On the value function for control problems with infinite horizon

J. Baumeister[1]

Department of Mathematics, Goethe University,
Robert-Mayer Str. 6-10, D-60054 Frankfurt a.M., Germany
E-mail: baumeist@math.uni-frankfurt.de

A. Leitão[1]

Department of Mathematics, Federal University of Santa Catarina,
P.O. Box 476, 88010-970 Florianopolis, Brazil
E-mail: aleitao@mtm.ufsc.br

G.N. Silva[2]

Department of Computational Sciences and Statistics, Universidade Estadual Paulista (UNESP),
15054-000 S. J. Rio Preto, Brazil
E-mail: gsilva@dcce.ibilce.unesp.br - Tel: +55 (17) 221-2201/221 2209 - Fax: 221-2203

In this paper we consider optimal control problems of infinite horizon type, whose control actions are given by L^1-functions. A characteristic of this type of problems is the utilization of a discount factor in the objective function. We provide an existence theorem, verify that the value function is locally Lipschitz and obtain necessary and sufficient conditions of optimality for the control problem in terms of upper Dini solutions of the Hamilton-Jakobi-Bellman inequality (equation). For a special class of control problems we prove that the value function is a viscosity solution of the Hamilton-Jakobi-Bellman (HJB) equation with certain decay condition. Finally, a "real life" example is provided to show the strength of our existence result.

Key Words: dynamic programming, optimal control, optimality conditions, infinite horizon, existence, viscosity solutions, upper Dini solutions

1. INTRODUCTION

[1]Work partially supported by CNPq and GMD.
[2]Corresponding author / Partially supported by FAPESP, CNPq and GMD

Given the initial values $(\tau, z) \in [0,\infty) \times \mathcal{R}^n$, let us consider the following optimal control problem:

$$\begin{cases} \text{Minimize } \; J(u;\tau,z) := \displaystyle\int_\tau^\infty e^{-\delta t} l(t, x(t), u(t))dt \\ \text{subject to} \\ u \in U_{ad}[\tau,\infty) := \big\{ w \in L^1_{loc}([\tau,\infty);\mathcal{R}^m) \mid w(t) \in \Omega \text{ a.e. in } [\tau,\infty) \big\} \\ x(t) = z + \displaystyle\int_\tau^t f(s, x(s), u(s))ds, \; t \in [\tau, \infty). \end{cases}$$

The problem above is called $P_\infty(\tau, z)$. Here $l : [0,\infty) \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}$, $f : [0,\infty) \times \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^n$ are given functions; $\Omega \subset \mathcal{R}^m$ is the set of admissible control actions; $\delta > 0$ denotes the discount rate. An admissible control strategy u is a L^1_{loc}-function satisfying the control constraint $u(t) \in \Omega$ a.e. in $[\tau,\infty)$. By L^1_{loc} we mean the space of locally integrable functions $w : [0,\infty) \to \mathcal{R}^n$. We call state trajectory an absolutely continuous functions $x : [0,\infty) \to \mathcal{R}^n$ that is a solution of the integral equation in $P_\infty(\tau, z)$.

A pair of functions $(x(\cdot), u(\cdot))$ consisting of an admissible control u and the corresponding state trajectory x is called an admissible process. We are specially interested in the set of admissible processes with finite costs:

$$\mathcal{D}_{\tau,z;\infty} := \Big\{ (x,u) \mid u \in U_{ad}[\tau,\infty); \;\; x(t) = z + \int_\tau^t f(r, x(r), u(r))dr; \; J(u;\tau,z) < \infty \Big\}.$$

We stress the fact that the cost functional in problem $P_\infty(\tau, z)$ is of infinite horizon type. Notice also that the problem is nonautonomous. Nonautonomous problems have to be considered when some linearization is done along a nonstationary trajectory.

This sort of control problem is particularly interesting for applications related to biological and economic sciences, since no reasonable bound can be placed on the time horizon. Several applications of this nature can be found in [6, 16, 17, 11], among others; see also Section 7.

We are mainly concerned with the investigation of properties of the value function for the family of control problems $P_\infty(\tau, z)$ and the implications of its properties in the theory of necessary and sufficient conditions of optimality. Among the properties of the value function is the fact that it is a solution of the Hamilton-Jakobi-Bellman (HJB) equation everywhere (in some sense). The difficulty to solve this equation comes from the fact that even for problems with regular data (C^1, for instance) the value function may not have any degree of regularity. So, it is necessary to seek for nonsmooth solutions for the HJB equation. In the literature, there are different successful definitions of solutions. The most popular are definitely solutions in the viscosity sense (see e.g., [9]). In our case, we first consider the concept of upper Dini solutions. We prove that the value function is an upper Dini solution of HJB–equation. Moreover, the value function is maximal among all upper Dini solutions of the HJB–equation. We also obtain some verification results. After that, we consider solutions in the sense of viscosity and prove the value function is also a viscosity solution of the HJB–equation. In all these results, a natural *end condition* for the value function is replaced with a decay condition at the infinity. For an account of different types of solutions of HJB–equation, see e.g., [3, 7].

As part of our developments we address the question of existence of optimal control processes for $P_\infty(\tau, z)$ since it seems that in the literature no result can be found for our situation; see Remark 2.1. To prove an existence result, one needs strong assumptions which cannot be left out. This means compactness of the set of admissible controls, some sort of convexity on the data and a bound on the trajectories. Unfortunately the boundedness of the admissible trajectories cannot be guaranteed by quantitative assumptions on the data alone as it would be desirable.

Section 2 is devoted to discussing the issue of existence of optimal processes for $P_\infty(\tau, z)$. In Section 3, we verify some properties of the value function, including (local) Lipschitz continuity. Upper Dini solutions of the HJB–equation is considered in Section 4, where the maximality of the value function among upper Dini solutions of the HJB–equation is provided. In Section 5 we derive two verification results using upper Dini solutions of the HJB–equation and in Section 6 we characterize the value function as a viscosity solution of the HJB-equation. In the final section we show that our existence result is applicable in a "real life" problem.

2. EXISTENCE OF OPTIMAL PROCESSES

In this section we discuss the issue of existence of an optimal control process for the $P_\infty(\tau, z)$. The existence result we provide here is under compactness of the set of admissible control actions and convexity of the so called extended velocity vector. It will be made clear shortly. We make the observation that the arguments of the proof are the usual in which we start with a minimizing sequence and under "compactness" of the trajectories on compact subintervals of $[\tau, \infty)$ we extract a subsequence, which converges to an absolutely continuous function uniformly on compact subintervals. This function, with the help of Filippov's lemma, is proved to be a state trajectory corresponding to some admissible control. Together, they form a process that is optimal for $P_\infty(\tau, z)$.

Let us consider problem $P_\infty(\tau, z)$ under the following assumptions:

A1) There exists $K_1 > 0$, such that for every fixed $t \in [0, \infty)$, $u \in \Omega$

$$|l(t, x, u) - l(t, z, u)| + |f(t, x, u) - f(t, z, u)| \le K_1 |x - z|,$$

holds for all $x, z \in \mathcal{R}^n$;

A2) There exists a positive scalar K_2 such that

$$|l(t,x,u)| + |f(t,x,u)| \leq K_2(1+|x|), \tag{1}$$

for all $t \in [0,\infty)$, $x \in \mathcal{R}^n$, $u \in \Omega$;

A3) The functions $l(\cdot,\cdot,\cdot)$, $f(\cdot,\cdot,\cdot)$ are continuous;

A4) Given $t \in [0,\infty)$, $u \in \Omega$ the functions $l(t,\cdot,u)$, $f(t,\cdot,u)$ are continuously differentiable and the partial derivatives l_x, f_x are continuous on $[0,\infty) \times \mathcal{R}^n \times \Omega$;

A5) Ω is compact;

A6) Given $(\tau,z) \in [0,\infty) \times \mathcal{R}^n$, there exists a bounded (invariant) set $\mathcal{S}_{\tau,z} \subset \mathcal{R}^n$ such that every admissible process $(x(\cdot),u(\cdot)) \in \mathcal{D}_{\tau,z;\infty}$ satisfy $x(t) \in \mathcal{S}_{\tau,z}$, for $t \geq \tau$.

THEOREM 2.1. *Suppose A1), A2), A3), A5) and A6) are satisfied. If, in addition, the set of points* $\hat{f}(t,x,\Omega) := \{(e^{-\delta t} l(t,x,v), f(t,x,v))^T \mid v \in \Omega\}$ *is a convex set in* $\mathcal{R}^{n+1}$*, then there exists an optimal process.*[2]

Proof. Let $(x_k,u_k) \in \mathcal{D}_{\tau,z;\infty}$, $k \in \mathcal{N}$, be a minimizing sequence for problem $P_\infty(\tau,z)$. So, there exists a real number c such that $J(u_k;\tau,z) = \int_\tau^\infty e^{-\delta t} l(t,x_k(t),u_k(t))dt := c_k \downarrow c$, as k goes to infinity. Notice that c is finite due to the assumptions above.

For any admissible control $v(\cdot)$ we define the extended corresponding trajectory in $\mathcal{R}^{n+1}$ by

$$\hat{x} = \begin{bmatrix} x^0 \\ x \end{bmatrix}, \quad \text{where } \dot{x}^0 = e^{-\delta t} l(t,x,v), \quad x^0(\tau) = 0.$$

In this case, $x^0(t) = \int_\tau^t e^{-\delta s} l(s,x(s),v(s))ds$ and $\int_\tau^\infty e^{-\delta s} l(s,x(s),v(s))ds = J(v;\tau,z)$.

Our extended systems is given by

$$\dot{\hat{x}}(t) = \hat{f}(t,x,v) = \begin{bmatrix} e^{-\delta t} l(t,x,v) \\ f(t,x,v) \end{bmatrix}, \quad \hat{x}(\tau) = \begin{bmatrix} 0 \\ z \end{bmatrix}.$$

Because of assumption *A2)* and Gronwall's lemma, there exists, for each $j \in \mathcal{N}$, a constant $M_j > 0$ such that $|\hat{x}(t)| < M_j$, $t \in [\tau,\tau+j]$ for all admissible processes $(x,u) \in \mathcal{D}_{\tau,z;\infty}$. Furthermore, for all k, we have the estimate

$$\|\dot{\hat{x}}_k(t)\| \leq K_2(1+M_j),\ t \in [\tau,\tau+j].$$

Hence, the sequence $\hat{x}_k$ is equicontinuous in each interval $[\tau,\tau+j]$. Using Arzelá-Ascoli theorem (see [1, Theorem 1, Section 0.3]) and a diagonal argument, we extract a subsequence (we do not relabel) $\hat{x}_k(\cdot)$

[2] The function $\hat{f}$ is called the extended velocity vector related to the control problem $P_\infty(\tau,z)$.

converging to some continuous function $\hat{x}(\cdot) : [\tau, \infty) \to \mathcal{R}^n$ uniformly over compact intervals.

Note that assumptions *A2)* and *A6)* yield, for all k,

$$\|\dot{x}_k^0(t)\| \leq K_2 e^{-\gamma t}(1 + \mathrm{diam}(\mathcal{S}_{\tau,z})),\ t \in [\tau, \infty). \tag{2}$$

Since the right hand side of (2) is in $L^1([\tau, \infty))$, it follows from (2) and [1, Theorem 4, Section 0.3] that we can extract a subsequence (we keep the notation) $\hat{x}_k(\cdot)$ such that $\dot{x}_k^0(\cdot)$ converges weakly to $\dot{x}^0(\cdot)$ in $L^1([\tau, \infty))$.

We can, by invoking hypotheses *A3)*, *A5)* and making use of standard arguments involving separation of convex sets by hyperplanes, conclude that

$$\dot{\hat{x}}(t) \in \hat{f}(t, x(t), \Omega) \text{ a.e. in } [\tau, \infty).$$

Now, choose $T > \tau$. By Filippov's lemma (see [10]) there exists a measurable function $v_T(\cdot) : [\tau, T] \to \Omega$ such that $\hat{x}(t) = z + \int_\tau^t \hat{f}(t, x(t), v_T(t))dt$ on $[\tau, T]$. Now, the weak convergence $\dot{x}_k^0(\cdot) \rightharpoonup \dot{x}^0(\cdot)$ in $L^1([\tau, \infty))$ yields

$$\Big| \int_\tau^T e^{-\delta t} l(t, x_k(t), u_k(t))dt - \int_\tau^T e^{-\delta t} l(t, x(t), v_T(t))dt \Big| < \frac{\epsilon}{3}, \tag{3}$$

for k sufficiently large.

Next we construct an admissible control $u(\cdot)$ defined on the whole interval $[\tau, \infty)$ in which $x(t) = z + \int_\tau^t f(s, x(s), u(s))ds$.

Let $I_j := [T + j, T + j + 1]$, for $j = 0, 1, 2, \ldots$ and consider the differential inclusions $\dot{x}(t) \in f(t, x(t), \Omega)$ defined on the intervals I_j. Again, by Filippov's lemma, there exists $v_j(\cdot) : I_j \to \Omega$ such that $x(t) = x(T + j) + \int_{T+j}^t f(s, x(s), v_j(s))ds$ for all $t \in I_j$. Let

$$u_T(t) := \begin{cases} v_T(t) & , t \in [\tau, T] \\ 0 & , t \geq T \end{cases} \quad \text{and} \quad u_j(t) := \begin{cases} v_j(t) & , t \in I_j \\ 0 & , t \notin I_j \end{cases}.$$

Defining $u(t) := u_T(t) + \sum_{j=0}^\infty u_j(t)$, it is easy to verify that $u(\cdot)$ is an admissible control corresponding to the output trajectory $x(\cdot)$. We claim that the process $(x(\cdot), u(\cdot))$ is optimal for problem $P_\infty(\tau, z)$. Indeed, consider the estimate

$$\begin{aligned}
&\Big| \int_\tau^\infty e^{-\delta t} l(t, x(t), u(t))dt - \int_\tau^\infty e^{-\delta t} l(t, x_k(t), u_k(t))dt \Big| \\
&\quad \leq \Big| \int_T^\infty e^{-\delta t} l(t, x(t), u(t))dt \Big| \\
&\quad + \Big| \int_\tau^T e^{-\delta t} l(t, x(t), u(t))dt - \int_\tau^T e^{-\delta t} l(t, x_k(t), u_k(t))dt \Big| \\
&\quad + \Big| \int_T^\infty e^{-\delta t} l(t, x_k(t), u_k(t))dt \Big|.
\end{aligned}$$

The first term on the right hand side can be made small by an appropriate choice of $T > \tau$. The second term can be estimated by using (3) and choosing k large enough. The last term is estimated by making use of the weak convergence of the sequence $\dot{x}_k^0(\cdot)$ to $\dot{x}^0(\cdot)$ in $L^1([\tau,\infty))$. ▮

REMARK 2.1. *There is an existence theorem for problems of infinite horizon type by Baum; see [2]. But, as one can see immediately, this result is not applicable in a situation where the dynamics is autonomous as it is the case in our example in Section 7.*

3. LIPSCHITZ CONTINUITY OF THE VALUE FUNCTION

Let us consider again the family of optimal control problems $P_\infty(\tau,z)$, for initial conditions $(\tau,z) \in \mathcal{R} \times \mathcal{R}^n$. We start this section defining a key function in the dynamic programming, namely the value function.

DEFINITION 3.1. The application

$$V : [0,\infty) \times \mathcal{R}^n \ni (\tau,z) \longmapsto \inf\{J(u;\tau,z) \,|\, (x,u) \in \mathcal{D}_{\tau,z;\infty}\} \in \mathcal{R} \tag{4}$$

is called the *value function* associated with the family of problems $P_\infty(\tau,z)$.

Next we define the Hamilton-function as the application

$$\begin{aligned} H : [0,\infty) \times \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^m \;&\longrightarrow\; \mathcal{R} \\ (t,x,\lambda,u) \;&\longmapsto\; \langle \lambda,\, f(t,x,u)\rangle \;+\; e^{-\delta t} l(t,x,u). \end{aligned} \tag{5}$$

We also define the (control independent) auxiliary function

$$\mathcal{H} : [0,\infty) \times \mathcal{R}^n \times \mathcal{R}^n \ni (t,x,\lambda) \;\longmapsto\; \min_{u\in\Omega}\{H(t,x,\lambda,u)\} \in \mathcal{R}. \tag{6}$$

In addition to *A1), ...,A5)* we consider the extra assumption

$\tilde{A}6)$ The Lipschitz constant K_1 in *A1)* and the discount rate δ satisfy $K_1 - \delta < 0$. Furthermore, given $(\tau,z) \in [0,\infty) \times \mathcal{R}^n$, there exists a neighborhood $\mathcal{V}_{\tau,z}$ of (τ,z) and a bounded (invariant) set $\mathcal{S}_{\tau,z} \subset \mathcal{R}^n$ such that for every initial condition $(\tilde{\tau},\tilde{z}) \in \mathcal{V}_{\tau,z}$, the admissible processes $(x(\cdot),u(\cdot)) \in \mathcal{D}_{\tilde{\tau},\tilde{z};\infty}$ satisfy $x(\cdot) \in \mathcal{S}_{\tau,z}$.

REMARK 3.2. *The assumption concerning the relationship between the Lipschitz constant K_1 and the discount rate δ is important; see [3, 20, 8].*

Under the extra assumption above it is possible to verify some important properties of the value function V, which we summarize in Lemma 3.2. First we prove an auxiliary lemma. The proof of these results don't contain new ideas but for the convenience of the reader we include the proof.

LEMMA 3.1. *Suppose the assumptions A1), A2) are satisfied. Let* (τ, z), $(\tau, z_1) \in [0, \infty) \times B_R(0)$ *and* $u \in U_{ad}[\tau, \infty)$. *Let* x, x_1 *be the associated states. Then*

$$|x(t) - x_1(t)| \leq e^{K_1(t-\tau)}|z - z_1|,\ t \geq \tau. \tag{7}$$

Furthermore, we have

$$|x(t) - z| \leq K_2(1 + R)e^{K_2(t-\tau)}|t - \tau|,\ t \geq \tau. \tag{8}$$

Proof. We start by proving the second assertion. Note that for $t \geq \tau$ we have

$$\begin{aligned} |x(t) - z| &\leq \int_\tau^t |f(q, x(q), u(q))| dq \leq \int_\tau^t K_2(1 + |x(q)|) dq \\ &\leq K_2|t - \tau| + \int_\tau^t K_2|x(q) - z| dq + \int_\tau^t K_2|z| dq \\ &\leq \int_\tau^t K_2(q)|x(q) - z| dq + K_2(1 + R)|t - \tau|. \end{aligned}$$

By the Lemma of Gronwall we obtain

$$|x(t) - z| \leq K_2(1 + R)e^{K_2(t-\tau)}|t - \tau|,\ t \geq \tau,$$

proving (8). To prove the first statement, observe that

$$x(t) - x_1(t) = \int_\tau^t \big[f(s, x(s), u(s)) - f(s, x_1(s), u(s))\big] ds + (z - z_1),\ t \geq \tau.$$

Thus, inequality (7) follows from the Lipschitz continuity of f and the lemma of Gronwall. ▌

We need to combine (admissible) controls defined in different time intervals. Let $u_1 \in U_{ad}[s_1, \infty]$, $u_2 \in U_{ad}[s_2, \infty]$, where $0 \leq s_1 \leq s_2 < \infty$. We define the *catenation*

$$u_1 \overset{s_2}{\sqcap} u_2 := \begin{cases} u_1(t) \ , t \in [s_1, s_2) \\ u_2(t) \ , t \in [s_2, \infty) \end{cases}.$$

LEMMA 3.2. *Let assumptions A1),..., A5) and* $\tilde{A}6)$ *hold. Then we have* *a) $V(\tau,z)$ is well-defined for all $(\tau,z)\in[0,\infty)\times\mathcal{R}^n$;*

b) The value function $V(\cdot,\cdot)$ is local Lipschitz continuous.

Proof. Assertion *a)* follow basically from *$\tilde{A}6)$* and Theorem 2.1. The assertion *b)* is the most interesting one. To prove it, it is enough to verify that for every $T>0$, $R>0$, there exists a constant C such that for all (s_1,z_1), $(s_2,z_2)\in[0,T]\times\overline{B}_R(0)$ the following inequality hold:

$$|V(s_1,z_1)-V(s_2,z_2)|\le C(|s_1-s_2|+|z_1-z_2|). \tag{9}$$

Since $[0,T]\times\overline{B}_R(0)$ is compact, it follows from *$\tilde{A}6)$* that there exists a bounded invariant set $\mathcal{S}_{T,R}\subset\mathcal{R}^n$ such that for every initial condition $(\tau,z)\in[0,T]\times\overline{B}_R(0)$, the admissible processes $(x(\cdot),u(\cdot))\in\mathcal{D}_{\tau,z;\infty}$ satisfy $x(\cdot)\in\mathcal{S}_{T,R}$, i.e.,

$$|x(t)|\le M_{T,R},\ \ t\in[\tau,\infty). \tag{10}$$

Let (s_1,z_1), $(s_2,z_2)\in[0,\infty)\times B_R(0)$ be given, with $s_1\le s_2$. Given $\varepsilon>0$, choose $u_1\in U_{ad}[s_1,\infty)$, $u_2\in U_{ad}[s_2,\infty)$ such that

$$V(s_i,z_i)\ \ge\ J(u_i;s_i,z_i)\ -\ \varepsilon\,,\ \ i=1,2\,.$$

Define $\widetilde{u}_1:=u_1\overset{s_2}{\sqcap}u_2$, $\widetilde{u}_2:=u_{1|[s_2,\infty)}$. Let x_1, x_2, $\widetilde{x}_1$, $\widetilde{x}_2$ be the corresponding states with $x_1(s_1)=\widetilde{x}(s_1)=z_1$, $x_2(s_2)=\widetilde{x}_2(s_2)=z_2$.[3] To we obtain

$$\begin{aligned} V(s_1,z_1)-V(s_2,z_2)\ &\le\ V(s_1,z_1)-J(u_2;s_2,z_2)+\varepsilon\\ &\le\ J(\widetilde{u}_1;s_1,z_1)-J(u_2;s_2,z_2)+\varepsilon \end{aligned}$$

and by symmetry

$$\begin{aligned} |V(s_1,z_1)-V(s_2,z_2)|\ &\le\ \varepsilon+|J(\widetilde{u}_1;s_1,z_1)-J(u_2;s_2,z_2)|\\ &\quad+|J(u_1;s_1,z_1)-J(\widetilde{u}_2;s_2,y_2)|. \end{aligned}$$

To obtain (9) we have to estimate the terms

$$|J(\widetilde{u}_1;s_1,z_1)-J(u_2;s_2,z_2)|\,,\ \ |J(u_1;s_1,z_1)-J(\widetilde{u}_2;s_2,y_2)|.$$

From the definition of J and assumptions *A2)*, *A3)* follows

$$|J(\widetilde{u}_1;s_1,z_1)-J(u_2;s_2,z_2)|\ \le\ \Big|\int_{s_1}^{s_2}e^{-\delta r}l(r,\widetilde{x}_1(r),\widetilde{u}_1(r))dr\Big|$$

[3]Notice that $J(\widetilde{u}_1;s_1,z_1)$ and/or $J(\widetilde{u}_2;s_2,z_2)$ may be infinite.

$$
\begin{aligned}
&+\Big|\int_{s_2}^{\infty} e^{-\delta r} l(r,\widetilde{x}_1(r),\widetilde{u}_1(r)) - e^{-\delta t} l(r,x_2(r),u_2(r))dr\Big| \\
\leq\ & K_2 \int_{s_1}^{s_2} e^{-\delta r}(1+|x_1(r)|)dr + K_1 \int_{s_2}^{\infty} e^{-\delta r}|\widetilde{x}_1(r)-x_2(r)|dr. \qquad (11)
\end{aligned}
$$

From the Gronwall lemma, the first term on the right hand side of (11) can be estimated by $\widetilde{C}|s_1-s_2|$, with some constant $\widetilde{C}$ depending on K_2, T and R. We now estimate the last term. It follows from (7) that

$$
\begin{aligned}
|\widetilde{x}_1(r)-x_2(r)| &\leq e^{K_1(r-s_2)}|\widetilde{x}_1(s_2)-z_2| \\
&\leq e^{K_1(r-s_2)}\big(|\widetilde{x}_1(s_2)-z_1|+|z_1-z_2|\big),
\end{aligned}
$$

for $r \geq s_2$. Furthermore, it follows from (8) that $|\widetilde{x}_1(s_2)-z_1| \leq C|s_2-s_1|$, with $C = K_2(1+R)e^{K_2T}$. Hence,

$$|\widetilde{x}_1(r)-x_2(r)| \leq Ce^{K_1(r-s_2)}(|s_1-s_2|+|z_1-z_2|),\ r > s_2.$$

Substituting in (11) and using the assumption $K_1 < \delta$, we obtain

$$|J(\widetilde{u}_1;s_1,z_1) - J(u_2;s_2,z_2)| \leq C\,(|s_1-s_2|+|z_1-z_2|).$$

The term $|J(u_1;s_1,z_1) - J(\widetilde{u}_2;s_2,y_2)|$ can be analogously estimated and it follows from (10) that

$$|V(s_1,z_1) - V(s_2,z_2)| \leq \varepsilon + C\,(|s_1-s_2|+|z_1-z_2|),$$

where the constant C is independent of z_1, z_2, s_1, s_2, ε and grows exponentially with T. By taking the limit $\varepsilon \downarrow 0$, the theorem follows. ∎

4. DINI SOLUTIONS OF THE HJB–EQUATION

We now discuss a first concept for a solution of the HJB–equation

$$\partial_t v(t,x) + \mathcal{H}(t,x,\partial_x v(t,x)) = 0\,,\ (t,x) \in (0,\infty)\times\mathcal{R}^n. \qquad (12)$$

As it is well known, we should not consider C^1-solutions of (12) since such solutions do not exist in general: no matter how regular the control problem is, the value function might not be differentiable everywhere. Thus, a concept of nonsmooth solution for the HJB–equation is necessary in this context and will be introduced in the sequel.

REMARK 4.3. *The HJB–equation (12) is of nonstationary type since the control problem is nonautonomous. If f, l do not depend on t explicitly, then, by a simple time transformation, the equation (12) can be transformed into a stationary equation.*

As explained in the introduction, we use the Dini solution concept in order to provide necessary and sufficient conditions of optimality for $P_\infty(0,z)$. For the rest of this section let Φ be a mapping from $[0,\infty) \times \mathcal{R}^n$ to $\mathcal{R}$. This mapping should be considered as candidate for the value function.

DEFINITION 4.1. Let $\Phi : [0,\infty) \times \mathcal{R}^n \longrightarrow \mathcal{R}$. The *lower Dini derivative* of Φ at (t,x) in the direction of $(1,h)$, where $h \in \mathcal{R}^n$, is defined by

$$D^-\Phi(t,x;1,h) := \liminf_{r\downarrow 0; w\to h} \frac{\Phi(t+r, x+rw) - \Phi(t,x)}{r}\,.$$

Before proceeding we make a remark on the definition of lower Dini directional derivatives. Recently, it has been widely accepted that a lower Dini directional derivative must be defined through the lower limit with changes in both variables (t,x), as it is defined above. However, since we are dealing with Lipschitz functions, the more general definition reduces to a limit on $r \downarrow 0$ alone, as according to [18, Lemma 5.1].

DEFINITION 4.2. We say that a function Φ is an *upper Dini solution* of the HJB–equation if it is locally Lipschitz and satisfies

a) $\forall\ (\tau,z) \in [0,\infty) \times \mathcal{R}^n$,

$$\inf_{u\in\Omega} \left\{ D^-\Phi(\tau,z;1,f(\tau,z,u)) + e^{-\delta\tau} l(\tau,z,u) \right\} \geq 0; \tag{13}$$

b) $\Phi(\tau,z) \to 0$, as $\tau \to \infty$, uniformly in each compact subset $K \subset \mathcal{R}^n$.

The next lemma provides necessary and sufficient conditions for a function Φ to satisfy (13).

LEMMA 4.1. *A function Φ satisfies (13) if and only if for all $(\tau,z) \in [0,\infty) \times \mathcal{R}^n$,*

$$\Phi(\tau,z) \leq \inf_{(x,u)\in\mathcal{D}_{\tau,z;\infty}} \left\{ \int_\tau^t e^{-\delta s} l(s,x(s),u(s))ds + \Phi(t,x(t)) \right\},\ t \geq \tau. \tag{14}$$

Proof. Let Φ be a upper Dini solution. Given $(\tau,z) \in [0,\infty) \times \mathcal{R}^n$ and $t > \tau$, take $(x(\cdot),u(\cdot)) \in \mathcal{D}_{\tau,z;\infty}$. Since the mapping $s \mapsto \Phi(s,x(s))$ is absolutely continuous on bounded intervals (see e.g. [18, Lemma 6.1]), we have

$$\begin{aligned}\Phi(t,x(t)) &= \Phi(\tau,z)+\int_\tau^t \frac{d}{dt}\Phi(s,x(s))ds \pm \int_\tau^t e^{-\delta s}l(s,x(s),u(s))ds\\ &= \Phi(\tau,z)+\int_\tau^t \Big[D^-\Phi(s,x(s);1,f(s,x(s),u(s)))\\ &\quad +e^{-\delta s}l(s,x(s),u(s))\Big]ds-\int_\tau^t e^{-\delta s}l(s,x(s),u(s))ds\\ &\geq \Phi(\tau,z)-\int_\tau^t e^{-\delta s}l(s,x(s),u(s))ds.\end{aligned}$$

Since $(x(\cdot),u(\cdot))$ is arbitrary, (14) follows. Conversely, let $(\tau,z)\in[0,\infty)\times\mathcal{R}^n$. For $w\in\Omega$, choose $(x(\cdot),u(\cdot))\in\mathcal{D}_{\tau,z;\infty}$ with $u(s)\equiv w$ in $[\tau,\tau+h]$, for small $h>0$. It follows from (14) that

$$h^{-1}[\Phi(\tau+h,x(\tau+h))-\Phi(\tau,z))]\geq -h^{-1}\int_\tau^{\tau+h} e^{-\delta s}l(s,x(s),w)ds.$$

Hence, by using Taylor expansion,

$$\begin{aligned}&h^{-1}[\Phi(\tau+h,z+hf(\tau,z,w)+o(h^2))-\Phi(\tau,z))]\\ &\qquad\qquad\geq -h^{-1}\int_\tau^{\tau+h} e^{-\delta s}l(s,x(s),w)ds.\end{aligned}$$

Taking the $\liminf$ as $h\downarrow 0$, we obtain

$$D^-\Phi(\tau,z;1,f(t,z,w))+e^{-\delta\tau}l(\tau,z,w)\geq 0.$$

Since $w\in\Omega$ is arbitrary, Φ is an upper Dini solution of the HJB–equation. ∎

In the sequel we state, without proof, the *dynamic programming principle* in the infinite horizon context. Its proof can be obtained with minor changes in the proofs of [12, Lemmas I.4.1 and I.4.2].

LEMMA 4.2. *Let $(x,u)\in\mathcal{D}_{\tau,z;\infty}$, where $(\tau,z)\in[0,\infty)\times\mathcal{R}^n$. Then we have:*

a) For $\tau\leq t_1\leq t_2<\infty$

$$V(t_1,x(t_1))\leq V(t_2,x(t_2))+\int_{t_1}^{t_2} e^{-\delta r}l(r,x(r),u(r))dr.$$

b) The process (x,u) is optimal if and only if the identity

$$V(t_1,x(t_1))= V(t_2,x(t_2))+\int_{t_1}^{t_2} e^{-\delta r}l(r,x(r),u(r))dr$$

holds for every $\tau\leq t_1\leq t_2<\infty$.

We are now in position to state the main result of this section, which relates the value function to the HJB-equation and provides a maximal property of the value function among upper Dini solutions of the HJB-equation.

PROPOSITION 4.1. *Under assumptions A1), ..., A6) we have:*

a) The value function $V(\cdot,\cdot)$ is an upper Dini solution of the HJB–equation;

b) $V(\cdot,\cdot)$ possesses the maximal property

$$V(\tau,z) = \max\{\Phi(\tau,z) \mid \Phi \text{ is a upper Dini solution}\};$$

c) $V(\cdot,\cdot)$ satisfies (13) and (14) with equality.

The second part of item *c)* above is known in the literature as Bellman principle of optimality.

Proof. We start proving *a)*. The Lipschitz continuity of V follows from Lemma 3.2. Let $(\tau,z) \in [0,\infty) \times \mathcal{R}^n$ be given. It follows from Lemma 4.2 *a)* that the value function satisfies

$$V(\tau,z) \le \int_\tau^t e^{-\delta s} l(s,x(s),u(s))ds + V(t,x(t))$$

for $t > \tau$ and $(x(\cdot),u(\cdot)) \in \mathcal{D}_{\tau,z;\infty}$, from where we deduce (14). Because of Lemma 4.1, V satisfies (13). Next we prove the decay condition in Definition 4.2. Since V is locally Lipschitz, it is enough to prove that $\lim_{\tau\to\infty} V(\tau,z) = 0$ for each $z \in \mathcal{R}^n$. Let $z \in \mathcal{R}^n$ and $\varepsilon > 0$ be given. For each $\tau \ge 0$, choose a process $(x_{\varepsilon,\tau}(\cdot),u_{\varepsilon,\tau}(\cdot)) \in \mathcal{D}_{\tau,z;\infty}$ satisfying

$$J(u_{\varepsilon,\tau};\tau,z) - \varepsilon \le V(\tau,z) \le J(u_{\varepsilon,\tau};\tau,z).$$

We now prove that

$$\lim_{\tau\to\infty} J(u_{\varepsilon,\tau};\tau,z) = 0. \tag{15}$$

Indeed, from *A2)* and the Gronwall lemma we obtain $|x_{\varepsilon,\tau}(t)| \le M$ for $t \ge \tau$. Hence,

$$J(u_{\varepsilon,\tau};\tau,z) \le \int_\tau^\infty e^{-\delta t} K_2(t)(1+M)dt.$$

Since $e^{-\delta t}K_2 \in L^1([0,\infty))$, (15) follows.

Next we prove *b)*. Let ϕ be an upper Dini solution of the HJB–equation, and $(x(\cdot),u(\cdot)) \in \mathcal{D}_{\tau,z;\infty}$. Define the set

$$A \ := \ \left\{ r \in [\tau,\infty) \mid \dot{x}(r) \text{ and } \frac{d}{dr}\phi(r,x(r)) \text{ exist} \right\}.$$

The set $[\tau,\infty)\setminus A$ has zero Lebesgue measure. So, for all $r\in A$,

$$\begin{aligned}\frac{d}{dr}\phi(r,x(r))&+e^{-\delta r}l(r,x(r),u(r))\\&=D^-\phi(r,x(r);1,\dot{x}(r))+e^{-\delta r}l(r,x(r),u(r))\\&\geq\inf_{u\in\Omega}\big\{D^-\phi(r,x(r);1,f(r,x(r),u)+e^{-\delta r}l(r,x(r),u)\big\}.\end{aligned}$$

Hence, it follows from (13) that

$$\frac{d}{dr}\phi(r,x(r))+e^{-\delta r}l(r,x(r),u(r))\geq 0,\ \ r\in A.$$

Note that for any $\varepsilon>0$, there exists $(x_\varepsilon(\cdot),u_\varepsilon(\cdot))\in\mathcal{D}_{\tau,z;\infty}$ such that

$$\begin{aligned}V(\tau,z)+\varepsilon&\geq\int_\tau^\infty e^{-\delta r}l(r,x_\varepsilon(r),u_\varepsilon(r))dr\\&=\int_\tau^\infty\Big\{\frac{d}{dr}\phi(r,x_\varepsilon(r))+e^{-\delta r}l(r,x_\varepsilon(r),u_\varepsilon(r))\Big\}dr+\phi(\tau,z)\\&\geq\phi(\tau,z)\end{aligned}$$

(Note that from *A2)* and the Gronwall lemma, $|x_\varepsilon(t)|\leq M$ for $t\geq\tau$). Since ε is arbitrary, $V(\tau,z)\geq\phi(\tau,z)$ holds. Therefore, the maximality of the value function follows. To prove *c)*, note that if $(x(\cdot),u(\cdot))$ is optimal for $P_\infty(\tau,z)$, then

$$V(\tau,z)=V(t,x(t))+\int_\tau^t e^{-\delta r}l(r,x(r),u(r))dr \tag{16}$$

for $t\geq\tau$ (by Lemma 4.2 *b)*). Now, by using similar arguments as in the proof of the converse of Lemma 4.1, one obtains (13) with equality. We have already proved that V satisfies (14). Then, the equality follows immediately from (16). ▌

We believe the following conjecture is true.

Conjecture 4.1. *V is the unique upper Dini solution of the HJB–equation satisfying (13) with equality.*

5. VERIFICATION RESULTS

This section presents two results that provide a link between Dini solutions of the HJB–equation and optimality conditions for $P_\infty(\tau,z)$.

Proposition 5.1. *Let $(\overline{x}(\cdot),\overline{u}(\cdot))$ be an admissible process for problem $P_\infty(\tau,z)$. If there exists an upper Dini solution Φ satisfying $\Phi(\tau,z)=J(\overline{u};\tau,z)$, then $(\overline{x}(\cdot),\overline{u}(\cdot))$ is an optimal process for $P_\infty(\tau,z)$*

(i.e., $\Phi(\tau, z) = V(\tau, z)$). Conversely, if $(\overline{x}(\cdot), \overline{u}(\cdot))$ is optimal for problem $P_\infty(\tau, z)$, then there exists an upper Dini solution satisfying $\Phi(\tau, z) = J(\overline{u}; \tau, z)$.[4]

Proof. Let $(\overline{x}(\cdot), \overline{u}(\cdot))$ be an admissible process for problem $P_\infty(\tau, z)$ and let Φ be an upper Dini solution satisfying $\Phi(\tau, z) = J(\overline{u}; \tau, z)$. Now, we choose an arbitrary process $(x(\cdot), u(\cdot)) \in \mathcal{D}_{\tau,z;\infty}$. Because of *A2)* and the Gronwall lemma, $|x(t)| \leq M$ for $t \geq \tau$. Then

$$\begin{aligned} J(u; \tau, z) &= \int_\tau^\infty e^{-\delta r} l(r, x(r), u(r)) dr \\ &= \int_t^\infty e^{-\delta r} l(r, x(r), u(r)) dr + \int_t^\infty \frac{d}{dr} \Phi(r, x(r)) dr + \Phi(t, \overline{x}(t)) \\ &\geq \Phi(\tau, z). \end{aligned}$$

(see the proof of Proposition 4.1 for details). Since the process $(x(\cdot), u(\cdot))$ was arbitrarily chosen, we have $J(u; \tau, z) \geq J(\overline{u}; \tau, z)$ for all $(x(\cdot), u(\cdot)) \in \mathcal{D}_{\tau,z;\infty}$, which provides the optimality of $(\overline{x}(\cdot), \overline{u}(\cdot))$.

To prove the converse it is enough to choose $\Phi = V$. ▮

PROPOSITION 5.2. *Let $(\tau, z) \in [0, \infty) \times \mathcal{R}^n$ and $(\overline{x}, \overline{u}) \in \mathcal{D}_{\tau,z;\infty}$ be given. If there exists an upper Dini solution Φ of the HJB–equation such that*

$$D^- \Phi(t, \overline{x}(t); 1, f(t, \overline{x}(t), \overline{u}(t))) + e^{-\delta t} l(t, \overline{x}(t), \overline{u}(t)) = 0, \ \ t \geq \tau, \tag{17}$$

then $(\overline{x}, \overline{u})$ is optimal for $P_\infty(\tau, z)$. Further, we have $\Phi(t, \overline{x}(t)) = V(t, \overline{x}(t))$, for $t \geq \tau$.

Proof. Let $t \in [s, \infty)$ and chose $(x, u) \in \mathcal{D}_{t,\overline{x}(t);\infty}$. Then we have

$$\begin{aligned} J(u; t, \overline{x}(t)) &= \int_t^\infty e^{-\delta r} l(r, x(r), u(r))\, dr \\ &= \int_t^\infty e^{-\delta r} l(r, x(r), u(r))\, dr + \Phi(t, x(t)) + \int_t^\infty \frac{d}{dt} \Phi(r, x(r)) dr \\ &= \Phi(t, \overline{x}(t)) + \int_t^\infty \left[D^- \Phi(r, x; 1, f(r, x, u)) + e^{-\delta r} l(r, x, u) \right] dr \\ &\geq \Phi(t, \overline{x}(t))\, . \end{aligned} \tag{18}$$

Similarly, we obtain

$$\Phi(t, \overline{x}(t)) = \Phi(t, \overline{x}(t)) + \int_t^\infty \Big[D^- \Phi(r, \overline{x}(r); 1, f(r, \overline{x}(r), \overline{u}(r)))$$

[4] In the literature the upper Dini solutions are also called verification functions, because they check the optimality of a process $(\overline{x}(\cdot), \overline{u}(\cdot))$.

$$
\begin{aligned}
&\quad +e^{-\delta r} l(r, \overline{x}(r), \overline{u}(r))\Big] dr \\
&= \int_t^\infty e^{-\delta r} l(r, \overline{x}(r), \overline{u}(r)) dr + \Phi(t, \overline{x}(t)) + \int_t^\infty \frac{d}{dt} \Phi(r, \overline{x}(r)) dr \\
&= \int_t^\infty e^{-\delta r} l(r, \overline{x}(r), \overline{u}(r)) dr \\
&= J(\overline{u}; t, \overline{x}(t))
\end{aligned} \tag{19}
$$

From (18), (19) and the fact that $(x, u) \in \mathcal{D}_{t,\overline{x}(t);\infty}$ was chosen arbitrarily, we obtain

$$
\Phi(t, \overline{x}(t)) = J(\overline{u}; t, \overline{x}(t)) = V(t, \overline{x}(t)), \tag{20}
$$

for every $t \in [\tau, \infty)$. Taking $t = \tau$ in (20), the optimality of the process $(\overline{x}, \overline{u})$ follows. ▮

6. VISCOSITY SOLUTIONS

In this section we consider solutions of the HJB–equation in the sense of viscosity and prove that the value function is a viscosity solution of the HJB–equation with the appropriate decay conditions.

We begin by introducing the concept of viscosity solutions for the HJB–equation.

Definition 6.1. A function $v : [0, \infty) \times \mathcal{R}^n \to \mathcal{R}$ is called *viscosity solution* of the partial differential equation

$$
\partial_t v(t, x) \,+\, \mathcal{H}(t, x, \partial_x v(t, x)) \;=\; 0\,,\; (t, x) \in (0, \infty) \times \mathcal{R}^n\,, \tag{21}
$$

when it satisfies the following conditions:

a) v is continuous and $v(\tau, z) \to 0$, as $\tau \to \infty$, uniformly in each compact $K \subset \mathcal{R}^n$;

b) For every *test function* $\phi \in C^1((0, \infty) \times \mathcal{R}^n; \mathcal{R})$, such that $v - \phi$ has a local maximum at $(t, x) \in (0, \infty) \times \mathcal{R}^n$, we have

$$
\partial_t \phi(t, x) \,+\, \mathcal{H}(t, x, \partial_x \phi(t, x)) \;\geq\; 0\,;
$$

c) For every *test function* $\phi \in C^1((0, \infty) \times \mathcal{R}^n; \mathcal{R})$, such that $v - \phi$ has a local minimum at $(t, x) \in (0, \infty) \times \mathcal{R}^n$, we have

$$
\partial_t \phi(t, x) \,+\, \mathcal{H}(t, x, \partial_x \phi(t, x)) \;\leq\; 0\,.
$$

If the function v satisfies only *a)* and *b)*, it is called *viscosity sub-solution*. If V satisfies only *a)* and *c)*, it is called *viscosity super-solution*.

We know from Lemma 3.2, that the value function is well defined and, in particular, continuous in $[0,\infty)\times\mathcal{R}^n$. We also know, from Proposition 4.1, that V satisfies the decay condition of item *a)* above and

$$V(\tau,z) = \inf_{(x,u)\in\mathcal{D}_{t,z;\infty}} \Big\{ \int_\tau^t e^{-\delta t} l(r,x(r),u(r))dr + V(t,x(t)) \Big\},\ t\in[s,\infty). \tag{22}$$

Given $t\in[\tau,\infty)$, we define the application $\mathcal{T}_{\tau,t}$, which associates to each function $\psi:\mathcal{R}^n\to\mathcal{R}$, the function $(\mathcal{T}_{\tau,t}\,\psi):\mathcal{R}^n\to\mathcal{R}$, defined by

$$(\mathcal{T}_{\tau,t}\,\psi)(z) := \inf_{(x,u)\in\mathcal{D}_{\tau,z;\infty}} \Big\{ \int_\tau^t e^{-\delta t} l(r,x(r),u(r))dr + \psi(x(t)) \Big\}.$$

From (22) it is easy to check the identity

$$V(\tau,z) = (\mathcal{T}_{\tau,t}\,V(t,\cdot))(z),\ t\ge\tau. \tag{23}$$

This is the starting point to characterize the value function as a viscosity solution of the HJB–equation, in the sense of definition 6.1.

THEOREM 6.1. *The value function V in Definition 3.1 is a viscosity solution of the partial differential equation (12).*

Proof. This proof follows the lines of a similar result in [12] for the finite horizon case. ▮

7. AN EXAMPLE: OPTIMAL TAXATION OF A FISHERY

In [6] a model for a fishery is described where the regulating authority is allowed to impose tax per unit of landed fish. Notice that also negative values are allowed (subsidies). The model is governed by the following quantities:

t	time
$x(t)$	population biomass at time t
$h(t)$	harvest rate at time t
$E(t)$	fishing effort at time t
$w(t)h(t)$	amount of tax delivered to the tax agency at time t

The model is based on the following assumptions:

- $h(t) = qE(t)x(t)$; q is a catch coefficient;

- $x'(t) = F(x(t)) - qE(t)x(t)$, $t \geq 0$, $x(0) = x_0$; F is the natural growth function;
- $(p - \frac{c}{x(t)} - w(t))h(t)$ is the income of the fishermen; $p \geq 0$ is the price of landed fish, $c \geq 0$ is the operating cost, w is a tax control;
- $E'(t) = kE(t)((p - w(t))x(t) - c)$, $t \geq 0$, $E(0) = E_0$; k is a scaling constant;
- existence of two biological equilibrium points:

$F(0) = F(\bar{x}) = 0$, $\bar{x} > 0$;

- properties of the production function:

$F \in C^2[0, \infty)$, $F(x) > 0$, $0 < x < \bar{x}$, $F''(x) < 0$, $0 \leq x \leq \bar{x}$;

- objective function:

$\int\limits_0^\infty e^{-\delta t}(px(t) - c)E(t)dt$; $\delta > 0$ is the instantaneous rate of discount.

To simplify the considerations we set $q := 1$ and $k := 1$. The logistics mapping $F(x) := x(1 - x)$ is an example of a production function which satisfies the model assumptions above.

It is the agency's objective to maximize the total present value of taxes. Therefore we should consider the following optimal control problem:

$$\begin{cases} \text{Maximize } \int\limits_\tau^\infty e^{-\delta t}(px(t) - c)E(t)dt \\ \text{subject to the constraints} \\ w_- \leq w(t) \leq w_+ \, , \, t \in [0, \infty) \, , \\ x' \; = \; F(x) - Ex \;\; , x(0) = x_0 \, , \\ E' \; = \; E((p - w(t))x - c), E(0) = E_0 \, . \end{cases}$$

The case in which $w_- < 0$ allows us to consider subsidies which would have the effect of increasing the rate of expansion of the fishery. It is reasonable to choose $w_+ = p$ since the tax should not be larger than the price for the fish. By changing the sign of objective function we obtain the following problem

$$\begin{cases} \text{Minimize } J(u; \tau, x_0, E_0) := \int\limits_\tau^\infty e^{-\delta t}(c - px(t))E(t)dt \\ \text{subject to the constraints} \\ u_- \leq u(t) \leq u_+, \text{ a.e. in } [0, \infty) \, , \\ x' \; = \; F(x) - Ex \; , x(0) = x_0 \, , \\ E' \; = \; E(u(t)x - c) \; , E(0) = E_0 \, . \end{cases}$$

Here we have $u_+ := p - w_-$, $u_- := p - w_+ = 0$.

We denote the problem above by $P(\tau, x_0, E_0)$. The value $v(\tau, x_0, E_0)$ of the problem $P(\tau, x_0, E_0)$ is the infimum in $P(\tau, x_0, E_0)$. It is the present value of the fishery. Clearly, the analysis of the problem should be different for the cases $u_+ = \infty$ and $u_+ < \infty$. For the modelling of economic problems see [6, 16, 11]. Here we consider the case $u_+ < \infty$ only. In the other case we would have to enlarge the set of admissible controls in order not to run into existence problems. For economic/biological reasons we assume that the initial values x_0, E_0 satisfy

$$0 < x_0 < \bar{x}\,,\, 0 < E_0\,.$$

Since the strip $(0, \bar{x}) \times (0, \infty)$ is invariant under the dynamics it is easy to see that the assumptions *A1), A2), A3), A4), A5)* are satisfied whatever the control action is.

Assuming that the problem has a solution we find in [6] the following description of the optimal scenario:

1. There is a point $(x^*, E^*) \in (0, \bar{x}) \times (0, \infty)$ such that for all initial values (x_0, E_0) with $x_0 \in (0, \overline{x}), E_0 > 0$, the associated optimal trajectory ends in (x^*, E^*).

2. The optimal controls are of bang–bang type; the values u_-, u_+ are chosen according to the so called **m**ost **r**apid **a**pproach **s**trategy (MRAS).

To be more specific, let Γ_+ be the max–control trajectory and Γ_- be the min–control trajectory:

$$\Gamma_+ \quad \left\{ \begin{array}{rl} x' = & -F(x) + Ex, x(0) = x^* \\ E' = & -E(u_+ x - c), E(0) = E^* \end{array} \right. ,$$

$$\Gamma_- \quad \left\{ \begin{array}{rl} x' = & -F(x) + Ex, x(0) = x^* \\ E' = & -E(u_- x - c), E(0) = E^* \end{array} \right. .$$

The construction of the trajectories concerning MRAS consists in

- choosing the min–control (u_-) or max–control (u_+), such that the system is driven to the curves Γ_+, Γ_-;
- following the trajectories Γ_+, Γ_-, until the system arrives in (x^*, E^*);
- staying in (x^*, E^*) by choosing a singular control u^*.

As it is shown in [2], the situation is somewhat more complicated and the conjecture MRAS does not lead to optimal trajectories: Γ_+, Γ_- are not the curves of approach but there are so called **switching curves** Σ_+, Σ_-, different from Γ_+, Γ_- and containing the switching points of the optimal controls. The numerical simulation data in [19] are in agreement with these results.

The equilibrium points of the dynamics of the problem play an important role in the following analysis. Let

$$x_+ := \frac{c}{u_+}\,,\, E_+ := \frac{F(x_+)}{x_+}\,. \tag{24}$$

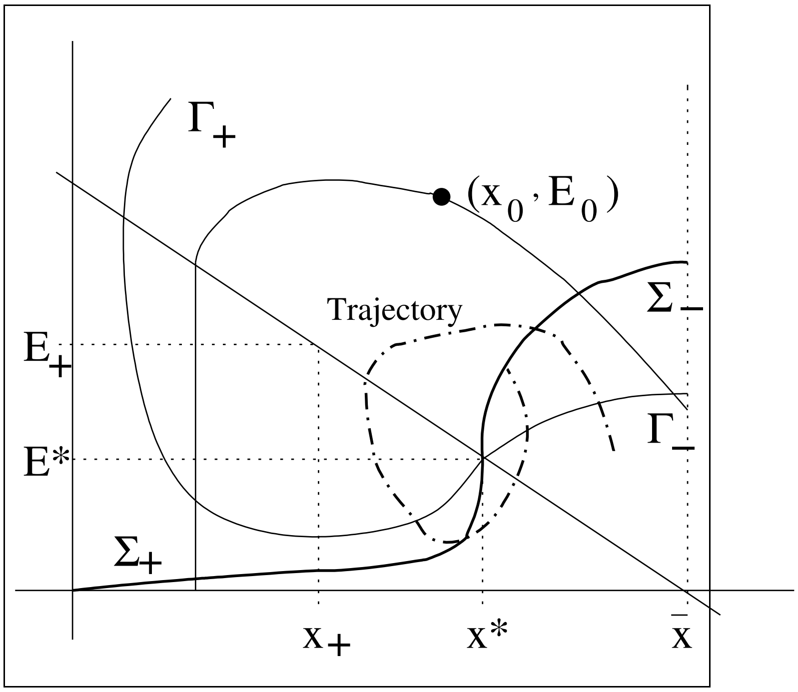

FIG. 1. The optimal scenario

Then the solution $(x(x_0,E_0;\cdot),E(x_0,E_0;\cdot))$ of

$$x' = F(x) - Ex\,,\ x(0) = x_0\,, \tag{25}$$

$$E' = E(u_+x - c)\,,\ E(0) = E_0\,, \tag{26}$$

exists for all times t and there exist $t_1 < 0 < t_2$ with

$$x(x_0,E_0;t_1) = \bar{x}\,,\ F(x(x_0,E_0;t_2)) - E(x_0,E_0;t_2)x(x_0,E_0;t_2) = 0\,. \tag{27}$$

Define

$$D(x_0,E_0) := \left\{(\xi,e)\middle|\xi = x(x_0,E_0;t)\,,\ 0 \le e \le E(x_0,E_0;t)\,,\ t_1 \le t \le t_2\right\}\,. \tag{28}$$

Clearly, $D(x_0,E_0)$ is an **invariant subset**: every trajectory starting in $D(x_0,E_0)$ stays there for all control actions for all times. Therefore we have shown that all admissible trajectories are bounded. Moreover, we can see that all admissible trajectories are locally uniformly bounded with respect to the initial data in the sense of $\tilde{A}6)$. Moreover, since the dynamics is linear in the control action the convexity property required in $\tilde{A}6)$ holds. Thus, our existence theorem is applicable.

The Hamilton functions $H, \mathcal{H}$ are given by

$$H(t,x,E,\lambda_1,\lambda_2,u) := \lambda_1(F(x)-Ex)+\lambda_2 E(ux-c)+e^{-\delta t}(c-px)E$$

and

$$\mathcal{H}(t,x,E,\lambda_1,\lambda_2) := \lambda_1(F(x)-Ex)+e^{-\delta t}(c-px)E-\lambda_2 Ec+Eu_+xh(\lambda_2)$$

respectively, where $h(\lambda) := \begin{cases} \lambda & \text{if } \lambda < 0 \\ 0 & \text{else} \end{cases}$.

The associated HJB–equation contains a function of $\partial_E v$ which is of the modules type; especially the equation is nonlinear.

Since $v(\tau,x_0,E_0) \le J(u_-;\tau,x_0,E_0)$ we see easily that $\lim\limits_{\tau\to\infty} v(\tau,x_0,E_0)=0$ holds for all admissible initial values x_0, E_0 locally uniformly; see Definition 4.2.

By setting

$$v(\tau,x_0,E_0) := e^{-\delta t} V(x_0,E_0)$$

we obtain the following stationary HJB–equation:

$$-\delta V + \widehat{\mathcal{H}}(x,E,\partial_x V,\partial_E V) = 0\,,\ (x,E)\in(0,\bar{x}\times(0,\infty)\,; \tag{29}$$

here $\widehat{\mathcal{H}}(x,E,\lambda_1,\lambda_2) := \lambda_1(F(x)-Ex)+(c-px)E-\lambda_2 Ec+Eu_+xh(\lambda_2)$.

To apply the verification results one should be able to solve equation (29). In [3] one can find some ideas for solving this equation numerically.

REFERENCES

1. Aubin, J.-P., Cellina, A., *Differential inclusions Set-valued maps and viability theory,* Springer-Verlag, Berlin, 1984

2. Baumeister, J.,*Optimal taxation of a fishery: Description of the optimal trajectories*, Preprint, Fachbereich Mathematik, Johann Wolfgang Goethe – Universität.

3. Bardi, M., Capuzzo-Dolcetta, I., *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations,* Birkhäuser Boston, 1997

4. Bellman, R., *Dynamic Programming,* Princeton University Press, Princeton, 1957

5. Carlson, D.A. e Haurie, A.B., *Infinite Horizon Optimal Control: Theory and Applications,* Springer–Verlag, Heidelberg, 1987

6. Clark, C.W., *Mathematical Bioeconomics: The Optimal Management of Renewable Resources. 2nd ed.,* John Wiley and Sons, New York, 1990

7. Clarke, F.H., Ledyaev, Yu. S., Stern, R.J. and Wolenski, P.R., *Nonsmooth analysis and control theory,* Springer–Verlag, New York, 1998

8. Colonius, F., *Asymptotic behavior of optimal control systems with low discount rates,* Mathematics of Operations research **14** (1989), 309–316

9. Crandall, M.G., Lions, P.-L., *Viscosity solutions of Hamilton–Jacobi equations,* Trans. Amer. Math. Soc. **277** (1983), 1–42

10. Filippov, A.F., *Differential Equations with Disconitnuous Right-Hand Sides,* Kluwer Academics, Dordrecht, 1988

11. Flaaten, *The economics of multispezies harvesting,* Studies in contemporary economics, Springer–Verlag, 1988

12. Fleming, W.H., Soner, H.M., *Controlled Markov Processes and Viscosity Solutions,* Springer–Verlag, 1993

13. Halkin, H., *Necessary conditions for optimal control problems with infinite horizon,* Econometrics **42** (1974), 267–272

14. Rischel, R.W., *An extended Pontryagin principle form control systems whose control laws contain measures,* SIAM J. Control and Optimization **3** (1965), 191–205

15. Rowland, J.D.L., Vinter, R.B., *Construction of optimal feedback controls,* Systems Control Lett. **16** (1991), 357–367

16. Seierstad, A., Sydsaetter, K., *Optimal Control Theory with Economic Applications,* North–Holland, Amsterdam, 1987

17. Sethi, P., Thompson, G., *Optimal Control Theory: Applications to Management Science. 2nd ed.,* Kluwer Academic Publishers, Boston, 2000

18. Vinter, R.B., Wolenski, P, *Hamilton-Jacobi theory for optimal control problems with data measurable in time,* SIAM J. Control Optim. **28** (1990), 1404–1419

19. Werner, C., *Der kürzeste Weg zum Gleichgewicht als Steuerungsprinzip – Ein Gegenbeispiel*, Thesis, Fachbereich Mathematik der J. W. Goethe–Universität

20. Wirth, F., *Convergence of the value functions of discounted infinite horizon optimal control problems with low discount rates,* Mathematics of Operations research **18** (1993), 1006–1019

21. Zabczyk, J., *Mathematical Control Theory: An Introduction,* Birkhäuser, Boston, 1992

22. Zhou, X.Y., *Verification theorems within the framework of viscosity solutions,* Journal of Math. Anal. and Appl. **177** (1993), 208–225