DECAY OF ALMOST PERIODIC SOLUTIONS OF CONSERVATION LAWS

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Abstract. We consider the asymptotic behavior of solutions of systems of inviscid or viscous conservation laws in one or several space variables, which are almost periodic in the space variables in a generalized sense introduced by Stepanoff and Wiener, which extends the original one of H. Bohr. We prove that if $u(x,t)$ is such a solution whose inclusion intervals at time t, with respect to $\varepsilon > 0$, satisfy $l_\varepsilon(t)/t \to 0$ as $t \to \infty$, and so that the scaling sequence $u^T(x,t) = u(Tx,Tt)$ is pre-compact as $T \to \infty$ in $L^1_{\text{loc}}(\mathbb{R}^{d+1}_+)$ then $u(x,t)$ decays to its mean value $\bar{u}$, which is independent of t, as $t \to \infty$. The decay considered here is in L^1_{loc} of the variable $\xi = x/t$, which implies, as we show, that $M_x(|u(x,t) - \bar{u}|) \to 0$, as $t \to \infty$, where M_x denotes taking the mean value with respect to x. In many cases we show that the solutions are almost periodic in the generalized sense if the initial data are. We also show, in these cases, how to reduce the condition on the growth of the inclusion intervals $l_\varepsilon(t)$ with t, as $t \to \infty$, for fixed $\varepsilon > 0$, to a condition on the growth of $l_\varepsilon(0)$ with ε, as $\varepsilon \to 0$, which amounts to impose restrictions only on the initial data. We show with a simple example the existence of almost periodic (non-periodic) functions whose inclusion intervals satisfy any prescribed growth condition as $\varepsilon \to 0$. The applications given here include inviscid and viscous scalar conservation laws in several space variables, some inviscid systems in chromatography and many viscous 2×2 systems such as those of nonlinear elasticity and Eulerian isentropic gas dynamics, with artificial viscosity, among others. In the case of the inviscid scalar equations and chromatography systems, the class of initial data for which decay results are proved includes, in particular, the L^∞ generalized limit periodic functions. Our procedures can be easily adapted to provide similar results for semilinear and kinetic relaxations of systems of conservation laws.

1. Introduction

The study of the asymptotic behavior of the solutions of nonlinear conservation laws goes back to the pioneering paper of E. Hopf [31] on the Burger's equation, which started the modern analytical theory of conservation laws and may be seen as its second major landmark after the foundational 1860 paper of Riemann [41]. In the referred paper Hopf introduces the vanishing viscosity method which means to add an artificial viscosity to the original equation, solve the approximating equation, and then send the viscosity coefficient to zero. By means of a tricky tranformation of the dependent variables, the now called Hopf-Cole transfomation, which transforms the viscous Burger's equation into the heat equation, he was able to obtain an explicit formula for the solutions. This was then used to prove the convergence of the vanishing viscosity solutions and also provided an explicit formula for the solution

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of the inviscid equation. The work of Hopf was followed by a series of papers of Oleinik surveyed in [40] establishing existence and uniqueness of solutions of scalar conservation laws in one space variable with strictly convex flux function, which satisfy an admissible (entropy) condition on the points of discontinuity introduced by her. Oleinik's entropy condition was not only crucial for the uniqueness of the solutions but alone can explain the asymptotic behavior of such solutions in two important representative cases: periodic and compact supported initial data (see [45]). However, the problem of the asymptotic behavior of the entropy solutions of scalar conservation laws with strictly convex flux function was first solved and to a large extent by Lax, in his well known paper [35]. Therein, Lax considered the general class of initial data $u_0 \in L^\infty(\mathbb{R})$ satisfying the condition that the limit

$$M(u_0) = \lim_{L\to\infty} \frac{1}{L}\int_a^{a+L} u_0(x)\,dx$$

exists uniformly in $a \in \mathbb{R}$. This includes the two special cases mentioned above. For this general class of initial data, Lax proves the decay of the solution in the L^∞ norm to $M(u_0)$ as $t \to \infty$. His analysis is heavily based in an explicit formula for the solution found by him, motivated by Hopf's formula. The decay property for such general class of initial data is still unknown for flux functions which are not strictly convex.

Concerning periodic initial data, an important progress was achieved by Glimm and Lax in their influencial paper [29]. Therein, they prove the global existence of entropy solution of the Cauchy problem for a general class of strictly hyperbolic genuinely nonlinear 2×2 systems of conservation laws, for L^∞ initial data of small oscillation. The solutions are constructed through Glimm scheme and the regularization property is also shown to be a consequence of stronger estimates for the interaction of waves holding for 2×2 systems, proved by Glimm in his celebrated paper [28]. For periodic initial data, they prove that the solution so obtained decays in the L^∞ norm at a rate $O(t^{-1})$. More recently, the study of the asymptotic structure of general periodic $BV_{\rm loc}$ entropy solutions of systems in the same class considered by Glimm-Lax, possessing the same decay property, was analized in detail by Dafermos [17], using his method of generalized characteristics. For scalar conservation laws in two space variables with $BV_{\rm loc}$ periodic initial data and a nonlinearity condition on the flux functions, the decay of the periodic entropy solutions in the $L^1_{\rm loc}$ norm was proved by Engquist and E [21].

In [8], Chen and Frid establish a connection between the decay of periodic entropy solutions, $u(x,t)$, in the $L^1_{\rm loc}(\mathbb{R}^d)$ metric as $t\to\infty$ and the pre-compactness in $L^1_{\rm loc}(\mathbb{R}^{d+1}_+)$ of the associated scaling sequence $u^T(x,t) = u(Tx,Tt)$, $T>0$. Here and in what follows we denote the L^p spaces with no reference to the range $\mathbb{R}^n$. They show that the pre-compactness of u^T in $L^1_{\rm loc}(\mathbb{R}^{d+1}_+)$ implies the decay in $L^1_{\rm loc}(\mathbb{R}^d)$ of those solutions, as $t\to\infty$. With the help of compactness results, such as those based on the compensated compactness theory (*e.g.* [19], [20], [6], [37], [38], [32], [7], [13], [26], etc.) and the one based on the kinetic formulation for scalar conservation laws in several space variables in [36], it was possible to obtain the decay in $L^1_{\rm loc}$ as $t\to\infty$ of large L^∞ periodic entropy solutions of many among the most representative systems of the theory, including, in particular, the Euler equations for isentropic gas dynamics, nonlinear elasticity and scalar conservation laws in several space variables with flux functions satisfying a nonlinearity condition. The result was also applied to obtain the decay of periodic solutions of systems of conservation laws with relaxation, in connection with results in [14], [15] also based on the compensated compactness theory. On the other hand, for *viscous* systems of conservation laws which are endowed with a strictly convex entropy, the decay of periodic solutions is in general easier and may be obtained by usual energy estimates as, for instance, those obtained in [30].

The purpose of this paper is first to establish an extension of the main result in [8] suitable for the study of the decay of generalized almost periodic solutions and then discuss several applications. The theory of almost periodic functions was founded by H. Bohr [4], in the context of continuous functions, and further extended to the context of measurable $L^p_{\rm loc}$ functions by Wiener [51], Stepanoff [46], Weyl [50], Besicovitch [1] and Besicovitch-Bohr [2] (see also [23]). For a complete account of this theory we refer also to the books of Bohr [3], Besicovitch [1] and Favard [22]. Here we will use a generalized concept of almost periodic functions which was introduced independently by Wiener and Stepanoff in the just referred papers. For definitions and properties used in this paper, concerning generalized almost periodic functions, see appendix A. We consider the asymptotic behavior of solutions $u(x,t)$ of systems of inviscid or viscous conservation laws in one or several space variables, which are almost periodic in the space variables x in the generalized sense of Stepanoff-Wiener, locally uniformly in the time variable $t\ge 0$. The latter means that if τ is an ε-almost period of $u(x,t)$, then it is also an ε-almost period of $u(x,s)$ for $0\le s\le t$. We prove that if the inclusion intervals of $u(x,t)$ at time t, with respect to $\varepsilon>0$, satisfy $l_\varepsilon(t)/t\to 0$ as $t\to\infty$, and the scaling sequence $u^T(x,t)=u(Tx,Tt)$ is pre-compact as $T\to\infty$ in $L^1_{\rm loc}(\mathbb{R}^{d+1}_+)$ then $u(x,t)$ decays to its mean value

$$\bar u=\lim_{L\to\infty}\frac{1}{(2L)^d}\int_{|x|_\infty\le L}u(x,t)\,dx, \tag{1.1}$$

which is independent of t, as $t\to\infty$. Here we denote $|x|_\infty=\max\{|x_i| : i=1,\dots,d\}$. The decay considered here is in $L^1_{\rm loc}$ of the variable $\xi=x/t$, which implies, as we show, that $M_x(|u(x,t)-\bar u|)\to 0$, as $t\to\infty$, where by M_x we denote the operation of taking average with respect to x, that is,

$$M_x(\psi(x,t))=\lim_{L\to\infty}\frac{1}{(2L)^d}\int_{|x|_\infty\le L}\psi(x,t)\,dx, \tag{1.2}$$

which always exists if $\psi(x,t)$ is (generalized) almost periodic in x. In many cases we show that the solutions are almost periodic in the generalized sense if the initial data are. We also show, in these cases, how to reduce the condition on the growth of the inclusion intervals $l_\varepsilon(t)$ with t, as $t\to\infty$, for fixed $\varepsilon>0$, to a condition on the growth of $l_\varepsilon(0)$ with ε, as $\varepsilon\to 0$, which amounts to impose restrictions only on the initial data. We show with a simple example the existence of almost periodic (non-periodic) functions whose inclusion intervals satisfy any prescribed growth condition as $\varepsilon\to 0$.

The applications given here include inviscid and viscous scalar conservation laws in several space variables, some inviscid systems in chromatography, and many viscous 2×2 systems such as those of nonlinear elasticity and Eulerian isentropic gas dynamics, with artificial viscosity, among others. In the case of inviscid scalar equations and chromatography systems, the class of initial data for which decay results are proved includes, in particular, the L^∞ generalized limit periodic functions.

We remark that, although we restrict our discussion to solutions uniformly bounded in L^∞ in order to keep a uniform treatment, most results presented here, in particular Theorem 2.1 and Theorem 5.1, can be suitably modified in order to be extended to the case of uniform bound in L^p, $1 \le p < \infty$. Specially the case $p = 2$ is of interest in connection with existence and compactness results obtained in this context (see, *e.g.*, [25]). We also remark that, following a procedure similar to the one in [8], the discusion about existence and decay of almost periodic solutions can be transported to the relaxation approximations. This extension becomes specially easier for the semilinear or kinetic approximations. For these relaxation approximations the proof of the almost periodicity of the solution is completely similar to the one for viscous approximations given in Theorem 5.1 of the present paper, based on Duhamel's principle. Also in connection with semilinear and kinetic approximations we mention the recent L^∞ uniform boundedness and compactness results of D. Serre [43]. See also Tzavaras [48] for compactness based on L^2 uniform estimates concerning the rate-type relaxation system in viscoelasticity.

The remaining of this paper is organized as follows. In section 2 we prove our general theorem on the decay of almost periodic solutions of inviscid or viscous multidimensional systems of conservation laws. In section 3 we analyze the case of the insvicid or viscous scalar conservation laws in several space variables. In section 4 we then discuss the application to some inviscid systems in chromatography. In section 5 we establish a general result concerning multidimensional viscous systems of conservation laws. In section 6 we comment on several applications of the theorem of section 5, which include well known systems such as those of Euler equations for isentropic gas dynamics and nonlinear elasticity with artificial viscosity. In appendix A we recall the definitions and some basic facts about almost periodic functions and show how to construct almost periodic (non-periodic) functions whose inclusion intervals with respect to $\varepsilon > 0$ satisfy any prescribed growth condition with ε as $\varepsilon \to 0$.

2. Decay of almost periodic solutions

We consider a multidimensional viscous or inviscid system of conservation laws

$$\partial_t u + \sum_{k=1}^{d} \partial_{x_k} f^k(u) = \sum_{k\,l} \partial^2_{x_k x_l} a_{k\,l}(u), \qquad x \in \mathbb{R}^d, t > 0, \tag{2.1}$$

where $u(x,t) \in \mathcal{U} \subseteq \mathbb{R}^n$, for some open set $\mathcal{U}$, and $f^k, a_{k\,l} : \mathcal{U} \to \mathbb{R}^n$ are smooth functions, for which an initial condition has been prescribed

$$u(x,0) = u_0(x). \tag{2.2}$$

A smooth function $\eta : \mathcal{U} \to \mathbb{R}$ is an entropy for (2.1) if there are smooth functions $q_k, b_{k\,l} : \mathcal{U} \to \mathbb{R}$, $k,l \in \{1,\dots,d\}$, called the associated entropy-fluxes and entropy-viscosities, respectively, such that

$$\nabla q_k = \nabla\eta \nabla f^k, \quad \nabla b_{k\,l}(u) = \nabla\eta\nabla a_{k\,l} \quad k,l \in \{1,\dots,d\}. \tag{2.3}$$

If η is strictly convex, (2.3) implies that the matrices ∇f^k are simultaneously symmetrizable by $\nabla^2\eta$ and, in particular, $\xi_1 \nabla f^1 + \cdots + \xi_d \nabla f^d$ is diagonalizable, for any $(\xi_1,\dots,\xi_d) \in \mathbb{R}^d$. The latter is the condition for the system (2.1) to be hyperbolic in the case where $a_{k\,l}(u) \equiv 0$, for all $k,l = 1,\dots,d$.

In this paper, we will only consider bounded measurable solutions, although the results hold also with slight adaptations in the more general case of $L^p_{\rm loc}$ solutions.

Definition 2.1. We say that $u\in L^\infty(\mathbb{R}^{d+1}_+)$ is an entropy solution (or simply a solution) of (2.1)-(2.2) if for any non-negative $\phi\in C^1_0(\mathbb{R}^{d+1})$ and for any convex entropy η, with associated entropy-fluxes and entropy-viscosities $q^k, b_{k\,l},\ k,l=1,\dots,d$, such that

$$\sum_{k\,l} v_k^\top \nabla^2\eta(u)\nabla a_{k\,l}(u)v_l\ge 0,\quad \text{for all } (v_1,\dots,v_d)\in(\mathbb{R}^n)^d, \tag{2.4}$$

one has

$$\iint_{\mathbb{R}^{d+1}_+}\{\eta(u)\phi_t+\sum q^k(u)\phi_{x_k}+\sum b_{k\,l}(u)\phi_{x_kx_l}\}\,dxdt+\int_{\mathbb{R}^d}\eta(u_0)\phi(x,0)\,dx\ge 0. \tag{2.5}$$

As usual, since the coordinate functions and their opposites $\pi_{i\pm}(u)=\pm u_i$, $i=1,\dots,n$, are obviously convex entropies with associated entropy-fluxes and entropy-viscosities $\pm f^k,\pm a_{k\,l}$, respectively, which trivially satisfy (2.4), the inequality (2.5) with $\eta(u)=\pi_{i\pm}(u)$, $i=1,\dots,n$, implies that u is a weak solution of (2.1)-(2.2), i.e., the equation

$$\iint_{\mathbb{R}^{d+1}_+}\{u\phi_t+\sum f^k(u)\phi_{x_k}+\sum a_{k\,l}(u)\phi_{x_kx_l}\}\,dxdt+\int_{\mathbb{R}^d}u_0(x)\phi(x,0)\,dx=0, \tag{2.6}$$

holds for any $\phi\in C^1_0(\mathbb{R}^{d+1})$. When $a_{k\,l}(u)\equiv 0$, $k,l=1,\dots,d$, entropy solutions are in general non-smooth, which is a basic fact in the theory of conservation laws (see, *e.g.*, [18], [42], [45]).

We are interested in the asymptotic behavior of solutions $u(x,t)$ of (2.1)-(2.2) which are generalized almost periodic functions, in the sense of Stepanoff-Wiener, which we abridge by saying that $u(x,t)$ is S-*a.p.*, in the x variable, locally uniformly in $t\ge 0$. For definitions and basic properties about generalized almost periodic functions see the appendix A. By locally uniformly in $t\ge0$ we mean that if τ is an ε-almost period of $u(x,t)$, then it is also an ε-almost period of $u(x,s)$, for $0\le s\le t$.

As in [8], we denote by $u^T(x,t)$, $T>0$, the scaling sequence associated with $u(x,t)$ defined by

$$u^T(x,t)=u(Tx,Tt). \tag{2.7}$$

Set

$$\bar u=\lim_{L\to\infty}\frac{1}{(2L)^d}\int_{|x|_\infty\le L}u_0(x)\,dx. \tag{2.8}$$

We may now state our general decay result for almost periodic solutions of (2.1)-(2.2).

Theorem 2.1. *Let $u(x,t)$ be a solution of (2.1)-(2.2) which is S-a.p. in x, locally uniformly in $t\ge0$. Let $l_\varepsilon(t)$ denote an inclusion interval of $u(x,t)$ with respect to $\varepsilon>0$. Assume the following:*

(i) *$l_\varepsilon(t)/t\to0$ as $t\to\infty$;*
(ii) *$u^T(x,t)$ is pre-compact in $L^1_{\rm loc}(\mathbb{R}^{d+1}_+)$ as $T\to\infty$.*

Then $u^T \to \bar u$ *as* $T\to\infty$ *in* $L^1_{\rm loc}(\mathbb{R}^{d+1}_+)$ *and one has*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T M_x(|u(x,t)-\bar u|)\,dt=0. \tag{2.9}$$

Moreover, in the inviscid case where $a_{k\,l}(u)\equiv 0$*, for all* k,l*, if* (2.1) *is endowed with a strictly convex entropy then one has* $u(\xi t,t)\to\bar u$ *in* $L^1_{\rm loc}(\mathbb{R}^d)$*, as* $t\to\infty$*. In particular, one has* $M_x(|u(x,t)-\bar u|)\to 0$ *as* $t\to\infty$*. The latter holds also in the viscous case,* $a_{k\,l}(u)\neq 0$*, for some* k,l*, provided that* (2.1) *is endowed with a strictly convex entropy satisfying* (2.4) *and* $\nabla_x u(x,t)$ *is uniformly bounded in* $\mathbb{R}^d\times[t_0,\infty)$ *for some* $t_0>0$.

Proof. The method of the proof is similar to the one in [8]. Consider an increasing sequence $\{T_k\}\subseteq(0,\infty)$, with $T_k\to\infty$, as $k\to\infty$, such that $u^{T_k}(x,t)$ converges in $L^1_{\rm loc}(\mathbb{R}^{d+1}_+)$ to a certain L^∞ function $\bar u(x,t)$. We will show that $\bar u(x,t)=\bar u$, a.e. in $\mathbb{R}^{d+1}_+$, where $\bar u$ is given by (2.8). We first show that, for almost all $t>0$, $\bar u(x,t)$ is independent of x. For that, given $\varepsilon>0$ and $t_0>0$, we consider the set

$$Q_{\varepsilon,t_0}=\{\frac{\tau}{T_k}\,:\,\tau \text{ is an } \varepsilon\text{-almost period of } u(\cdot,T_kt_0)\}.$$

We notice that Q_{ε,t_0} is dense in $\mathbb{R}^d$. This is clear from the fact that, since $l_\varepsilon(T_kt_0)/T_k\to 0$ as $T_k\to\infty$, given any cube with edge of lenght $\delta>0$, if $l_\varepsilon(T_kt_0)/T_k<\delta$, we can find a vector $\tau/T_k\in Q_{\varepsilon,t_0}$ inside this cube. We will show that for any $y\in\mathbb{R}^d$ we have $\bar u(x+y,t)=\bar u(x,t)$, for a.e. $(x,t)\in\mathbb{R}^{d+1}_+$. Now, let $\phi(x,t)$ be any continuous function with compact support contained in $[-L_0,L_0]^d\times[0,t_0]$, and let $y\in\mathbb{R}^d$ be given. By passing to a subsequence if necessary, we can find $y_k\in Q_{\varepsilon,t_0}$ such that $y_k\to y$,as $k\to\infty$, and y_k is an ε-almost period of $u^{T_k}(x,t)$ for $0\le t\le t_0$. We then have,

$$\begin{aligned}\int_{\mathbb{R}^{d+1}_+}\bar u(x+y,t)\phi(x,t)\,dxdt&=\lim_{k\to\infty}\int_{\mathbb{R}^{d+1}_+}u^{T_k}(x+y,t)\phi(x,t)\,dxdt\\ &=\lim_{k\to\infty}\int_{\mathbb{R}^{d+1}_+}u^{T_k}(x,t)\phi(x-y,t)\,dxdt\\ &=\lim_{k\to\infty}\int_{\mathbb{R}^{d+1}_+}u^{T_k}(x,t)\phi(x-y_k,t)\,dxdt\\ &=\lim_{k\to\infty}\int_{\mathbb{R}^{d+1}_+}u^{T_k}(x+y_k,t)\phi(x,t)\,dxdt\\ &\le\lim_{k\to\infty}\int_{\mathbb{R}^{d+1}_+}u^{T_k}(x,t)\phi(x,t)\,dxdt+C(L_0,t_0)\|\phi\|_\infty\varepsilon\\ &=\int_{\mathbb{R}^{d+1}_+}\bar u(x,t)\phi(x,t)\,dxdt+C(L_0,t_0)\|\phi\|_\infty\varepsilon,\end{aligned}$$

and similarly we get

$$\int_{\mathbb{R}^{d+1}_+}u(x+y,t)\phi(x,t)\,dxdt\ge\int_{\mathbb{R}^{d+1}_+}\bar u(x,t)\phi(x,t)\,dxdt-C(L_0,t_0)\|\phi\|_\infty\varepsilon,$$

where $C(L_0, t_0)$ is a positive constant depending only on L_0, t_0. Since $\varepsilon > 0$ is arbitrary, we get

$$\int_{\mathbb{R}^{d+1}_+} \bar{u}(x+y,t)\phi(x,t)\,dxdt = \int_{\mathbb{R}^{d+1}_+} \bar{u}(x,t)\phi(x,t)\,dxdt$$

The function ϕ being also arbitrary we finally get $\bar{u}(x+y,t) = \bar{u}(x,t)$, provided that (x,t) and $(x+y,t)$ are Lebesgue points of $\bar{u}$. In particular, since $y \in \mathbb{R}^d$ is arbitrary, we get $\bar{u}(x_1,t) = \bar{u}(x_2,t)$ whenever (x_1,t) and (x_2,t) are Lebesgue points of $\bar{u}(x,t)$ and so $\bar{u}(x,t) = \bar{u}(t)$, for a.e. $(x,t) \in \mathbb{R}^{d+1}_+$, for a certain bounded measurable function $\bar{u}(t)$ depending only on t.

Now, since $\bar{u}$ is a weak solution of the inviscid correspondent of (2.1)-(2.2) ($a_{k\,l}(u) \equiv 0$, $k,l = 1,\dots,d$) with initial data $u_0(x) \equiv \bar{u}$, $x \in \mathbb{R}^d$, we easily see that $\bar{u}(t) = \bar{u}$, for a.e. $t \geq 0$. Hence, we arrived at $u^T \to \bar{u}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}_+)$, as $T \to \infty$. As usual, the latter implies

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{|\xi|_\infty \leq c} |u(\xi t,t) - \bar{u}|\,d\xi dt = 0, \tag{2.10}$$

for any $c > 0$. So, (2.9) will follow from (2.15) below, letting $\varepsilon \to 0$.

Now let us prove the last part of the statement. We take the Dafermos' quadratic entropy $\alpha(u,\bar{u})$ associated with a strictly convex entropy $\eta(u)$, given by

$$\alpha(u,\bar{u}) = \eta(u) - \eta(\bar{u}) - \nabla\eta(\bar{u})(u - \bar{u}),$$

with associated entropy-fluxes $\beta^k(u,\bar{u})$ and entropy-viscosities $\gamma_{k\,l}(u,\bar{u})$ given by

$$\begin{aligned}\beta^k(u,\bar{u}) &= q^k(u) - q^k(\bar{u}) - \nabla\eta(\bar{u})(f^k(u) - f^k(\bar{u})),\\ \gamma_{k\,l}(u,\bar{u}) &= b_{k\,l}(u) - b_{k\,l}(\bar{u}) - \nabla\eta(\bar{u})(a_{k\,l}(u) - a_{k\,l}(\bar{u})).\end{aligned}$$

Clearly, we have

$$\partial_t\alpha(u,\bar{u}) + \sum \partial_{x_k}\beta^k(u,\bar{u}) \leq \sum \partial^2_{x_k x_l}\gamma_{k\,l}(u,\bar{u}), \tag{2.11}$$

in the sense of distributions. From (2.11), as in [10] pp.317-319, for the inviscid case, and pp.352-353, for the viscous case (see also [11], pp.38-41), using also the boundedness of $\nabla_x u$, for $t \geq t_0 > 0$, in the latter case, it follows that, for any fixed $c > 0$, the function

$$Y(t) = \int_{|\xi|_\infty \leq c} \alpha(u(\xi t,t),\bar{u})\,d\xi,$$

is in $BV_{\mathrm{loc}}(0,\infty)$ and satisfies

$$\frac{dY}{dt}(t) \leq \frac{C}{t},$$

in the sense of measures, for some $C > 0$. The above inequality, together with (2.10), which we may write in the form

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T Y(t)\,dt = 0,$$

leads, as in the proof of Theorem 2.3 of [10], to the conclusion that

$$\operatorname*{ess\,lim}_{t\to\infty} \int_{|\xi|_\infty \leq c} |u(\xi t,t) - \bar{u}|\,d\xi = 0, \tag{2.12}$$

for any $c > 0$.

To prove the decay in terms of mean values, let us partition $\mathbb{R}^d$ in a net of d-dimensional cubes with edges of length $3l_\varepsilon(t)$ parallel to the axes. Denote by $\mathcal{S}_t$

the set of such cubes contained in $\{x \in \mathbb{R}^d : |x| \le ct\}$, for certain fixed $c > 0$. Clearly, for each $I \in \mathcal{S}_t$ there is an ε-almost period τ_I such that $I - \tau_I \supset [0, 2l_\varepsilon(t)]^d$. Let $N(t)$ be the number of elements of $\mathcal{S}_t$. Since, $l_\varepsilon(t)/t \to 0$ as $t \to \infty$, we have $N(t) > 0$ and $(2c)^d/2 \le 3^d N(t) l_\varepsilon(t)^d/t^d \le (2c)^d$, for t sufficiently large. Hence, we have

(2.13)

$$\begin{aligned}
\int_{|\xi|\le c} |u(\xi t, t) - \bar u|\, d\xi &= \frac{1}{t^d}\int_{|x|\le ct} |u(x,t)-\bar u|\,dx \ge \frac{1}{t^d}\sum_{I\in\mathcal{S}_t}\int_I |u(x,t)-\bar u|\,dx \\
&\ge \frac{1}{t^d}\sum_{I\in\mathcal{S}_t}\int_I |u(x-\tau_I,t)-\bar u|\,dx - \frac{1}{t^d}\sum_{I\in\mathcal{S}_t}\int_I |u(x,t)-u(x-\tau_I,t)|\,dx \\
&\ge -\varepsilon\frac{[(3l_\varepsilon(t))^d]N(t)}{t^d} + 3^d\frac{(2l_\varepsilon(t))^dN(t)}{t^d}\frac{1}{(2l_\varepsilon(t))^d}\int_{[0,2l_\varepsilon(t)]^d}|u(x,t)-\bar u|\,dx \\
&\ge -\varepsilon(2c)^d + \frac{(4c)^d}{2}\frac{1}{l_\varepsilon(t)^d}\int_{[0,2l_\varepsilon(t)]^d}|u(x,t)-\bar u|\,dx
\end{aligned}$$

Now, we partition $\mathbb{R}^d$ in a net of cubes with edges of length $l_\varepsilon(t)$ parallel to the axes. For each such cube I' there exists an ε-almost period $\tau_{I'}$ such that $I' - \tau_{I'} \subseteq [0, 2l_\varepsilon]^d$. Hence, we get

$$\begin{aligned}
M_x(|u(x,t)-\bar u|) &= \lim_{s\to\infty}\frac{1}{(2sl_\varepsilon(t))^d}\int_{[-sl_\varepsilon(t),sl_\varepsilon(t)]^d}|u(x,t)-\bar u|\,dx \\
&\le \lim_{s\to\infty}\frac{1}{(2sl_\varepsilon(t))^d}\sum_{I'\subseteq[-sl_\varepsilon(t),sl_\varepsilon(t)]^d}\int_{I'}|u(x-\tau_{I'},t)-\bar u|\,dx \\
&+ \lim_{s\to\infty}\frac{1}{(2sl_\varepsilon(t))^d}\sum_{I'\subseteq[-sl_\varepsilon(t),sl_\varepsilon(t)]^d}\int_{I'}|u(x,t)-u(x-\tau_{I'},t)|\,dx \\
&\le \frac{1}{l_\varepsilon(t)^d}\int_{[0,2l_\varepsilon(t)]^d}|u(x,t)-\bar u|\,dx + 2^d\varepsilon
\end{aligned} \tag{2.14}$$

So, from (2.13) and (2.14) we obtain

$$\int_{|\xi|\le c}|u(\xi t,t)-\bar u|\,d\xi \ge \frac{(4c)^d}{2}M_x(|u(x,t)-\bar u|) - (4c)^d\varepsilon, \tag{2.15}$$

which, together with (2.12), gives

$$\operatorname*{ess\,lim\,sup}_{t\to\infty} M_x(|u(x,t)-\bar u|) \le 2\varepsilon,$$

and, since ε can be taken arbitrarily small, we conclude

$$\operatorname*{ess\,lim}_{t\to\infty} M_x(|u(x,t)-\bar u|) = 0,$$

as desired.

□

Remark 2.1. The first part of the statement of Theorem 2.1 holds also for almost periodic solutions (in a suitable sense) of the more general class of viscous systems of the form

$$\partial_t u + \sum_k \partial_{x_k} f^k(u) = \sum_{k,l}\partial_{x_k}\left(B_{kl}(u)\partial_{x_l}u\right), \tag{2.16}$$

as is clear from the proof. The second part also holds provided that we define the notion of a strictly convex entropy η for (2.16) to mean now that η is strictly convex, there exist functions q^k such that

$$\nabla q^k(u) = \nabla\eta(u)\nabla f^k(u),$$

and

$$\sum_{k,l} \nabla^2\eta(u)(B_{k\,l}(u)v_k, v_l) \geq 0, \quad \text{for all } (v_1,\dots,v_d)\in(\mathbb{R}^n)^d.$$

3. Scalar conservation laws in several space variables

In this section we consider the initial value problem for a scalar conservation law in several space variables

$$\partial_t u + \sum_{k=1}^d \partial_{x_k} f^k(u) = 0, \tag{3.1}$$

$$u(x,0) = u_0(x), \tag{3.2}$$

where the $f^k(u)$ are smooth functions and u_0 is a bounded S-*a.p.* function defined in $\mathbb{R}^d$. We are going to apply Theorem 2.1 to obtain the decay of the entropy solution of (3.1)-(3.2) provided that u_0 satisfies a suitable condition on the growth of its inclusion intervals $l_\varepsilon(0)$ as $\varepsilon\to 0$. Existence and L^1_{loc} stability of entropy solutions of (3.1)-(3.2), with $u_0\in L^\infty(\mathbb{R}^d)$, was proved by Kruzkov [33]. We now establish the following theorem.

Theorem 3.1. *Let $f(u) = (f^1(u),\dots,f^d(u))$ satisfy the nonlinearity condition*

$$\mathit{meas}\{u\in\mathbb{R} : \tau+\kappa\cdot f'(u)=0\}=0, \quad \forall(\tau,\kappa)\in\mathbb{R}^{d+1}, s.t.\ \tau^2+|\kappa|^2=1. \tag{3.3}$$

Assume $u_0\in L^\infty(\mathbb{R}^d)$ is S-a.p. and there exists a sequence of S-a.p. functions $u_{0,\nu}$, $\nu\in\mathbb{N}$, belonging to $L^\infty(\mathbb{R}^d)$ such that $M_x(|u_0(x)-u_{0,\nu}(x)|)\to 0$, as $\nu\to\infty$, and so that, for each $\nu\in\mathbb{N}$, the inclusion intervals of $u_{0,\nu}$ with respect to ε, $l_\varepsilon^\nu(0)$, satisfy $\varepsilon^{1/d}l_\varepsilon^\nu(0)\to 0$, as $\varepsilon\to 0$. Let $u(x,t)$ be the unique entropy solution of (3.1)-(3.2). *Then $u(\xi t,t)\to\bar u$ as $t\to\infty$ in $L^1_{\text{loc}}(\mathbb{R}^d)$, and, in particular, $M_x(|u(x,t)-\bar u|)\to 0$ as $t\to\infty$.*

Proof. Let us first consider the case in which the inclusion intervals of u_0 with respect to $\varepsilon>0$, $l_\varepsilon(0)$, themselves satisfy the growth condition $\varepsilon^{1/d}l_\varepsilon(0)\to 0$, as $\varepsilon\to 0$. To apply Theorem 2.1 we need first to show that $u(x,t)$ is S-*a.p.* in x locally uniformly in t, and $l_\varepsilon(t)/t\to 0$ as $t\to\infty$. For $x\in\mathbb{R}^d$ and $K>0$, let us denote by I_x^K the d-dimensional cube $[x_1,x_1+K]\times\cdots\times[x_d,x_d+K]$, and set $I_x^1=I_x$. Let

$$K_0=\sup\{|f'(u)|_\infty : u\in[-\|u_0\|_\infty,\|u_0\|_\infty]\},$$

and $x_0=(1,\dots,1)\in\mathbb{R}^d$. Kruzkov's stability theorem [33] gives, for a.e. $t>0$,

$$\begin{aligned}\int_{I_x}|u(y+\tau,t)-u(y,t)|\,dy &\leq \int_{I_{x-tK_0x_0}^{2K_0t}}|u_0(y+\tau)-u_0(y)|\,dy\\ &\leq (2K_0t+1)^d\sup_{x\in\mathbb{R}^d}\int_{I_x}|u_0(y+\tau)-u_0(y)|\,dy.\end{aligned} \tag{3.4}$$

So, for any $\varepsilon>0$, if τ is an almost period of u_0 associated with $\varepsilon/(2K_0t+1)^d$, then it is an ε-almost period of $u(x,t)$, and we may take $l_\varepsilon(t)=l_{\varepsilon/(2K_0t+1)^d}(0)$ as inclusion intervals of $u(x,t)$ with respect to ε. Hence, from the assumption on the growth of $l_\varepsilon(0)$, as $\varepsilon \to 0$, we conclude that $l_\varepsilon(t)/t \to 0$ as $t \to \infty$. Now, the other hypothesis in Theorem 2.1, namely, the compactness of the scaling sequence $u^T(x,t) = u(Tx,Tt)$ follows from (3.3) by the compactness result of Lions-Perthame-Tadmor in [36]. Hence, we can apply Theorem 2.1 to obtain the desired conclusion.

Now, let us consider the general case in which $M_x(|u_0 - u_{0,\nu}|) \to 0$, as $\nu \to \infty$, for a sequence $u_{0,\nu}$ of *S-a.p.* functions belonging to $L^\infty(\mathbb{R}^d)$, whose inclusion intervals with respect to $\varepsilon > 0$, $l_\varepsilon^\nu(0)$, satisfy the growth condition $\varepsilon^{1/d} l_\varepsilon^\nu(0) \to 0$, as $\varepsilon \to 0$. Let $\mathcal{K}$ be any compact in $\mathbb{R}_+^{d+1}$. Hence, again by Kruzkov's L^1 stability theorem [33], one obtains positive constants $C(\mathcal{K}), L(\mathcal{K})$ depending only on $\mathcal{K}$ such that, for each $\nu \in \mathbb{N}$,

$$\begin{aligned}
\int_{\mathcal{K}} |u^T(x,t) - \bar u|\, dxdt &\le \int_{\mathcal{K}} |u^T(x,t) - u^{\nu\, T}(x,t)|\, dxdt \\
&\quad + \int_{\mathcal{K}} |u^{\nu\, T}(x,t) - \overline{u^\nu}|\, dxdt + \operatorname{meas}(\mathcal{K})|\bar u - \overline{u^\nu}| \\
&\le \frac{C(\mathcal{K})}{T^d} \int_{|y| \le L(\mathcal{K})T} |u_0(y) - u_{0,\nu}(y)|\, dy \\
&\quad + \int_{\mathcal{K}} |u^{\nu\, T}(x,t) - \overline{u^\nu}|\, dxdt + \operatorname{meas}(\mathcal{K})|\bar u - \overline{u^\nu}|.
\end{aligned}$$

Now, since Theorem 2.1 applies to each u^ν, as it was shown above, we get

$$\limsup_{T\to\infty} \int_{\mathcal{K}} |u^T(x,t) - \bar u|\, dxdt \le \tilde C(\mathcal{K}) M_x(|u_0(x) - u_{0,\nu}(x)|) + \operatorname{meas}(\mathcal{K})|\bar u - \overline{u^\nu}|,$$

and so, letting $\nu \to \infty$, since by assumption $M_x(|u_0(x) - u_{0,\nu}(x)|) \to 0$, as $\nu \to \infty$, and, a fortiori, also $|\bar u - \overline{u^\nu}| \to 0$, as $\nu \to 0$, we get

$$\lim_{T\to\infty} \int_{\mathcal{K}} |u^T(x,t) - \bar u|\, dxdt = 0.$$

Since $\mathcal{K}$ is arbitrary we have $u^T \to \bar u$ in $L^1_{\mathrm{loc}}(\mathbb{R}_+^{d+1})$, as $T \to \infty$. The conclusion then follows as in the proof of Theorem 2.1.

□

Remark 3.1. Clearly, the hypothesis of Theorem 3.1 concerning the initial data $u_0(x)$ is satisfied by any generalized limit periodic function belonging to $L^\infty(\mathbb{R}^d)$, that is, any *S-a.p.* function in $L^\infty(\mathbb{R}^d)$ which is limit of L^∞ purely periodic functions in the sense of the norm $\|\psi\|_W = M_x(|\psi|)$, for $\psi \in$ *S-a.p.* .

4. Applications to some inviscid systems in chromatography

In this section we analyze the application of Theorem 2.1 to some special inviscid systems of conservation laws for which the compactness of the solution operator and the L^1 stability with respect to initial data have been proved in recent works. Namely, we are going to consider the initial value problem

$$\partial_t u + \partial_x f(u) = 0, \tag{4.1}$$

$$u(x,0) = u_0(x), \tag{4.2}$$

where (4.1) is the $n \times n$ chromatography system. The analysis here is very similar to the case of scalar conservation laws analized in section 3. For this system one has

$$f_i(u) = \frac{k_i u_i}{1 + u_1 + \cdots + u_n}, \qquad i = 1, \dots, n, \tag{4.3}$$

where k_i are given numbers with

$$0 < k_1 < k_2 < \cdots < k_n.$$

These systems belong to the so called Temple class which is characterized by the following two properties: (1) There exists a complete set of Riemann invariants defined everywhere in the domain of f, $\mathcal{U} \subseteq \mathbb{R}^n$, that is, a set of functions $\{\omega_1(u), \dots, \omega_n(u)\}$ satisfying $\nabla\omega_i(u) = l_i(u)$, where the $l_i(u)$ are n linearly independent left eigenvectors of $\nabla f(u)$, $i = 1, \dots, n$; (2) the level sets $\{u \in \mathcal{U} : \omega_i(u) = \text{constant}\}$ are hyperplanes (*cf.* [42]). Recently, Bressan and Goatin [5] constructed a continuous semigroup of solutions on a domain of L^∞ functions, for systems (4.1) in the Temple class, which are strictly hyperbolic and *genuinely nonlinear*, where the trajectories depend Lipschitz continuously on the initial data in the L^1 metric. In [5], the initial data are supposed to take values in a domain $E \subseteq \mathcal{U}$ of the form

$$E = \{u \in \mathcal{U} : \omega_i(u) \in [a_i, b_i],\ i = 1, \dots, n\},$$

in which the following strong hyperbolicity condition holds:

(SH) Given any n vectors $u^1, \dots, u^n \in E$, the eigenvalues $\lambda_1(u), \dots, \lambda_n(u)$ of $\nabla f(u)$ at these points are such that $\lambda_1(u^1) < \lambda_2(u^2) < \cdots < \lambda_n(u^n)$. Moreover, the right eigenvectors $r_1(u^1), r_2(u^2), \dots, r_n(u^n)$ are linearly independent.

As remarked in [5] the above assumption is automatically satisfied if the system is strictly hyperbolic and E is contained in a small neighborhood of a given point. As in the case of scalar equations analized in section 3, (3.4) is also valid in the present situation as long as u takes its values in a region E as above satisfying the condition (SH).

Concerning compactness of the solution operator of (4.1),(4.2), (4.3), we recall this has been proved by James, Peng and Perthame [32], where the compactness is achieved through compensated compactness [47, 39, 19] and a kinetic formulation for the chromatography system. So, combining the L^1 stability theorem in [5], the compactness theorem in [32], and our Theorem 2.1 we arrive at the following result.

Theorem 4.1. *Consider the problem* (4.1), (4.2), (4.3). *Assume* (4.1), (4.3) *is strictly hyperbolic and genuinely nonlinear and that* $u_0 \in L^\infty(\mathbb{R}^d)$ *is S-a.p. . Suppose that there exists a sequence of S-a.p. functions* $u_{0,\nu}$, $\nu \in \mathbb{N}$, *belonging to* $L^\infty(\mathbb{R}^d)$ *such that* $M_x(|u_0(x) - u_{0,\nu}(x)|) \to 0$, *as* $\nu \to \infty$, *and so that, for each* $\nu \in \mathbb{N}$, *the inclusion intervals of* $u_{0,\nu}$ *with respect to* ε, $l_\varepsilon^\nu(0)$, *satisfy* $\varepsilon l_\varepsilon^\nu(0) \to 0$, *as* $\varepsilon \to 0$. *Suppose also that* u_0 *and all* $u_{0,\nu}$ *take their values in a region* E *where (SH) is satisfied. Let* $u(x,t)$ *be the unique entropy solution of* (3.1)-(3.2). *Then* $u(\xi t, t) \to \bar{u}$ *as* $t \to \infty$ *in* $L^1_{\text{loc}}(\mathbb{R}^d)$, *and, in particular,* $M_x(|u(x,t) - \bar{u}|) \to 0$ *as* $t \to \infty$.

5. Viscous systems of conservation laws in several space variables

In this section we consider viscous systems of conservation laws of the form

$$\partial_t u + \sum_{k=1}^{d} \partial_{x_k} f^k(u) = \Delta u, \qquad t > 0,\ x = (x_1, \dots, x_d) \in \mathbb{R}^d, \tag{5.1}$$

where $u(x,t)$ assumes values in a domain $\mathcal{U} \subseteq \mathbb{R}^n$, $f^k : \mathcal{U} \to \mathbb{R}^n$ are smooth functions, $k = 1, \dots, d$ and Δ denotes the Laplacian operator in $\mathbb{R}^d$. Let be given initial data

$$u(x,0) = u_0(x), \tag{5.2}$$

where $u_0 \in L^\infty(\mathbb{R}^d)$ is S-*a.p.* , and takes its values in a closed region $\overline{\Omega} \subseteq \mathcal{U}$ which is positively invariant under the flow generated by (5.1). Such regions were characterized in [16] and their existence is known for many particular systems (see examples in the next section). In the simplest case of scalar equations ($d = 1$) invariant compact intervals are obtained from the well known maximum principle.

For flux functions f^k, $k = 1, \dots, d$, which are smooth and Lipschitz continuous over the positively invariant closed region $\overline{\Omega}$, the existence and uniqueness of global smooth solutions of (5.1)-(5.2) is well known and can be constructed through the procedures in [30]. Concerning these solutions we have the following result.

Theorem 5.1. *Let $\overline{\Omega}$ be a positively invariant closed region for* (5.1)*, f^k, $k = 1, \dots, d$, be smooth over $\mathcal{U} \supset \overline{\Omega}$ and Lipschitz continuous over $\overline{\Omega}$, and $u_0 \in L^\infty(\mathbb{R}^d)$ be S-a.p. assuming its values in $\overline{\Omega}$. Let $u(x,t)$ be the classical solution of* (5.1)*-*(5.2) *which is defined and smooth in $\mathbb{R}^d \times (0,\infty)$. Then $u(x,t)$ is S-a.p. in x, locally uniformly in $t \in [0,\infty)$, and its inclusion intervals with respect to $\varepsilon > 0$, $l_\varepsilon(t)$, satisfy $l_\varepsilon(t)/t \to 0$, as $t \to \infty$, provided the inclusion intervals of u_0, $l_\varepsilon(0)$, satisfy $(\log \varepsilon)^{-1} l_\varepsilon(0) \to 0$ as $\varepsilon \to 0$. Moreover, if $\overline{\Omega}$ is bounded then, for any $t_0 > 0$, $\nabla_x u$ is uniformly bounded for $t \ge t_0$.*

Proof. Let $K(x,t)$ be the heat kernel, that is,

$$K(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp(-|x|^2/4t).$$

As is well known one has

$$\|D^\alpha K(\cdot,t)\|_1 \le \frac{C(\alpha)}{t^{|\alpha|/2}},$$

where, as usual, $\alpha = (i_1, \dots, i_d)$, $D^\alpha = \partial_{x_1}^{i_1} \dots \partial_{x_d}^{i_d}$ and $|\alpha| = i_1 + \dots + i_d$, $i_1, \dots, i_d \in \mathbb{N}$. Let $u(x,t)$ be the global solution of (5.1)-(5.2) which is smooth in $\mathbb{R}^d \times (0,\infty)$ and uniformly bounded in $\mathbb{R}^d \times [0,T]$, for any $T > 0$. For any $t_0 \ge 0$, by Duhamel's formula, it satisfies the representation

$$u(t) = K(t - t_0) * u(t_0) - \sum_{k=1}^d \int_{t_0}^t K_{x_k}(t-s) * f^k(u(s))\,ds. \tag{5.3}$$

For some $\tau > 0$, consider the operator $\mathcal{L} : L^\infty(\mathbb{R}^d \times [t_0, t_0+\sigma]) \to L^\infty(\mathbb{R}^d \times [t_0, t_0+\sigma])$ given by

$$\mathcal{L}(v)(t) = K(t - t_0) * u(t_0) - \sum_{k=1}^d \int_{t_0}^t K_{x_k}(t-s) * f^k(v(s))\,ds. \tag{5.4}$$

Let C_0 be a Lipschitz constant for f^k in $\overline{\Omega}$, $k = 1, \dots, d$. One easily verifies that if σ is such that

$$dC_0\sqrt{\sigma} < 1, \tag{5.5}$$

then $\mathcal{L}$ is a contraction in $L^\infty(\mathbb{R}^d \times [t_0, t_0+\sigma])$. If $v(\cdot,t)$ is uniformly S-*a.p.* in x for $t \in [t_0, t_0+\sigma]$ then $\mathcal{L}(v)(\cdot,t)$ also is and there is a constant C_1, independent of t_0, such that if

$$d_S(v_\tau(t), v(t)) \le C_1 d_S(u_\tau(t_0), u(t_0)), \tag{5.6}$$

then the same inequality holds for $\mathcal{L}(v)$, where, for $\tau \in \mathbb{R}^d$ and $g, h : \mathbb{R}^d \to \mathbb{R}^n$, we denote $g_\tau(x) = g(x+\tau)$, and

$$d_S(g,h) = \sup_{x\in\mathbb{R}^d} \int_{I_x} |g(y) - h(y)|\, dy.$$

To prove the second of the latter assertions we use (5.4) and obtain that

$$d_S(\mathcal{L}(v)_\tau(t), \mathcal{L}(v)(t)) \le d_S(u_\tau(t_0), u(t_0))(1 + dC_0C_1\sqrt{\sigma}) \le C_1 d_S(u_\tau(t_0), u(t_0)),$$

if $C_1 \ge 1/(1 - dC_0\sqrt{\sigma})$. The first assertion follows similarly. Assuming that σ satisfies (5.5), we see that if τ is an almost period of $u(t_0)$ with respect to ε then τ is an almost period of $u(t)$, for $t_0 \le t \le t_0 + \sigma$, with respect to $C_1\varepsilon$. Indeed, we define the sequence $v^\nu \in L^\infty(\mathbb{R}^d \times [t_0, t_0+\sigma])$, $\nu \in \mathbb{N}$, such that $v^1(t) \equiv u(t_0)$ and $v^{\nu+1} = \mathcal{L}(v^\nu)$. Then, for all $\nu \in \mathbb{N}$, one has

$$d_S(v_\tau^\nu(t), v^\nu(t)) \le C_1 d_S(u_\tau(t_0), u(t_0)),$$

and, since u is the only fixed point of $\mathcal{L}$, $v^\nu \to u$ in $L^\infty(\mathbb{R}^d \times [t_0, t_0+\sigma])$ and the assertion follows. We then easily obtain by induction that if τ is an almost period of u_0 with respect to ε, then τ is an almost period of $u(t)$ with respect to $C_1^{(t/\sigma)+1}\varepsilon$. Hence,

$$l_\varepsilon(t) \equiv l_{\varepsilon/C_1^{(t/\sigma)+1}}(0) \tag{5.7}$$

is an inclusion interval for $u(\cdot,t)$ with respect to ε. Therefore, if $(\log\varepsilon)^{-1} l_\varepsilon(0) \to 0$ as $\varepsilon \to 0$, we see from (5.7) that $l_\varepsilon(t)/t \to 0$ as $t \to \infty$.

Now, let us prove the uniform boundedness of $\nabla_x u$ in $\mathbb{R}^d \times [\delta, \infty)$ for any $\delta > 0$ (*cf.* proposition 2.1 in [24]). It suffices to take $\delta = \sigma$, with σ satisfying (5.5). If $\overline{\Omega}$ is bounded, we have $\|u(t)\|_\infty \le r$, for all $t > 0$, for some $r > 0$. Returning to (5.4), we find that there exists a $C_2 > 0$, depending only on $C(\alpha), r, C_0, d$, $|\alpha| = 1$, such that, if v satisfies

$$\|\partial_{x_j} v(t)\|_\infty \le \frac{C_2}{\sqrt{t - t_0}},$$

then so does $\mathcal{L}(v)$. Hence, $u(t)$ also satisfies the above inequality by the same fixed point argument used above, so

$$\|\partial_{x_j} u(t)\|_\infty \le \frac{C_2}{\sqrt{t-t_0}}, \qquad t_0 < t \le t_0 + \sigma. \tag{5.8}$$

Cover $[\sigma, \infty)$ with overlapping intervals of the form $J_m = [\frac{m\sigma}{2}, \frac{(m+2)\sigma}{2}]$, $m = 0, 1, 2, \dots$. In each interval J_m, (5.8) holds with $t_0 = \frac{m\sigma}{2}$ and, for any $t \ge \sigma$, there is a $m_0 \ge 1$ such that $t \in J_{m_0} \cap J_{m_0+1}$, so that from (5.8) applied in the interval J_{m_0} we get

$$\|\partial_{x_j} v(t)\|_\infty \le \frac{\sqrt{2}C_2}{\sqrt{\sigma}},$$

and the assertion follows.

□

6. Applications

In this section we comment on some of applications of Theorem 5.1 in connection with Theorem 2.1 in order to obtain the decay of almost periodic solutions of viscous systems of conservation laws. Theorem 5.1 is applicable to the viscous pertubation of all hyperbolic systems of conservation laws for which the existence of compact positively invariant regions is known. If in addition the compactness of the scaling sequence u^T is known, an application of Theorem 2.1 immediately gives the decay of the solution of the perturbed viscous system. This includes, in particular, the viscous perturbations of all systems for which the decay of periodic solutions was obtained in [8]. We just mention a few examples below.

6.1. **Viscous scalar conservation laws.** If in (5.1) $n=1$ then as is well known there exists a unique uniformly bounded solution of (5.1)-(5.2), $u(x,t)$, the uniform boundedness being consequence of the usual maximum principle. Hence, if the initial data $u_0(x)$ is S-*a.p.* one obtains by Theorem 5.1 that $u(x,t)$ is S-*a.p.* and $l_\varepsilon(t)/t \to 0$ as $t\to\infty$, provided $(\log\varepsilon)^{-1} l_\varepsilon(0)\to 0$ as $\varepsilon\to 0$. Now, again using a compactness result in [36] we obtain the compactness of the scaling sequence $u^T(x,t)$ and so we can apply Theorem 2.1 to conclude the decay of $u(x,t)$ to $\bar u$, as $t\to\infty$, in particular that $M_x(|u(x,t)-\bar u|)\to 0$ as $t\to\infty$. We observe the curious fact that the restriction over the growth of the inclusion intervals of the initial data as $\varepsilon\to 0$ is stronger in this case than in the inviscid case.

6.2. **Nonlinear elasticity with artificial viscosity.** Consider the 2×2 one-dimensional system of nonlinear elasticity with artificial viscosity given by

$$\begin{cases} \partial_t u_1 - \partial_x u_2 = \partial_x^2 u_1, \\ \partial_t u_2 - \partial_x \sigma(u_1) = \partial_x^2 u_2, \end{cases} \tag{6.1}$$

with $\sigma'(v)>0$ and $v\sigma''(v)>0$ if $v\neq 0$. As is well known (see, *e.g.*, [42]) this system admits a family of bounded positively invariant regions which may include any bounded set in $\mathbb{R}^2$. Using the principle of invariant regions in [16] one may, in a standard way, extend the unique local solution to a unique globally defined uniformly bounded solution of (6.1)-(5.2). Hence, Theorem 5.1 is applicable and one obtains that the solution is S-*a.p.* and satisfies $l_\varepsilon(t)/t\to 0$ as $t\to\infty$ as long as $(\log\varepsilon)^{-1}l_\varepsilon(0)\to 0$ as $\varepsilon\to 0$. Now, DiPerna's compactness theorem in [19] implies that the scaling sequence u^T is compact in $L^1_{\text{loc}}(\mathbb{R}^2_+)$. Again, we can apply Theorem 2.1 and obtain, in particular, $M_x(|u(x,t)-\bar u|)\to 0$ as $t\to\infty$. We obtain the same result for a number of other viscous systems which are also endowed with bounded positively invariant regions and for which compensated compactness has beeing successfully applied such as the $n\times n$ system of chromatography with Langmuir coordinates [32], the quadractic systems in [13], the conjugate type systems in [26], etc..

6.3. **Isentropic gas dynamics with artificial viscosity.** Let us consider the 2×2 one-dimensional system of isentropic gas dynamics, for ideal polytropic gases, with an artificial viscosity given by

$$\begin{cases} \partial_t\rho + \partial_x m = \partial_x^2\rho, \\ \partial_t m + \partial_x(\frac{m^2}{\rho} + p(\rho)) = \partial_x^2 m, \quad p(\rho)=\kappa\rho^\gamma, \end{cases} \tag{6.2}$$

with $\gamma > 1$. This system is also endowed with a family of positively invariant regions given by $-C\rho + \rho\int^{\rho}(\sqrt{p'(\rho)}/\rho)\,d\rho \le m \le C\rho - \rho\int^{\rho}(\sqrt{p'(\rho)}/\rho)\,d\rho$, with $C > 0$ (*cf.* [20]). If the initial data satisfies $\rho_0(x) > \delta > 0$ and $m_0(x) \le C_0\rho_0(x)$, for some $C_0 > 0$, the existence of a unique local solution may be proved in a standard way; this solution then can be extended as long as $\rho(x,t) > 0$. The proof that the vacuum ($\rho = 0$) is not assumed in finite time is then a decisive point for the global existence of a solution to (6.2)-(5.2). The proof of this property given in [20], which assumes square integrability of $(\rho_0 - \bar{\rho}, m)$, for a certain $\bar{\rho}$, is not adequate here since we want to consider almost periodic initial data. Nevertheless, the proof that ρ remains bounded away from vacuum given in [12] does not make use of square integrability of the initial data and can be easily adapted to the case of a Cauchy problem. Indeed, it is based on the fact that $v = 1/\rho$ satisfies

$$\partial_t v - \partial_x^2 v \le \partial_x(zv) + \frac{z^2 v}{4},$$

where $z = m/\rho$, and so the maximum principle implies that $v \le g$, where g satisfies

$$\partial_t g - \partial_x^2 g = \partial_x(zg) + \frac{z^2 g}{4},$$

and $g(x,0) = v(x,0)$. Then our problem is reduced to the proof that g is uniformly bounded in $(0,T)$ by some $C(T)$ provided that z is uniformly bounded, which is the case if (ρ, m) belongs to one of the positively invariant regions. The latter is achieved using the integral representation of g through Duhamel's principle, and its contractive property for small time intervals as in the proof of Theorem 5.1. In this way the global existence of a solution to (6.2)-(5.2) follows. Now, since the flux function of (6.2) is Lipschitz continuous over any one of the invariant regions, one can still apply Theorem 5.1, so that the solution will be *S-a.p.* if so is the initial condition and inclusion intervals will satisfy the growth condition with time as long as the inclusion intervals of the initial data satisfy the corresponding growth condition in the statement of Theorem 5.1. Since the compactness of the scaling sequence can be obtained from the compactness results in [20] ($\gamma = 1 + 2/(2m+1)$, $m = 2, 3, \dots$), [6] ($1 < \gamma \le 5/3$), [37] ($\gamma \ge 3$) and [38] ($5/3 < \gamma < 3$), we can apply Theorem 2.1 to obtain, in particular, that $M_x(|u(x,t) - \bar{u}|) \to 0$, as $t \to \infty$, with $u(x,t) = (\rho(x,t), m(x,t))$.

Appendix A. Almost periodic functions

In this section we recall the definition and some basic properties of the almost periodic functions that are needed in the paper. This class of functions was introduced by H. Bohr [3] in the context of continuous functions defined in the real line. According to the original definition, a function $f : \mathbb{R} \to \mathbb{R}$ (or $f : \mathbb{R} \to \mathbb{C}$) is called *almost periodic* if, given $\varepsilon > 0$, there exists a number $l_\varepsilon > 0$, called inclusion interval with respect to ε, such that for all $x_0 \in \mathbb{R}$ there exists a number τ, with $x_0 \le \tau \le x_0 + l_\varepsilon$, called an ε-almost period or translation number with respect to ε, such that

$$\sup_{x\in\mathbb{R}} |f(x+\tau) - f(x)| \le \varepsilon.$$

The fundamental result of the theory developed by H. Bohr asserts that the almost periodic functions can be uniformly approximated by trigonometric polynomials, that is, finite linear combinations of functions of the form $\sin\lambda x$, $\cos\lambda x$, with $\lambda\in\mathbb{R}$. In particular, the limit (mean value of f)

$$M_x(f)=\lim_{L\to\infty}\frac{1}{L}\int_a^{a+L} f(x)\,dx,$$

exists uniformly with respect to $a\in\mathbb{R}$.

The definition and all the properties of the almost periodic functions in $\mathbb{R}$ can be immediately extended to continuous functions of several variables, $f:\mathbb{R}^d\to\mathbb{R}$. So, denoting $I_x^K=[x_1,x_1+K]\times\ldots\times[x_d,x_d+K]$, $K>0$, a continuous function $f:\mathbb{R}^d\to\mathbb{R}$ is said to be almost periodic if, given $\varepsilon>0$, there exists a number $l_\varepsilon>0$, called inclusion interval, such that for all $x_0\in\mathbb{R}^n$ one can find a vector $\tau\in I_{x_0}^{l_\varepsilon}$, called an ε-almost period, such that

$$\sup_{x\in\mathbb{R}^d}|f(x+\tau)-f(x)|\le\varepsilon.$$

A particular subclass of the almost periodic functions is that of the *limit-periodic functions*, that is, those continuous functions which can be uniformly aproximated by continuous periodic functions.

The concept of almost periodic functions was generalized by Stepanoff [46], Wiener [51], Weyl [50] and Besicovitch [1]. According to a definition due to Stepanoff and Wiener, used in this paper, a function $f:\mathbb{R}^d\to\mathbb{R}$ (or $f:\mathbb{R}\to\mathbb{C}$) in $L^1_{\rm loc}(\mathbb{R}^d)$ is a *generalized almost periodic function*, or briefly *S-a.p.* function, if, given $\varepsilon>0$, there exists a number $l_\varepsilon>0$, still called inclusion interval, such that for all $x_0\in\mathbb{R}^d$, there exists a vector $\tau\in I_{x_0}^{l_\varepsilon}$, called ε-almost period, such that

$$\sup_{x\in\mathbb{R}^d}\int_{I_x}|f(y+\tau)-f(y)|\,dy\le\varepsilon,$$

where $I_x=I_x^1$. The fundamental property of the *S-a.p.* functions is that they can be approximated by trigonometrical polynomials in the metric d_S given by

$$d_S(f,g)=\sup_{x\in\mathbb{R}^d}\int_{I_x}|f(y)-g(y)|\,dy.$$

In particular, the limit (mean value of f)

$$M_x(f)=\lim_{L\to\infty}\frac{1}{L^d}\int_{I_a^L} f(x)\,dx,$$

exists uniformly in $a\in\mathbb{R}^d$, and so we may also write

$$M_x(f)=\lim_{L\to\infty}\frac{1}{(2L)^d}\int_{|x|_\infty\le L} f(x)\,dx,$$

where $|x|_\infty=\max_{1\le j\le d}|x_j|$.

We close this brief recall by showing that given any decreasing sequence $\varepsilon_k\downarrow 0$, as $k\to\infty$, with

$$\sum_{j=k+1}^{\infty}\varepsilon_j\le\varepsilon_k,\qquad k=1,2,\ldots,$$

it is possible to construct a classical (non-periodic) almost periodic function (actually, limit-periodic function) whose inclusion intervals satisfy $l_{\varepsilon_k}=2(3^k)$. This, in particular, shows that there exist (non-periodic) almost periodic functions whose inclusion intervals satisfy whatever growth rate as $\varepsilon\to 0$ one may wish to prescribe. The construction is trivial. One starts with an interval, for instance,

$(-1,1)$, take a function ϕ_0 in $C_0(-1,1)$, and set $f=\phi_0$ in $(-1,1)$. Then, we take $\phi_{0-}\in C_0((-3,-1))$ and $\phi_{0+}\in C_0((1,3))$, such that $\|\phi_0(\cdot\pm 2)-\phi_{0\pm}\|_\infty<\varepsilon_1/2$, define $\phi_1=\phi_{0-}+\phi_0+\phi_{0+}$ and set $f=\phi_1$ in $(-3,3)$. Similarly, we take $\phi_{1-}\in C_0((-9,-3))$, $\phi_{1+}\in C_0((3,9))$, such that $\|\phi_1(\cdot\pm 6)-\phi_{1\pm}\|_\infty<\varepsilon_2/2$, define $\phi_2=\phi_{1-}+\phi_1+\phi_{1+}$ and set $f=\phi_2$ in $(-9,9)$. In this way we can define inductively f in the whole real line. Specifically, assuming that $f=\phi_k$ in $(-3^k,3^k)$ with $\phi_k\in C_0((-3^k,3^k))$, we take $\phi_{k-}\in C_0((-3^{k+1},-3^k))$ and $\phi_{k+}\in C_0((3^k,3^{k+1}))$, such that $\|\phi_k(\cdot\pm 2(3^k))-\phi_{k\pm}\|_\infty<\varepsilon_{k+1}/2$, define $\phi_{k+1}=\phi_{k-}+\phi_k+\phi_{k+}$ and set $f=\phi_{k+1}$ in $(-3^{k+1},3^{k+1})$. It is easy to see that the so constructed function is almost periodic (actually, limit periodic) and satisfies $l_{\varepsilon_k}=2(3^k)$. For instance, if one wants to have $(\log\varepsilon)^{-1}l_\varepsilon\to 0$ as $\varepsilon\to 0$, it suffices to choose, say, $\varepsilon_k=e^{-k(3^k)}$.

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