Dynamical Systems: Moving into the Next Century *

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It was Voltaire, I think, who said that "even the most skillful future-teller cannot always be wrong." A long list of, old and new, misguided prophecies about natural phenomena or the evolution of human societies, leaves no doubt about his point...

Anticipating the evolution of a scientific field should be a somewhat less hazardous goal, though. For scientific knowledge has a kind of internal coherence, new challenges often originating from the solution of older ones. Which is, perhaps, even truer for Mathematics, a science that has kept a remarkable unity throughout its history. Hilbert's famous speech at the 1900 International Congress of Mathematicians did go a long way into predicting and, to a substantial extent, influencing main directions of mathematical research in the twentieth century. However, he could not foresee the birth and extraordinary development of Dynamical Systems, even if two of his problems (16th and 21st) were related to it.

Poincaré, whose fundamental work in Celestial Mechanics was founding Dynamics as a mathematical discipline, of course knew that *cette étude aura par elle-même un intérêt du premier ordre.* His legacy was taken over a few decades later by Birkhoff, who also clarified important issues raised by Boltzmann and Maxwell. By the middle of the century, Kolmogorov, Arnold, and Moser were settling a major problem going back to Laplace and Leverrier, not to mention Newton, and precisely formulated by Poincaré: convergence of the Lindsted series obtained by formal solution of the gravitation equations. The implications went well beyond the problem of the stability of the Solar system, that was the initial motivation. Kolmogorov, Arnold, Moser theory remained, to present days, one of the most active areas in Dynamical Systems, through major contributions from Herman, Mather, Rüssman, Zehnder, and many other mathematicians.

Gradient-like systems and, short afterwards, the "horseshoe" were the first attempts to provide a general paradigm, valid for most dynamical systems. Introduced by Smale at the beginning of the sixties, and developed through the work of his students and collaborators, as well as of mathematicians in the Soviet Union, the theory of uniformly hyperbolic systems also aimed at characterizing the notion of structural stability, proposed by Andronov and Pontryagin in the thirties. This objective was achieved (essentially, stability is tantamount to uniform hyperbolicity),

* Work written while visiting Princeton University. The author is partially supported by Pronex-Dynamical Systems and Faperj, Brazil.

for systems that are once differentiable, the final outstanding parts of the proof being provided by Mañé in the mid-eighties, and by Hayashi some ten years later.

Hyperbolicity was a main idea in Anosov's solution of the classical problem of ergodicity for the geodesic flow of manifolds with negative curvature; the surface case had been done by Hedlund and Hopf. Furthermore, bringing ideas from statistical mechanics into play in the domain of smooth dynamics, Sinai, Ruelle, and Bowen proved that uniformly hyperbolic systems admit a very precise description in statistical terms (typical evolution of most initial states), notwithstanding the great complexity they may exhibit at the level of individual orbits.

On the other hand, it was soon clear that uniform hyperbolicity is too rigid a property, that can not account for *most* dynamical systems: there are smooth maps and flows that are not even approximated by hyperbolic ones. A new, more general, paradigm was needed...

Subsequent progress occurred in several fronts: bifurcation theory, especially homoclinic phenomena (Newhouse, Palis, Takens, Yoccoz, Moreira); specific models of non-hyperbolic behavior, such as the Hénon maps (Benedicks, Carleson, Mora, Viana, Young) or the Lorenz flows (Afraimovich, Shilnikov, Guckenheimer, Williams); various extensions of the notion of hyperbolic system, especially Pesin's theory of non-uniform hyperbolicity (Pesin, Katok, Ledrappier, Young); low-dimensional systems, like maps of the interval (Jakobson, Milnor, Thurston, de Melo, van Strien, Mañé, Sullivan, Yoccoz, McMullen, Lyubich, Swiatek) or the complex sphere (Douady, Hubbard, Sad, and many of the previous).

As the twentieth century is drawing to an end, time seems ripe for a new attempt at developing a global theory, suitable for the majority of dynamical systems. A new view of such a theory has been emerging, and some very exciting progress is taking place right now. Let me tell you a bit about it, and some of the challenges we shall be facing along the way.

1 Basic set-up

I focus on two main models of evolution law (see the last section for brief comments on other models). The first one corresponds to transformations $f : M \to M$ on a space M, the points of which describe the different states of the system. The orbit of each $x_0 \in M$ is the sequence $(x_n)_n$ defined by $x_n = f(x_{n-1})$ for $n \geq 1$. Most of the time I assume that f is invertible, and then one also defines $x_n = f^{-1}(x_{n+1})$ for $n < 0$.

Another model are continuous-time flows $f^t : M \to M$, $t \in \mathbb{R}$, that is, one-parameter families of transformations satisfying $f^{t+s} = f^t \circ f^s$ for $t, s \in \mathbb{R}$, and $f^0 = id$. The orbit of $x_0 \in M$ is the curve $x_t = f^t(x_0)$, where $t \in \mathbb{R}$. Assuming the flow depends smoothly on time t, there is an associated vector field F on M, defined by

$$F(x) = \frac{d}{dt} f^t(x)\Big|_{t=0}.$$

Indeed, it will always be understood that M is a (compact) manifold, and the system is smooth, both in time and in space M.

In either setting, the overall objective is twofold:

- to describe the behavior of most orbits for most systems, specially when time goes to infinity
- to understand whether this behavior is stable under small modifications of the evolution of the law

I want to stress that description of *all* orbits or systems is not a realistic goal, in general, because there are too many forms of exceptional behavior. Also, concerning the second problem above, one should note that mathematical models are, really, only approximations to the phenomena they are meant to describe.

2 Attractors: finiteness

An *attractor* is a compact invariant set $\Lambda \subset M$ whose *basin of attraction*

$$B(\Lambda) := \{\text{points in } M \text{ whose forward orbits converge to } \Lambda\}$$

has positive Lebesgue probability (volume). As I shall comment upon later, it is possible for a dynamical system to have an infinite number of attractors, which renders the description of the dynamics rather difficult.

The following conjecture of Palis is at the basis of an ambitious program put forward in [44] to bypass this, and other stumbling blocks on the way towards an understanding of complex dynamical behavior: *Can any dynamical system (flow or diffeomorphism) be approximated by another having only a finite number of attractors, whose basins of attraction include almost every orbit ?* Moreover, these attractors should have nice ergodic properties, including existence of physical measures and stochastic stability (stability under small random noise). I shall return to this later.

The finiteness conjecture has been established by Lyubich [34] for real quadratic maps $x \mapsto a + x^2$, in a very strong form: for almost every parameter value of a there is a unique attractor, which is either periodic or "chaotic" (non-uniformly hyperbolic). There is an ongoing extension for rather general families of unimodal maps of the interval, by Lyubich, de Melo, and de Melo. Before that, Swiatek with the aid of Graczyk [26], and Lyubich [35], had shown that parameters corresponding to occurrence of a periodic attractor are dense. Recently, Kozlovsky [33] extended this last result for very general families of unimodal maps. On the other hand, a pioneer theorem of Jakobson [30] stated that non-uniformly hyperbolic dynamics corresponds to a set of parameters with positive Lebesgue probability.

There has also been some remarkable progress for systems in higher dimensions, part of which is touched upon in the next sections. Moreover, finiteness of attractors, and corresponding statistical properties, were proven for "general" dynamical systems with random noise [3].

3 Infinitely many sinks

Systems with an infinite number of attractors do exist: by Newhouse [43], coexistence of infinitely many attracting periodic orbits is generic (Baire second category) in certain open subsets of the space $\mathrm{Diff}^k(M^2)$ of C^k diffeomorphisms, for any surface M^2 and any $k \geq 2$. These open sets exhibit several other complicated phenomena, such as homoclinic tangencies [51], with all their dynamical consequences [45, Chapter 7], and also super-exponential growth of the number of periodic orbits [31].

Newhouse's results have been extended to arbitrary dimensions [46], to conservative systems [24], and to certain holomorphic maps [19]. A version for non-hyperbolic (Hénon-like) attractors was given in [20], and another mechanism yielding infinitely many sinks was described in [12], that applies also to $\mathrm{Diff}^1(M^d)$ for dimension $d \geq 3$.

Nevertheless, three decades after the initial results, this phenomenon remains essentially as little understood as ever. It is not even known whether coexistence of infinitely many periodic attractors may occur *robustly*, that is, for a whole open set of systems (not just a generic subset of it). But, in light of the conjectures mentioned before, one does not expect this to be possible. Even more: *Does coexistence of infinitely many attractors correspond to a zero measure set in parameter space* (zero Lebesgue measure on typical families with a finite or countable number of parameters) *?* In any case, *is there a symbolic description for systems with an infinite number of attractors ?*

4 Dynamical decompositions

The *limit set* of a diffeomorphism $f : M \to M$ is the set of accumulation points of all the orbits

$$L := \text{closure}\left(\{\lim_k f^{n_k}(x) : x \in M \text{ and } n_k \to \pm\infty\}\right).$$

The diffeomorphism is called *uniformly hyperbolic* (or Axiom A [42,54]) if L is a *uniformly hyperbolic set*, that is, if there exists a splitting of the tangent space over it $T_L M = E^u \oplus E^s$ into two sub-bundles E^u and E^s, such that the derivative of f preserves both sub-bundles, expanding E^u and contracting E^s, at uniform rates. In that case, by [54], the limit set may be broken into a finite number of *basic pieces*, two-by-two disjoint,

$$L = \Lambda_1 \cup \cdots \cup \Lambda_N , \tag{1}$$

each of which is compact, invariant, and *dynamically indecomposable:* it contains dense orbits. Since attractors are included among the basic pieces (not exclusively), existence of a finite number of attractors is an immediate consequence. *Is there a corresponding decomposition for very general (non-hyperbolic) systems ?*

There has been some very encouraging progress in this direction. The hint comes from the fact that each basic piece Λ_j in (1) is a *robust set*:

there are neighborhoods U of Λ_j and $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$, such that $\Lambda_j = \Lambda_f$, and Λ_g is dynamically indecomposable for every system $g \in \mathcal{U}$, where

$$\Lambda_g := \{x \in M : g^n(x) \in U \text{ for every } n \in \mathbb{Z}\}$$

is the maximal invariant set of g inside U. It turns out that we can say a great deal about robust sets of diffeomorphisms. If M is two-dimensional, robustness implies uniform hyperbolicity [36]. In general, Bonatti, Díaz, Pujals, Ures [13,21] prove that every robust set satisfies a weaker (but still uniform) property of hyperbolicity: dominated splitting, or even partial hyperbolicity.

A *dominated splitting* for a compact invariant set $\Lambda \subset M$ is a decomposition $T_\Lambda M = E^1 \oplus E^2$ of the tangent space, such that the derivative preserves both sub-bundles E^1 and E^2, and is more expanding/less contracting on the former than on the latter: there is $\sigma > 1$ such that

$$\|Df(x)v^1\| \geq \sigma \|Df(x)v^2\| \tag{2}$$

for every $x \in \Lambda$ and any norm-1 vectors $v^1 \in E^1$ and $v^2 \in E^2$. If Df either expands E^1 or contracts E^2, then we call Λ *partially hyperbolic*. One reason such properties are so important is that they yield very useful geometric information about the dynamics on Λ, like the existence of invariant foliations.

Corresponding results have also been proved for flows, at least in dimension 3. Robust sets that do not contain equilibrium points are hyperbolic [28]. Most interesting, by a result of Morales, Pacifico, and Pujals [41], robust sets containing equilibria are, necessarily, attractors or repellers, and they are partially hyperbolic of Lorenz type. This theory developed in [41] fits nicely with the conclusions of Tucker [55], who has recently proved the long standing conjecture that the famous equations of Lorenz contain a “strange” attractor.

In either case, discrete-time or continuous-time, a partially hyperbolic invariant set is not necessarily robust. *What are the appropriate additional conditions that ensure robustness ?* Related to this, Pujals, Sambarino have just shown that, for surface diffeomorphisms, invariant sets with a dominated splitting admit a dynamical decomposition into finitely many basic pieces.

Another related problem concerns the *shadowing property*. The classical statement, for uniformly hyperbolic systems [15], asserts that near any *pseudo-orbit*, that is, any sequence $(x_n)_n$ such that $\mathrm{dist}(f(x_n), x_{n+1})$ is small for all n, there exists a true orbit of the system. Such a strong statement can not be expected to hold in any reasonable generality outside the hyperbolic context, see [11,61]. On the other hand, properties of *finite-time shadowing for most orbits* are implicit in some important situations, e.g. in the proof of stochastic stability of non-hyperbolic systems such as the Hénon maps [7]. *Is there a useful shadowing lemma for very general non-uniformly hyperbolic systems ?*

5 Non-uniform hyperbolicity

In Pesin theory one assumes that an invariant probability μ has been fixed, one interesting case being μ = volume (Lebesgue measure). The values $\lambda_1(x), \ldots, \lambda_m(x)$ taken by

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n(x)v\|$$

when v runs over all non-zero tangent vectors at x are the *Lyapunov exponents* of f at $x \in M$. The limit exists for every $v \in T_xM \setminus \{0\}$ and μ-almost every $x \in M$, by a theorem of Oseledets. If μ is ergodic then the $\lambda_j(\cdot)$, $1 \le j \le m$, are constant over a full μ-measure set. Similar notions and facts hold for flows.

Non-uniform hyperbolicity, meaning non-zero Lyapunov exponents, ensures that the system shares some of the key features of uniformly hyperbolic systems, such as local stable and unstable manifolds that are smooth embedded disks [48]. In contrast, little is known about systems with zero exponents. *Are Lyapunov exponents of flows and diffeomorphisms, typically, non-zero ?*

The answer can not be unconditionally affirmative, not for conservative systems at least. Herman [60] constructed open sets of smooth volume-preserving maps admitting positive volume invariant sets that consist of invariant tori restricted to which the diffeomorphism acts as a rigid translation:

$$f(\theta_1, \ldots, \theta_d) = (\theta_1 + \omega_1, \ldots, \theta_d + \omega_d) \quad \text{for all } (\theta_1, \ldots, \theta_d). \tag{3}$$

For general (dissipative) systems it is not known whether Lyapunov exponents can be robustly zero. On the other hand, abundance of non-uniform hyperbolicity was proved in a few important models, such as the Hénon family [6], and its multidimensional counterparts in [58]. Recently, Dolgopyat proved genericity of non-zero Lyapunov exponents among volume-preserving, strongly partially hyperbolic diffeomorphisms in dimension 3 (a definition will appear later).

It is useful to place these problems in a more general setting, that of *linear co-cycles* over a map (or a flow). Here one considers, together with the transformation $f: M \to M$, a map $A: M \to SL(k, \mathbb{R})$. The n-th iterate of A is defined by $A^n(x) = A(f^n(x)) \cdots A(f(x))A(x)$, for all $n \ge 1$. Assuming f is invertible, one also defines $A^{-n}(x)$ as the inverse of $A^n(f^{-n}(x))$. The Lyapunov exponents of (f, A) at a point $x \in M$ are the values of

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(x)v\|$$

over all $v \in T_xM \setminus \{0\}$ (Oseledets theorem applies in this setting).

Several approaches have been devised for studying Lyapunov exponents in this framework, e.g. Furstenberg [25], Herman [29], and Kotani [32]. Roughly speaking, mild conditions are often sufficient to ensure that the exponents are non-zero. For instance, for products of random

$SL(2,\mathbb{R})$ matrices (i.i.d. with probability distribution ν) [25] requires only the non-existence of a probability measure in the projective space $\mathbb{RP}^1$ invariant under *every* matrix in the support of ν.

However, this should be contrasted with the following dichotomy that was recently proved by Bochi: generic (Baire second category) continuous $SL(2,\mathbb{R})$ co-cycles over any ergodic aperiodic transformation $f : (M,\mu) \to (M,\mu)$ either are *uniformly hyperbolic* or have *zero Lyapunov exponents* (aperiodicity just means that the periodic points of f are not a full μ-measure set). A corresponding statement for diffeomorphisms was claimed by Mañé several years ago, but no proof was ever available: *C^1-generic volume-preserving surface diffeomorphisms either are uniformly hyperbolic or else have zero Lyapunov exponents at Lebesgue almost every point ?* What about symplectic diffeomorphisms in any dimension ?

6 Physical measures

Given $f : M \to M$ and $x \in M$, the *time-average* of a function $\varphi : M \to \mathbb{R}$ on the orbit of x is

$$\lim_{n\to+\infty} \frac{1}{n+1} \left(\varphi(x) + \varphi(f(x)) + \cdots + \varphi(f^n(x))\right).$$

Assuming the limit exists for every continuous function φ, it defines a probability measure μ_x on the Borel σ-algebra of M: the integral of φ with respect to μ_x is just the time-average of φ on the orbit of x. This measure describes the asymptotic behavior of x in quantitative terms:

$$\mu_x(D) = \text{ average time the orbit of } x \text{ spends in } D,$$

for any measurable $D \subset M$ whose boundary has zero μ_x-measure. A probability μ is a *physical (or Sinai-Ruelle-Bowen) measure* for f if $\mu = \mu_x$ for a set of initial states x with positive Lebesgue measure. This set is denoted $B(\mu)$ and called the *basin* of μ. These notions extend to flows in a straightforward way.

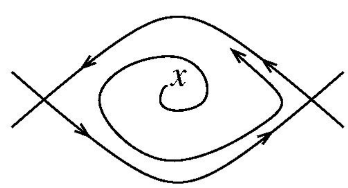

Fig. 1. A system without time-averages

The ergodic theorem of Birkhoff ensures that time-averages exist at almost every point, with respect to any finite invariant measure. This is especially meaningful for conservative systems. In general, *time-averages may fail to exist for large sets of initial states $x \in M$,* in the sense of Lebesgue measure. One such example, a planar flow with a double saddle connection, is depicted in Figure 1. *For most dynamical systems: Do time-averages converge Lebesgue almost everywhere ? Is almost every point in the basin of some physical measure ? Are the physical measures finitely many ?*

Proving existence or finiteness of physical measures is usually quite hard, but this has been achieved in a handful of very important cases. Besides the classical results of Sinai, Ruelle, and Bowen [16,52,53] for hyperbolic systems, let me mention Jakobson [30] for unimodal maps of the interval, Benedicks, Young [9] for the non-uniformly hyperbolic Hénon maps constructed by Benedicks, Carleson [6], and Alves, Bonatti, Viana [1,2,14] for robust classes of partially hyperbolic maps. Especially, [2] assumes only some general facts of non-uniform hyperbolicity (non-zero Lyapunov exponents), raising the conjecture: *Does non-uniform hyperbolicity at Lebesgue almost every point imply existence of physical measures ?*

7 Homoclinic phenomena

A recent result of Pujals, Sambarino [51] states that every surface diffeomorphism can be C^1-approximated (uniform approximation, both of the map and of the first derivative) by another which either is uniformly hyperbolic or has a homoclinic tangency. A *homoclinic tangency* is a non-transverse intersection between the stable manifold and the unstable manifold of some hyperbolic saddle. See Figure 2, which corresponds to a case where the stable and unstable manifolds also have transverse intersections. Thus, in 2 dimensions homoclinic tangencies are *the* obstruction to hyperbolicity, and an understanding of non-hyperbolic behavior must rely on understanding the forms of complicated dynamics occurring near homoclinic tangencies.

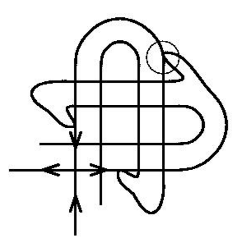

Fig. 2. A homoclinic tangency

There is an intimate relation between the phenomenon of homoclinic tangencies and the Hénon family of maps $f(x,y) = (1 - ax^2 + by, x)$. In fact, perturbations of the Hénon family model much of what goes on when a homoclinic tangency is unfolded along a parameterized family of surface diffeomorphisms; see [45, Chapter 3]. The proof, by Benedicks, Carleson [6], that Hénon maps have a non-uniformly hyperbolic attractor for a set of parameter values with positive Lebesgue measure opened the way to a very complete picture of the dynamics of a large class of non-hyperbolic systems. See [7–10], as well as [22,40,57] for an insertion of these results in the general context of bifurcations.

Germane to this, there is an important recent work of Palis, Yoccoz [47], proving that non-uniform hyperbolicity is prevalent in parameter space (relative Lebesgue measure close to 1) near homoclinic tangencies of surface diffeomorphisms. See Figure 2. They need an assumption about the fractal dimension of the hyperbolic set involved in the tangency, which is related to the fact that, as of today, the theory of Hénon-like maps is restricted to the strongly dissipative case. *Can this theory be extended to mildly dissipative diffeomorphisms ?*

Let me point out that, by [56], Newhouse's construction [43] of open subsets of $\mathrm{Diff}^k(M^2)$, $k \geq 2$, corresponding to non-hyperbolic dynamics does not apply to C^1 diffeomorphisms. In fact, it is not known whether hyperbolic (Axiom A) diffeomorphisms are dense in the space of C^1 diffeomorphisms on a surface (that is not true in higher dimensions, by a result of Abraham and Smale).

The following generalization of [51] to arbitrary dimension was also conjectured by Palis [44]: *Can any diffeomorphism be C^r-approximated, any $r \geq 1$, by another which either is hyperbolic or else has a homoclinic tangency or a heteroclinic cycle ?* By *heteroclinic cycle* one means a finite set of hyperbolic periodic points, cyclically related by intersections between stable and unstable manifolds, such that not all their stable manifolds have the same dimension. The proof of this conjecture is mostly open, even for $r = 1$.

C^r-approximation results are, very often, extremely hard when $r > 1$. For instance: *Given a recurrent point x of a diffeomorphism f, is there a C^r-nearby diffeomorphism for which x is a periodic point ?* Pugh's *C^1-closing lemma* [49] asserts that the answer is affirmative, if one takes $r = 1$. This was extended for flows, and some special classes of systems, like symplectic maps and Hamiltonian flows, in a joint work with Robinson. In contrast, little could be proved for $r > 1$ (even for $r = 1 + \epsilon$!), apart from partial results for flows on surfaces, by Peixoto and, recently, Gutierrez [27]. For instance, it is not known whether diffeomorphisms with some periodic point are dense in $\mathrm{Diff}^2(\mathbb{T}^2)$. But Herman [60] proved that the C^∞-closing lemma is, actually, false for symplectic maps and Hamiltonian flows on some manifolds. In the meantime, there were some remarkable improvements of Pugh's result, such as the *ergodic C^1-closing lemma* of Mañé [37], and the *C^1-connecting lemma* of Hayashi [28].

8 Conservative systems

The importance of elliptic dynamical behavior can not be overestimated, especially in the context of conservative systems, that is, symplectic (or just volume-preserving) maps and Hamiltonian flows. The theorem of Kolmogorov, Arnold, Moser for symplectic maps states that near every non-degenerate elliptic periodic point (all the eigenvalues have norm 1) there exist positive volume sets consisting of invariant tori, restricted to which the map acts as (3). In particular, in the presence of such an elliptic point the system can not be ergodic. In the special case when the ambient manifold M is 2-dimensional, the tori are Jordan curves around the elliptic point.

Important results, notably by Herman and Rüssmann [60], have considerably deepened the scope of this theory. In 2-dimensions, Zehnder [62] proved that hyperbolic behavior is present in the regions between KAM curves, in the form of transverse homoclinic intersections associated to hyperbolic periodic points. And the theory of Aubry-Mather sets [5,38] completed a remarkably rich picture of the dynamics near the elliptic point, providing a substitute for the "missing" curves.

In global terms, the balance between elliptic and hyperbolic dynamics is still far from understood. For the *standard family* of area-preserving maps of the 2-torus

$$f_\kappa(x, y) = (-y + 2x + \kappa \sin(2\pi x), x) \mod \mathbb{Z}^2,$$

it has been shown by Duarte [23] that elliptic points and KAM invariant curves are abundant for generic (Baire second category) sufficiently large parameters $\kappa \in \mathbb{R}$. However, it is widely believed that the upper hand should belong to non-uniform hyperbolicity, from a probabilistic point of view: *Is there a positive Lebesgue measure set of values of κ for which f_κ (i) has non-zero Lyapunov exponents on a positive (respectively, full) Lebesgue measure set of points ? (ii) has no elliptic periodic points, and is even ergodic ?* Kosygin and Sinai have announced that they can construct an uncountable set of parameters κ for which (i) and (ii) are valid. More recently, they also announced that their methods yield a positive Lebesgue measure set, with full density at $\kappa = \infty$.

For symplectic maps with $d \geq 2$ degrees of freedom (the ambient space M has dimension $2d \geq 4$) KAM tori do not bound regions in the dynamical space, and so orbits not contained in any invariant torus may escape to infinity. Ever since this possibility was raised in [4], there have been several attempts at proving that *Arnold diffusion* does occur in generic situations. Currently, the broadest results seem to be given by Mather's variational methods [39], see also [59].

The accessibility approach to proving ergodicity of volume-preserving systems was introduced by Brin-Pesin [17], and brought to a whole new level of generality by a series of papers of Pugh, Shub, Wilkinson, Burns, Niţică, Török, and Dolgopyat, over the last few years; see the survey [18] for updated information. In a few words, one assumes the diffeomorphism

f to be *strongly partially hyperbolic*: the tangent space splits into three invariant sub-bundles $TM = E^u \oplus E^c \oplus E^s$ such that $Df|E^u$ is an expansion, $Df|E^s$ is a contraction, and $Df|E^c$ is between the two, in the sense of (2). Then, *accessibility* means that any two points may be connected by a piecewise smooth path tangent to either E^u or E^s (except at a finite number of corner points).

While this notion is useful also for general (dissipative) systems, it has proven itself especially effective in the conservative case. In particular, a result of Pugh, Shub [50] states that, for volume-preserving diffeomorphisms, accessibility together with a number of technical hypotheses implies stable ergodicity (all volume-preserving maps in a neighborhood of f are ergodic). Results in the converse direction have been obtained as well. However, it should be noted that partial hyperbolicity is not necessary for stable ergodicity [14].

9 Concluding remarks

One thing I find particularly fascinating about Dynamical Systems is its being a meeting ground for mathematical fields and experimental sciences alike. Analysis (real and complex), Topology (differential and algebraic), Probability and Measure Theory, Geometry (in various flavors), Number Theory, all have lent their methods to Dynamics (often, they benefited from it). And concrete problems in such areas as Mechanics (classical or fluid), Electromagnetism, Thermodynamics, Demography, Information Theory, or Economics, lead to some of the questions that shaped this domain of Mathematics. In particular, the evolution models that I discussed more closely, and which have been the aim of most efforts in this field, can be traced back to its origins in the qualitative theory of differential equations and, more specifically, in Celestial Mechanics.

Other natural phenomena are better described by such mathematical models as partial differential equations, or stochastic maps and flows. One prize example is Fluid Mechanics, especially the problem of turbulence. Although some very basic questions remain open, e.g., the very existence of global solutions for the Navier-Stokes equation, ideas such as I discussed before should be useful in extending the theory to such infinite-dimensional models. One direction along which such an extension can be attempted for dissipative partial differential equations is by first reducing the system to an invariant *inertial manifold*, finite-dimensional and attracting the solution through *almost every* initial condition.

Of course, new challenges keep being posed. Right now, one is beginning to dare try to understand the dynamics of extremely complex biological systems, such as ecological environments or the human brain. Will our fundamental paradigms apply to such situations, or will we have to devise fundamentally new tools and ideas ? It would be very nice to hang around long enough and find out...

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