Medial Axes and Mean Curvature Motion II: Singularities

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What happens to the medial axis of a curve that evolves through MCM (Mean Curvature Motion)? We explore some theoretical results regarding properties of both medial axes and curvature motions. Specifically, using singularity theory, we present all possible topological transitions of a medial axis whose originating curve undergoes MCM. All calculations are presented in a clear and organized fashion and are easily generalized for other front motions. A companion article deals with non-singular points of the medial axis through direct calculations.

1. INTRODUCTION

1.1. Organization of the paper

The main definitions are placed right in the introduction, after which we can pose the question that is the motivation of this work: what happens to the medial axis (or symmetry set) of a curve as this curve evolves through Mean Curvature Motion? In this article, we focus our attention in the singularities of the medial axis (through which the interesting topological transitions occur); a companion paper [19] examines non-singular points.

Section 2 gives some pointers for the basics of singularity theory that will be used. A series of papers by Bruce, Giblin and others [5][10] used these ideas to provide a general classification of the possible kind of general singularities that might happen in the medial axis and in symmetry sets. Section 3 only presents those results and reorganizes them in a way suitable for our purposes.

In [4], Bruce and Giblin provide all topological transitions that a symmetry set can suffer as its originating curve changes. Section 4 extends their method to find all possible topological transitions that a symmetry set can go as the curve evolves through time for a **specific time evolution of the curve**; we actually pinpoint the possible directions of such transitions based on the particular curve evolution scheme that was chosen. We show the power of this method applying it to Mean Curvature Motion in a very straight-forward manner.

The actual calculations behind the main results of section 4 are delayed until section 5. This way, one could just apply the main results to various evolution schemes and Medial Axes without paying attention to the details of the method. However, if one chooses to work with objects similar to the Medial Axis, one will definitely be concerned with such computations. We hope the reader will find them well organized and clearly presented. Besides, many readers will find easier to understand the abstract concepts of section 4 through their application in section 5.

This work is part of the doctoral thesis [20]; for another look at evolving medial axes, besides the previously mentioned [19], we also recommend the article by August [1].

1.2. Mean Curvature Motion

Front deformations determined by differential equations depending on local properties have been a subject of study for a long time; one instance of such deformations is given by the Mean Curvature Motion (MCM, for short), here stated for curves in $\mathbb{R}^2$.

Definition 1.1. Let $\mathcal{C}(t)$, $0 \leq t \leq T_{MAX}$, denote a family of closed $\mathbf{C}^2$ curves embedded in $\mathbf{R}^2$ parametrized by $Q(s,t)$; We say that $\mathcal{C}(t)$ satisfies the MEAN CURVATURE EVOLUTION EQUATION if

$$Q_t(s,t) = K(s,t)N(s,t)$$

where $K(s,t)$ and $N(s,t)$ are, respectively, the curvature of $\mathcal{C}\,(t)$ and its “inwards” unit normal vector at the point $Q(s,t)$.

MCM has received much attention in the last 15 years, specially after Gage [9] [8] and Grayson [12] proved that the MCM for curves embedded in $\mathbb{R}^2$ has some fascinating properties (mainly, any embedded curve remains embedded and converges to a “circular point”). For curves that are not that smooth, the management of singularities can be handled by introducing solutions in a weak form, as presented by Chen [6] and Evans [7].

It is not surprising that a curve evolution scheme with such geometric appeal would eventually find applications in shape representation. After setting up the necessary basic Mathematics of the Mean Curvature Evolution, Kimia [13] [14] [15] presents an attractive shape classification method based on curvature evolutions.

1.3. Medial Axis

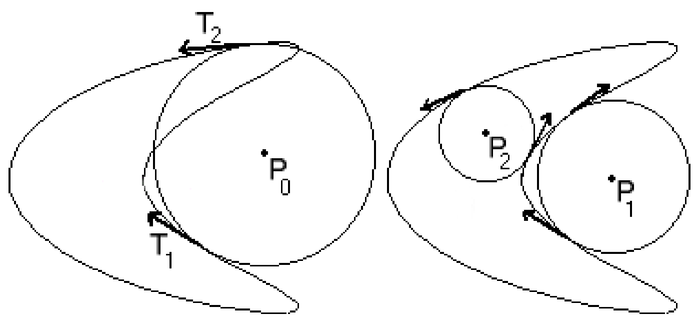
The medial axis plays a role in convex set characterization as shown in 1935 by Motzkin [17]. In the early 60s, H. Blum [2] suggested the use of the “medial axis function” as a possible shape descriptor. Yu [21] addresses the famous “instability” of skeletons by boundary deformations and proposes a way to regularize them. Other mathematical properties of medial axes can be found in the books by Giblin [10], [11] and Serra [18] (the latter also mentions some unresolved questions on the subject). There are a number of closely related definitions for medial axes and related objects. Here is our favorite definition for the medial axes:

Definition 1.2. The MEDIAL AXIS of a closed curve $\mathcal{C}$ is (the closure of) the set of the centers of circles that are maximal and contained inside or outside $\mathcal{C}$.

To be honest, we will really use the following object:

DEFINITION 1.3. The SYMMETRY SET of a curve $\mathcal{C}$ is (the closure of) the set of the centers of circles that are tangent to $\mathcal{C}$ at two (or more) distinct points.

The symmetry set is basically an extension of the medial axis where we don't really care if the circle is globally maximal. The closure operation will essentially give us some additional centers of circles that are tangent to $\mathcal{C}$ at only one point, but with higher order of tangency. Note that if the orientations of the tangent vectors are opposite with relation to the circle, then we have a "bad" point of the symmetry set that can never be part of the medial axis. We will be much more interested in points like P_1 and P_2 in the figure below.



Bad symmetry set point   Good symmetry set points

2. A BIT OF SINGULARITY THEORY

2.1. Level bifurcation sets

DEFINITION 2.1. Given a smooth manifold M and a function $f : \mathbf{R}^r \times M \to \mathbf{R}$, we define the LEVEL BIFURCATION SET OF f as

$$LB_f = \{x \in \mathbf{R}^r \mid f(x,\cdot) \text{ has two singularities at distinct points } y_1,\, y_2 \in M \text{ such that } f(x,y_1) = f(x,y_2)\}$$

We can factor f through $F = (\pi_1, f) : \mathbf{R}^r \times M \to \mathbf{R}^r \times \mathbf{R}$ and the projection $\pi_2 : \mathbf{R}^r \times \mathbf{R} \to \mathbf{R}$. As in [5], note that LB_f is now the projection π_1 of the self-intersection of the critical set Σ_F of F.

DEFINITION 2.2. We define the EXTENDED DISTANCE TRANSFORM of a curve $\mathcal{C}$ parametrized by $(a(s), b(s))$ as the function $E_{\mathcal{C}} : \mathbf{R}^3 \to \mathbf{R}$ given by

$$2E(x, y, s) = (x - a(s))^2 + (y - b(s))^2$$

The connection between the symmetry set and the extended distance transform should now be clear. Indeed, the symmetry set is essentially

$$\zeta_{\mathcal{C}} = \{P \in \mathbf{R}^2 \mid E_{\mathcal{C}}(P,\cdot) \text{ has two singularities} \\ \text{at distinct points } Q_1,\ Q_2 \in \mathcal{C} \text{ such that } E_{\mathcal{C}}(P,Q_1) = E_{\mathcal{C}}(P,Q_2)$$

We say "essentially" because there is also a closure step to be taken. If M is compact, everything is nice and we should have

Proposition 2.1. *The symmetry set of a curve $\mathcal{C}$ is the closure of the level bifurcation set of its extended distance transform, and, therefore, can be obtained by projecting the closure of the self-intersection of*

$$\Sigma_F = (\pi_1, E_{\mathcal{C}}) : \mathbf{R}^r \times M \to \mathbf{R}^r \times \mathbf{R}$$

Proof. There is nothing to be justified, except for a change between "closure of projection" and "projection of closure" that is clearly valid in this case. ■

2.2. Some results from catastrophe theory

We start this section with some definitions:

Definition 2.3. A function $F : \mathbf{R} \times \mathbf{R}^r \to \mathbf{R}$ is called a r-parameter unfolding of the function $f = F(\cdot, 0) : \mathbf{R} \to \mathbf{R}$.

Remark. This definition is usually applied only to the germ of the function F around $(x, 0_r) \in \mathbf{R} \times \mathbf{R}^r$ (and therefore it depends only on the germ of f around $x \in \mathbf{R}$). Usually, x is a singularity of f; adding the other r parameters to create F can be seen as an effort to "solve" that singularity.

Definition 2.4. A morphism between two unfoldings $F : \mathbf{R} \times \mathbf{R}^r \to \mathbf{R}$ and $G : \mathbf{R} \times \mathbf{R}^s \to \mathbf{R}$ at a point $x \in \mathbf{R}$, is a triple of smooth functions $s : \mathbf{R} \times \mathbf{R}^s \to \mathbf{R}$, $A : \mathbf{R}^s \to \mathbf{R}^r$ and $B : \mathbf{R}^s \to \mathbf{R}$ such that

$$\begin{aligned} G(x,P) &= F\left(s(x,P), A(P)\right) + B(P) \\ s(x,0_s) &= x; \\ A(0_s) &= 0_r \end{aligned}$$

Definition 2.5. In the case above, we say that the unfolding G is induced from the unfolding F by (s, A, B).

Remark. Note that in this case

$$g(x) = G(x,0_s) = F(s(x,0_s), A(0_s)) + B(0_s) = F(x,0_r) + B(0_s) = f(x) + B(0_s)$$

that is, f and g are the same function (modulo a constant).

Remark. Again, this should usually be thought as a morphism between the germs of F and G instead. In this case, all functions s, A and B must be defined only locally, and they are local differentiable functions instead.

DEFINITION 2.6. A particular unfolding F is called VERSAL if any other unfolding G of $f = F(\cdot, 0)$ can be induced from F by a suitable morphism.

We are now ready to state a couple of results from catastrophe theory that will be very useful to us: the existence of some canonical forms for a versal unfolding of a function f.

THEOREM 2.1. *The versal unfoldings associated to a function f depend only on the order of the singularity $f(x)$. Indeed, we can explicitly write such versal unfoldings in general as*

$$\begin{aligned}
\text{Order 1} &\rightarrow F(x,a) = \pm x^2 \text{ is versal} \\
\text{Order 2} &\rightarrow F(x,a,b) = x^3 + ax \text{ is versal} \\
\text{Order 3} &\rightarrow F(x,a,b,c) = \pm x^4 + ax^2 + bx \text{ is versal} \\
&\dots
\end{aligned}$$

Remark. By order i here, we mean that $f'(x_0) = f''(x_0) = \dots f^{(i)}(x_0) = 0$ but $f^{(i+1)}(x_0) \neq 0$.

Proof. Check Brocker's book [3], for example, for a much more general version of this. ■

These results can be extended to "multi-germs", that is, we can deal with several functions (or germs) at a time. A morphism between two unfoldings $\{F_1, F_2, \dots\}$ and $\{G_1, G_2, \dots\}$ of a multi-germ $\{f_1, f_2, \dots\}$ would then be a series s_i, A, B such that

$$G_i(x,P) = F_i\left(s_i(x,P), A(P)\right) + B(P)$$

Note that in the definition above we require A and B to be the same for all functions, but s can possibly change for each pair. The good news is that we still have canonical forms for multi-germs: they are similar to the previous case, except that we can't get rid of all constant terms now since the term $B(P)$ must be the same for all F_i, G_i (we can get rid of one of them, though). This gets much clearer through an example.

EXAMPLE 2.1. If f_1 has an order 4 singularity at x_1, f_2 has an order 3 singularity at x_2 and f_3 has an order 1 singularity at x_3, then any unfolding of $\{f_1, f_2, f_3\}$ can be induced from

$$G_1 = x^5 + a_1x^3 + a_2x^2 + a_3x$$
$$G_2 = \pm x^4 + a_4x^2 + a_5x + a_6$$
$$G_3 = \pm x^2 + a_7$$

(after a suitable translation in the x direction). That is, $\{G_1, G_2, G_3\}$ is a versal unfolding.

Now we combine this with another powerful proposition.

PROPOSITION 2.2. *For a general compact smooth M submanifold of $\mathbf{R}^n$ ($n \leq 5$), for any fixed $Q_0 \in \mathbf{R}^n$, and for any collection of points $P_i \in M$, the multi-germ represented by the function $F_i : M \times \mathbf{R}^n \to \mathbf{R}$ given by $E_M(P,Q) = |P - Q|^2$ (around P_i) is a versal unfolding of the multi-germ represented by $f_i(Q) = F(P_i, Q)$.*

Proof. This statement has been taken from [5]; it is a consequence of results in [16]. ■

Moreover, according to [5], the (level) bifurcation sets of isomorphic multi-germs must be equivalent by a local diffeomorphism of the plane. The path is now clear: if we want to look at the symmetry set, we consider it as the level bifurcation set of the extended distance transform. In general, this is a versal unfolding, but so is the canonical one given by some simple expressions like, for example, the set of G_i's on the example above. That will indicate that both these unfoldings are isomorphic, and so their level bifurcation sets are locally diffeomorphic. But the level bifurcation set of the G_i's is easy to obtain! We will apply this process for all possible **generic** cases of singularities that we expect to find when looking at multi-germs based on the extended distance transform.

3. SYMMETRY SET CANONICAL FORMS

We are now ready to face the only possible (generic) canonical forms that the symmetry set has to offer. There are 5 of them.

3.1. Case A_3

This corresponds to points of the symmetry set that are centers of circles that are tangent to the curve at one point only, but with order of tangency 3 there. From singularity theory, we obtain the canonical form

$$G(u, a, b) = \pm u^4 + au^2 + bu$$

The level bifurcation set of G consists of the pairs (a, b) for which there are distinct u, v such that

$$G(u, a, b) = G(v, a, b)$$
$$G_u(u, a, b) = G_v(v, a, b) = 0$$

that translates to

$$\begin{aligned}
\pm u^4 + au^2 + bu &= \pm v^4 + av^2 + bv \\
\pm 4u^3 + 2au + b &= 0 \\
\pm 4v^3 + 2av + b &= 0
\end{aligned}$$

Solve the last two equations for a and b (using that $u \neq v$) to obtain

$$\begin{aligned}
\pm a &= -2v^2 - 2uv - 2u^2 \\
\pm b &= 4uv^2 + 4vu^2
\end{aligned}$$

and then substitute them into the first equation

$$\begin{aligned}
-u^4 + 2u^2v^2 + 2u^3v = -v^4 + 2v^3u + 2u^2v^2 \Rightarrow \\
(v+u)\,(-u+v)^3 = 0 \Rightarrow u = -v
\end{aligned}$$

Therefore, we must have $a = \mp 2u^2$ and $b = 0$. Whatever this sign is, this is the parametrization of a half-line. This indicates that the medial axis close to a A_3 point is a curve that suddenly stops and reverses. Compare this with the analysis by Teixeira in [19] – there, we declared this situation to be a "stumper" since the points $Q(s_1)$ and $Q(s_2)$ on the curve that correspond to this symmetry set point "came together". Reversing the direction on the symmetry set consists in "switching" the points $Q(s_1)$ and $Q(s_2)$ as you keep moving them along the curve.

3.2. Case A_2A_1

This is a center of a circle that is double-tangent to the curve at a point and tangent to the curve at another one. Note that the even order of tangency indicates that this will never be in the medial axis proper (the corresponding circle crosses the curve). The canonical form here is

$$\begin{aligned}
G_1(u,a,b) &= u^3 + au + b \\
G_2(u,a,b) &= \pm u^2
\end{aligned}$$

and there are three potential contributions for the level bifurcation set. One corresponds to

$$\begin{aligned}
G_1(u,a,b) &= G_1(v,a,b) \\
G_{1u}(u,a,b) &= G_{1v}(v,a,b) = 0
\end{aligned}$$

i.e.,

$$\begin{aligned}
u^3 + au + b &= v^3 + av + b \\
3u^2 + a &= 0 \\
3v^2 + a &= 0
\end{aligned}$$

However, this has no non-trivial solutions. Actually, it is clear that

$$\begin{aligned}
G_2(u,a,b) &= G_2(v,a,b) \\
G_{2u}(u,a,b) &= G_{2v}(v,a,b) = 0
\end{aligned}$$

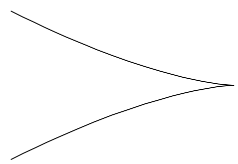
has no non-trivial solution either, so we are left with only one case

$$G_1(u,a,b) = G_2(v,a,b)$$
$$G_{1u}(u,a,b) = G_{2v}(v,a,b) = 0$$

that is,

$$u^3 + au + b = \pm v^2$$
$$3u^2 + a = 0$$
$$\pm 2v = 0$$

what implies $a = -3u^2$ and $b = 2u^3$. This is the parametric graph of the following cusp



And therefore this will be the shape of the symmetry set around a A_2A_1 point. Once more, we note that such behavior can never be found in the Medial Axis because of the A_2 contact.

3.3. Case $A_1A_1A_1$

Now, the canonical form is simply

$$G_1(u,a,b) = \pm u^2$$
$$G_2(u,a,b) = \pm u^2 + a$$
$$G_3(u,a,b) = \pm u^2 + b$$

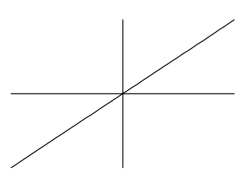
None of the G_iG_i pairs contribute with anything, but we can cross them like this

$$G_2(u,a,b) = G_3(u,a,b)$$
$$G_{2u}(u,a,b) = G_{3v}(v,a,b) = 0$$

i.e.,

$$\pm u^2 + a = \pm v^2 + b$$
$$\pm u = \pm v = 0$$

so clearly $a = b$. Similarly, G_1G_2 gives us $a = 0$ and G_1G_3 gives us $b = 0$. Therefore, the local picture of the symmetry set is



Note that if you are traveling the symmetry set, there is always a natural way of passing through the singularity.

3.4. Case A_2

This case is more generic than the previous ones, but never corresponds to a point in the medial axis (due to the even order of tangency). The calculation is already done in the A_2A_1 case: it corresponds to an isolated point (unless you allow $u = v$ in that calculation, in which case you detect the evolute of the curve passing through that isolated point). With our definition of symmetry set (that requires $u \neq v$), this will never be a point of the symmetry set either.

3.5. Case A_1A_1

Again, this case is more generic than the 3 first ones (there must be a 1D-set of such points). The calculation is all done in the A_2A_1 case, and this will correspond to a nice simple line. Therefore, this case corresponds to points where the symmetry set is smooth.

4. EVOLUTION OF THE MEDIAL AXIS CLOSE TO ITS SINGULARITIES

Here we repeat the arguments of the previous section for some 1-parameter families of level bifurcation sets, exactly in the spirit of [4]. Actually, a great part of the results below can be found in that paper for a generic 1-parameter family of curves; the difference is that we present our calculations following an organized (and hopefully simple) method, and we need some more work in finding out what transitions are allowed for our specific curve deformation. Our consistent method can be easily applied to other kinds of movement and we provide lots of nice pictures.

In short, the final result is the specification of what kind of transitions can occur to the singularities of the Medial Axis as the original curve evolves through a specific motion.

4.1. Introducing time

We would like to use the same ideas that appeared in the last section. However, we have now a time-changing extended distance transform given by

$$2F(s,t,x,y) = (x - a(s,t))^2 + (y - b(s,t))^2$$

where $(a(s,t), b(s,t))$ is a parametrization of the curve $\mathcal{C}$ as it evolves through, for example, Mean Curvature Motion. The same catastrophe theory applies if we just think of F as a 3-parameter unfolding this time. That means that our canonical forms G will have an extra parameter, but we can still write

$$G(u,a,b,c) = F(s(u,a,b,c), A(a,b,c)) + B(a,b,c)$$
$$A(a,b,c) = (t(a,b,c), x(a,b,c), y(a,b,c))$$

Now, the level bifurcation set of G (some strange looking surface) will be locally diffeomorphic to the corresponding level bifurcation set of $F(\cdot,t,x,y)$; this is another surface, an amalgamation of all level bifurcation sets of $F(\cdot,t_0,x,y)$ for each t_0. We need to separate this amalgamation in each one of its components based on the particular value of t_0, that is, we want to slice it into the different symmetry sets for each t_0!

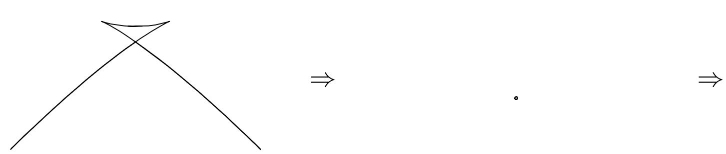
Well, instead of slicing the unknown level bifurcation-surface of F (that lives in the txy space) using the easy-to-see $t = t_0$ planes, we will slice the easy-to-obtain level bifurcation-surface of G (that lives in the abc space) using the strange-looking surfaces $t(a, b, c) = t_0$. In particular, if we have an interesting singularity at $a = b = c = 0$ and $t = 0$, the $t = 0$ plane will correspond to the 0-level surface of $t(a, b, c)$ in the abc space. Intersect it with the level bifurcation-surface of G and we have a picture that is diffeomorphic to the symmetry set of F at $t = 0$ (at least locally, around the point $(x(0, 0, 0), y(0, 0, 0))$). If we are interested only in the local picture of such symmetry set, we don't really need to know the whole $t(a, b, c) = 0$ surface – just its tangent plane at $(0, 0, 0)$ will suffice, that is, we want to get our hands on the plane $t_a(0, 0, 0)a + t_b(0, 0, 0)b + t_c(0, 0, 0)c = 0$ in the abc space.

As time passes, how does this slice change? Well, the level set moves to $t(a, b, c) = \varepsilon$, say, and this new slice of the level bifurcation set of G is (locally diffeomorphic to) the new symmetry set (level bifurcation set of F). Again, since we only care about the local picture close to $(0, 0, 0)$, we might as well imagine the level set $t = 0$ moving in the direction of the gradient of t, that is, (t_a, t_b, t_c) calculated at $a = b = c = 0$! So the slices $at_a + bt_b + ct_c = -\varepsilon, 0, \varepsilon$ of the level bifurcation set of G should be locally diffeomorphic to the symmetry sets at times $-\varepsilon, 0, \varepsilon$!

The method we follow in this section should now be clear: given a singularity F, we write down its canonical form G, calculate its level bifurcation set and find all possible generic ways of slicing it close to their origin. Then we find out what (t_a, t_b, t_c) is at $a = b = c = 0$, an indication of the direction of the time level sets. Finally, by comparing these results, we know what generic transitions the symmetry set (slices $at_a + bt_b + ct_c = -\varepsilon, 0, \varepsilon$) can go through for the Mean Curvature Motion case. We present these calculations in a systematic way that is (hopefully) fairly self-explanatory and seemingly extendable to more complex singularities or motions. The summary of our results is the following:

As a curve evolves through $Q_t = HN$, its medial axis can go through one of the following generic transitions:

- Dove-tail (case A_4)

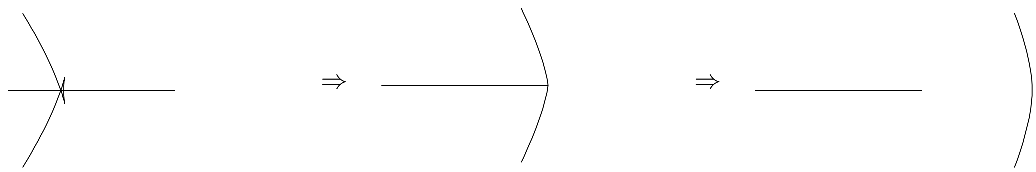


A dove-tail... ...shrinks to a point... ...and vanishes.

This transition happens when we have a point Q on the curve at which $K_s = K_{ss} = 0$; the medial axis radius at the singularity is $r = 1/K$. The direction of the transition above is correct if, at Q

$$K_{sss} \left(H_{sss} + K^2 H_s \right) > 0$$

- Dove-tail (case A_3A_1)



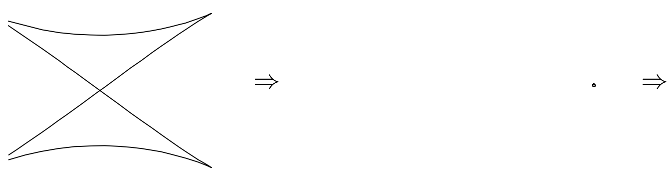
A dove-tail and a half-line... ...simplify to... ...a curve and a half-line.

The transition happens when we have points Q_1 and Q_2 on the curve such that $K_{1s} = 0$ and $r = 1/K_1$ for the medial axis point corresponding to Q_1 and Q_2. The direction above is correct if

$$\frac{H_{1ss}}{K_1}(1-\cos\alpha) + H_{1s}\sin\alpha + K_1(H_1 - H_2) < 0$$

where α is the angle from T_1 to T_2 (the tangent vectors at Q_1 and Q_2).

- Moths (case A_2A_2)

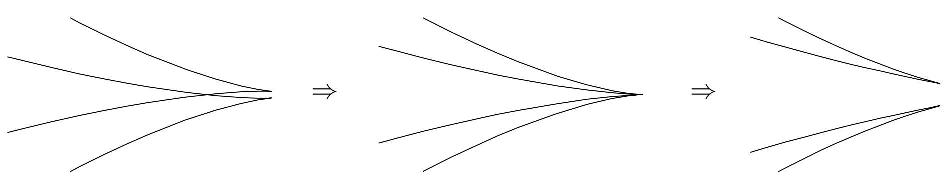


A moth... ...shrinks to a point... ...and vanishes.

Moths happen whenever we have a bi-osculating circle tangent to the curve at Q_1 and Q_2 such that the curvature functions at these points satisfy $K_{1s}K_{2s} > 0$. The direction above is correct (the moth vanishes) whenever

$$(K_{1s} + K_{2s})\left(H_{1s} + H_{2s} + (H_1 - H_2)\frac{K\sin\alpha}{1-\cos\alpha}\right) > 0$$

- Wvv transition (case A_2A_2)

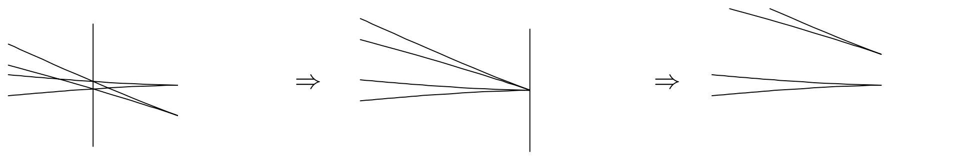


Two interlaced cusps... ...touch by the vertex... ...and separate.

This happens whenever we have a bi-osculating circle tangent to the curve at Q_1 and Q_2 but the curvatures at these points satisfy $K_{1s}K_{2s} < 0$. Once again,.the direction shown above is correct (cusps separate) whenever

$$(K_{1s} + K_{2s})\left(H_{1s} + H_{2s} + (H_1 - H_2)\frac{K\sin\alpha}{1-\cos\alpha}\right) > 0$$

- Two cusps and a line (case $A_2A_1A_1$)



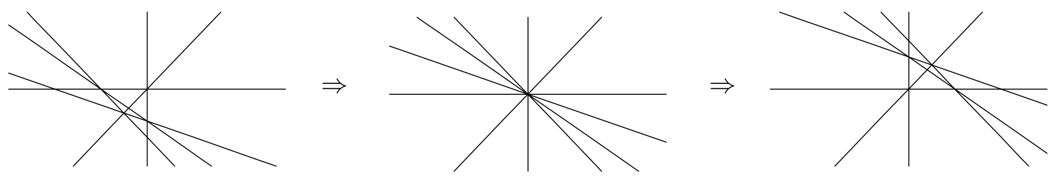
A line across 2 cusps... ...touch... ...and separate.

This happens whenever the point P at the medial axis is center of a circle that is tangent to the curve at points Q_2 and Q_3 and osculating at a point Q_1. The direction above (separation) is correct if

$$\frac{\sin(\beta-\alpha)+\sin\alpha-\sin\beta}{K_{1s}}\left(\begin{array}{c}(K_1(H_2-H_1)+H_{1s}\sin\alpha)(1-\cos\beta)+\\+(K_1(H_3-H_1)-H_{1s}\sin\beta)(1-\cos\alpha)\end{array}\right)<0$$

where α is the angle from T_1 to T_2 and β is the angle from T_1 to T_3.

- Flipping quadrilateral (case $A_1A_1A_1A_1$)



Quadrilateral and diagonals... ...shift towards a vertex... ...and go through.

This case happens whenever a point P of the medial axis is center of a circle that is tangent to the curve at 4 points Q_1, Q_2, Q_3 and Q_4. Since the two sides of the transition are virtually identical, there is no telling which direction the transition goes through.

4.2. Applications

4.2.1. Mean Curvature Motion

It is immediate to apply the results above to the Mean Curvature Motion, using $H = K$. Then

- For dove-tails, since

$$K_{sss}\left(H_{sss}+K^2H_s\right)=K_{sss}^2+0>0$$

whenever $K_s = 0$, the direction is always vanishing;

- For dove-tails, the direction of the transition will depend on

$$\frac{K_{1ss}}{K_1}(1-\cos\alpha)+K_1(K_1-K_2)<0$$

since again $H_{1s} = K_{1s} = 0$ in this case. Both transitions can occur. This seems to be the only case where MCM can really complicate the symmetry set;

- For moths and wvv transitions, the direction depends on

$$(K_{1s}+K_{2s})\left(K_{1s}+K_{2s}+(K_1-K_2)\,\frac{K\sin\alpha}{1-\cos\alpha}\right)>0$$

but since $K_1=K_2$ in this case, the condition is always satisfied (i.e., moths disappear and W's turn into VV's for the MCM);

- Two cusps and a line can go either way, since the sign of

$$\frac{\sin(\beta-\alpha)+\sin\alpha-\sin\beta}{K_{1s}}\left(\begin{array}{l}(K_1(K_2-K_1)+K_{1s}\sin\alpha)(1-\cos\beta)+\\+(K_1(K_3-K_1)-K_{1s}\sin\beta)(1-\cos\alpha)\end{array}\right)$$

can go either way.

- Finally, quadrilaterals flip, and both directions are equivalent.

4.2.2. Reaction Equation

It is instructive to see what happens if H is a constant (independent of s). Note that all "discriminant" expressions above turn out to be zero if we do so! That is not a surprise: the symmetry set is, in a way, the points in the shockwave of $Q_t=N$. There is a tendency for the symmetry set not to go through these transitions: if a singularity exists at $t=0$, it will still be there at other times. If you have a dove-tail, it stays there; and so on.

5. COMPUTATIONS

In this section, we present the computations that led to the main result on the previous section. We start with a collection of results that will be used in all cases.

5.1. Preparations

If we use

$$\begin{aligned}G(u,a,b,c)&=F(s(u,a,b,c),A(a,b,c))+B(a,b,c)\\A(a,b,c)&=(t(a,b,c),x(a,b,c),y(a,b,c))\end{aligned}$$

and the notation

$$dA=\begin{pmatrix}t_a & x_a & y_a\\ t_b & x_b & y_b\\ t_c & x_c & y_c\end{pmatrix}$$

We obtain the following relations between $G_{(u)}$, $F_{(s)}$ and $s_{(u)}$

$$\begin{aligned}
\frac{\partial G}{\partial u} &= F_s s_u \\
\frac{\partial^2 G}{\partial u^2} &= F_{ss} s_u^2 + F_s s_{uu} \\
\frac{\partial^3 G}{\partial u^3} &= F_{sss} s_u^3 + 3F_{ss} s_{uu} s_u + F_s s_{uuu} \\
\frac{\partial^4 G}{\partial u^4} &= F_{ssss} s_u^4 + 6F_{sss} s_{uu} s_u^2 + F_{ss}(4s_u s_{uuu} + 3s_{uu}^2) + F_s s_{uuuu} \\
\frac{\partial^5 G}{\partial u^5} &= F_{sssss} s_u^5 + 10F_{ssss} s_{uu} s_u^3 + F_{sss}(10s_u^2 s_{uuu} + 15 s_u s_{uu}^2) + F_{ss}(...) + F_s s_{uuuuu} \\
\frac{\partial^6 G}{\partial u^6} &= F_{ssssss} s_u^6 + 15F_{sssss} s_{uu} s_u^4 + F_{ssss}(20s_u^3 s_{uuu} + 45 s_u^2 s_{uu}^2) + F_{sss}(...) + F_{ss}(...) + F_s(...) \\
\frac{\partial^7 G}{\partial u^7} &= F_{sssssss} s_u^7 + 21F_{ssssss} s_{uu} s_u^5 + F_{sssss}(35s_u^4 s_{uuu} + 105 s_u^3 s_{uu}^2) + ...
\end{aligned} \tag{1}$$

and some more relations involving the derivatives of these expressions with relation to a, b, c:

$$\begin{aligned}
\nabla\Delta G &= dA\nabla\Delta F + F_s \nabla s \\
\nabla G_u &= dA\nabla F_s s_u + F_{ss} s_u \nabla s + F_s \nabla s_u \\
\nabla G_{uu} &= dA\left(\nabla F_{ss} s_u^2 + \nabla F_s s_{uu}\right) + \left(F_{sss} s_u^2 + F_{ss} s_{uu}\right)\nabla s + F_{ss}\nabla(s_u^2) + F_s \nabla s_{uu} \\
\nabla G_{uuu} &= dA\left(\nabla F_{sss} s_u^3 + 3\nabla F_{ss} s_{uu} s_u + \nabla F_s s_{uuu}\right) + \left(F_{ssss} s_u^3 + 3F_{sss} s_{uu} s_u + F_{ss} s_{uuu}\right)\nabla s + \\
&\quad + F_{sss}\nabla(s_u^3) + 3F_{ss}\nabla(s_{uu} s_u) + F_s \nabla(s_{uuu})
\end{aligned} \tag{2}$$

where the symbol Δ should be read as a "difference" ($\Delta F = F_1 - F_2$) and ∇ is the "spatial" gradient, a vector with three components that are the derivatives with relation to the *abc* variables. The difference will be useful whenever we have two or more canonical forms – we can subtract them to get rid of our $B(a, b, c)$ term, and then the a, b, c derivatives give us the first equation on the previous list (this will become clear later on). In general, we will have to pick enough (three) equations from the set 2 in order to mount a linear system that will allow us to calculate dA and, consequently, (t_a, t_b, t_c). Now, we can probably calculate some of the $s_{(u)}$ terms from the set 1, but what about the other terms?

Well, remember that our function F is the extended distance transform

$$\begin{aligned}
2F &= \langle Q(s,t) - (x,y), Q(s,t) - (x,y)\rangle \\
F_t &= \langle Q - (x,y), Q_t\rangle
\end{aligned}$$

Now, fix $t = 0$ and suppose $Q(s, 0)$ is arc-length parametrized in order to find the following 7 expressions for $F_{(s)}$

$$
\begin{aligned}
\frac{\partial F}{\partial s} &= \langle Q - (x,y), T \rangle \\
\frac{\partial^2 F}{\partial s^2} &= \langle Q - (x,y), KN \rangle + 1 \\
\frac{\partial^3 F}{\partial s^3} &= \left\langle Q - (x,y), -K^2 T + K_s N \right\rangle \\
\frac{\partial^4 F}{\partial s^4} &= \left\langle Q - (x,y), -3KK_s T + \left(-K^3 + K_{ss}\right) N \right\rangle - K^2 \qquad (3) \\
\frac{\partial^5 F}{\partial s^5} &= \left\langle Q - (x,y), \left(K^4 - 4KK_{ss} - 3K_s^2\right) T + \left(-6K^2 K_s + K_{sss}\right) N \right\rangle - 5KK_s \\
\frac{\partial^6 F}{\partial s^6} &= \left\langle Q - (x,y), \left(10K^3 K_s - 5KK_{sss} - 10K_s K_{ss}\right) T \right\rangle + \\
&+ \left\langle Q - (x,y), \left(K^5 - 10K^2 K_{ss} - 15KK_s^2 + K_{ssss}\right) N \right\rangle + \\
&+ K^4 - 9KK_{ss} - 8K_s^2 \\
\frac{\partial^7 F}{\partial s^7} &= \left\langle Q - (x,y), (...) T + \left(15K^4 K_s - 15K^2 K_{sss} - 60KK_s K_{ss} - 15K_s^3 + K_{sssss}\right) N \right\rangle + \\
&+ 14K^3 K_s - 14KK_{sss} - 35K_s K_{ss}
\end{aligned}
$$

and the corresponding expressions for the $P = (x, y)$ derivatives are

$$
\begin{aligned}
F_P &= P - Q(s,t) \\
F_{sP} &= -T \\
F_{ssP} &= -KN \\
F_{sssP} &= K^2 T - K_s N
\end{aligned}
$$

We will also need some of the $F_{t(s)}$ for $t = 0$

$$
\begin{aligned}
F_t &= \langle Q - (x,y), Q_t \rangle \\
F_{ts} &= \langle Q - (x,y), Q_{ts} \rangle + \langle T, Q_t \rangle \\
F_{tss} &= \langle Q - (x,y), Q_{tss} \rangle + 2\langle T, Q_{ts} \rangle + \langle KN, Q_t \rangle \\
F_{tsss} &= \langle Q - (x,y), Q_{tsss} \rangle + 3\langle T, Q_{tss} \rangle + 3\langle KN, Q_{ts} \rangle + \left\langle -K^2 T + K_s N, Q_t \right\rangle
\end{aligned}
$$

So if you know the curve evolution you are about to use, you can fill in for the values of Q_t and the derivatives $Q_{t(s)}$. Usually we have an evolution like $Q_t = HN$ (usually, H is a function of the curvature and its derivatives) and

$$
\begin{aligned}
Q_t &= HN \\
Q_{ts} &= -HKT + H_s N \\
Q_{tss} &= (-2H_s K - HK_s) T + \left(H_{ss} - HK^2\right) N \\
Q_{tsss} &= \left(-3H_{ss} K - 3H_s K_s - HK_{ss} + HK^3\right) T + \left(H_{sss} - 3H_s K^2 - 3HKK_s\right) N
\end{aligned}
$$

and using $Q-(x,y)=-rN$ (what holds whenever we have at least A_1 tangency) we end up with

$$\begin{aligned} F_t &= -rH \\ F_{st} &= -rH_s \\ F_{sst} &= -r\left(H_{ss}-HK^2\right)-HK \\ F_{ssst} &= -r\left(H_{sss}-3H_sK^2-3HKK_s\right)-3H_sK-2HK_s \end{aligned} \tag{4}$$

Note that whenever $r=1/K$, these translate simply to

$$\begin{aligned} F_t &= -\frac{H}{K} \\ F_{st} &= -\frac{H_s}{K} \\ F_{sst} &= -\frac{H_{ss}}{K} \\ F_{ssst} &= -\frac{H_{sss}}{K} \end{aligned} \tag{5}$$

We are now ready to face all the cases for all possible canonical forms; again, there are 5 of them.

5.2. Case A_4

In this case, we have a circle of center P that is tangent to the curve at a point Q with order of tangency 4. The canonical form has only one polynomial

$$G(u,a,b,c)=u^5+au^3+bu^2+cu$$

5.2.1. Level bifurcation set of the canonical form

We are looking for the points (a,b,c) for which there are distinct u and v such that

$$\begin{aligned} G(u,a,b,c) &= G(v,a,b,c) \\ G_u(u,a,b,c) &= G_v(v,a,b,c)=0 \end{aligned}$$

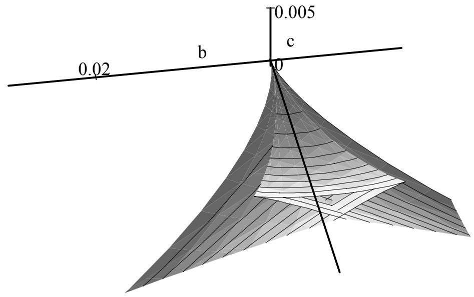
Now, those equations immediately translate to

$$\begin{aligned} u^5+au^3+bu^2+cu &= v^5+av^3+bv^2+cv \\ 5u^4+3au^2+2bu+c &= 0 \\ 5v^4+3av^2+2bv+c &= 0 \end{aligned}$$

and, solving for a, b and c

$$\begin{aligned} a &= -3u^2-4vu-3v^2 \\ b &= 2\left(u+v\right)\left(u^2+3uv+v^2\right) \\ c &= -uv\left(4u^2+7uv+4v^2\right) \end{aligned}$$

We plot this surface below



A_4 level bifurcation set

Note that the "leftmost" edge in this picture is not a cropping effect, i.e., the surface actually "ends" there (and the same applies for its symmetric with relation to the ac-plane). Indeed, a normal vector to this surface is obtained through

$$\begin{aligned} X_u &= \left(-6u - 4v, 6u^2 + 16uv + 8v^2, -12u^2v - 14uv^2 - 4v^3\right) = \\ &= 2\,(2v + 3u)\,(-1, u + 2v, -v\,(2u + v)) \\ X_v &= \left(-4u - 6v, 8u^2 + 16uv + 6v^2, -4u^3 - 14u^2v - 12uv^2\right) = \\ &= 2(2u + 3v)\,(-1, 2u + v, -u\,(u + 2v)) \\ X_u \times X_v &= -4\,(2v + 3u)\,(2u + 3v)\,(u - v)\left(v^2 + vu + u^2, v + u, 1\right) \end{aligned}$$

so the non-regular points are given by $u = v$, $2u + 3v = 0$ and $2v + 3u = 0$. Otherwise, as $u, v \to 0$ the unit normal vector approaches $\pm(0, 0, 1)$. Each one of these special cases gives us a curve:

$$\begin{aligned} u = v &\Rightarrow \left(-10u^2, 20u^3, -15u^4\right) \Rightarrow \vec{T} = \left(-20u, 60u^2, -60u^3\right) \parallel (1,0,0) \text{ as } u \to 0 \\ 2u = -3v &\Rightarrow \left(-\frac{15}{4}v^2, \frac{5}{4}v^3, \frac{15}{4}v^4\right) \Rightarrow \vec{T} = \left(-\frac{15}{2}v, \frac{15}{4}v^2, 15v^3\right) \parallel (1,0,0) \text{ as } u \to 0 \\ 2v = -3u &\Rightarrow \left(-\frac{15}{4}u^2, \frac{5}{4}u^3, \frac{15}{4}u^4\right) \Rightarrow \vec{T} = \left(-\frac{15}{2}u, \frac{15}{4}u^2, 15u^3\right) \parallel (1,0,0) \text{ as } u \to 0 \end{aligned}$$

The first case corresponds to both edges named before, while the two latter cases (that are the same curve) correspond to the other two top edges. We should also consider possible self-intersections

$$\begin{aligned} -3u_1^2 - 4v_1u_1 - 3v_1^2 &= -3u_2^2 - 4v_2u_2 - 3v_2^2 \\ 2\,(u_1 + v_1)\left(u_1^2 + 3u_1v_1 + v_1^2\right) &= 2\,(u_2 + v_2)\left(u_2^2 + 3u_2v_2 + v_2^2\right) \\ -u_1v_1\left(4u_1^2 + 7u_1v_1 + 4v_1^2\right) &= -u_2v_2\left(4u_2^2 + 7u_2v_2 + 4v_2^2\right) \end{aligned}$$

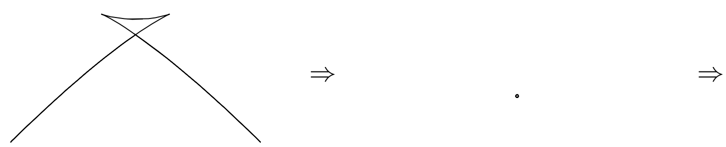
Besides the trivial solution $u_1 = u_2$ and $v_1 = v_2$, we also have a solution $u_2 = v_1$ and $u_1 = v_2$ that indicates the fact that the surface is "doubled" (all sheets are covered twice) – note that the normals have the same direction in both these cases. The last solution of that system can be expressed as

$$\{u_1, v_1\} = \left\{\frac{1 \pm \sqrt{5}}{2}t\right\} \text{ and } \{u_2, v_2\} = \left\{\frac{-1 \pm \sqrt{5}}{2}t\right\}$$

in any of the possible orders (there are four pairs of parameters for each point). Note that two of these points have a normal pointing towards $\left(2t^2, t, 1\right)$ and the other two have a normal in the $\left(2t^2, -t, 1\right)$ direction, indicating a transversal self-intersection:

$$\{u_1, v_1\} = \left\{\frac{1 \pm \sqrt{5}}{2} t\right\} \Rightarrow \left(-5t^2, 0, 5t^4\right) \Rightarrow \vec{T} = \left(-10t, 0, 20t^3\right) \parallel (1,0,0) \text{ as } t \to 0$$

Note that all edges and the self-intersection all have the same direction close to $(0,0,0)$; therefore, when shifting our plane in the direction (t_a, t_b, t_c), the only expression that matters is the sign of $(1,0,0) \cdot (t_a, t_b, t_c) = t_a$. The general shape of the transition can then be obtained from, for example, picking $(t_a, t_b, t_c) = (1,0,0)$. Indeed here are the general shapes for the cross-sections at $a = -\varepsilon$, $a = 0$ and $a = \varepsilon > 0$:



Note that the picture on the left side is actually bounded (there is no "cropping effect", the curve actually stops at the lower corners).

5.2.2. *Level sets for the time function*

The A_4 case requires $F_s = F_{ss} = F_{sss} = F_{ssss} = 0$:

$$\begin{aligned} P - Q &= rN \\ -rK + 1 &= 0 \\ -rK_s &= 0 \\ -r\left(K_{ss} - K^3\right) - K^2 &= 0 \end{aligned}$$

Not surprisingly, this implies $r = 1/K$ and $K_s = K_{ss} = 0$ at the point Q on the curve, so

$$F_{sssss} = -r\left(-6K^2K_s + K_{sss}\right) - 5KK_s = -\frac{K_{sss}}{K}$$

$$F_{ssssss} = -r\left(K^5 - 10K^2K_{ss} - 15KK_s^2 + K_{ssss}\right) + K^4 - 9KK_{ss} - 8K_s^2 = -\frac{K_{ssss}}{K}$$

$$F_{sssssss} = \begin{array}{c} -r\left(15K^4K_s - 15K^2K_{sss} - 60KK_sK_{ss} - 15K_s^3 + K_{sssss}\right) + \\ +14K^3K_s - 14KK_{sss} - 35K_sK_{ss} \end{array} =$$

$$= KK_{sss} - \frac{K_{sssss}}{K}$$

Now, from the $G_{(u)}$ expressions and using that $F_s = F_{ss} = F_{sss} = F_{ssss} = 0$ at the point at hand, we could obtain s_u, s_{uu} and s_{uuu}

$$\begin{aligned} 120 &= F_{sssss} s_u^5 \\ 0 &= F_{ssssss} s_u^2 + 15 F_{sssss} s_{uu} \\ 0 &= F_{sssssss} s_u^4 + 21 F_{ssssss} s_{uu} s_u^2 + F_{sssss}(35 s_u s_{uuu} + 105 s_{uu}^2) \end{aligned}$$

i.e.,

$$\begin{aligned} s_u^5 &= -120 \frac{K}{K_{sss}} \\ s_{uu} &= -\frac{K_{ssss}}{15 K_{sss}} s_u^2 \\ s_{uuu} &= \ldots \end{aligned}$$

Now, the linear system we look for is given by equations based on ∇G_{uuu}, ∇G_{uu} and ∇G_u. Indeed, using again that $F_s = F_{ss} = F_{sss} = F_{ssss} = 0$ at $(u, a, b, c) = (0, 0, 0, 0)$, we can mount the linear system from Equations 2

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = dA \begin{pmatrix} F_{ssst} s_u^3 + 3F_{sst} s_{uu} s_u + F_{st} s_{uuu} & F_{sst} s_u^2 + F_{st} s_{uu} & F_{st} s_u \\ F_{sssx} s_u^3 + 3F_{ssx} s_{uu} s_u + F_{sx} s_{uuu} & F_{ssx} s_u^2 + F_{sx} s_{uu} & F_{sx} s_u \\ F_{sssy} s_u^3 + 3F_{ssy} s_{uu} s_u + F_{sy} s_{uuu} & F_{ssy} s_u^2 + F_{sy} s_{uu} & F_{sy} s_u \end{pmatrix}$$

We could easily find then t_a, t_b and t_c, but we only care about t_a We start with the following determinant

$$\begin{aligned} D &= \begin{vmatrix} F_{ssst} s_u^3 + 3F_{sst} s_{uu} s_u + F_{st} s_{uuu} & F_{sst} s_u^2 + F_{st} s_{uu} & F_{st} s_u \\ F_{sssx} s_u^3 + 3F_{ssx} s_{uu} s_u + F_{sx} s_{uuu} & F_{ssx} s_u^2 + F_{sx} s_{uu} & F_{sx} s_u \\ F_{sssy} s_u^3 + 3F_{ssy} s_{uu} s_u + F_{sy} s_{uuu} & F_{ssy} s_u^2 + F_{sy} s_{uu} & F_{sy} s_u \end{vmatrix} = \\ &= \begin{vmatrix} F_{ssst} s_u^3 & F_{sst} s_u^2 & F_{st} s_u \\ F_{sssx} s_u^3 & F_{ssx} s_u^2 & F_{sx} s_u \\ F_{sssy} s_u^3 & F_{ssy} s_u^2 & F_{sy} s_u \end{vmatrix} = s_u^6 \begin{vmatrix} F_{ssst} & F_{sst} & F_{st} \\ F_{sssx} & F_{ssx} & F_{sx} \\ F_{sssy} & F_{ssy} & F_{sy} \end{vmatrix} \\ &= s_u^6 \begin{vmatrix} F_{ssst} & F_{sst} & F_{st} \\ K^2 T & -KN & -T \end{vmatrix} = -s_u^6 K \left(F_{ssst} + K^2 F_{st} \right) \end{aligned}$$

We also need the determinant of the following minor (that is independent of the time evolution of the curve)

$$\begin{aligned} D_a &= \begin{vmatrix} F_{ssx} s_u^2 + F_{sx} s_{uu} & F_{sx} s_u \\ F_{ssy} s_u^2 + F_{sy} s_{uu} & F_{sy} s_u \end{vmatrix} = s_u^3 \begin{vmatrix} F_{ssx} & F_{sx} \\ F_{ssy} & F_{sy} \end{vmatrix} = \\ &= s_u^3 \begin{vmatrix} -KN & -T \end{vmatrix} = -K s_u^3 \end{aligned}$$

and then

$$t_a = 6 \frac{D_a}{D} = \frac{6}{s_u^3 \left(F_{ssst} + K^2 F_{st} \right)}$$

So the destruction of dove-tails is tied to the sign of t_a; using the expression for s_u, we see that dove-tails disappear if it can be shown that

$$K K_{sss} \left(F_{ssst} + K^2 F_{st} \right) < 0$$

Finally, as $r = 1/K$, we can use Equations 5 for evolutions of the kind $Q_t = HN$; then, the simplification of the symmetry set happens if the following inequality holds whenever $K_s = K_{ss} = 0$

$$K_{sss}\left(H_{sss} + K^2 H_s\right) > 0$$

5.3. Case 2: A_3A_1

We have a circle of center P that is tangent to the curve at two points Q_1 and Q_2 with order of tangency 3 and 1, respectively. There are four possible pairs of canonical forms

$$\begin{aligned} G_1(u,a,b,c) &= \pm u^4 + au^2 + bu + c \\ G_2(u,a,b,c) &= \pm u^2 \end{aligned}$$

5.3.1. Level bifurcation set of the canonical form

We are now looking for (a,b,c) for which there are distinct u and v such that

$$\begin{aligned} G_1(u,a,b,c) &= G_1(v,a,b,c) \\ G_{1u}(u,a,b,c) &= G_{1v}(v,a,b,c) = 0 \end{aligned}$$

or

$$\begin{aligned} G_1(u,a,b,c) &= G_2(v,a,b,c) \\ G_{1u}(u,a,b,c) &= G_{2v}(v,a,b,c) = 0 \end{aligned}$$

The first case is

$$\begin{aligned} \pm u^4 + au^2 + bu + c &= \pm v^4 + av^2 + bv + c \\ \pm 4u^3 + 2au + b &= 0 \\ \pm 4v^3 + 2av + b &= 0 \end{aligned}$$

Use that $u \neq v$ and solve for (a,b,c); we are forced to conclude that $u = -v$ and

$$\begin{aligned} a &= \mp 2u^2 \\ b &= 0 \\ c &= c \end{aligned}$$

that is, the first part of the level bifurcation set is half the ac-plane.

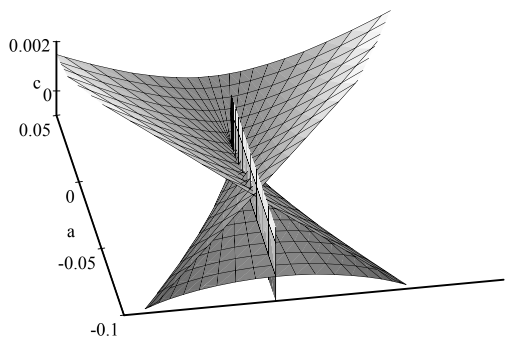
The second case means

$$\begin{aligned} \pm u^4 + au^2 + bu + c &= \pm v^2 \\ \pm 4u^3 + 2au + b &= 0 \\ \pm 2v &= 0 \end{aligned}$$

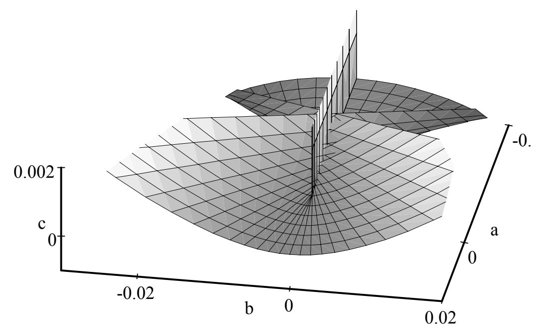
so $v = 0$,

$$\begin{aligned} a &= a \\ b &= \mp 4u^3 - 2au \\ c &= \pm 3u^4 + au^2 \end{aligned}$$

and we are left with another dove-tail looking surface (note that the sign used for G_2 does not affect the level bifurcation set at all). We use the top sign to draw both surfaces together below, from two different points of view



A_3A_1 bifurcation set - View 1



A_3A_1 bifurcation set - View 2

The bottom signs produce basically the same picture, but with a different orientation (indeed, the relation $(a, b, c) \to (-a, -b, -c)$ takes the level bifurcation set produced by one sign by the set produced by the other). We will work with the "top sign" surface until we need to do otherwise.

What are the edges of these surfaces? Clearly, the half-plane has an edge along $(0, 0, 1)$. For the dove-tail part, write

$$\begin{aligned} X_a &= \left(1, -2u, u^2\right) \\ X_u &= \left(0, -12u^2 - 2a, 12u^3 + 2au\right) \\ X_a \times X_u &= -2\left(6u^2 + a\right)\left(u^2, u, 1\right) \end{aligned}$$

so we have a critical curve at $a = -6u^2$; otherwise, the normal points towards $(0, 0, 1)$ as $a, u \to 0$. This critical curve is

$$a = -6u^2 \Rightarrow \left(-6u^2, 8u^3, -3u^4\right) \Rightarrow \vec{T} = \left(-12u, 24u^2, -12u^3\right) \parallel (1, 0, 0) \text{ as } u \to 0$$

Finally, we should mention the self-intersections of the level bifurcation set. The intersection between the half-plane and the dove-tail are given by

$$\begin{aligned} a &= -2u^2 \\ -4u_1^3 - 2au_1 &= 0 \\ 3u_1^4 + au_1^2 &= c \end{aligned}$$

whose solutions are

$$u_1 = 0 \Rightarrow \left(-2u^2, 0, 0\right) \Rightarrow \vec{T} = (-4u, 0, 0) \parallel (1, 0, 0) \text{ as } u \to 0$$
$$u_1 = \pm u \Rightarrow \left(-2u^2, 0, u^4\right) \Rightarrow \vec{T} = \left(-4u, 0, 4u^3\right) \parallel (1, 0, 0) \text{ as } u \to 0$$

while the self-intersections of the dove-tail are

$$\begin{aligned} a_1 &= a_2 = a \\ -4u_1^3 - 2au_1 &= -4u_2^3 - 2au_2 \\ 3u_1^4 + au_1^2 &= 3u_2^4 + au_2^2 \end{aligned}$$

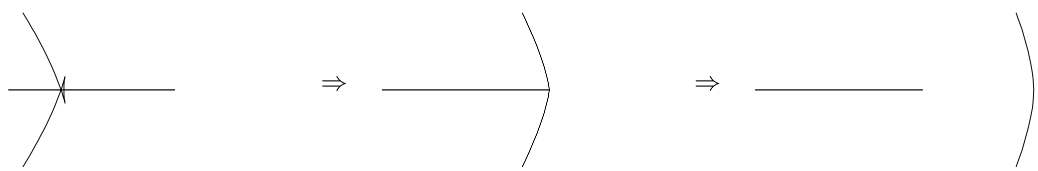
whose only non-trivial solution is

$$u_1 = -u_2 = -u, \; a = -2u^2 \Rightarrow \left(-2u^2, 0, u^4\right)$$

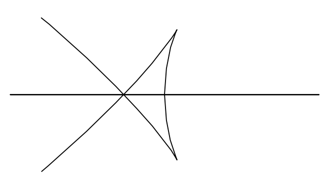
Not surprisingly, this belongs to the half-plane above. They are actually $A_1A_1A_1$ points; indeed, $G_1 = t^4 - 2u^2t^2 + u^4 = (t-u)^2(t+u)^2$ has two minima at the 0-level and $G_2 = t^2$ has a third minimum also at the 0-level.

Since the only directions involved in the edges and self-intersections are $(1, 0, 0)$ and $(0, 0, 1)$, the only expressions that matter are (the signs of) t_a and t_c. We choose two representatives to obtain the shape of the cross-sections, say, $(1, 0, 1)$ and $(1, 0, -1)$.

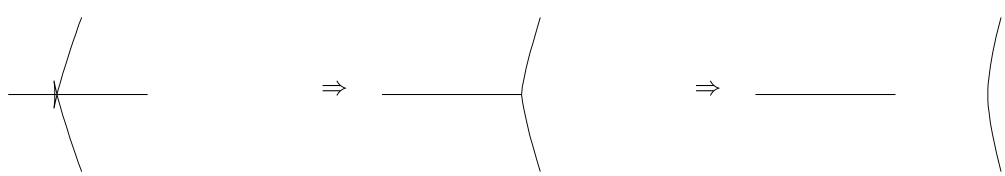
For the first, look at the cross-sections given by $a + c = -\varepsilon, 0, \varepsilon$ (where again $\varepsilon > 0$). The symmetry sets look like



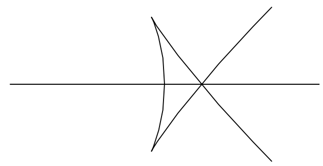
Note that the horizontal line actually ends at the right side for all pictures. All the other ends are effects of image cropping. In particular, the left-side picture deserves a close-up



Now, let's look at the cross-sections given by $a - c = -\varepsilon, 0, \varepsilon$. We have



Again, the horizontal line actually ends at its right side, and the close-up for the first picture reveals a similar situation:



Summarizing, $t_a > 0$ indicates a "simplification" of the symmetry set that separates its two components and transforms the dove-tail into a smooth curve; the sign of $t_a t_c$ determines the relative orientation of those two components.

Finally, if we use the "bottom sign" surface, we must revert the orientations of the transitions above, i.e., then $t_a < 0$ is the simplification while the relative orientation of the components still comes from the sign of $t_a t_c$.

5.3.2. Level sets for the time function

The $A_3 A_1$ case requires $F_s = F_{ss} = F_{sss} = 0$ at (P, Q_1), what implies $r = 1/K_1$ and $K_{1s} = 0$ at that point. So, for Q_1:

$$
\begin{aligned}
F_{ssss} &= -r\left(-K^3 + K_{ss}\right) - K^2 = -\frac{K_{1ss}}{K_1} \\
F_{sssss} &= -r\left(-6K^2 K_s + K_{sss}\right) - 5KK_s = -\frac{K_{1sss}}{K_1}
\end{aligned}
$$

Now, from the $G_{(s)}$ expressions and using that $F_s = F_{ss} = 0$ at Q, we obtain s_u

$$
\begin{aligned}
\pm 24 &= F_{ssss} s_u^4 \\
0 &= F_{sssss} s_u^2 + 10 F_{ssss} s_{uu}
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
s_u^4 &= \mp 24 \frac{K_1}{K_{1ss}} \\
s_{uu} &= -\frac{K_{1sss}}{10 K_{1ss}} s_u^2
\end{aligned}
$$

In particular, this tells us when to use each sign; if $\frac{K_1}{K_{1ss}} < 0$, we must use the top sign, and vice-versa. Similarly, we only have $F_s = 0$ at (P, Q_2), what implies $P - Q_2 = rN_2$.

Now, the linear system we look for is given by equations based on ∇G_{1uu}, ∇G_{1u} and $\nabla \Delta G$. From the set of equations 2, using $F_s = F_{ss} = F_{sss} = 0$ at Q_1 and $F_s = 0$ at Q_2 we mount the following linear system at $(u, a, b, c) = (0, 0, 0, 0)$:

$$
\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = dA \begin{pmatrix} F_{1sst} s_u^2 + F_{1st} s_{uu} & F_{1st} s_u & \Delta F_t \\ F_{1ssx} s_u^2 + F_{1sx} s_{uu} & F_{1sx} s_u & \Delta F_x \\ F_{1ssy} s_u^2 + F_{1sy} s_{uu} & F_{1sy} s_u & \Delta F_y \end{pmatrix}
$$

We only care about t_a and t_c. The time evolution dependent determinant is

$$D = \begin{vmatrix} F_{1sst}s_u^2 + F_{1st}s_{uu} & F_{1st}s_u & \Delta F_t \\ F_{1ssx}s_u^2 + F_{1sx}s_{uu} & F_{1sx}s_u & \Delta F_x \\ F_{1ssy}s_u^2 + F_{1sy}s_{uu} & F_{1sy}s_u & \Delta F_y \end{vmatrix} =$$

$$= \begin{vmatrix} F_{1sst}s_u^2 & F_{1st}s_u & \Delta F_t \\ F_{1ssx}s_u^2 & F_{1sx}s_u & \Delta F_x \\ F_{1ssy}s_u^2 & F_{1sy}s_u & \Delta F_y \end{vmatrix} = s_u^3 \begin{vmatrix} F_{1sst} & F_{1st} & \Delta F_t \\ F_{1ssx} & F_{1sx} & \Delta F_x \\ F_{1ssy} & F_{1sy} & \Delta F_y \end{vmatrix}$$

The other important determinants are independent of the time evolution

$$D_a = \begin{vmatrix} F_{1sx}s_u & \Delta F_x \\ F_{1sy}s_u & \Delta F_y \end{vmatrix} = s_u \begin{vmatrix} F_{1sx} & \Delta F_x \\ F_{1sy} & \Delta F_y \end{vmatrix}$$

$$D_c = \begin{vmatrix} F_{1ssx}s_u^2 + F_{1sx}s_{uu} & F_{1sx}s_u \\ F_{1ssy}s_u^2 + F_{1sy}s_{uu} & F_{1sy}s_u \end{vmatrix} = s_u^3 \begin{vmatrix} F_{1ssx} & F_{1sx} \\ F_{1ssy} & F_{1sy} \end{vmatrix}$$

Now, if we let α be the angle from T_1 to T_2

$$\begin{vmatrix} F_{1ssx} & F_{1sx} \\ F_{1ssy} & F_{1sy} \end{vmatrix} = \begin{vmatrix} -K_1 N_1 & -T_1 \end{vmatrix} = -K_1$$

$$\begin{vmatrix} F_{1ssx} & \Delta F_x \\ F_{1ssy} & \Delta F_y \end{vmatrix} = \begin{vmatrix} -K_1 N_1 & rN_1 - rN_2 \end{vmatrix} = \sin\alpha$$

$$\begin{vmatrix} F_{1sx} & \Delta F_x \\ F_{1sy} & \Delta F_y \end{vmatrix} = \begin{vmatrix} -T_1 & rN_1 - rN_2 \end{vmatrix} = -\frac{1-\cos\alpha}{K_1}$$

so we have

$$D = -s_u^3 \left(F_{1sst}\frac{1-\cos\alpha}{K_1} + F_{1st}\sin\alpha + K_1 \Delta F_t \right)$$

$$D_a = -s_u \frac{1-\cos\alpha}{K_1}$$

$$D_c = -s_u^3 K_1$$

and then

$$t_a = 2\frac{D_a}{D} = \frac{2}{s_u^2}\frac{1-\cos\alpha}{K_1}\left(F_{1sst}\frac{1-\cos\alpha}{K_1} + F_{1st}\sin\alpha + K_1 \Delta F_t \right)^{-1}$$

$$t_c = \frac{D_c}{D} = K_1 \left(F_{1sst}\frac{1-\cos\alpha}{K_1} + F_{1st}\sin\alpha + K_1 \Delta F_t \right)^{-1}$$

Note that

$$t_a t_c = 2\frac{1-\cos\alpha}{s_u^2}\left(F_{1sst}\frac{1-\cos\alpha}{K_1} + F_{1st}\sin\alpha + K_1 \Delta F_t \right)^{-2} > 0$$

so the only possible orientation of the two components of the symmetry set is the first one (the “T” looking one, as opposite to the “Y” looking orientation).

Now, the direction of the transition depends on the sign of $t_a > 0$; in other words, simplification occurs if we can show that

$$F_{1sst}\,(1-\cos\alpha) + F_{1st}K_1\sin\alpha + K_1^2\Delta F_t > 0$$

Since $r = 1/K_1$, Equations 4 turn to

$$\Delta F_t = -\frac{H_1 - H_2}{K_1}$$
$$F_{1ts} = -\frac{H_{1s}}{K_1}$$
$$F_{1tss} = -\frac{H_{1ss}}{K_1}$$

so the simplification of the symmetry set happens if the following inequality holds whenever $K_{1s} = 0$

$$\frac{H_{1ss}}{K_1}\,(1-\cos\alpha) + H_{1s}\sin\alpha + K_1\,(H_1 - H_2) < 0$$

where α is the angle from T_1 to T_2.

5.4. Case 3: A_2A_2

In this case, we have a circle of center P that is tangent to the curve at two points Q_1 and Q_2 with orders of tangency 2 and 2, respectively (a bi-osculating circle). The canonical form has two polynomials

$$G_1(u,a,b,c) = u^3 + au + c$$
$$G_2(u,a,b,c) = u^3 + bu$$

5.4.1. Level bifurcation set of the canonical form

Three cases to consider now: we are now looking for (a,b,c) for which there are distinct u and v such that

$$G_1(u,a,b,c) = G_1(v,a,b,c)$$
$$G_{1u}(u,a,b,c) = G_{1v}(v,a,b,c) = 0$$

or

$$G_2(u,a,b,c) = G_2(v,a,b,c)$$
$$G_{2u}(u,a,b,c) = G_{2v}(v,a,b,c) = 0$$

or

$$G_1(u,a,b,c) = G_2(v,a,b,c)$$
$$G_{1u}(u,a,b,c) = G_{2v}(v,a,b,c) = 0$$

The first case is

$$\begin{aligned} u^3 + au + c &= v^3 + av + c \\ 3u^2 + a &= 0 \\ 3v^2 + a &= 0 \end{aligned}$$

but this forces $u = v = 0$, so there is nothing to consider here.

The second case is similar; indeed,

$$\begin{aligned} u^3 + bu &= v^3 + bv \\ 3u^2 + b &= 0 \\ 3v^2 + b &= 0 \end{aligned}$$

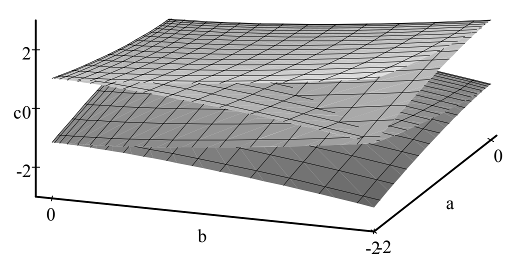
also implies $u = v = 0$, so nothing to be done here either. We are left with only one case

$$\begin{aligned} u^3 + au + c &= v^3 + bv \\ 3u^2 + a &= 0 \\ 3v^2 + b &= 0 \end{aligned}$$

Solving for (a, b, c), we find

$$\begin{aligned} a &= -3u^2 \\ b &= -3v^2 \\ c &= 2u^3 - 2v^3 \end{aligned}$$

whose graph is the next picture



A_2A_2 bifurcation set

You can almost think of these as 4 different surfaces that are almost flat, each pair intersecting along a different curve. Our normal vector comes from

$$\begin{aligned} X_u &= \left[-6u, 0, 6u^2\right] \\ X_v &= \left[0, -6v, -6v^2\right] \\ X_u \times X_v &= 36uv\,[u, -v, 1] \end{aligned}$$

So the "edges" correspond to $u = 0$ and $v = 0$; otherwise, the normal points towards $(0,0,1)$ as $u, v \to 0$. Also, the self-intersection is given by

$$\begin{aligned} -3u_1^2 &= -3u_2^2 \\ -3v_1^2 &= -3v_2^2 \\ 2u_1^3 - 2v_1^3 &= 2u_2^3 - 2v_2^3 \end{aligned}$$

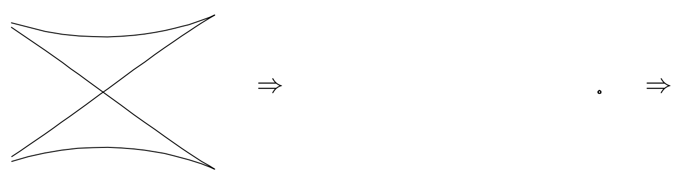
whose only non-trivial solution is $u_1 = -u_2 = v_1 = -v_2$.

The critical curves are then the two edges and the self-intersection

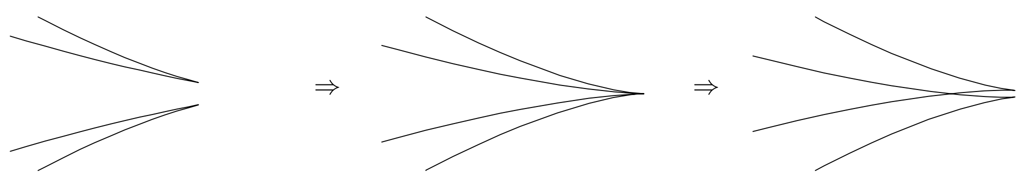
$$\begin{aligned} u = 0 &\Rightarrow \left(0, -3v^2, -2v^3\right) \Rightarrow \vec{T} = \left(0, -6v, -6v^2\right) \parallel (0,1,0) \text{ as } v \to 0 \\ v = 0 &\Rightarrow \left(-3u^2, 0, 2u^3\right) \Rightarrow \vec{T} = \left(-6u, 0, 6u^2\right) \parallel (1,0,0) \text{ as } u \to 0 \\ u = v &\Rightarrow \left(-3u^2, -3u^2, 0\right) \Rightarrow \vec{T} = (-6u, -6u, 0) \parallel (1,1,0) \text{ as } u \to 0 \end{aligned}$$

and the expressions that matter are (the signs of) t_a, t_b and $t_a + t_b$. Since the $t_a t_b$ plane is divided in 6 regions when we consider the signs of those expressions, we choose three representatives to obtain the shape of the cross-sections, say, $(1,1,0)$, $(1,-2,0)$ and $(2,-1,0)$.

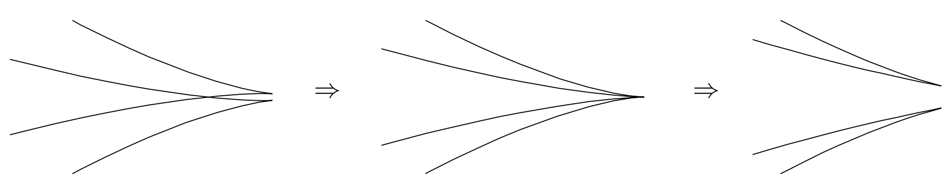
For the first, look at the cross-sections given by $a + b = -\varepsilon, 0, \varepsilon$. The sequence looks like "the disappearing moth"



While the cross-sections given by $a - 2b = -\varepsilon, 0, \varepsilon$ are the following "vvw" transition



Finally, the cross-sections given by $2a - b = -\varepsilon, 0, \varepsilon$ are (not surprisingly) similar to the ones we have just obtained, a "wvv" transition



Summarizing, we have

- $t_a, t_b > 0 \Rightarrow$Disappearing moth;
- $t_a, t_b < 0 \Rightarrow$Appearing moth;
- $t_a t_b < 0$ and $t_a + t_b > 0 \Rightarrow$A wvv transition;
- $t_a t_b < 0$ and $t_a + t_b < 0 \Rightarrow$A vvw transition;

5.4.2. Level sets for the time function

We can treat both polynomials in similar ways: we require $F_s = F_{ss} = 0$ at (P, Q_1) and (P, Q_2), what implies $r = 1/K_1 = 1/K_2$. We will write F, G, Q, K,... for a while, bringing the correct indexes back later. So, at Q:

$$F_{sss} = -rK_s = -\frac{K_s}{K}$$

Now, from the G_{uuu} expressions and using that $F_s = F_{ss} = 0$ at Q, we obtain s_u

$$6 = F_{sss}s_u^3 \Rightarrow s_u^3 = -\frac{6K}{K_s}$$

Now, the linear system we are looking for is given by equations based on ∇G_{1u}, ∇G_{2u} and $\nabla \Delta G$. From 2, using $F_s = F_{ss} = 0$ at Q_1 and Q_2 we mount the following linear system at $(u, a, b, c) = (0, 0, 0, 0)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = dA \begin{pmatrix} F_{1st}s_{1u} & F_{2st}s_{2u} & \Delta F_t \\ F_{1sx}s_{1u} & F_{2sx}s_{2u} & \Delta F_x \\ F_{1sy}s_{1u} & F_{2sy}s_{2u} & \Delta F_y \end{pmatrix}$$

The important values for us are t_a and t_b. Here is the evolution dependent determinant

$$D = \begin{vmatrix} F_{1st}s_{1u} & F_{2st}s_{2u} & \Delta F_t \\ F_{1sx}s_{1u} & F_{2sx}s_{2u} & \Delta F_x \\ F_{1sy}s_{1u} & F_{2sy}s_{2u} & \Delta F_y \end{vmatrix} = s_{1u}s_{2u} \begin{vmatrix} F_{1st} & F_{2st} & \Delta F_t \\ F_{1sx} & F_{2sx} & \Delta F_x \\ F_{1sy} & F_{2sy} & \Delta F_y \end{vmatrix}$$

The following minor determinants are independent of the particular time evolution of the curve

$$D_a = \begin{vmatrix} F_{2sx}s_{2u} & \Delta F_x \\ F_{2sy}s_{2u} & \Delta F_y \end{vmatrix} = s_{2u} \begin{vmatrix} F_{2sx} & \Delta F_x \\ F_{2sy} & \Delta F_y \end{vmatrix}$$

$$D_b = \begin{vmatrix} F_{1sx}s_{1u} & \Delta F_x \\ F_{1sy}s_{1u} & \Delta F_y \end{vmatrix} = s_{1u} \begin{vmatrix} F_{1sx} & \Delta F_x \\ F_{1sy} & \Delta F_y \end{vmatrix}$$

But since

$$\begin{vmatrix} F_{1sx} & \Delta F_x \\ F_{1sy} & \Delta F_y \end{vmatrix} = \begin{vmatrix} -T_1 & rN_1 - rN_2 \end{vmatrix} = -\frac{1-\cos\alpha}{K}$$

$$\begin{vmatrix} F_{2sx} & \Delta F_x \\ F_{2sy} & \Delta F_y \end{vmatrix} = \begin{vmatrix} -T_2 & rN_1 - rN_2 \end{vmatrix} = \frac{1-\cos\alpha}{K}$$

$$\begin{vmatrix} F_{1sx} & F_{2sx} \\ F_{1sy} & F_{2sy} \end{vmatrix} = \begin{vmatrix} -T_1 & -T_2 \end{vmatrix} = \sin\alpha$$

we have

$$D = s_{1u}s_{2u}\frac{1-\cos\alpha}{K}\left(F_{1st} + F_{2st} + \Delta F_t K \frac{\sin\alpha}{1-\cos\alpha}\right)$$

$$D_a = s_{2u}\frac{1-\cos\alpha}{K}$$

$$D_b = -s_{1u}\frac{1-\cos\alpha}{K}$$

and then

$$t_a = \frac{D_a}{D} = -\sqrt[3]{\frac{K_{1s}}{6K}} \left(F_{1st} + F_{2st} + \Delta F_t \frac{K \sin\alpha}{1-\cos\alpha} \right)^{-1}$$
$$t_b = -\frac{D_b}{D} = -\sqrt[3]{\frac{K_{2s}}{6K}} \left(F_{1st} + F_{2st} + \Delta F_t \frac{K \sin\alpha}{1-\cos\alpha} \right)^{-1}$$

So we have the following options:

- Disappearing moth

$$K_{1s}K_{2s} > 0 \text{ and } K_{1s} \left(\frac{F_{1st} + F_{2st}}{K} + \Delta F_t \frac{\sin\alpha}{1-\cos\alpha} \right) < 0$$

- Appearing moth

$$K_{1s}K_{2s} > 0 \text{ but } K_{1s} \left(\frac{F_{1st} + F_{2st}}{K} + \Delta F_t \frac{\sin\alpha}{1-\cos\alpha} \right) > 0$$

- WVV transition

$$K_{1s}K_{2s} < 0 \text{ and } (K_{1s} + K_{2s}) \left(\frac{F_{1st} + F_{2st}}{K} + \Delta F_t \frac{\sin\alpha}{1-\cos\alpha} \right) < 0$$

- VVW transition

$$K_{1s}K_{2s} < 0 \text{ but } (K_{1s} + K_{2s}) \left(\frac{F_{1st} + F_{2st}}{K} + \Delta F_t \frac{\sin\alpha}{1-\cos\alpha} \right) > 0$$

We again use Equations 5

$$\Delta F_t = -\frac{H_1 - H_2}{K}$$
$$F_{1st} = -\frac{H_{1s}}{K};\ F_{2st} = -\frac{H_{2s}}{K}$$

to rewrite the “simplification clause” as

$$(K_{1s} + K_{2s}) \left(H_{1s} + H_{2s} + (H_1 - H_2) \frac{K \sin\alpha}{1-\cos\alpha} \right) > 0$$

whenever $K_1 = K_2 = K$; here, we consider the disappearing moth and the WVV transition as “simplifying”. Once more α is the angle between T_1 and T_2.

5.5. Case 4: $A_2A_1A_1$

In this case, we have a circle of center P that is tangent to the curve at three points Q_1, Q_2 and Q_3 with orders of tangency 2, 1 and 1, respectively. The canonical form has three polynomials

$$G_1(u, a, b, c) = u^3 + au$$
$$G_2(u, a, b, c) = \pm u^2 + b$$
$$G_3(u, a, b, c) = \pm u^2 + c$$

5.5.1. Level bifurcation set of the canonical form

There are 6 potential cases that can contribute for the level bifurcation set here. The first one corresponds to pairs of distinct u, v such that

$$G_1(u,a,b,c) = G_1(v,a,b,c)$$
$$G_{1u}(u,a,b,c) = G_{1v}(v,a,b,c) = 0$$

i.e.,

$$u^3 + au = v^3 + av$$
$$3u^2 + a = 0$$
$$3v^2 + a = 0$$

and there are no solutions with $u \neq v$.

Secondly, we should consider the "G_2G_2" case

$$G_2(u,a,b,c) = G_2(v,a,b,c)$$
$$G_{2u}(u,a,b,c) = G_{2v}(v,a,b,c) = 0$$

However, this has been done before (look at the A_3A_1 case) and no level bifurcation surface was generated then. The "G_3G_3" case is similar and produces nothing as well, and the "G_2G_3" case is

$$G_2(u,a,b,c) = G_3(v,a,b,c)$$
$$G_{2u}(u,a,b,c) = G_{3v}(v,a,b,c) = 0$$

or

$$\pm u^2 + b = \pm v^2 + c$$
$$\pm 2u = 0$$
$$\pm 2v = 0$$

giving us the whole plane $b = c$ no matter what the signs are. Finally, the "G_1G_2" and "G_1G_3" cases are similar. The first is

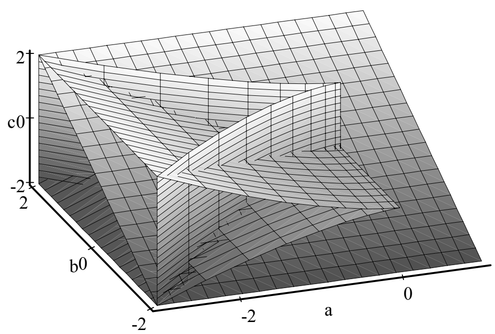
$$G_1(u,a,b,c) = G_2(v,a,b,c)$$
$$G_{1u}(u,a,b,c) = G_{2v}(v,a,b,c) = 0$$

or

$$u^3 + au = \pm v^2 + b$$
$$3u^2 + a = 0$$
$$\pm 2v = 0$$

This gives us $[a,b,c] = \left[-3u^2, -2u^3, c\right]$. Similarly, pairing up G_1 and G_3 gives us another surface parametrized by $[a,b,c] = [-3u^2, b, -2u^3]$.

Summarizing, the level bifurcation set is given by a plane $b = c$ and two cusp-like surfaces $4a^3 = -27b^2$ and $4a^3 = -27c^2$, graphed below



$A_2A_1A_1$ level bifurcation set

Clearly, the only edges are the edges of the cusps in the $(0,1,0)$ and $(0,0,1)$ directions. None of the 3 individual surfaces have self-intersections, but we can find intersections between each two of them. The cusp-plane intersections are actually the same curve as seen below

$$b = -2u^3 \Rightarrow (a,b,c) = \left(-3u^2, -2u^3, -2u^3\right) \Rightarrow \vec{T} = \left(-6u, -6u^2, -6u^2\right) \parallel (1,0,0) \text{ as } u \to 0$$
$$c = -2u^3 \Rightarrow (a,b,c) = \left(-3u^2, -2u^3, -2u^3\right) \Rightarrow \vec{T} = \left(-6u, -6u^2, -6u^2\right) \parallel (1,0,0) \text{ as } u \to 0$$

while the cusp-cusp intersection is given by

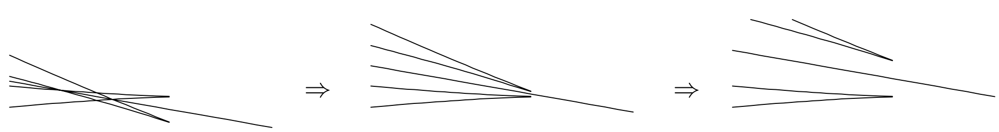
$$\begin{aligned} -3u_1^2 &= -3u_2^2 \\ -2u_1^3 &= b \\ c &= -2u_2^3 \end{aligned}$$

leading to two curves (one of them is, of course, the previous one)

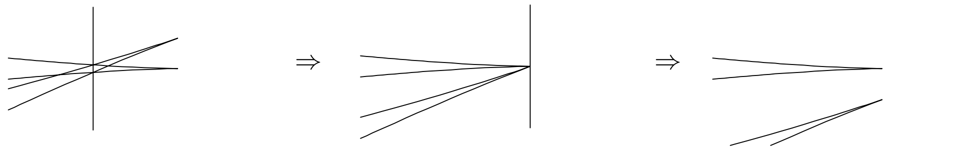
$$\pm u_1 = u_2 = u \Rightarrow (a,b,c) = \left(-3u^2, \mp 2u^3, -2u^3\right) \Rightarrow \vec{T} = \left(-6u, \mp 6u^2, -6u^2\right) \parallel (1,0,0) \text{ as } u \to 0$$

Collecting all the information, we found three edge/self-intersection directions given by the three axes. This indicates that the signs of t_a, t_b and t_c all matter, and we should choose 4 representatives, say, $(1,1,1)$, $(1,-1,1)$, $(1,1,-1)$ and $(1,-1,-1)$.

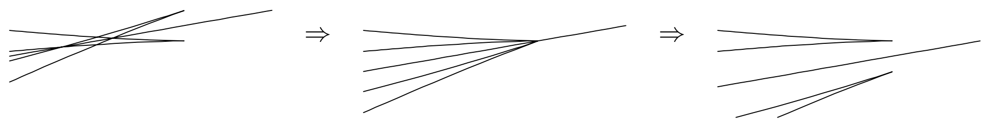
The first sequence of cross-sections is $a + b + c = -\varepsilon, 0, \varepsilon$



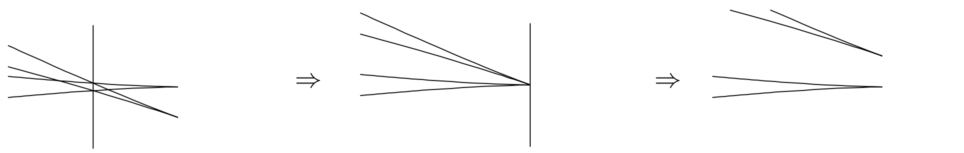
The second is $a - b + c = -\varepsilon, 0, \varepsilon$



The cross-sections $a - b - c = -\varepsilon, 0, \varepsilon$ give us



Finally, we have $a + b - c = -\varepsilon, 0, \varepsilon$



Summarizing, we have

- $t_a > 0 \Rightarrow$Two cusps and a line with two triple intersections separate;
- $t_a < 0 \Rightarrow$Two cusps and a line entangle into two triple intersections;
- $t_b t_c > 0 \Rightarrow$Line is "between" cusps;
- $t_b t_c < 0 \Rightarrow$Line is "across" cusps;

5.5.2. Level sets for the time function

The equations we get for G_1 are exactly the same as the ones we got in the A_2A_2 case, i.e., we have $F_s = F_{ss} = 0$ at (P, Q_1) and $r = 1/K_1$, and

$$s_u^3 = -\frac{6K}{K_s}$$

Now, the linear system we are looking for is given by equations based on ∇G_{1u}, $\nabla\Delta_{12}G$ and $\nabla\Delta_{13}G$ (here we use $\Delta_{ij}Z$ to represent $Z_j - Z_i$). From 2, using $F_s = F_{ss} = 0$ at Q_1, we mount the following linear system at $(u, a, b, c) = (0, 0, 0, 0)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = dA \begin{pmatrix} F_{1st}s_{1u} & \Delta_{12}F_t & \Delta_{13}F_t \\ F_{1sx}s_{1u} & \Delta_{12}F_x & \Delta_{13}F_x \\ F_{1sy}s_{1u} & \Delta_{12}F_y & \Delta_{13}F_y \end{pmatrix}$$

Now

$$D = \begin{vmatrix} F_{1st}s_{1u} & \Delta_{12}F_t & \Delta_{13}F_t \\ F_{1sx}s_{1u} & \Delta_{12}F_x & \Delta_{13}F_x \\ F_{1sy}s_{1u} & \Delta_{12}F_y & \Delta_{13}F_y \end{vmatrix}$$

Let α be the angle between T_1 and T_2 and β the angle between T_1 and T_3 (and $\beta-\alpha$ the angle between T_2 and T_3). Then we have the minors

$$D_a = \begin{vmatrix} \Delta_{12}F_x & \Delta_{13}F_x \\ \Delta_{12}F_y & \Delta_{13}F_y \end{vmatrix} = \begin{vmatrix} rN_2 - rN_1 & rN_3 - rN_1 \end{vmatrix} = \frac{\sin(\beta-\alpha)+\sin\alpha-\sin\beta}{K_1^2}$$
$$D_b = \begin{vmatrix} F_{1sx}s_{1u} & \Delta_{12}F_x \\ F_{1sy}s_{1u} & \Delta_{12}F_y \end{vmatrix} = s_{1u}\begin{vmatrix} -T_1 & rN_2 - rN_1 \end{vmatrix} = s_{1u}\frac{1-\cos\alpha}{K_1}$$
$$D_c = \begin{vmatrix} F_{1sx}s_{1u} & \Delta_{13}F_x \\ F_{1sy}s_{1u} & \Delta_{13}F_y \end{vmatrix} = s_{1u}\begin{vmatrix} -T_1 & rN_3 - rN_1 \end{vmatrix} = s_{1u}\frac{1-\cos\beta}{K_1}$$

and then

$$D = s_{1u}\left(F_{1st}\frac{\sin(\beta-\alpha)+\sin\alpha-\sin\beta}{K_1^2} + \Delta_{12}F_t\frac{1-\cos\beta}{K_1} + \Delta_{13}F_t\frac{1-\cos\alpha}{K_1}\right)$$

From these expressions. we can get

$$t_a = \frac{D_a}{D};\ t_b = -\frac{D_b}{D};\ t_c = \frac{D_c}{D}$$

Note that the

$$t_b t_c = -\frac{s_{1u}^2}{K_1^2}\frac{(1-\cos\alpha)(1-\cos\beta)}{D^2} < 0$$

so we expect the line to be always "across" the cusps. The separation of the line and the cusps happens when

$$\frac{\sin(\beta-\alpha)+\sin\alpha-\sin\beta}{K_{1s}}\begin{pmatrix} F_{1st}\frac{\sin(\beta-\alpha)+\sin\alpha-\sin\beta}{K_1}+ \\ +\Delta_{12}F_t(1-\cos\beta)+\Delta_{13}F_t(1-\cos\alpha) \end{pmatrix} > 0$$

Using Equations 4

$$\Delta_{12}F_t = -\frac{H_2-H_1}{K_1};\ \Delta_{13}F_t = -\frac{H_3-H_1}{K_1}$$
$$F_{1st} = -\frac{H_{1s}}{K_1}$$

we rewrite the separation condition as

$$\frac{\sin(\beta-\alpha)+\sin\alpha-\sin\beta}{K_{1s}}\begin{pmatrix} (K_1(H_2-H_1)+H_{1s}\sin\alpha)(1-\cos\beta)+ \\ +(K_1(H_3-H_1)-H_{1s}\sin\beta)(1-\cos\alpha) \end{pmatrix} < 0$$

Note that the pictures on both sides of the transition are very similar, so it is hard to talk about "simplification" here.

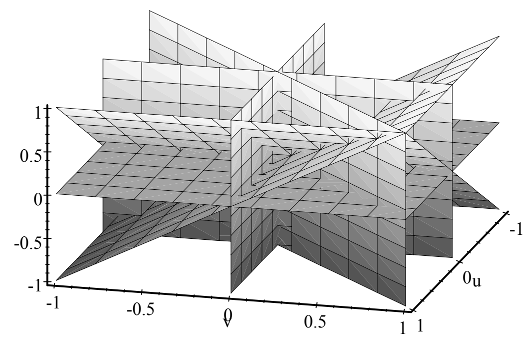
5.6. Case 5: $A_1A_1A_1A_1$

This is simply a circle of center P that is tangent to the curve at four points. The canonical form has four polynomials

$$G_1(u,a,b,c) = \pm u^2$$
$$G_2(u,a,b,c) = \pm u^2 + a$$
$$G_3(u,a,b,c) = \pm u^2 + b$$
$$G_4(u,a,b,c) = \pm u^2 + c$$

5.6.1. Level bifurcation set of the canonical form

We have done all the work for this case before. The "G_iG_i" cases are known not to provide any level bifurcation surfaces. The cases "G_2G_3", "G_2G_4" and "G_3G_4" gives us the planes $a = b$, $a = c$ and $b = c$ respectively, while "G_1G_2", "G_1G_3" and "G_1G_4" gives us the planes $a = 0$, $b = 0$ and $c = 0$. The level bifurcation set is this "paper cut hell":



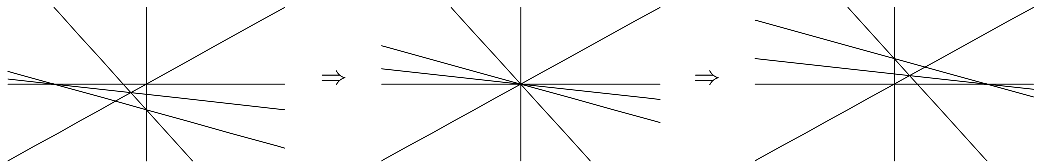
$A_1A_1A_1A_1$ bifurcation set

It is not hard to see that the self-intersections of this level bifurcation set corresponds to lines in the directions $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$, so, at first, all signs of t_a, t_b, t_c, t_a+t_b, t_a+t_c, t_b+t_c and $t_a+t_b+t_c$ matter. However, since the level bifurcation set above is invariant through permutations of the variables a, b and c, we will only consider the cases

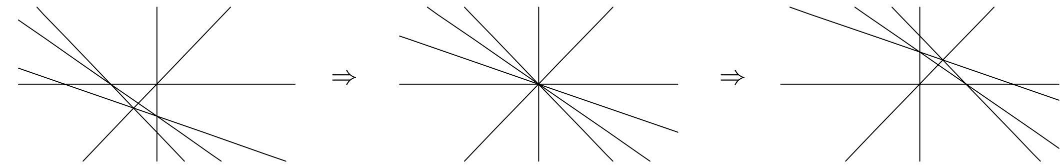
Case	t_a	t_b	t_c	t_a+t_b	t_a+t_c	t_b+t_c	$t_a+t_b+t_c$	Representative
I	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$(1,2,3)$
II	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$(2,3,-1)$
III	$+$	$+$	$-$	$+$	$+$	$-$	$+$	$(3,1,-2)$
IV	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$(2,3,4)$
V	$+$	$+$	$-$	$+$	$-$	$-$	$-$	$(1,2,4)$

The pictures for each case are

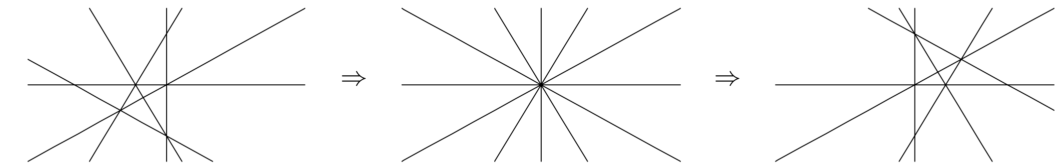
I:



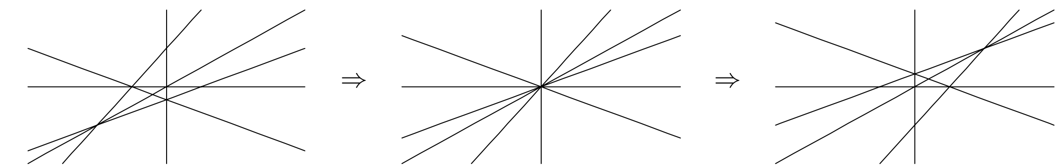
II:



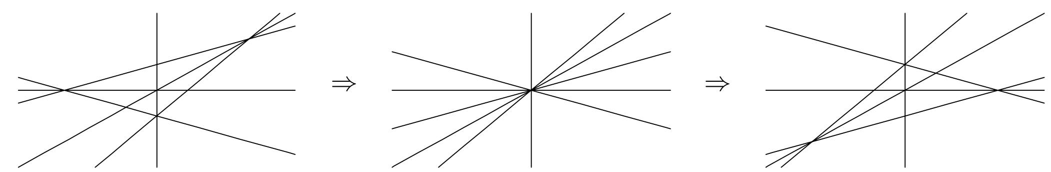
ÏII:



IV:



V:



Note that the two sides of each transition are indistinguishable. Also, when reversing the signs of all expressions t_a, t_b, t_c, $t_a + t_b$, $t_a + t_c$, $t_b + t_c$ and $t_a + t_b + t_c$, just read the transitions above in the right-to-left direction.

5.6.2. Level sets for the time function

In this case, we can obtain a linear system immediately from the differences $\nabla\Delta_{12}G$, $\nabla\Delta_{13}G$ and $\nabla\Delta_{14}G$. Indeed, at $(u, a, b, c) = (0, 0, 0, 0)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = dA \begin{pmatrix} \Delta_{12}F_t & \Delta_{13}F_t & \Delta_{14}F_t \\ \Delta_{12}F_x & \Delta_{13}F_x & \Delta_{14}F_x \\ \Delta_{12}F_y & \Delta_{13}F_y & \Delta_{14}F_y \end{pmatrix}$$

Now, the minors independent of the evolution are

$$D_a = \begin{vmatrix} \Delta_{13}F_x & \Delta_{14}F_x \\ \Delta_{13}F_y & \Delta_{14}F_y \end{vmatrix} = r^2\left(\sin\left(\gamma-\beta\right)+\sin\beta-\sin\gamma\right) = 4r^2\sin\left(\frac{\gamma-\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\sin\left(\frac{\beta}{2}\right)$$
$$D_b = \begin{vmatrix} \Delta_{12}F_x & \Delta_{14}F_x \\ \Delta_{12}F_y & \Delta_{14}F_y \end{vmatrix} = r^2\left(\sin\left(\gamma-\alpha\right)+\sin\alpha-\sin\gamma\right) = 4r^2\sin\left(\frac{\gamma-\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\gamma}{2}\right)$$
$$D_c = \begin{vmatrix} \Delta_{12}F_x & \Delta_{13}F_x \\ \Delta_{12}F_y & \Delta_{13}F_y \end{vmatrix} = r^2\left(\sin\left(\beta-\alpha\right)+\sin\alpha-\sin\beta\right) = 4r^2\sin\left(\frac{\beta-\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$$

and the big determinant is

$$D = \Delta_{12}F_t\ D_c - \Delta_{13}F_t\ D_b + \Delta_{14}F_t\ D_c$$

where we use now α, β and γ for the three pairs of points. It is not worth expanding

$$t_a = \frac{D_a}{D}$$
$$t_b = -\frac{D_b}{D}$$
$$t_c = \frac{D_c}{D}$$

However, we should note that

$$t_a+t_b+t_c = \frac{D_a-D_b+D_c}{D} = \frac{r^2}{D}\left(\sin\left(\gamma-\beta\right)-\sin\left(\gamma-\alpha\right)+\sin\left(\beta-\alpha\right)\right) = $$
$$= \frac{4r^2}{D}\sin\left(\frac{\beta-\alpha}{2}\right)\sin\left(\frac{\gamma-\alpha}{2}\right)\sin\left(\frac{\gamma-\beta}{2}\right)$$

and therefore

$$t_at_bt_c\left(t_a+t_b+t_c\right) =$$
$$= -\left(\frac{r^4}{D^2}\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\sin\left(\frac{\beta-\alpha}{2}\right)\sin\left(\frac{\gamma-\alpha}{2}\right)\sin\left(\frac{\gamma-\beta}{2}\right)\right)^2 < 0$$

so only transitions II, III and IV are possible. Since these 3 pictures are locally diffeomorphic (described as "take a convex quadrilateral and its diagonals, shrink it towards one of its vertices and then start expanding the quadrilateral out of this vertex on the other side), we can state that they correspond to the only possible transition in this case.

As we had seen before, both sides of the $A_1A_1A_1A_1$ transition are indistinguishable, so there is nothing to say in this case about the direction of the transition, no matter what time evolution we are dealing with.

5.7. Further Work

It is a bit surprising that the medial axis can get more "complicated" as its originating curve goes through MCM (as in the case A_3A_1 above). Using the calculations above, we wish to examine if there are other curve motions that have the property of always simplifying the medial axis. Preliminary results indicate that this is not the case.

ACKNOWLEDGMENT

This work is part of Ralph Teixeira's doctoral thesis [20], written at Harvard University; such thesis was advised by Prof. David Mumford and Prof. Peter Giblin, and partially sponsored by the Brazilian agency CNPq (Conselho Nacional de Pesquisa), through process 201205/92. To all of them, our thanks.

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