follows the ideas used in [17], [18] and [19] for studying these properties in the case of the BO equation.

4.1 Isotropic weights

We begin by studying the solutions of Eqn.(13) in the spaces $\mathcal{F}_r^s$, $s \geq 0$, $r > 0$, defined in Section 2. Recall that in (16) we set , $G_\mu(t,\xi,\eta) = e^{t(i|\xi|\eta - \mu(\xi^2+\eta^2))}$. We will need:

Lemma 4.1 *For the function G_μ we have:*

$$\partial_\xi G_\mu = t(isgn(\xi)\eta - 2\mu\xi)G_\mu \tag{27}$$

$$
\begin{aligned}
\partial_\xi^2 G_\mu &= [t^2(-\eta^2+4\mu^2\xi^2-4\mu i|\xi|\eta)-2\mu t]G_\mu \qquad (28)\\
&\quad +2it\eta\delta(\xi)e^{-t\mu\eta^2}\\
\partial_\xi^{(2k)} G_\mu &= G_\mu \sum_{j=0}^{k} t^{2k-j} g_{2k,2(k-j)}(\mu,\xi,\eta sgn(\xi)) \qquad (29)\\
&\quad +e^{-t\mu\eta^2}\sum_{j=0}^{k-1} f_{2k,j}(t,\mu,\eta)\delta^{(2j)}(\xi)\\
\partial_\xi^{(2k-1)} G_\mu &= G_\mu \sum_{j=0}^{k-1} t^{2k-1-j} g_{2k-1,2(k-j)-1}(\mu,\xi,\eta sgn(\xi)) \qquad (30)\\
&\quad +e^{-t\mu\eta^2}\sum_{j=1}^{k-1} f_{2k-1,j}(t,\mu,\eta)\delta^{(2j-1)}(\xi)\\
\partial_\eta G_\mu &= -t(i|\xi|+2\mu\eta)G_\mu \qquad (31)\\
\partial_\eta^2 G_\mu &= [t^2(-\xi^2+4i\mu|\xi|\eta+4\mu^2\eta^2)-2\mu t]G_\mu \qquad (32)\\
\partial_\eta^k G_\mu &= G_\mu \sum_{j=0}^{[k/2]} t^{k-j} h_{k,k-2j}(\mu,|\xi|,\eta) \qquad (33)
\end{aligned}
$$

where $k \in \mathbb{N}$, $h_{j,m}$ and $g_{j,m}$ are homogeneous polynomials of degree m in the last two variables for any $j \in \mathbb{N}$ and $f_{j,m}(t,\mu,\eta)$ is a polynomial of degree m in t for μ and η fixed and also a polynomial of the same degree in η for μ and t fixed.

Proof: Using the Leibniz rule it is easy to show by an induction argument that:

$$
\begin{aligned}
f_\xi^{(2k)} &= (it\eta)^{2k} f + 2i\sum_{j=0}^{k-1}(-1)^j (t\eta)^{2k-2j-1}\delta^{(2j)}\\
f_\xi^{(2k+1)} &= (it\eta\ sgn(\xi))^{2k+1} f + 2i\sum_{j=1}^{k}(-1)^{j-1}(t\eta)^{2k-2j+1}\delta^{(2j-1)}
\end{aligned}
$$

for any $k \in \mathbb{Z}^+$, where $f(t,\xi,\eta) = e^{-\mu t(\xi^2+\eta^2)}$. Set $g_\mu(t,\xi,\eta) = e^{it|\xi|\eta}$. Then $\frac{\partial^{(k)}}{\partial\xi^k} g_\mu(t,\xi,\eta) = p_k(t,\xi,\eta) g_\mu(t,\xi,\eta)$, where p_k is a polynomial of degree k in each of the variables t and ξ with its higher order term of the form $(-2\mu t\xi)^4$. $G_\mu(t,\xi,\eta) = g_\mu(t,\xi,\eta) f(t,\xi,\eta)$ and then a further application of the Leibniz rule (with the derivatives of g_μ taken in a generalized sense) leads to the result. The calculations for the variable η can be handled similarly. $\square$

In order to prove the global well-posedness of (13)-(14) in $\mathcal{F}^s_r$, it suffices to prove the persistence and boundedness of $E_\mu(t)$ in these spaces.

Theorem 4.1 *(a) Let $\phi \in \mathcal{F}^s_1$, $s \geq 1$, $t \geq 0$ and $\mu \geq 0$, then $E_\mu(t) \in \mathcal{B}(\mathcal{F}^s_1)$ and it holds:*

$$\|E_\mu(t)\phi\|_{s,1} \leq p_1(\mu,t)\|\phi\|_{s,1}, \tag{34}$$

where p_1 is a polynomial of first degree in t with positive coefficients depending on μ. Moreover, $E_\mu(\cdot)\phi \in C([0,+\infty);\mathcal{F}^s_1)$.
(b) Let $\phi \in \mathcal{F}^s_k$, $s \geq k \geq 2$, $k \in \mathbb{N}$, $t \geq 0$, $\mu \geq 0$ then $E_\mu(t) \in \mathcal{B}(\mathcal{F}^s_k)$ if and only if

$$\int x^{l-1}\phi(x,y)dx = 0, \ \ \forall y \in \mathbb{R}, \ l = 1,..,k-1. \tag{35}$$

This is a necessary and sufficient condition for the existence of solution to (13)-(14) in $\mathcal{F}^s_k$. If that condition is satisfied, then $E_\mu(\cdot)\phi \in C([0,+\infty);\mathcal{F}^s_k)$ and an estimate similar to that in (34) holds.

Proof: For simplicity's sake, we write $u(t) = u^\mu(t) = E_\mu(t)\phi$.
(a)Using (27) we have that:

$$\hat{u}_\xi = (-t(isgn(\xi)\eta + 2\mu\xi)\hat{\phi} + \hat{\phi}_\xi)G_\mu \tag{36}$$

$$\hat{u}_\eta = (-t(i|\xi| + 2\mu\eta)\hat{\phi} + \hat{\phi}_\eta)G_\mu. \tag{37}$$

Then (34) follows using Plancherel and the Leibniz rule.

(b) To fix ideas take $k = 2$, $\phi \in \mathcal{F}_2^s$, $s \geq 2$. Now compute one more derivative:

$$\begin{aligned} \hat{u}_{\xi\xi} &= [\hat{\phi}(t^2(-\eta^2 + 4\mu^2\xi^2 + 4i\mu|\xi|\eta) - 2t\mu) - \\ &\quad \text{-}2\hat{\phi}_\xi t(isgn(\xi)\eta - 2\mu\xi) + \hat{\phi}_{\xi\xi}]G_\mu - 2it\delta(\xi)\eta\hat{\phi}(0,\eta)e^{-t\mu\eta^2} \end{aligned} \tag{38}$$

$$\hat{u}_{\eta\eta} = [\hat{\phi}(t^2(4\mu^2\eta^2 - \xi^2) - 2t\mu) - 2\hat{\phi}_\eta t(2\mu\eta + i|\xi|) + \hat{\phi}_{\eta\eta}]G_\mu. \tag{39}$$

Then by Plancherel's theorem and Proposition 2.1 one gets that $x^2 u \in L^2$ if and only if $\hat{\phi}(0,\eta) = 0$, $\forall \eta \in \mathbb{R}$. In such a case elementary calculations lead to the boundedness of $E_\mu(t)$ in $\mathcal{F}_2^s$. It is easy to see that $\hat{\phi}(0,\eta) = 0$, $\forall \eta \in \mathbb{R}$ if and only if $\int \phi(x,y)dx = 0$,$\forall y \in \mathbb{R}$.

For $k > 2$, one computes $\partial_\xi^k \hat{u}$ and $\partial_\eta^k \hat{u}$ with the help of Lemma 4.1. (35) is equivalent to the vanishing of the coefficients multiplying the delta function and its derivatives in the expression for these quantities. $\square$

The preceding theorem allows us to obtain a unique continuation principle.

Corollary 4.1 *Let $\phi \in \mathcal{S}(\mathbb{R}^2)$ and $\phi(.,y) \in C_0^\infty(\mathbb{R})$ for any $y \in \mathbb{R}$. Take $\mu \geq 0$ and suppose that there exist positive numbers R and t such that*

$$supp\, u(t,.,y) = supp\, (E_\mu(t)\phi)(.,y) \subset (-R,R),\ \forall y \in \mathbb{R}.$$

Then ϕ must vanish.

Proof: If the hypotheses hold, then $x^k u(t) \in L^2$, $\forall k \in \mathbb{Z}^+$, thus $\partial_\xi^k \hat{u}(t) \in L^2$ and then for the singular part in the expression of $\partial_\xi^k \hat{u}(t)$ to vanish, condition (35) must be satisfied for any $k \in \mathbb{Z}^+$. But then denoting $\mathcal{F}^{(x)}$ the Fourier transform in the variable x, we conclude that $\frac{d^k}{d\xi^k}\mathcal{F}^{(x)}\phi(.,y)(0) = 0,\ \forall k \in \mathbb{Z}^+$ and the result follows because the support of $\phi(.,y)$ is compact so that $\mathcal{F}^{(x)}\phi(.,y)$ is analytic for any $y \in \mathbb{R}$. $\square$

The following result deals with the persistence and gain of regularity of the solutions of (13)-(14) in some weighted spaces.

Proposition 4.1 *Let $s \geq 1,\ \lambda \geq 0,\ \gamma \in [0,1]$ and $\mu > 0$. Then for any $\phi \in \mathcal{F}^s_\gamma$, $E_\mu(t) \in \mathcal{B}(\mathcal{F}^s_\gamma, \mathcal{F}^{s+\lambda}_\gamma)$ for all $t > 0$, and satisfies the estimate:*

$$\|E_\mu(t)\phi\|_{s+\lambda,\gamma} \leq K_\gamma(t;\mu,\lambda)\|\phi\|_{s,\gamma},$$

with K_γ continuous. Moreover $E_\mu(.)\phi \in C((0,+\infty),\ \mathcal{F}^{s+\lambda}_\gamma)$.

Proof: By Lemma 3.2 it suffices to prove:

$$\|E_\mu(t)\phi\|_{0,\gamma} \leq Q_\gamma(t;\mu,\lambda)\|\phi\|_{0,\gamma}, \tag{40}$$

for any $\phi \in \mathcal{F}^s_\gamma$, Q_γ continuous. This inequality holds for $\gamma = 0$, again by Lemma 3.2.

Combining the Fourier transform and the relations (36)-(37) and (17) one gets (40) for $\gamma = 1$.

Thus, since $\mathcal{S} \subset \mathcal{F}^s_1$ and $\mathcal{S}$ is dense in L^2_1, $E_\mu(t)$ can be extended to a bounded operator in L^2_1 and the result follows for $\gamma \in (0,1)$ by Stein and Weiss's interpolation theorem on L^p spaces with varying measures (see [8]).$\square$

Regarding the nonlinear problem we have the following result:

Theorem 4.2 *Let $\phi \in H^s(w^2)$, $s > 2$ and w be a weight with all its first and second derivatives bounded and such that*

$$|w(x,y)| \leq c_\lambda e^{\lambda(x^2+y^2)},\ \forall (x,y) \in \mathbb{R}^2,\ \forall \lambda \in (0,\lambda^*), \tag{41}$$

for some $\lambda^* > 0$, $c_\lambda > 0$. Then the solution u^μ, $\mu \geq 0$ of Eqn.(12) with initial condition ϕ belongs to $C([0, T_s^\mu); H^s(w^2))$.

For the proof we will need the following lemma:

Lemma 4.2 *For w as in Theorem 4.2 set $w_\lambda(x,y) = w(x,y)e^{-\lambda(x^2+y^2)}$. Then there exist constants c_j, $j = 1, ..., 4$ independent of λ such that*

$$|\nabla w_\lambda|_\infty \leq c_1 |\nabla w|_\infty + c_2 \tag{42}$$

$$|D^\alpha w_\lambda|_\infty \leq c_3 |\nabla w|_\infty + |D^\alpha w|_\infty + c_4 \tag{43}$$

for any multindex $\alpha = (\alpha_1, \alpha_2)$ such that $|\alpha| = 2$.

Proof: It is easy to see that:

$$w_{\lambda,x}(x,y) = e^{-\lambda(x^2+y^2)}(w_x(x,y) - 2\lambda x w(x,y)).$$

Then straightforward calculations using Lemma 3.1 and

$$|w(x,y) - w(0,0)| \leq |\nabla w|_\infty \sqrt{x^2+y^2}, \ \forall (x,y) \in \mathbb{R}^2$$

lead to estimate (43). To prove (43) one may follow an analogous procedure using the expressions of the second derivatives of w_λ. $\Box$

Proof of Theorem 4.2.

In view of Theorem 3.1 it suffices to prove that $\|wu(t)\|$ remains bounded as long as $t \in [0, T]$ for any $T \in (0, T_s^\mu)$.

Define $w_\lambda(x,y) = w(x,y)e^{-\lambda(x^2+y^2)}$ and denote u^μ by u. Take a $T \in (0, T_s^\mu)$. By the hypotheses, $\|w_\lambda u(t)\| < \infty$ and $\|w_\lambda u_t(t)\| < \infty$, $\forall t \in (0, T]$

so we can multiply Eqn.(12) by w_λ and take the inner product in L^2 with $w_\lambda u$ to obtain:

$$\frac{d}{dt}\|w_\lambda u\|^2 = 2(w_\lambda u, w_\lambda H u_{xy} - w_\lambda u^p u_y + \mu w_\lambda \Delta u) \tag{44}$$

Notice that we have:

$$\begin{aligned}
(w_\lambda u, w_\lambda H u_{xy}) &= (w_\lambda u, [w_\lambda, H]\partial^2_{xy} u) + (w_\lambda u, H[w_\lambda, \partial^2_{xy}]u) \\
(w_\lambda u, w_\lambda \Delta u) &= (w_\lambda u, [w_\lambda, \Delta]u) - \|\nabla w_\lambda u\|^2 \\
\|w_\lambda u^p u_y\| &\leq |u^{p-1} u_y|_\infty \|w_\lambda u\| \leq c_s M_s^p \|w_\lambda u\|
\end{aligned}$$

where $M_s = M_s(T) = \max_{t\in[0,T]} \|u(t)\|_s$. Then by Lemma 4.2 and (11) one obtains:

$$\frac{d}{dt}\|w_\lambda u\|^2 \leq (C_1 M_s)^2 + (1 + c_s M_s^p)\|w_\lambda u\|^2 \tag{45}$$

and Gronwall's inequality implies

$$\|w_\lambda u\|^2 \leq e^{(1+c_s M_s^p)T}(\|w_\lambda \phi\|^2 + T(C_1 M_s)^2).$$

The Monotone Convergence Theorem yields:

$$\|wu\|^2 \leq e^{(1+c_s M_s^p)T}(\|w\phi\|^2 + T(C_1 M_s)^2). \tag{46}$$

Thus, we have proved that $u(t) \in H^s(w^2)$, $\forall t \in (0, T_s^\mu)$. The continuity can be derived from analogous estimates.$\Box$

From the local existence result one may derive the continuous dependence in weighted Sobolev spaces.

Corollary 4.2 *Under the assumptions in the preceding theorem the continuous dependence of solutions of Eqn.(12) also holds in the spaces* $H^s(w^2)$.

Proof: For $\mu \geq 0$, denote u, v two solutions of Eqn.(12) with (possibly different) initial values ϕ, ψ in $H^s(w^2)$, $s > 2$ and set $z = v - u$. By the previous theorem, they remain in $H^s(w^2)$ as long as they do in H^s. Let $T > 0$ be such that both solutions exist for $t \in [0, T]$. Then z satisfies:

$$z_t - Hz_{xy} + v^p z_y + (v^p - u^p)u_y = \mu \Delta z \tag{47}$$

Proceeding as in the previous proof:

$$\begin{aligned} \frac{d}{dt}\|wz\|^2 &= 2(wz, wHz_{xy} - wv^p z_y - w(v^p - u^p)u_y + \mu w \Delta z) && (48) \\ &\leq c\|z\|_s^2 + c(\|v\|_s^p + q(\|u\|_s, \|v\|_s)\|u\|_s)\|wz\|^2, && (49) \end{aligned}$$

where $q(m,n) = m^{p-1} + m^{p-2}q + \cdots + n^{p-1}$, $n, m > 0$.

Suppose now that there exists a sequence $\{\phi_n\} \in H^s(w^2)$ such that $\phi_n \to \phi$ in $H^s(w^2)$. Then by the continuous dependence in H^s, for any $T \in (0, T_s^\mu)$, we can find N such that the solutions u_n associated to ϕ_n are defined in $[0, T]$ for $n \geq N$ and $\sup_{t\in[0,T]} \|u_n(t) - u(t)\|_s \to 0$ as $n \to +\infty$. Let $K > 0$ be such that $\sup_{t\in[0,T]} \|u(t)\|_s \leq K$ and $\sup_{t\in[0,T]} \|u_n(t)\|_s \leq K$, $\forall n \geq N$. Then for any $\epsilon > 0$, (49) implies

$$\frac{d}{dt}\|w(u_n - u)\|^2 \leq \epsilon + 2cK^p\|w(u_n - u)\|^2. \tag{50}$$

For n large enough. By Gronwall's inequality we have:

$$\|w(u_n - u)\|^2 \leq e^{2cK^pT}\|w(\phi_n - \phi)\|^2 + \epsilon T e^{2cK^pT} \tag{51}$$

so the result follows. $\square$

Observe that the weights $w_\gamma(x,y) = (1 + x^2 + y^2)^{\gamma/2}$, $\gamma \in [0,1]$ satisfy the hypotheses of the preceding two theorems. Then we have local well-posedness for Eqn.(12) in the spaces $\mathcal{F}_\gamma^s$ for $\gamma \in [0,1]$, $s > 2$. This result is analogous to that obtained in [18] for BO. Nevertheless, $\nabla w_\gamma \in L^\infty$ if and only if $\gamma \leq 1$, so that the theorems just mentioned do not allow application to stronger weights. More than that, **there is no persistence** if $\gamma \geq 2$ and p is odd.

Theorem 4.3 *Let $s > 2$ and p odd. Then if $u \in C([0,T];\mathcal{F}^s_{2,0})$ is the solution of Eqn.(12), with $\mu \geq 0$, corresponding to an initial condition $\phi \in \mathcal{F}^s_{2,0}$, u must vanish identically.*

Proof: Suppose that u satisfies the hypothesis of the theorem. It is easy to see that its Fourier transform $\hat{u}$ must be continuous in the variable ξ.

After multiplying by x^2 in Eqn.(12) we obtain

$$\partial_t(x^2 u) = x^2 \mu \Delta u - x^2 H u_{xy} - x^2 u^p u_y \tag{52}$$

and since

$$\|x^2 u^p u_y\| \leq |u^{p-1} u_y|_\infty \|x^2 u\| \leq c\|u\|_s^p \|x^2 u\|,$$

then $x^2 u^p u_y \in L^2$.

Analogously it can be proved that this term belongs to $C([0,T];L^2)$. Taking the Fourier transform in Eqn.(52) and integrating with respect to t we conclude that for any $t \in (0,T]$, the function $(\xi,\eta) \mapsto \delta(\xi)\eta \int_0^t \hat{u}(\tau,\xi,\eta)d\tau$ is measurable. Then we must have:

$$\int_0^t \hat{u}(\tau,0,\cdot)d\tau = 0, \ \ \forall t \in [0,T],$$

and this in turn implies $\hat{u}(t,0,.) = 0, \ \ \forall t \in [0,T]$. By the boundedness of the Fourier transform one gets, for $t \in [0,T]$, $\xi \in \mathbb{R}$ and $a.e.$ $\eta \in \mathbb{R}$:

$$\hat{u}(t,\xi,\eta) \ = \ \hat{u}(0,\xi,\eta) + \int_0^t [Hu_{xy} - u^p u_y + \mu\Delta u]\hat{}(\tau,\xi,\eta)d\tau$$

$$= \quad \hat{u}(0,\xi,\eta) + \int_0^t [i|\xi|\eta\hat{u} - i\frac{\eta}{p+1}\widehat{u^{p+1}} - \mu(\xi^2+\eta^2)\hat{u}](\tau,\xi,\eta)d\tau.$$

Evaluating at $\xi = 0$, we obtain

$$\int_0^t \widehat{u^{p+1}}(\tau,0,\eta)d\tau = 0,\ \forall t \in [0,T],\ \eta - a.e.$$

Then it is easy to see that $\int u^{p+1}(t,x,y)dx = 0,\ \forall y \in \mathbb{R},\ \forall t \in [0,T]$ and the result follows at once. $\square$

Remark 4.1 *For the values $1 < \gamma < 2$ we do not know how to prove whether persistence holds in the spaces $\mathcal{F}^s_\gamma,\ s > 2$.*

As for the linear equation we have a unique continuation's result.

Corollary 4.3 *Let $u \in C([0,T];H^s),\ s > 2$, be a solution of Eqn.(12), with $\mu \geq 0$ and p odd. If there exists $R > 0$ such that*

$$supp\, u(t,.,y) \subset (-R,R),\ \forall y \in \mathbb{R}$$

and for each $t \in [t_1,t_2],\ t_1 < t_2,\ t_1,t_2 \in [0,T]$, then u must vanish identically in the interval $[t_1,t_2]$.

Proof: For the proof it suffices to observe that if u satisfies the hypotheses of the corollary, then $u(t)$ must belong to the space $\mathcal{F}^s_{2,0},\ s > 2$ for $t \in [t_1,t_2]$. Then apply Theorem 4.3 using $u(t_1)$ as initial data. $\square$

4.2 Anisotropic weights

The difficulties with the persistence in $\mathcal{F}^s_2$ are closely connected with the appearence of the delta function $\delta(\xi)$. One could then use nonisotropic weights to analyse the behavior of the solutions considering higher decay rates only in the variable y. Thus, we will investigate in this section the IVP's associated to Eqn.(12) and Eqn.(13) in the spaces $\mathcal{F}^s_{1,k}$, $k \in \mathbb{N}$, $s \geq 0$.

For the linear problem we obtain:

Theorem 4.4 *Let $k \in \mathbb{N}$, $s \geq k$, $t \geq 0$, $\mu \geq 0$, and take $\phi \in \mathcal{F}^s_{1,k}$. Then $E_\mu(t) \in \mathcal{B}(\mathcal{F}^s_{1,k})$ and satisfies the estimate:*

$$\|E_\mu(t)\phi\|_{\mathcal{F}^s_{1,k}} \leq p_k(t;\mu)\|\phi\|_{\mathcal{F}^s_{1,k}}, \tag{53}$$

where $p_k(t;\mu)$ is a polynomial of k-th degree in t with positive coefficients depending only on μ. Moreover, $E_\mu(.)\phi \in C([0,+\infty);\mathcal{F}^s_{1,k})$.

Proof: By virtue of Proposition 2.2, Lemma 4.1 and the Leibniz rule one gets:

$$\begin{aligned}
\|y^k E_\mu(t)\phi\| &\leq \sum_{l=0}^{k} |c_l| \| \sum_{j=0}^{[l/2]} t^{l-j} h_{l,l-2j}(\mu,|\xi|,\eta)\partial_\eta^{(k-l)}\hat{\phi}\| \\
&\leq \sum_{l=0}^{k} c_{l,\mu} \sum_{j=0}^{[l/2]} t^{l-j} \|(\xi^{l-2j}+\eta^{l-2j})\partial_\eta^{(k-l)}\hat{\phi}\| \\
&\leq (\sum_{l=0}^{k} d_{l,\mu}(1+t^l))\|\phi\|_{\mathcal{F}^k_{1,k}}.
\end{aligned}$$

(53) follows from the above expression. The continuity of $E_\mu(\cdot)\phi$ can be proved making similar calculations. □

The following lemma will be very useful:

Lemma 4.3 *Let $k \in \mathbb{N}$, $\mu > 0$, $t > 0$ and α multindex, then:*

$$\|D^\alpha E_\mu(t)\phi\|_{\mathcal{F}^0_{0,k}} \leq c(\mu,|\alpha|)(t^{-|\alpha|/2} + t^{k-|\alpha|/2})\|\phi\|_{\mathcal{F}^0_{0,k}},\ \forall\phi \in \mathcal{F}^0_{0,k}. \tag{54}$$

Proof: Using Leibniz's rule we obtain:

$$\|y^k D^\alpha E_\mu(t)\phi\| \leq \sum_{j=0}^{\min\{k,\alpha_2\}} c_{\alpha_2,j}\|(\xi^2+\eta^2)^{(|\alpha|-j)/2}\partial_\eta^{(k-j)}(G_\mu(t,\xi,\eta)\hat{\phi})\|, \quad (55)$$

where $\alpha = (\alpha_1, \alpha_2)$. Set $Z = \|(\xi^2+\eta^2)^{(|\alpha|-j)/2}\partial_\eta^{(k-j)}(G_\mu(t,\xi,\eta)\hat{\phi})\|$. Then by Lemmas 3.1 and 4.1 we get:

$$\begin{aligned}
Z &\leq \|(\xi^2+\eta^2)^{(|\alpha|-j)/2}\sum_{m=0}^{k-j}\sum_{l=0}^{[m/2]} c_{m,l,\mu} t^{m-l}(\xi^2+\eta^2)^{\frac{m-2l}{2}}|G_\mu(t,\xi,\eta)\partial_\eta^{(k-j-m)}\hat{\phi}|\| \\
&\leq \sum_{m=0}^{k-j} c_{m,\mu}\sum_{l=0}^{[m/2]} t^{m-l}\sup_{(\xi,\eta)\in\mathbb{R}^2}\{(\xi^2+\eta^2)^{\frac{|\alpha|-j+m-2l}{2}}e^{-\mu t(\xi^2+\eta^2)}\}\|\partial_\eta^{(k-j-m)}\hat{\phi}\| \\
&\leq \sum_{m=0}^{k-j} c(m,\mu,|\alpha|)\sum_{l=0}^{[m/2]} t^{\frac{m+j-|\alpha|}{2}}\|\partial_\eta^{(k-j-m)}\hat{\phi}\| \\
&\leq c(\mu,|\alpha|)(t^{-|\alpha|/2}+t^{k-|\alpha|/2})\|(1+y^{2k})^{1/2}\phi\|.
\end{aligned}$$

Taking this expression into (55), the result follows.□

Theorem 4.5 *Let $k \in \mathbb{N}$, $s \geq 0$, $\lambda \geq 0$, $\mu > 0$ and $t > 0$. Then for any $\phi \in \mathcal{F}^s_{1,k}$:*

a)$E_\mu(t) \in \mathcal{B}(\mathcal{F}^s_{1,k}, \mathcal{F}^{s+\lambda}_{1,k})$ and satisfies the estimate:

$$\|E_\mu(t)\phi\|_{\mathcal{F}^{s+\lambda}_{1,k}} \leq K(t;\mu,\lambda)\|\phi\|_{\mathcal{F}^s_{1,k}} \quad (56)$$

where K is a continuous function such that $K \sim O(t^{-\lambda/2})$ as $t \to 0^+$. Moreover, $E_\mu(\cdot)\phi \in C((0,+\infty);\mathcal{F}^{s+\lambda}_{1,k})$.

b)For any multindex α, $D^\alpha E_\mu(t) \in \mathcal{B}(\mathcal{F}^s_{1,k})$ and the following inequality holds:

$$\|D^\alpha E_\mu(t)\phi\|_{\mathcal{F}^s_{1,k}} \leq Q(t;\mu,|\alpha|)\|\phi\|_{\mathcal{F}^s_{1,k}}, \quad (57)$$

where Q is continuous, $Q \sim O(t^{-|\alpha|/2})$ as $t \to 0^+$. It is also true that $D^\alpha E_\mu(\cdot)\phi \in C((0,+\infty);\mathcal{F}^s_{1,k})$.

Proof: The results are direct consequences of Lemmas 3.2 and 4.3 and Theorem 4.1.□

The next theorem shows the persistence of the solutions of Eqn.(12) in the spaces $\mathcal{F}^k_{1,k}$.

Theorem 4.6 *Let $u^\mu \in C([0,T^\mu_k); H^k)$, $k \in \mathbb{N}$, $k \geq 3$ be the maximal solution of Eqn.(12), $\mu \geq 0$ with initial data $\phi \in \mathcal{F}^k_{1,k}$. Then $u^\mu \in C([0,T^\mu_k); \mathcal{F}^k_{1,k})$.*

Proof: Take $\mu > 0$ and write $u^\mu = u$ for simplicity. Similarly as in the proof of the Theorem 3.1, using Proposition 2.3 and Theorem 4.5 one obtains a local solution to Eqn.(12) in the complete metric space

$$\mathcal{H}_k(T^*) = \{f \in C([0,T^*]; \mathcal{F}^k_{1,k}) : \sup_{t\in[0,T^*]} \|f(t) - E_\mu(t)\phi\|_{\mathcal{F}^k_{1,k}} \leq \|\phi\|_{\mathcal{F}^k_{1,k}}\},$$

(analogous to $\mathcal{X}_k(\tilde{T}^\mu)$), for some $T^* > 0$. By the uniqueness in H^k, this solution must be u and by Theorem 4.2 it only remains to prove that it belongs to $\mathcal{F}^0_{0,k}$ for $t \in [0,T^\mu_k)$. It suffices to estimate $\|y^k u(t)\|$, $\forall t \in [0,T^*]$.

Note that $y^k u^p u_y \in L^2$ since

$$\|y^k u^p u_y\| \leq c(k)\|u\|^p_k \|y^k u\| \leq c(k) M^p_k \|y^k u\|,$$

where $M_k = \sup_{t\in[0,T^*]} \|u(t)\|_k$.

Using the integral equation (23) and Lemma 4.3 one can show that for $|\alpha| = 2$, $y^k D^\alpha u(t) \in L^2$,$\forall t \in (0,T^*]$. Thus we have that $y^k u_t(t) \in L^2$,$\forall t \in (0,T^*]$.

Using Proposition 2.2 and integration by parts, one obtains:

$$\partial_t \|y^k u\|^2 \leq d_1(\mu,k)\|u\|^2_k + d_2(k)\|y^k u\|^2 \tag{58}$$

where d_1 is monotone increasing in μ. By Gronwall's inequality:

$$\|y^k u(t)\|^2 \leq e^{d_2(k)T^*}(\|y^k\phi\|^2 + T^* d_1(\mu, k) M_k^2). \tag{59}$$

That allows us to extend the solution u to its interval of existence in H^k.

For the case $\mu = 0$, observe that by (59) and Theorem 3.1, there exists $T > 0$ such that

$$\|y^k u^\mu(t)\|^2 \leq K(T, k, \mu^*, \|\phi\|_{\mathcal{F}^k_{1,k}}), \tag{60}$$

for any $\mu \in (0, \mu^*)$, $\mu^* > 0$ arbitrary and for any $t \in [0, T]$. Then for any $t \in [0, T]$, there exists a sequence μ_n such that u^{μ_n} converges weakly in $L^2(w^2)$, with $w(x, y) = y^k$. But we know that $u^\mu \to u^0$ in L^2 uniformly in $[0, T]$, and then (59) follows also for $\mu = 0$ by the weak lower semicontinuity of the norm of $L^2(w^2)$ and an extension argument yields the result. □

5 Long time behavior of small solutions and non-linear data scattering

We will now investigate for which values of p Eqn.(12) is globally well-posed for small initial data.

For convenience denote $E_0(t)$ by $E(t)$. For any $f \in \mathcal{S}(\mathbb{R}^2)$:

$$E(t)f(x, y) = \frac{1}{2\pi}\int e^{i(|\xi|\eta t + x\xi + y\eta)}\hat{f}(\xi, \eta)d\xi d\eta = \frac{1}{2\pi}I(t) * f(x, y) \tag{61}$$

where $I(t) = (e^{i|\xi|\eta t})\check{}$. We can compute the kernel $I(t)$ explicitly:

Lemma 5.1 *For any real numbers x, y, t, we have:*

$$I(t)(x, y) = \begin{cases} \frac{2}{t}\cos(\frac{xy}{t})\chi_{\{yt \leq 0\}} & \text{if } t \neq 0 \\ 2\pi\delta(x, y) & \text{if } t = 0. \end{cases}$$

Proof: Set $f(\xi,\eta) = e^{i\xi\eta}\chi_{\{\xi\geq 0\}}$ and note that $e^{i|\xi|\eta} = f(\xi,\eta) + f(-\xi,\eta)$. Straightforward computations show that f satisfies:

$$
\begin{aligned}
f_\xi &= i\eta f + \delta_\xi \otimes \mathbf{1}_\eta && (62)\\
f_\eta &= i\xi f && (63)
\end{aligned}
$$

in $\mathcal{S}'$. Using the properties of the Fourier Transform, one sees that if $g = \check{f}$ then:

$$
\begin{aligned}
g_y &= -ixg - \mathbf{1}_x \otimes \delta_y && (64)\\
g_x &= -iyg. && (65)
\end{aligned}
$$

By linearity, it is easy to see that the solutions have the form $g = g_0 + ce^{-ixy}$, for $c \in \mathbb{R}$, g_0 being any particular solution. Since $g_0(x,y) = e^{-ixy}\chi_{\{y\leq 0\}}$ satisfies (64)-(65) then $\check{f}(x,y) = e^{-ixy}\chi_{\{y\leq 0\}} + ce^{-ixy}$ for some $c \in \mathbb{R}$ and

$$I(t)(x,y) = \frac{2}{t}\cos\left(\frac{xy}{t}\right)\left(\chi_{\{yt\leq 0\}} + c\right),$$

for $t \neq 0$.

The constant c is determined by the fact that $I(t) \to 2\pi\delta$ in $\mathcal{S}'$ as $t \to 0$. Indeed, some calculations using the Fourier Transform in one variable show that:

$$\lim_{t\to 0} < I(t), \phi >= 2\pi\phi(0,0) + 4\pi c\phi(0,0).$$

Then $c = 0$ and the result is proved. □

The expression just derived allows us to obtain the following $L^p - L^q$ estimate for the semigroup $E(t)$:

Proposition 5.1 *For any $f \in L^1 \cap L^2$ we have*

$$|E(t)f|_{\frac{2}{1-\theta}} \leq c|t|^{-\theta}|f|_{\frac{2}{1+\theta}} \qquad (66)$$

where $\theta \in [0,1]$.

Proof: The result follows from Young's inequality for the convolution (see [8]), Lemma 5.1 and interpolation. $\square$

Using the Sobolev's Embbeding Theorem one gets:

Proposition 5.2 *For* $s > 1$ *and* $f \in L^1 \cap H^s$ *we have:*

$$|E(t)f|_\infty \leq c(1+|t|)^{-1}(|f|_1 + \|f\|_s) \tag{67}$$

Now we present the main result of the section.

Theorem 5.1 *For* $p \geq 3$ *and* $s > 3$ *there exist* $\delta > 0$ *and* $R = R(\delta) > 0$ *such that if* $\phi \in L^1_1 \cap H^s$ *with*

$$|\phi|_{1,1} + \|\phi\|_s < \delta,$$

then Eqn.(4) has a unique global solution u *in the class* $C_b(\mathbb{R}, H^s)$ *with*

$$\sup_{t\in\mathbb{R}}(1+|t|)|u(t)|_{1,\infty} \leq R. \tag{68}$$

For the proof we will need the following lemma:

Lemma 5.2 *For* $t \geq 0$ *let*

$$J(t) = (1+t)\int_0^t \frac{1}{(1+t-\tau)(1+\tau)^{p-1}}d\tau.$$

Then we have:

i) $J(t) = O(1)$ *as* $t \sim +\infty$ *for* $p \geq 3$*;*

ii) $J(t) \to +\infty$ *as* $t \to +\infty$ *for* $p = 1, 2$.

Proof: One can compute the integral explicitly obtaining:

$$\frac{J(t)}{1+t} = 2(2+t)^{1-p}\ln(1+t) + \sum_{j=2}^{p-1} \frac{1}{1-j}(2+t)^{j-p}\left(\frac{1}{(1+t)^{j-1}} - 1\right).$$

From this expression the result follows easily.□

Proof of Theorem 5.1.

Observe that by virtue of (20) it suffices to show that (68) holds. Indeed, in such a case and using the hypotheses we have:

$$\int_0^t |u_y|_\infty |u|_\infty^{p-1} d\tau \leq \int_0^t |u|_{1,\infty}^p d\tau \leq R^p \int_0^t (1+|t|)^{-p} d\tau \leq const. \qquad (69)$$

To prove (68) take $T \in (0, T_s)$. Let us denote $K(T) = \sup_{t\in[0,T]}(1 + |t|)|u(t)|_{1,\infty}$. The hypotheses of the theorem allow us to use Proposition 5.2, Lemma 5.2 and (20) in the integral equation (23), obtaining:

$$\begin{aligned}(1+t)|u(t)|_{1,\infty} &\leq c\delta + c(1+t)\int_0^t (1+t-\tau)^{-1}|u(\tau)|_\infty^{p-1}\|u(\tau)\|_s^2 d\tau \\ &\leq c\delta + c\delta^2 K(T)^{p-1} e^{cK(t)^p},\end{aligned}$$

where $t \in [0, T]$.

One can choose $\delta > 0$ small enough such that the function $x \mapsto c\delta + c\delta^2 x^{p-1} e^{cx^p} - x$ has a positive zero. Let $R = R(\delta)$ be the first positive zero of this function. Then the estimates just obtained imply that $K(T) \leq R$. Using again that the set of solutions is invariant under the transformation $(t, x, y) \mapsto (-t, x, -y)$ and an extension argument the theorem follows. □

It turns out that the global solutions just found are also scattering states.

Theorem 5.2 *Under the assumptions of the preceding theorem there exist $\phi_\pm \in H^s$ such that*

$$\|u(t) - E(t)\phi_\pm\|_r \to 0$$

when $t \to \pm\infty$, *for* $r \in [s-1, s)$.

Proof: The proof uses standard arguments (see [29], [33], ([34], Chapter 6)). $\square$

6 Solitary waves

If we look for travelling wave-type solutions of Eqn.(4) in the form:

$$u(x,y,t) = v(x-bt, y-ct), \quad b \geq 0, \;\; c \geq 0, \;\; b^2+c^2 > 0,$$

then the function v must obey

$$-bv_x - cv_y - Hv_{xy} + v^p v_y = 0. \tag{70}$$

Using the Fourier trasform it is easy to see that it can not have nonvanishing integrable solutions.

Put $b = 0$. The equation to analyse in this case is the following:

$$-cv_y - Hv_{xy} + v^p v_y = 0. \tag{71}$$

Assuming that v and Hv_x vanish at infinity, we get:

$$-cv - Hv_x + \frac{1}{p+1} v^{p+1} = 0. \tag{72}$$

For any $\phi \in H^{1/2}$ consider the functional:

$$\mathcal{L}(\phi) = c\int \phi^2 dxdy + \int \phi H \phi_x dxdy - \frac{2}{(p+1)(p+2)} \int \phi^{p+2} dxdy. \tag{73}$$

By the Sobolev's Embedding Theorem (see [8]):

$$H^{1/2}(\mathbb{R}^n) \subset L^{p+2}(\mathbb{R}^n), \; \forall p \in [0, 2/(n-1)], \; n > 1,$$

one can only guarantee that $\mathcal{L}$ is well defined and continuous for $p \in \{1,2\}$ (we are considering $p \in \mathbb{N}$). Notice that the solutions of Eqn.(72) are the critical points of this functional. We have:

Theorem 6.1 *For any $p \in \{1,2\}$, Eqn.(72) has no nonvanishing solutions in $H^{1/2}$.*

Proof: Consider first $p = 1$. In [6] it was proved that any nonvanishing solution for:

$$-f(x) - Hf_x(x) + f^2(x) = 0,$$

which satisfies the boundary condition $f(x) \to 0$ as $|x| \to 0$, must be of the form:

$$f_a(x) = \frac{2}{1+(x-a)^2},\ a \in \mathbb{R}.$$

Using this result one gets, after a simple change of variables, that the desired solution of Eqn.(72) with $p = 1$ must have the form:

$$v\,(x,y) = \frac{4c}{1+c^2\,(x+g\,(y))^2} \tag{74}$$

for some function g. Since for any $c > 0$ the function defined by (74) is not square integrable, then the result follows for the case $p = 1$.

To study the case $p = 2$, we shall use a technique presented in [32] for obtaining Pohozaev identities.

Suppose that $v \in H^{1/2}$ is a solution for Eqn.(72). For any $\lambda > 0$ set $v_\lambda(x,y) = \lambda^{1/2} v(\lambda x, \lambda y)$. It defines a curve in $H^{1/2}$ and we have

$$\mathcal{L}(v_\lambda) = \lambda^{-1} c \int v^2 dxdy + \int vHv_x dxdy - \frac{1}{3}\int v^4 dxdy. \tag{75}$$

Since v is a critical point for $\mathcal{L}$ and $v_1 = v$, we must have that

$$\frac{d}{d\lambda}\Big|_{\lambda=1}\mathcal{L}(v_\lambda) = \left\langle \mathcal{L}'(v), \frac{d}{d\lambda}\Big|_{\lambda=1} v_\lambda \right\rangle = 0.$$

On the other hand, from (73) we obtain,

$$\frac{d}{d\lambda}\Big|_{\lambda=1}\mathcal{L}(v_\lambda) = -c\int v^2 dxdy.$$

Thus, the solution must vanish. $\square$

Remark 6.1 *Observe that the result for $p = 1$ does not contradict the existence result in [37], since the operator $H\partial_x$ is degenerate in $\mathbb{R}^2$, that is, there is no positive constant c such that $|(H\partial_x f)\hat{}(\xi,\eta)| \geq c\sqrt{\xi^2+\eta^2}|\hat{f}(\xi,\eta)|$.*

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