On the Module of Relatively Exact 1-Forms Modulo an Integrable Foliation in the Projective Space

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Abstract

Let $\mathcal{F}$ be a foliation in $\mathbb{C}P(2)$ with a noncomposite first integral g and let ω_1 be a relatively exact meromorphic 1-form modulo $\mathcal{F}$. We prove that if every g-fiber is connected and every $\mathcal{F}$-invariant component of the pole divisor of ω_1 is an irreducible g-fiber then ω_1 is of the form $df + \omega$, where f is a meromorphic function on $\mathbb{C}P(2)$ and ω is a meromorphic 1-form which induces the foliation $\mathcal{F}$.

0 Introduction

In my original work about deformation of a foliation with a first integral, I have described a certain class of relatively exact 1-forms modulo that foliation. Next, Paulo Sad showed me an article [2], which motivated me to generalize the method used in my work, which originally comes from Yu. Ilyashenko's article [4], and classify all relatively exact 1-forms modulo an integrable foliation. This article is the result of this generalization. We have started our definitions with foliations in the projective 2-space $\mathbb{C}P(2)$, which in our case are always holomorphic and singular, to say that the classification considered here can also be asked for any foliation.

A foliation $\mathcal{F}$ in $\mathbb{C}P(2)$ is called integrable if there exists a meromorphic function g on $\mathbb{C}P(2)$ such that all g-fibers are invariant by $\mathcal{F}$. The meromorphic function g is said to be the first integral of $\mathcal{F}$. g is called noncomposite if a general g-fiber is irreducible. Poincaré in his article [9] has studied integrable foliations in $\mathbb{C}P(2)$ and has proved (p. 52-53) that every integrable foliation $\mathcal{F}$ in $\mathbb{C}P(2)$ has a noncomposite first integral. It is easy to see that g is noncomposite, if and only if, g cannot be factored as a composite

$$(1) \qquad \mathbb{C}P(2) \xrightarrow{g'} \overline{\mathbb{C}} \xrightarrow{i} \overline{\mathbb{C}}$$

where i is a holomorphic map of degree greater than one. It can also be shown that any other first integral g of $\mathcal{F}$ is factored as (1), where i is a holomorphic function and g' is a given noncomposite meromorphic function.

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Definition 0.1 Let $\mathcal{F}$ be a foliation in $\mathbb{C}P(2)$. A meromorphic 1-form ω_1 on $\mathbb{C}P(2)$ is called relatively exact modulo $\mathcal{F}$ if the restriction of ω_1 to each leaf L of $\mathcal{F}$ is exact i.e., there is a meromorphic function f on L such that $\omega_1 \mid_L = d(f)$.

The reader can easily verify that

Proposition 0.1 *A meromorphic 1-form ω_1 is relatively exact modulo $\mathcal{F}$ if and only if*

$$\int_\delta \omega_1 = 0 \tag{2}$$

for all closed cycles in the leaves of $\mathcal{F}$, where this integral is well-defined.

Notations 0.1 Fix a foliation $\mathcal{F}$ in $\mathbb{C}P(2)$. For any meromorphic 1-form ω_1 on $\mathbb{C}P(2)$, we denote by

$$pol(\omega_1) = D = \sum_{s=1}^{r} n_s D_s \tag{3}$$

the pole divisor of ω_1. For us D is not more than a finite union of codimension one irreducible subvarieties of $\mathbb{C}P(2)$ with associated positive integer numbers, n_s's, to them. By the degree of divisor D we mean

$$deg(D) = \sum_{s=1}^{r} n_s deg(D_s)$$

Define

$$\check{D} = \sum_{D_s \text{ is } \mathcal{F}\text{-invariant}} D_s, \quad \tilde{D} = \sum_{D_s \text{ is not } \mathcal{F}\text{-invariant}} D_s$$

In what follows when we say that a meromorphic object (function, 1-form, ...) Z has the pole divisor D, we mean that

$$pol(Z) + D \geq 0$$

Our main theorem in this article is the following

Theorem 0.1 *Let $\mathcal{F}$ be an integrable foliation in $\mathbb{C}P(2)$ with a noncomposite first integral g whose fibers are connected. Let also ω_1 be a relatively exact meromorphic 1-form modulo $\mathcal{F}$ such that every $\mathcal{F}$-invariant component of $pol(\omega_1)$ is an irreducible g-fiber. Then ω_1 has the form*

$$\omega_1 = df + \omega \tag{4}$$

where f is a meromorphic function on $\mathbb{C}P(2)$ with pole divisor $D - \tilde{D}$ and ω is a meromorphic 1-form in $\mathbb{C}P(2)$ which induces $\mathcal{F}$ and has the pole divisor $D + \check{D}$.

Note that for us a g-fiber does not contain the indeterminacy points of g i.e., the points in $(g)_\infty \cap (g)_\infty$. The above theorem is no more true if we assume that some g-fibers are disconnected. P. Bonnet in [2] gives the example $g = x(1+xy)$ in $\mathbb{C}^2$ having the disconnected fiber $g = 0$. The 1-forms $y^{k+1}dx + xy^k dy,\ k > 0$ are relatively exact modulo g but they are not of the form (4). Without the hypothesis of connectedness of g-fibers the classification of relatively exact polynomial 1-forms modulo a polynomial is done in [2]. It will be interesting to combine the methods of that article and this one to have a complete classification for fibrations in $\mathbb{C}P(2)$.

Let $\mathcal{F}$ be a foliation in $\mathbb{C}P(2)$ with a first integral of the type $\frac{F^p}{G^q}$, where F and G are two relatively prime irreducible polynomials in $\mathbb{C}^2$, $\frac{deg(F)}{deg(G)} = \frac{q}{p}$ and $g.c.d.(p,q) = 1$. A polynomial 1-form ω_1 in $\mathbb{C}^2$ can be considered as a meromorphic 1-form in $\mathbb{C}P(2)$ with pole at the line at infinity H_∞. We define the degree of ω_1 to be $mul(\omega_1, H_\infty) - 2$, where $mul(\omega_1, H_\infty)$ is the multiplicity of H_∞ as a pole of ω_1. Suppose that all $\frac{F^p}{G^q}$-fibers are connected. This hypothesis is satisfied assuming some generic conditions over F and G. The following corollary was the starting point of this article.

Corollary 0.0.1 *In the above situation, let ω_1 be a polynomial holomorphic 1-form in $\mathbb{C}^2$ of degree $d = deg(FG) - 2$ and let $\frac{\omega_1}{FG}$ be relatively exact modulo $\mathcal{F}$. Then ω_1 has the form*

$$pGdP - qPdG + pQdF - qFdQ$$

where P and Q are two polynomials of the same degree of F and G, respectively.

Proof: The fact that $deg(\omega_1) = deg(FG) - 2 = d$ implies that the line at infinity is not a pole of $\frac{\omega_1}{FG}$. In other words $\frac{\omega_1}{FG}$ has the pole divisor $\{F = 0\} + \{G = 0\}$. By Theorem 0.1 there exist polynomials B and R of degree $d + 2$ such that

$$\frac{\omega_1}{FG} = d(\frac{B}{FG}) - \frac{R}{FG}(\frac{pGdF - qFdG}{FG}) \Rightarrow$$

$$\omega_1 = \frac{FGdB - Bd(FG) - R(pGdF - qFdG)}{FG} \tag{5}$$

This implies that

$$F \mid B + pR,\ G \mid B - qR \Rightarrow$$

$$B + pR = -(p+q)FQ,\ B - qR = (p+q)GP \ \Rightarrow$$

$$B = pGP - qFQ,\ R = -GP - FQ$$

where P and Q are two polynomials of the degree $deg(F)$ and $deg(G)$, respectively. Substituting this in (5) we obtain the desired form of ω_1. $\diamond$

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1 Multivalued Functions

We start this section with a precise definition of multivalued functions.

Definition 1.1 Let $\tau : M \to N$ be a holomorphic map between two complex manifolds M and N of the same dimension. Every meromorphic function f on M is called a multivalued function on N. Roughly speaking, the image of a point $p \in N$ under f is the set $f(\tau^{-1}(p))$. The map τ is called the ramification map of the multivalued function f and the set of critical values of τ is called the ramification set of f. If τ is a map of degree d then we say that f is a d-valued function.

In what follows we have always the intuitional concept of multivalued function and to study such function we will have to construct a ramification map τ for it.

Fix a foliation $\mathcal{F}$ and an effective divisor $D = \sum_{i=1}^{r} n_i D_i$ in $\mathbb{C}P(2)$ i.e., a divisor with positive n_i's. Recall that

$$\check{D} = \sum_{D_s \text{ is } \mathcal{F}\text{-invariant}} D_s, \quad \tilde{D} = \sum_{D_s \text{ is not } \mathcal{F}\text{-invariant}} D_s$$

In what follows by D we mean both the divisor and the union of the components of the divisor; the meaning being clear from the text.
Let R be the union of the singularities of $\mathcal{F}$ and D and the tangency points between $\mathcal{F}$ and $\tilde{D}$. Let also $\tilde{L}$ be a line in $\mathbb{C}P(2)$ such that

1. $\tilde{L}$ doesn't pass through any point of R;
2. $\tilde{L}$ has $d = deg(\mathcal{F})$ distinct tangency points of order two with $\mathcal{F}$ in distinct leaves of $\mathcal{F}$;
3. $\tilde{L}$ intersects $\tilde{D}$ transversally;
4. The tangency points in 2 and the intersection points in 3 are in distinct leaves of $\mathcal{F}$.

Let Ω_D be the space of all relatively exact meromorphic 1-form modulo $\mathcal{F}$ with pole divisor D and $\omega_1 \in \Omega_D$. For any two points a, b in the leaf L, let $\delta_{a,b}$ be a path in L which connects a to b. If the leaf L is not contained in D then $L \cap D$ is a discrete set in L and we can assume that $\delta_{a,b}$ does not pass through the points of $L \cap D$, except in the initial and end points. By the equality (2) we know that

$$\int_{\delta_{a,b}} \omega_1$$

does not depend on the choosing of the path which connects a to b. For any $p \in \mathbb{C}P(2)$ let

$$L_p \cap \tilde{L} = \{p_i \mid i \in I\}$$

where L_p is the leaf through p. Define the multivalued function f_{ω_1} as follows:

$$f_{\omega_1}(p)_i = \int_{\delta_{p,p_i}} \omega_1 \tag{6}$$

When there is no confusion we drop the subscript ω_1. We will not need to construct a ramification map for f (if there exists any). One can Observe that if p_i is not a tangency point between $\mathcal{F}$ and $\tilde{L}$ then the branch f_i of the multivalued function f is a well-defined meromorphic function in a small neighborhood of p.

Lemma 1.1 *Keeping the above notations we have*

1. *If $p, p_i \in \mathbb{C}P(2)\backslash(D \cup R)$, then f_i is holomorphic;*
2. *If D_s is not a $\mathcal{F}$-invariant component of D, $p \in D_s\backslash R$ and $p_i \in \mathbb{C}P(2)\backslash(D \cup R)$, then f_i has pole of order $n_s - 1$ at D_s;*
3. *If D_s is a $\mathcal{F}$-invariant component of D and $p \in D_s\backslash R$, then f_i has pole of order n_s at D_s.*

Proof: The first affirmation is trivial. There is a coordinate system (x, y) around p such that in this coordinate system $p = (0, 0)$ and $\mathcal{F}$ is given by $y = const.$. If D_s is not $\mathcal{F}$-invariant then we can also assume that D_s in this coordinate is given by $x = 0$. In this coordinate ω_1 has the form

$$\omega_1 = \frac{A(x, y)}{x^{n_s}}dx + B(x, y)dy$$

If D_s is $\mathcal{F}$-invariant then it is given by $y = 0$ and ω_1 has the form

$$\omega_1 = \frac{A(x, y)}{y^{n_s}}dx + B(x, y)dy$$

A is a holomorphic function on the definition domain of the coordinate in both cases.
Let $\Sigma_a = \{(a, y) \mid y \in (\mathbb{C}, 0)\}$ be a small transverse section at $(a, 0)$ to $\mathcal{F}$. We have

$$f(x, y)_i = \int_a^x \frac{A(\xi, y)}{\xi^{n_s}}d\xi + \int_{\delta_{(a,y),p_i}} \omega_1 \tag{7}$$

in the first case

$$f(x, y)_i = \frac{1}{y^{n_s}} \int_a^x A(\xi, y)d\xi + \int_{\delta_{(a,y),p_i}} \omega_1 \tag{8}$$

in the second case. In formula (7) the first term is a meromorphic function of order $n_s - 1$ and the second term is a holomorphic function. This proves our second

affirmation.
For the second case, we can cover δ_{p,p_i} with a finite number of coordinate systems with the mentioned property, and then by (8) we can write f_i as a sum of terms with pole of multiplicity n_s at D_s. $\diamond$

The following equality is a direct consequence of the definition

$$f(p)_i - f(p)_j = \int_{\delta p_i, p_j} \omega_1 \quad i,j \in I \tag{9}$$

Observe that when $\int_{\delta p_i, p_j} \omega_1 = 0$ for all i, j, then the function f_{ω_1} is univalued.

Notations 1.1 Keeping the notations used in Theorem 0.1, Let $\mathcal{A}$ be the set of meromorphic functions on $\mathbb{C}P(2)$ with pole divisor $D - \tilde{D}$, $\mathcal{B}$ be the set of meromorphic 1-forms which induce the foliation $\mathcal{F}$ and have the pole divisor $D + \check{D}$ and $\mathcal{C}$ be the set of all $(f, \omega) \in \mathcal{A} \times \mathcal{B}$ such that $df + \omega$ has pole divisor D. Define the operator

$$T : \mathcal{C} \to \Omega_D$$

$$T(f, \omega) = df + \omega$$

One can see that the assertion of Theorem 0.1 is equivalent to this fact that T is surjective.
Let $\mathcal{D}$ be the set of all 1-forms $\omega_1 \in \Omega_D$ for which all values in (9) are zero i.e, for any two points $p_i, p_j \in \tilde{L}$ which are in the same leaf of $\mathcal{F}$

$$\int_{\delta p_i, p_j} \omega_1 = 0$$

Lemma 1.2 *If $\omega_1 \in \mathcal{D}$ then $f_{\omega_1} \in \mathcal{A}$ and there exists $\omega \in \mathcal{B}$ such that*

$$\omega_1 = df_{\omega_1} + \omega$$

Proof: By (9) and the hypothesis we know that f_{ω_1} is one valued meromorphic function on $\mathbb{C}P(2) \backslash R$. R is a finite set and therefore by Levi's Theorem f_{ω_1} extends to $\mathbb{C}P(2)$. Now the first assertion is a consequence of Lemma 1.1. Let ω' be a 1-form which induces the foliation $\mathcal{F}$, then by the definition of f_{ω_1} we have

$$df_{\omega_1} \wedge \omega' = \omega_1 \wedge \omega'$$

which implies that $df_{\omega_1} - \omega_1 \in \mathcal{B}$ and this proves our second affirmation.$\diamond$

Lemma 1.3 *Let $\mathcal{F}$ be an integrable foliation with a noncomposite first integral g. suppose that every $\mathcal{F}$-invariant component of D is an irreducible fiber of g. Then the dimension of the vector space $\mathcal{D}$ is*

$$dim(Im(T)) - deg(D - \tilde{D}) + \sum_{s=1}^{r} [\frac{n_s}{m_s}]$$

where m_s is the multiplicity of g along D_s. If D_s is not $\mathcal{F}$-invariant put $m_s = \infty$.

Proof: One can see easily that,

$$ker(T) = \{(f, -df) \mid f \in \mathcal{A},\ f \text{ is a first integral of } \mathcal{F}\ \}$$

Let us compute $dim(ker(T))$. If $f \in \mathcal{A}$ is a first integral of $\mathcal{F}$ then there exists a holomorphic map $i : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $f = i \circ g$. Also the pole divisor of f contains only $\mathcal{F}$-invariant components of D. Let D_s be an invariant component of D, m_s be the multiplicity of g along D_s, a_s be the value of g associated to D_s and q_s be the multiplicity of i at a_s. Since $f \in \mathcal{A}$ we have

$$q_s m_s \leq n_s$$

We know that the set $i^{-1}(\infty) = \{a_1, a_2, \ldots, a_{r_1}\}$ consists of values associated to the invariant components of D. We can consider i as a meromorphic function on $\overline{\mathbb{C}}$ with poles at $a_1, a_2, \ldots, a_{r_1}$.
For every meromorphic function i' on $\overline{\mathbb{C}}$ with poles $a_1, a_2, \cdots, a_{r_1}$ and with multiplicities $[\frac{n_1}{m_1}], [\frac{n_2}{m_2}], \ldots, [\frac{n_{r_1}}{m_{r_1}}]$, respectively, the fact that $g^{-1}(a_s) = D_s$ is irreducible for all s, implies that

$$(i' \circ g\ ,\ d(i' \circ g)) \in ker(T)$$

We conclude that the dimension of $ker(T)$ is the dimension of the space of such meromorphic functions on $\overline{\mathbb{C}}$, which is

$$dim(ker(T)) = \sum_{s=1}^{r}[\frac{n_s}{m_s}] + 1 \tag{10}$$

Every first integral of $\mathcal{F}$ which is identically zero on a line, is identically zero. In other words $ker(T) \cap \{(f, \omega) \in \mathcal{C} \mid f\mid_{\tilde{L}} = 0\} = \{0\}$, and so

$$dim(\mathcal{D}) = dim(T^{-1}(\mathcal{D})) =$$

$$dim\{(f, \omega) \in \mathcal{C} \mid f\mid_{\tilde{L}} = 0\} = dim(\mathcal{C}) - (deg(D - \tilde{D}) + 1) \tag{11}$$

Remark 1.1 In above equality we have assumed that the dimension of the following vector space

$$\{f\mid_{\tilde{L}} \mid f \in \mathcal{A}\ \&\ \exists \omega \in \mathcal{B},\ (f, \omega) \in \mathcal{C}\}$$

is $deg(D - \tilde{D}) + 1$. The proof of this fact is done in Appendix A.

The equality $dim(\mathcal{C}) = dim(Im(T)) + dim(ker(T))$ and (10) prove our affirmation. $\diamond$

The values in (9) are associated to each leaf and we can consider them as the values of a multivalued function e which is defined on the space of the leaves of $\mathcal{F}$. If $e \equiv 0$ then f is one valued. That is why e is called the branching function of f. In the next section we will study the multivalued function e for integrable foliations by constructing the related ramification map.

2 Ramification Map of e

Let $\mathcal{F}$ be an integrable foliation with a noncomposite first integral $g : \mathbb{C}P(2) \to \overline{\mathbb{C}}$. Define

$$I = g\mid_{\tilde{L}} : \tilde{L} \to \overline{\mathbb{C}}$$

$$V = \{(x, y) \in \tilde{L} \times \tilde{L} \mid I(x) = I(y)\}$$

$$\tilde{I} = I \circ \pi_1 = I \circ \pi_2 : V \to \overline{\mathbb{C}}$$

where π_1 and π_2 are projections on the first and second coordinates, respectively. The first trivial irreducible component of V is the diagonal variety

$$\Delta = \{(x, x) \mid x \in \tilde{L}\}$$

Let b be a regular value of I and V_1 be an irreducible component of V. The set $G_1 = I^{-1}(b) \times I^{-1}(b) \cap V_1$ defines a directed graph on the set $I^{-1}(b) = \{a_1, a_2, \ldots, a_r\}$. Recall that the directed graph G_1 is defined as follows: the vertexes are the points $a_i \in I^{-1}(b)$. If $(a_i, a_j) \in I^{-1}(b) \times I^{-1}(b) \cap V_1$ then we put an edge directed from a_i to a_j. Later we will need the following simple fact:

Lemma 2.1 G_1 has a cycle.

Proof: Let C denote the set of critical values of I. For any $(a_1, a_2) \in G_1$ let λ be a path in $\tilde{L}\backslash I^{-1}(C)$ which connects a_1 to a_2. The monodromy of I along λ (see [1] or [8])

$$h_\lambda : I^{-1}(b) \to I^{-1}(b)$$

takes a_1 to a_2 and a_2 to another point a_3 in $I^{-1}(b)$. The pair (a_2, a_3) is also in the component V_1. Repeat this method to find a sequence of pairs

$$(a_1, a_2), (a_2, a_3), (a_3, a_4), \ldots \in G_1$$

G_1 is a finite graph and so we must have a closed cycle in G_1.$\diamond$

In the sequel by L_c we mean the fiber $g^{-1}(c)\backslash A(g)$, where $A(g)$ is the set in which g is not defined. Let X be a topological space, $x \in X$ and f be a function on X, we denote by (X, x), $(X, x)^*$ and (f, x), respectively a neighborhood of x in X, a punctured neighborhood of x in X and the germ of f at x.

Definition 2.1 Let m and n be two positive integers with $g.c.d.(m, n) = 1$ and let V be an algebraic variety of dimension one in a complex manifold of dimension two. The point $p \in V$ is called a (m, n)-singularity if there is a coordinate system (x, y) in a neighborhood of p such that in this coordinate $p = (0, 0)$ and (V, p) is given by the equation $x^n = y^m$. Note that if one of the numbers n or m is one then p is an smooth point of V.

Let $z_1, z_2, \ldots, z_d$ be tangency points between $\mathcal{F}$ and $\tilde{L}$. The reader can verify easily that

Lemma 2.2 *The following statements are true:*

1. *$(V,(z_i,z_i))$ has two irreducible smooth components; one of those components is $(\Delta,(z_i,z_i))$.*

2. *Let $a \in A \cap \tilde{L}$ and $b \in B \cap \tilde{L}$, where A and B are two irreducible components of L_c with multiplicities m and n, respectively (maybe $A = B$ and $a = b$). Then $(V,(a,b))$ has g.c.d.$(m,n) = d'$ irreducible components each of them with $(m/d', n/d')$-singularity at (a,b). These are all irreducible components of V in some neighborhood of (a,b);*

3. *1 and 2 give us all singularities of V.*

Let
$$\pi : S \to V$$
be a normalization map of the variety V, where S is a Riemann surface (non connected). The pullback
$$\tau = \tilde{I} \circ \pi : S \to \overline{\mathbb{C}}$$
of the function $\tilde{I}$ is a holomorphic function. We can see easily that

Lemma 2.3 *Keeping the notations used in Lemma 2.2, the set $\pi^{-1}(a,b)$ consists of d' points which are critical points of τ of order d'. All other critical points of τ are obtained in a similar way.*

Notations 2.1 Let L_c be an irreducible g-fiber which intersects $\tilde{L}$ transversally and let a, b be two points in $L_c \cap \tilde{L}$. Let also $\delta_{a,b}$ be a path in L_c which connects a to b and let
$$h_{\delta_{a,b}} : (\tilde{L}, a) \to (\tilde{L}, b)$$
be the related holonomy. There is a unique point $z(\delta_{a,b}) \in \pi^{-1}(a,b)$ such that
$$(S, z(\delta_{a,b}))^* = \{\pi^{-1}(x, h_{\delta_{a,b}}(x)) \mid x \in (\tilde{L}, a)^*\}$$
Also for every $z \in \pi^{-1}(a,b)$ there exists a path $\delta_{a,b}$ such that $z = z(\delta_{a,b})$.
If L_c has multiplicity one then all $\delta_{a,b}$'s induce the same holonomy and so the point $z(\delta_{a,b})$ does not depend on the choice of the path $\delta_{a,b}$. In this case $\pi^{-1}(a,b) = \{z(\delta_{a,b})\}$ and we can use the notation (a,b) instead of $z(\delta_{a,b})$.
In the case where L_c is tangent to $\tilde{L}$ at a (but not at b), then we can also define a holonomy map $h_{\delta_{a,b}}$ which is no more a local biholomorphism. The point a is a critical point of order two of $h_{\delta_{a,b}}$.

For any $c \in \overline{\mathbb{C}}$ let

$$L_c \cap \tilde{L} = \{p_i(c) \mid i \in I\}$$

Define the k^2-valued meromorphic function $e = e_{\omega_1}$ on $\overline{\mathbb{C}}$, which is one valued on S, as follows

$$e(c)_{i,j} = e(z(\delta_{p_i(c),p_j(c)})) = \int_{\delta p_i(c),p_j(c)} \omega_1 \quad i,j = 1,2,\ldots,k \tag{12}$$

Here k is the degree of a generic leaf of $\mathcal{F}$. The diagonal branch $e\mid_\Delta$ of e is identically zero and τ is the ramification map of the multivalued function e.
From now on, we suppose that the $\mathcal{F}$-invariant components of D are irreducible g-fibers.
Pole Divisor of e: Let D_s be a component of the divisor D and $a \in D_s \cap \tilde{L}$ and b be any other point in $L_{g(a)} \cap \tilde{L}$. e has a pole at $z(\delta_{a,b})$. In some neighborhood of $z(\delta_{a,b})$ we have:

$$e = f_i \circ \pi_1 \circ \pi \tag{13}$$

Fix a coordinate system around $z(\delta_{a,b})$ such that $\pi_1 \circ \pi$ has the form $z \to z^s$. By a pole coefficient of e at $z(\delta_{a,b})$, we mean a coefficient of some $z^n, n < 0$ in the the Laurent series of e at $z(\delta_{a,b})$. Consider two cases

- D_s is not invariant by $\mathcal{F}$;
  According to Lemma 1.1 e has a pole of order $n_s - 1$ in $z(\delta_{a,b})$. Note that $\pi_1 \circ \pi$ is a local biholomorphism around $z(\delta_{a,b})$.
- D_s is invariant by $\mathcal{F}$;
  Let m_s be the multiplicity of g along D_s. Then $z(\delta_{a,b})$ is a critical point of order m_s of $\pi_1 \circ \pi$, and so by (13) we conclude that e has a pole of order $m_s n_s$ at $z(\delta_{a,b})$, but note that from the $n_s m_s$ pole coefficients only n_s of them are not zero.

All pole points of e are obtained in a similar way. Note that all g-fibers are connected and so the above cases produce all possible pole points of e.
Let a, b, c be three points in the fiber L_c. Then the relation $\delta_{a,b} + \delta_{b,c} = \delta_{a,c}$ implies that

$$(e, z(\delta_{a,b})) + (e, z(\delta_{b,c})) = (e, z(\delta_{a,c})) \tag{14}$$

Recall that $(e, z(\delta_{a,b}))$ is the germ of e at $z(\delta_{a,b})$. Define the vector space

$$\Phi_D = \{e_{\omega_1} \mid \omega_1 \in \Omega_D\}$$

Lemma 2.4 *Suppose that the $\mathcal{F}$-invariant components of D are irreducible g-fibers. Then the dimension of the vector space Φ_D is at most*

$$deg(D - \tilde{D}) - \sum_{s=1}^{r} [\frac{n_s}{m_s}]$$

where m_s is the multiplicity of g along D_s.

Proof: We will consider independent pole coefficients of $e \in \Phi_D$ in the fixed coordinate system. In fact, we will prove that e has at most

$$deg(D-\tilde{D}) - \sum_{s=1}^{r}[\frac{n_s}{m_s}]$$

number of independent pole coefficients. An element $e \in \Phi_D$ depends only on these coefficients, because if these coefficients are zero then e is holomorphic on S. But S is compact and so e must be constant in each component of S. The definition of e, the relation (14) and Lemma 2.1 imply that these constants must be zero, and so $e \equiv 0$.
The relations in (14) say that many poles of e are not independent. Let $a_1 \in D_s \cap \tilde{L} \cap L_c$ and

$$L_c \cap \tilde{L} = \{a_1, a_2, \ldots, a_k\}$$

We have the relations

$$(e, z(\delta_{a_i,a_j})) + (e, z(\delta_{a_j,a_k})) = (e, z(\delta_{a_i,a_k})),\ i,j,k = 1,2,\cdots,k,\ k = deg(L_c)$$

Consider two cases

- D_s is not invariant by $\mathcal{F}$;
  $(e, z(\delta_{a_i,a_j})), i,j \geq 2$ are holomorphic functions. Fix the path δ_{a_1,a_2}, where a_2 is any other point in $L_c \cap \tilde{L}$. We have only the following independent poles

  $$(e, z(\delta_{a_1,a_2})),\ a_1 \in D_s \cap \tilde{L} \tag{15}$$

  This gives us $(n_s - 1)deg(D_s)$ independent pole coefficients of e.

- D_s is invariant by $\mathcal{F}$;
  Fix the paths $\delta_{a_1,a_i},\ i = 2,3,\ldots,deg(D_s)$ and the closed path δ in D_s with initial and end point a_1 and such that each point in $\pi^{-1}(a_1,a_1)$ has the form $z(i\delta)$ for some $0 \leq i < m_s$, where m_s is the multiplicity of g along D_s.
  By a similar argument we have at most the following independent poles

  $$(e, z(\delta)),\ (e, z(\delta_{a_1,a_i})),\ i = 2,\ldots,deg(D_s) \tag{16}$$

  But we can still find zero coefficients in the pole of $(e, z(\delta))$ as follows:
  Let z be a coordinate system in $(\tilde{L}, a_1)$ in which $h = h_\delta$ is given by $z \to \alpha z$, where $\alpha^{m_s} = 1$. Then

  $$h_\delta^{m_s} = Id \Rightarrow$$

  $$\sum_{i=1}^{m_i-1} e \circ (\pi_1 \circ \pi)^{-1} \circ h^i = \sum_{i=1}^{m_i-1} f_i \circ h^i \equiv 0$$

  But $h^i(z) = \alpha^i z$ and $\sum_{i=1}^{m_s-1} \alpha^{ki} = 0$ if and only if k is not a multiple of m_s. This implies that $(e, z(\delta))$ has $n_s - [\frac{n_s}{m_s}]$ non-zero pole coefficients and so we have $n_s deg(D_s) - [\frac{n_s}{m_s}]$ independent pole coefficients.

The lemma is proved.$\diamond$

Proof of Theorem 0.1: This is a direct consequence of Lemmas 2.4 and 1.3 as follows: Define the linear map

$$i : \Omega_D \to \Phi_D$$

$$i(\omega_1) = e_{\omega_1}$$

we know that $\mathcal{D} = ker(i)$ and so

$$dim(\Omega_D) = dim(\Phi_D) + dim(\mathcal{D}) \leq dim(Im(T))$$

but $Im(T) \subset \Omega_D$ and so $\Omega_D = Im(T)$.$\diamond$

Challenge: The main obstacle in the generalization of Lemma 2.4 to any foliation is the definition of S. By the above arguments we suggest the following definition:
Two paths $\delta_{a,b}$, $\delta'_{a,b}$ in the leaf L of $\mathcal{F}$ are equivalent

$$\delta_{a,b} \simeq \delta'_{a,b}$$

if $\delta_{a,b} - \delta'_{a,b}$ has trivial holonomy. This is an equivalency relation. Define

$$S = \{\delta_{a,b} \mid a, b \in L \cap \tilde{L},\ L \text{ is a leaf of } \mathcal{F}\}/\simeq$$

The reader can verify that S has a natural structure of a Riemann surface.

3 Logarithmic Foliations

Let $\mathcal{F}$ be a foliation in $\mathbb{C}P(2)$ given by the polynomial 1-form

$$\omega(f,\lambda) = \omega(f_1,\ldots,f_r,\lambda_1,\ldots,\lambda_r) = f_1\cdots f_r \sum_{i=1}^{r} \lambda_i \frac{df_i}{f_i}$$

where f_i's are irreducible polynomials in $\mathbb{C}^2$ and $\sum deg(f_i)\lambda_i = 0$. $\mathcal{F}$ is called a logarithmic foliation and has the multivalued first integral $F = log(f_1^{\lambda_1}\cdots f_r^{\lambda_r})$ in $U = \mathbb{C}P(2)\backslash(\cup_{i=1}^{r}\{f_i = 0\})$. Under some generic conditions on f_i's and λ_i's we have the following proposition:

Proposition 3.1 *Let $\bar{f}_i$ be a polynomial in $\mathbb{C}^2$ with the same degree of f_i. Let also $\bar{\lambda}_i$'s be complex numbers with $\sum \bar{\lambda}_i deg(f_i) = 0$. Then the 1-forms*

$$1.\ \frac{\omega(f_1,\ldots,\widehat{f_i,\bar{f}_i},\ldots,f_r,\lambda)}{f_1f_2\cdots f_r},\quad 2.\ \frac{\omega(f,\bar{\lambda})}{f_1f_2\cdots f_r}$$

are relatively exact modulo $\mathcal{F}$, where $\widehat{f_i,\bar{f}_i}$ means that we have replaced f_i with $\bar{f}_i$.

Proof: The reader can verify easily that

$$\frac{\omega(f_1,\ldots,\widehat{f_i,\bar{f}_i},\ldots,f_r,\lambda)}{f_1f_2\cdots f_r}=\frac{\bar{f}_i}{f_i}(\frac{\omega(f,\lambda)}{f_1f_2\cdots f_r})+\lambda_i d(\frac{\bar{f}_i}{f_i})$$

which shows that (1) is relatively exact modulo $\mathcal{F}$. For the second 1-form we have

$$\frac{\omega(f,\bar{\lambda})}{f_1f_2\cdots f_r}=d(log(f_1^{\bar{\lambda}_1}f_2^{\bar{\lambda}_2}\cdots f_r^{\bar{\lambda}_r}))=d(g)$$

The function g is multivalued on U. We prove that g is a univalued meromorphic function on each leaf of $\mathcal{F}$. Let $\tau:\tilde{U}\to U$ be the universal covering map of U. This map is a ramification map of both multivalued functions F and g. Denote by $\tilde{F}$ and $\tilde{g}$ the holomorphic maps on $\tilde{U}$ associated to F and g. The map τ is generically a biholomorphism between $\tilde{F}^{-1}(c)=L_c, c\neq 0,\infty$ and its image $L=\tau(\tilde{F}^{-1}(c))$, which is a leaf of $\mathcal{F}$. We have

$$\tau^*(g\mid L)=\tilde{g}\mid_{L_c}$$

which implies that g restricted to the leaves of $\mathcal{F}$ is univalued function. $\diamond$

Question: Are the 1-forms in Proposition 3.1 all relatively exact 1-forms modulo $\mathcal{F}$ with the pole divisor $\sum_i\{f_i=0\}$?

A Proof of Remark 1.1

Choose an affine coordinate $\mathbb{C}^2$ in $\mathbb{C}P(2)$ such that the line at infinity $\mathbb{C}P(2)\backslash\mathbb{C}^2$ is not invariant by $\mathcal{F}$. In this coordinate let

$$\mathcal{F}:\omega_0=Pdy-Qdx=0$$

$$\check{D}_i: F_i=0,\ i=1,2,\ldots,r_1$$

$$\tilde{D}_i: R_i=0,\ i=1,2,\ldots,r_2$$

where $\check{D}_i$'s and $\tilde{D}_i$'s denote $\mathcal{F}$-invariant and not $\mathcal{F}$-invariant components of D, respectively, and F_i's and R_i's are polynomials in $\mathbb{C}^2$. The following notations will be used

$$F^n=F_1^{n_1}F_2^{n_2}\cdots F_{r_1}^{n_{r_1}},\ F=F_1F_2\cdots F_{r_1},\ ndF=\sum_{i=1}^{r_1}n_sF_1F_2\cdots\hat{F}_i\cdots F_{r_1}dF_i$$

In this coordinate we have

$$f=\frac{B}{F^nR^{m-1}},\ \omega=\frac{A}{F^{n+1}R^m}\omega_0$$

where B and A are two polynomials and

$$deg(B) = \alpha - deg(R),\ deg(A) = \alpha + deg(F) - (d+2)$$

where $\alpha = deg(F^n R^m)$ and $d = deg(\mathcal{F})$. Moreover $\omega_1 = df + \omega$, then

$$\omega_1 = df + \omega = d(\frac{B}{F^n R^{m-1}}) + \frac{A}{F^{n+1} R^m}\omega_0 = \frac{FRdB - B(R.ndF + F.mdR) + A\omega_0}{F^{n+1} R^m}$$

We know that $Pol(\omega_1) = D$, and so

$$F \mid BR.ndF - A\omega \Rightarrow$$

$$F_i \mid n_i BR\frac{F}{F_i}dF_i - A\omega_0,\ i = 1, 2, \ldots, r_1 \tag{17}$$

Let $\mathcal{C}$ be the set of polynomials B of degree $\alpha - deg(R)$ which satisfy (17) with some A of degree $\alpha + deg(F) - (d+2)$. We want to prove that

$$\{B \mid_{\check{L}} \mid B \in \mathcal{C}\}$$

has the dimension $\alpha - deg(R) + 1$.
Let us verify this fact for a foliation with a noncomposite first integral $g : \mathbb{C}P(2) \to \overline{\mathbb{C}}$. In [9] p. 56-57, as a consequence of a theorem of Halphen, it has been proved that among the components $\check{D}_i, \imath = 1, 2, \ldots, r_1$ at most only two of them can have multiplicities greater than one, namely

$$\check{D}_1 : F = 0$$

$$\check{D}_2 : G = 0$$

Let g has multiplicity p, q along $\check{D}_1$, $\check{D}_2$, respectively. Then $g.c.d.(p, q) = 1$ and $g = \frac{F^p}{G^q}$(up to multiplication by a constant). The other $\check{D}_i$'s are given by

$$F^p - c_i G^q = 0$$

In this case we can assume that the invariant part of the divisor D is

$$n_1\{F = 0\} + n_2\{G = 0\}$$

because if $F^p - cG^q$ appears in $pol(\omega_1)$ then we have

$$\omega_1 = \frac{\omega_1'}{F^p - cG^q} = (\frac{F^p}{G^q} - c)^{-1}\frac{\omega_1'}{G^q}$$

$(\frac{F^p}{G^q} - c)$ is constant on the leaves of $\mathcal{F}$ and so $\frac{\omega_1'}{G^q}$ is relatively exact modulo $\mathcal{F}$.
Now the relations in (17) are

$$F \mid n_1 BR - pA,\ G \mid n_2 BR + qA \Rightarrow$$

$$n_1BR - pA = (n_1q + n_2p)F\bar{G},\ n_2BR + qA = (n_1q + n_2p)G\bar{F}$$
$$deg(\bar{F}) = \alpha - deg(G),\ deg(\bar{G}) = \alpha - deg(F)$$

which implies that

$$B = \frac{qF\bar{G} + pG\bar{F}}{R}$$

All elements of $\mathcal{C}$ has the above form and it is easy to verify our affirmation in this case.$\diamond$

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