# PHYSICAL MEASURES AND ABSOLUTE CONTINUITY FOR ONE-DIMENSIONAL CENTER DIRECTION

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Abstract. For a class of partially hyperbolic $C^k$, $k > 1$ diffeomorphisms with circle center leaves we prove existence and finiteness of physical (or Sinai-Ruelle-Bowen) measures, whose basins cover a full Lebesgue measure subset of the ambient manifold. Our conditions contain an open and dense subset of all $C^k$ partially hyperbolic skew-products on compact circle bundles.

Our arguments blend ideas from the theory of Gibbs states for diffeomorphisms with mostly contracting center direction together with recent progress in the theory of cocycles over hyperbolic systems that call into play geometric properties of invariant foliations such as absolute continuity. Recent results show that absolute continuity of the center foliation is often a rigid property among volume preserving systems. We prove that this is not at all the case in the dissipative setting, where absolute continuity can even be robust.

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## 1. Introduction

Let $f : N \to N$ be a diffeomorphism on some compact Riemannian manifold $N$. An invariant probability $\mu$ is a *physical (Sinai, Ruelle, Bowen) measure* for $f$ if the set of points $z \in N$ for which

$$\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^i(z)} \to \mu \quad \text{(in the weak* sense)} \tag{1}$$

has positive volume. This set is denoted $B(\mu)$ and called the *basin* of $\mu$. A program for investigating the physical measures of partially hyperbolic diffeomorphisms was initiated by Alves, Bonatti, Viana in [5, 19], who proved existence and finiteness when $f$ is either "mostly expanding" (asymptotic forward expansion) or "mostly contracting" (asymptotic forward contraction) along the center direction.

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In this paper we analyze the existence and finiteness problem without a priori conditions on the behavior along the center direction, in the case when the center bundle has dimension 1. Our results are illustrated by the following example.

Suppose $N = M \times S^1$, for some compact manifold $M$, and $f_0 : N \to N$ is a partially hyperbolic *skew-product*

$$f_0 : M \times S^1 \to M \times S^1, \quad f_0(x, \theta) = (g_0(x), h_0(x, \theta)) \tag{2}$$

with center bundle $E^c$ coinciding with the vertical direction $\{0\} \times TS^1$ at every point. This implies $g_0$ is an Anosov diffeomorphism, and we also take it to be transitive (all known Anosov diffeomorphisms being transitive). Assume $f_0$ is of class $C^k$ for some $k > 1$, not necessarily an integer.

**Theorem A.** *There exists a $C^k$ neighborhood $\mathcal{U}_0$ of $f_0$ such that for every $f \in \mathcal{U}_0$ which is accessible and whose center stable foliation is absolutely continuous there exists a finite number of physical measures. These measures are ergodic, the union of their basins has full volume in $N$, and the center Lyapunov exponents are either negative or zero. In the latter (zero) case the physical measure is unique.*

The subset of accessible diffeomorphisms is $C^1$ open and $C^k$ dense in the neighborhood of $f_0$ (Theorem 1.5 of Niţică, Török [32]). Absolute continuity is also quite common in this context as we are going to see. That is surprising, since Avila, Viana, Wilkinson [14] have recently shown that absolute continuity of the center foliation is a rigid property for *volume preserving* perturbations of skew-products. In contrast, here we prove

**Theorem B.** *Suppose $f_0$ exhibits some periodic vertical leaf $\ell$ such that $f_0^{per(\ell)} \mid \ell$ is Morse-Smale with a unique periodic attractor and repeller. Then $f_0$ is in the closure of an open set $\mathcal{V}$ of $C^k$ diffeomorphisms such that for every $f \in \mathcal{V}$,*

- *the center stable, the center unstable, and the center foliation are absolutely continuous*
- *both $f$ and its inverse have a unique physical measure, whose basin has full Lebesgue measure in $N$.*

Then the same is true for $f_0 = g_0 \times \mathrm{id}$, since it is $C^k$ approximated by diffeomorphisms as in the hypothesis of the theorem.

Although we are primarily interested in general (dissipative) diffeomorphisms, our methods also shed some light on the issue of absolute continuity in the volume preserving context. Let $\lambda^c(f)$ denote the integrated center Lyapunov exponent of $f$ relative to the Lebesgue measure.

**Theorem C.** *For any small $C^1$ neighborhood $\mathcal{W}$ of $f_0 = g_0 \times \mathrm{id}$ in the space of volume preserving diffeomorphisms of $N$,*

1. *the subset $\mathcal{W}_0$ of diffeomorphisms $f \in \mathcal{W}$ such that $\lambda^c(f) \neq 0$ is $C^1$ open and dense in $\mathcal{W}$;*
2. *if $f \in \mathcal{W}_0$ and $\lambda^c(f) > 0$ then the center foliation and the center stable foliation are not (even upper leafwise) absolutely continuous;*
3. *there exists a non-empty $C^1$ open set $\mathcal{W}_1 \subset \{f \in \mathcal{W}_0 : \lambda^c(f) > 0\}$ such that the center unstable foliation of every $g \in \mathcal{W}_1$ is absolutely continuous.*

*Claims (2) and (3) remain true when $\lambda^c(f) < 0$, if one exchanges center stable with center unstable. Every $C^k$, $k > 1$ diffeomorphism $f \in \mathcal{W}_1$ has a $C^k$ neighborhood $\mathcal{W}_f$ in the space of all (possibly dissipative) diffeomorphisms where the center unstable foliation remains absolutely continuous.*

Theorems A and B follow from more detailed statements that we present in the next section, where we also recall the main notions involved. A discussion of the volume preserving case is given in Section 7.3, including the proof of Theorem C and a (partly conjectural) scenario.

## 2. Statement of results

Let $\mathcal{P}^k_*(N)$ be the space of partially hyperbolic, dynamically coherent, $C^k$ diffeomorphisms whose center leaves are compact, with any dimension, and form a fiber bundle. Unless otherwise stated, we always assume $k > 1$. Most of our results concern the subspace $\mathcal{P}^k_1(N)$ of diffeomorphisms with 1-dimensional center dimension. Let us begin by recalling the notions involved in these definitions.

### 2.1. **Basic concepts.**

A diffeomorphism $f : N \to N$ is *partially hyperbolic* if there exists a continuous $Df$-invariant splitting $TN = E^u \oplus E^c \oplus E^s$ and there exist constants $C > 0$ and $\lambda < 1$ such that

(a) $\|Df_x^{-n}(v^u)\| \le C\lambda^n$ and $\|Df_x^n(v^s)\| \le C\lambda^n$
(b) $\|Df_x^{-n}(v^u)\| \le C\lambda^n \|Df_x^{-n}(v_c)\|$ and $\|Df_x^n(v^s)\| \le C\lambda^n \|Df_x^n(v_c)\|$

for all unit vectors $v^u \in E^u_x$, $v^c \in E^c_x$, $v^s \in E^s_x$, and all $x \in N$ and $n \ge 0$. Condition (a) means that the derivative $Df$ is uniformly expanding along $E^u$ and uniformly contracting along $E^s$. Condition (b) means that the behavior of $Df$ along the *center bundle* $E^c$ is dominated by the behavior along the other two factors. Here all three bundles are assumed to have positive dimension.

The bundles $E^u$ and $E^s$ are always integrable: there exist foliations $\mathcal{W}^u$ and $\mathcal{W}^s$ of $N$ tangent to $E^u$ and $E^s$, respectively, at every point. In fact these foliations are unique. Moreover, they are *absolutely continuous*, meaning that the projections along the leaves between any two cross-sections preserve the class of sets with zero volume inside the cross-section. See [20, 28, 42]. A diffeomorphism $f : N \to N$ is *dynamically coherent* if the bundles $E^{cu} = E^c \oplus E^u$ and $E^{cs} = E^c \oplus E^s$ also admit integral foliations, $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$. Then, intersecting their leaves one obtains a *center foliation* $\mathcal{W}^c$ tangent at every point to the center bundle $E^c$.

We say that the center leaves *form a fiber bundle* over the leaf space $N/\mathcal{W}^c$ if for any $\mathcal{W}^c(x) \in N/\mathcal{W}^c$ there is a neighborhood $V \subset N/\mathcal{W}^c$ of $\mathcal{W}^c(x)$ and a homeomorphism

$$h_x : V \times \mathcal{W}^c(x) \to \pi_c^{-1}(V)$$

smooth along the verticals $\{\ell\} \times \mathcal{W}^c(x)$ and mapping each vertical onto the corresponding center leaf $\ell$.

*Remark* 2.1. The fiber bundle condition is probably not necessary. Indeed, when the diffeomorphisms are volume preserving, Avila, Viana, Wilkinson [14] prove that if $\dim E^c = 1$ and the generic center leaves are circles then the center leaves form a fiber bundle *up to a finite cover*. In particular, all leaves are circles. Our arguments extend easily to this situation.

A partially hyperbolic diffeomorphism $f : N \to N$ is *accessible* if any points $z$, $w \in N$ can be joined by a piecewise smooth curve $\gamma$ such that every smooth leg of $\gamma$ is tangent to either $E^u$ or $E^s$ at every point. Equivalently, every smooth leg of the curve $\gamma$ is contained in a leaf of either $\mathcal{W}^u$ or $\mathcal{W}^s$.

The *center Lyapunov exponent* $\lambda^c(\mu)$ of an $f$-invariant probability measure $\mu$ is defined by

$$\lambda^c(\mu) = \int \lambda^c(z)\, d\mu(z) \quad \text{where } \lambda^c(z) = \lim_{n\to\infty} \frac{1}{n} \log |Df^n \mid E^c_z|. \tag{3}$$

By the ergodic theorem, this may be rewritten

$$\lambda^c(\mu) = \int \log |Df \mid E^c_z|\, d\mu(z). \tag{4}$$

If $\mu$ is ergodic then $\lambda^c(\mu) = \lambda^c(z)$ for $\mu$-almost every $z$.

Finally, the center direction is *mostly contracting* (Bonatti, Viana [19]) if

$$\limsup_{n\to+\infty} \frac{1}{n} \log \|Df^n \mid E^c_x\| < 0. \tag{5}$$

for a positive volume measure subset of any disk inside a strong unstable leaf. It was shown by Andersson [9] that this is a $C^k$, $k > 1$ open property.

2.2. **The leaf space.** Let $d$ be the Riemannian distance on $N$. We endow the leaf space $N/\mathcal{W}^c$ with the distance defined by

$$d_c(\xi, \eta) = \sup_{x\in\xi} \inf_{y\in\eta} d(x, y) + \sup_{y\in\eta} \inf_{x\in\xi} d(x, y) \quad \text{for each } \xi, \eta \in N/\mathcal{W}^c.$$

The quotient map $\pi_c : (N, d) \to (N/\mathcal{W}^c, d_c)$ is continuous and onto. In particular, the metric space $(N/\mathcal{W}^c, d_c)$ is compact.

Let $f_c : N/\mathcal{W}^c \to N/\mathcal{W}^c$ be the map induced by $f$ on the quotient space $N/\mathcal{W}^c$. The stable set of a point $\xi \in N/\mathcal{W}^c$ for $f_c$ is defined by

$$W^s(\xi) = \{\eta \in N/\mathcal{W}^c : d_c(f_c^n(\xi), f_c^n(\eta)) \to 0 \text{ when } n \to +\infty\}$$

and the local stable set of size $\varepsilon > 0$ is defined by

$$W^s_\varepsilon(\xi) = \{\eta \in N/\mathcal{W}^c : d_c(f_c^n(\xi), f_c^n(\eta)) \le \varepsilon \text{ for all } n \ge 0\}.$$

The unstable set and local unstable set of size $\varepsilon > 0$ are defined in the same way, for backward iterates. It follows from the definitions that there exist constants $K$, $\tau$, $\varepsilon$, $\delta > 0$ such that

(1) $d_c(f_c^n(\eta_1), f_c^n(\eta_2)) \le Ke^{-\tau n} d_c(\eta_1, \eta_2)$ for all $\eta_1, \eta_2 \in W^s_\varepsilon(\xi)$, $n \ge 0$;
(2) $d_c(f_c^{-n}(\zeta_1), f_c^{-n}(\zeta_2)) \le Ke^{-\tau n} d_c(\zeta_1, \zeta_2)$ for all $\zeta_1, \zeta_2 \in W^u_\varepsilon(\xi)$, $n \ge 0$;
(3) if $d_c(\xi_1, \xi_2) \le \delta$ then $W^s_\varepsilon(\xi_1)$ and $W^u_\varepsilon(\xi_2)$ intersect at exactly one point, denoted $[\xi_1\, \xi_2]$ and this point depends continuously on $(\xi_1, \xi_2)$.

This means that $f_c$ is a hyperbolic homeomorphism (in the sense of Viana [48]). We denote $\mathcal{W}^c(\Lambda) = \pi_c^{-1}(\Lambda)$, for any subset $\Lambda$ of $N/\mathcal{W}^c$.

By Anosov's closing lemma [10], periodic points are dense in the non-wandering set of $f_c$. Smale's spectral decomposition theorem [45], the non-wandering set splits into a finite number of compact, invariant, transitive, pairwise disjoint subsets. Among these basic pieces of the non-wandering set, the *attractors* $\Lambda_i$, $i = 1, \dots, k$ of $f_c$ are characterized by the fact that

$$\Lambda_i = \bigcap_{n=0}^{\infty} f_c^n(U_i)$$

for some neighborhood $U_i$ of $\Lambda_i$ and it is transitive. The union of the stable sets $W^s(\Lambda_i)$, $i = 1, \dots, k$ is an open dense subset of $N/\mathcal{W}^c$. Every attractor $\Lambda_i$ consists of entire unstable sets, and so $\mathcal{W}^c(\Lambda_i)$ is $\mathcal{W}^u$-saturated, that is, it consists of entire strong unstable leaves of $f$. Additionally, every $\Lambda_i$ has finitely many connected components $\Lambda_{i,j}$, $j = 1, \dots, n_i$ that are mapped to one another cyclically. The unstable set $W^u(x)$ of every $x \in \Lambda_{i,j}$ is contained and dense in $\Lambda_{i,j}$. In particular, $\mathcal{W}^c(\Lambda_{i,j})$ is also $\mathcal{W}^u$-saturated. If $f_c$ is transitive, there is a unique attractor $\Lambda_1 = N/\mathcal{W}^c$.

We say $f$ is *accessible on* $\Lambda_i$ if, for every $j$, any points $z$, $w \in \mathcal{W}^c(\Lambda_{i,j})$ can be joined by a piecewise smooth curve $\gamma$ such that every smooth leg of $\gamma$ is tangent to either $E^u$ or $E^s$ at every point and the corner points belong to the same $\mathcal{W}^c(\Lambda_{i,j})$. The center direction of $f \mid \mathcal{W}^c(\Lambda_i)$ is *mostly contracting* if (5) holds for a positive volume measure subset of any disk inside a strong unstable leaf contained in $\mathcal{W}^c(\Lambda_i)$.

2.3. **Physical measures.** We are ready to state our main result on existence and finiteness of physical measures:

**Theorem D.** *If $f \in \mathcal{P}_1^k(N)$, $k > 1$ is accessible on every attractor and the center stable foliation is absolutely continuous then, for each attractor $\Lambda_i$, either*

(a) *there is a Lipschitz metric on each leaf of $\mathcal{W}^c(\Lambda_i)$, depending continuously on the leaf and invariant under $f$; then $f$ admits a unique physical measure, which is ergodic, whose basin has full volume in the stable set of $\mathcal{W}^c(\Lambda_i)$, and whose center Lyapunov exponent vanishes;*

(b) *or the center direction of $f \mid \mathcal{W}^c(\Lambda_i)$ is mostly contracting; then $f \mid \mathcal{W}^c(\Lambda_i)$ has finitely many physical measures, they are ergodic for $f$ and Bernoulli for some iterate, the union of their basins is a full volume subset of the stable set of $\mathcal{W}^c(\Lambda_i)$, and their center Lyapunov exponents are negative.*

*The union of the basins of these physical measures has full volume in $N$.*

To see that Theorem A is contained in Theorem D let us to note that, for every $k \geq 1$, any $C^k$ partially hyperbolic skew-product $f_0$ is in the interior of $\mathcal{P}_1^k(N)$. Indeed, partial hyperbolicity is well known to be a $C^1$ open property and the stability theorem for normally hyperbolic foliations (Hirsch, Pugh, Shub [28]) gives that every $f$ in a $C^1$ neighborhood of $f_0$ admits an invariant $\mathcal{W}_f^*$ foliation, for each $* \in \{cu, cs, c\}$, and there exists a homeomorphism mapping the leaves of $\mathcal{W}_f^*$ diffeomorphically to the leaves of $\mathcal{W}_{f_0}^*$. In particular, the center leaves of $f$ form a circle fiber bundle.

*Remark* 2.2. When the center fiber bundle is trivial, as happens near skew-products, part (a) of the Theorem D gives that $f \mid \mathcal{W}^c(\Lambda_i)$ is topologically conjugate to a rotation extension

$$\Lambda_i \times \mathbb{R}/\mathbb{Z} \to \Lambda_i \times \mathbb{R}/\mathbb{Z}, \quad (x, \theta) \mapsto (f_c(x), \theta + \omega(x)).$$

To see this, fix some consistent orientation of the center leaves and any continuous section $\sigma : N/\mathcal{W}^c \to N$ of the center foliation, that is, any continuous map such that $\sigma(\ell) \in \ell$ for every $\ell \in N/\mathcal{W}^c$. Then define

$$h : \mathcal{W}^c(\Lambda_i) \to \Lambda_i \times \mathbb{R}/\mathbb{Z}, \quad h(z) = (\pi_c(z), |\sigma(\pi_c(z)), z|)$$

where $|\sigma(\pi_c(z)), z|$ denotes the length, with respect to the $f$-invariant Lipschitz metric, of the (oriented) curve segment from $\sigma(\pi_c(z))$ to $z$ inside the center leaf. This map sends the center leaves of $f$ to verticals $\{w\} \times \mathbb{R}/\mathbb{Z}$, mapping the $f$-invariant Lipschitz metric on the center leaves to the standard metric on $\mathbb{R}/\mathbb{Z}$. Then $h \circ f \circ h^{-1}$ preserves the standard metric measure on the verticals, and so it is a rotation extension, as stated. Observe that, in addition, both $h$ and its inverse are Lipschitz on every leaf.

Explicit bounds on the number of physical measures can be given in many cases. For instance, we will see in Theorem 5.3 that if $f$ admits some periodic center leaf $\ell$ restricted to which $f$ is Morse-Smale then the number of physical measures over the attractor containing $\pi_c(\ell)$ is bounded by the number of periodic orbits on $\ell$. Notice that we must have alternative (b) of Theorem D in this case, since alternative (a) is incompatible with the existence of hyperbolic periodic points.

We also want to analyze the dependence of the physical measures on the dynamics. For this, we assume $N = M \times S^1$ and restrict ourselves to the subset $\mathcal{S}^k(N) \subset \mathcal{P}_1^k(N)$ of skew-product maps. We prove in Theorem 5.6 that there is an open and dense subset of diffeomorphisms $f \in \mathcal{S}^k(N)$ with mostly contracting center direction, such that the number of physical measures is locally constant and the physical measures vary continuously with the diffeomorphism. This property of *statistical stability* has been studied in a number of recent works, including Alves, Viana [8], Vásquez [47], Andersson [9].

As mentioned before, existence and finiteness of physical measures for partially hyperbolic diffeomorphisms was proved by Alves, Bonatti, Viana [5, 19], under certain assumptions of weak hyperbolicity along the center direction. Substantial improvements followed, by Alves, Luzzatto, Pinheiro [6, 7], Alves, Araujo [4], Vasquez [47], Pinheiro [36], and Andersson [9], among others. Perturbations of certain skew-products over hyperbolic maps have been studied by Alves [2, 3], Buzzi, Sestier, Tsujii [23], and Gouezel [25]. In a remarkable recent paper, Tsujii [46] proved that generic (dense $G_\delta$) partially hyperbolic surface *endomorphisms* do admit finitely many physical measures, such that the union of their basins has full Lebesgue measure. His approach is very different from the one in the present paper and it is not clear how it could be extended to diffeomorphisms in higher dimensions, even in the case of one-dimensional center bundle.

2.4. **Absolute continuity.** It has been pointed out by Shub, Wilkinson [44] that foliations tangent to the center subbundle $E^c$ are often *not* absolutely continuous. In fact, Ruelle, Wilkinson [41] showed that the disintegration of Lebesgue measure along the leaves is often atomic. Moreover, Avila, Viana, Wilkinson [14] observed recently that for certain classes of volume preserving diffeomorphisms, including perturbations of skew-products (2), absolute continuity of the center foliation is a rigid property: it implies that the center foliation is actually smooth, and the map is smoothly conjugate to a rigid model.

However, we prove that this is not at all the case in our dissipative setting:

**Theorem E.** *There is an open set $\mathcal{U} \subset \mathcal{P}_1^k(N)$, $k > 1$, such that the center stable, the center unstable, and the center foliation are absolutely continuous for every $f \in \mathcal{U}$. Moreover, $\mathcal{U}$ may be chosen to accumulate on every skew-product map $f_0$ that admits a periodic vertical fiber restricted to which the map is Morse-Smale with a unique periodic attractor and repeller.*

Two weaker forms of absolute continuity are considered by Avila, Viana, Wilkinson [14]. Let vol denote Lebesgue measure in the ambient manifold and $\mathrm{vol}_L$ be Lebesgue measure restricted to some submanifold $L$. A foliation $\mathcal{F}$ on $N$ is (*lower*) *leafwise absolutely continuous* if for every zero vol-measure set $Y \subset N$ and vol-almost every $z \in M$, the leaf $L$ through $z$ meets $Y$ in a zero $\mathrm{vol}_L$-measure set. Similarly, $\mathcal{F}$ is *upper leafwise absolutely continuous* if $\mathrm{vol}_L(Y) = 0$ for every leaf $L$ through a full measure subset of points $z \in M$ implies $\mathrm{vol}(Y) = 0$. Absolute continuity implies both lower and upper leafwise absolute continuity (see [14, 21]); the converse is not true in general. We will see in Proposition 6.2 that the center stable foliation of a partially hyperbolic, dynamically coherent diffeomorphism with mostly contracting center direction is always upper leafwise absolutely continuous. This does not extend to lower leafwise absolutely continuity, in general: robust counter-examples will appear in [49]; see also Example 6.1 for a related construction. However, as stated before, full absolute continuity of the center foliation does hold on some open subsets of diffeomorphisms with mostly contracting center.

## 3. Gibbs $u$-states

Let $f : N \to N$ be a partially hyperbolic diffeomorphism. In what follows we denote $I_r = [-r, r]$ for $r > 0$ and $d_* = \dim E^*$ for each $* \in \{u, cu, c, cs, s\}$. We use $\mathrm{vol}^*$ to represent the volume measure induced by the restriction of the Riemannian structure on the leaves of the foliation $\mathcal{W}^*$ for each $* \in \{u, cu, c, cs, s\}$.

Following Pesin, Sinai [34] and Alves, Bonatti, Viana [5, 19] (see also [17, Chapter 11]), we call *Gibbs u-state* any invariant probability measure $m$ whose conditional probabilities (Rokhlin [40]) along strong unstable leaves are absolutely continuous with respect to the volume measure $\mathrm{vol}^u$ on the leaf. More precisely, let

$$\Phi : I_1^{d_u} \times I_1^{d_{cs}} \to N$$

be any *foliated box* for the strong unstable foliation. By this we mean that $\Phi$ is a homeomorphism and maps every horizontal plaque $I_1^{d_u} \times \{\eta\}$ diffeomorphically to a disk inside some strong unstable leaf. Pulling $m$ back under $\Phi$ one obtains a measure $m_\Phi$ on $I_1^{d_u} \times I_1^{d_{cs}}$. The definition of Gibbs $u$-state means that there exists a measurable function $\alpha_\Phi(\cdot\,,\cdot) \geq 0$ and a measure $m_\Phi^{cs}$ on $I_1^{d_{cs}}$ such that

$$m_\Phi(A) = \int_A \alpha_\Phi(\xi, \zeta)\, d\xi\, dm_\Phi^{cs}(\zeta) \tag{6}$$

for every measurable set $A \subset I_1^{d_u} \times I_1^{d_{cs}}$.

Proofs for the following basic properties of Gibbs $u$-states can be found in Section 11.2 of Bonatti, Díaz, Viana [17]:

**Proposition 3.1.** *Let $f : N \to N$ be a partially hyperbolic diffeomorphism.*

(1) *The densities of a Gibbs $u$-state with respect to Lebesgue measure along strong unstable plaques are positive and bounded from zero and infinity.*
(2) *The support of every Gibbs $u$-state is $\mathcal{W}^u$-saturated, that is, it consists of entire strong unstable leaves.*
(3) *The set of Gibbs $u$-states is non-empty, weak* compact, and convex. Ergodic components of Gibbs $u$-states are Gibbs $u$-states.*
(4) *Every physical measure of $f$ is a Gibbs $u$-state and, conversely, every ergodic $u$-state whose center Lyapunov exponents are negative is a physical measure.*

Now let $f \in \mathcal{P}_*^k(N)$. Recall that $\pi_c : N \to N/\mathcal{W}^c$ denotes the natural quotient map and $f_c : N/\mathcal{W}^c \to N/\mathcal{W}^c$ is the hyperbolic homeomorphism induced by $f$ in the leaf space. Given small neighborhoods $V_\xi^s \subset W_\varepsilon^s(\xi)$ and $V_\xi^u \subset W_\varepsilon^u(\xi)$ inside the corresponding stable and unstable sets, the map

$$(\eta, \zeta) \mapsto [\eta, \zeta] \tag{7}$$

defines a homeomorphism between $V_\xi^u \times V_\xi^s$ and some neighborhood $V_\xi$ of $\xi$. A probability measure $\mu$ on $N/\mathcal{W}^c$ has *local product structure* if for $\mu$-almost every point $\xi$ and any such product neighborhood $V_\xi$ the restriction $\mu \mid V_\xi$ is equivalent to a product $\nu^u \times \nu^s$, where $\nu^u$ is a measure on $V_\xi^u$ and $\nu^s$ is a measure on $V_\xi^s$.

In the sequel we prove three additional facts about Gibbs $u$-states that are important for our arguments.

**Proposition 3.2.** *Take $f \in \mathcal{P}_*^k(N)$, $k > 1$ such that the center stable foliation is absolutely continuous. For every ergodic Gibbs $u$-state $m$ the support of the projection $(\pi_c)_*(m)$ coincides with some attractor of $f_c$. In particular, periodic points are dense in the support of $(\pi_c)_*(m)$.*

*Moreover, any two such projections with the same support must coincide. In particular, the set of projections of all ergodic Gibbs $u$-states of $f$ down to $N/\mathcal{W}^c$ is finite.*

**Proposition 3.3.** *Take $f \in \mathcal{P}_*^k(N)$, $k > 1$ such that the center stable foliation is absolutely continuous. If $m$ is a Gibbs $u$-state for $f$ then $\mu = (\pi_c)_*(m)$ has local product structure.*

*Remark* 3.4. Suppose $f$ is volume preserving. The Lebesgue measure vol is both an $s$-state and a $u$-state, because the strong stable foliation and the strong unstable foliation are both absolutely continuous. Thus, Proposition 3.3 implies that $(\pi_c)_*(m)$ has local product structure if *either* $\mathcal{W}^{cu}$ *or* $\mathcal{W}^{cs}$ is absolutely continuous.

**Proposition 3.5.** *Let $f \in \mathcal{P}^k_*(N)$, $k > 1$ and $\Lambda$ be an attractor of $f_c$. Suppose the center stable foliation of $f$ is absolutely continuous and $f$ is accessible on $\Lambda$. Then every ergodic Gibbs $u$-state of $f$ supported in $\mathcal{W}^c(\Lambda)$ has some non-positive center Lyapunov exponent.*

As a special case, we get that if $f \in \mathcal{P}^k_1(N)$, $k > 1$ is accessible on an attractor $\Lambda$ of $f_c$ and the center stable foliation is absolutely continuous, then the (unique) center Lyapunov exponent of every ergodic Gibbs $u$-state supported in $\mathcal{W}^c(\Lambda)$ is non-positive.

The proofs of these propositions are given in Sections 3.1 through 3.3.

3.1. **Finiteness in leaf space.** Here we prove Proposition 3.2. Let $m_1$ be any ergodic Gibbs $u$-state and $\mu_1 = (\pi_c)_*(m_1)$. Notice that $\mu_1$ is ergodic and so its support is a transitive set for $f_c$. Moreover, $\operatorname{supp}\mu_1 = \pi_c(\operatorname{supp} m_1)$ consists of entire unstable sets, because the support of $m_1$ is $\mathcal{W}^u$-saturated (Proposition 3.1). Thus, $\operatorname{supp}\mu_1$ is an attractor $\Lambda$ of $f_c$. As pointed out before, periodic points are dense in each attractor of $f_c$.

Now we only have to show that if $\mu_2 = (\pi_c)_* m_2$ for another ergodic Gibbs $u$-state $m_2$ and $\operatorname{supp}\mu_2 = \Lambda = \operatorname{supp}\mu_1$ then $\mu_1 = \mu_2$. For this, take $x_c \in \Lambda$, let $U_c$ be a neighborhood of $x_c$ in the quotient space $N/\mathcal{W}^c$, and let $U = \pi_c^{-1}(U_c)$. Then $U$ has positive $m_i$-measure for $i = 1, 2$. So, since the $m_i$ are ergodic Gibbs $u$-states, there are disks $D_i \subset U$, $i = 1, 2$ contained in strong unstable leaves and such that Lebesgue almost every point in $D_i$ is in the basin $B(m_i)$ of $m_i$. Moreover, these disks may be chosen such that the center stable foliation induces a holonomy map $h^{cs} : D_1 \to D_2$. Since the center stable foliation is absolutely continuous, it follows that $h^{cs}$ maps some point $x_1 \in D_1 \cap B(m_1)$ to a point $x_2 \in D_2 \cap B(m_2)$ in the basin of $m_2$. Then $x_1$ and $x_2$ belong to the same center stable leaf of $f$, and so their projections $\pi_c(x_1)$ and $\pi_c(x_2)$ belong to the same stable set of $f_c$. Notice that $\pi_c(B(m_i)) \subset B(\mu_i)$ for $i = 1, 2$, and so each point $\pi(x_i) \in B(\mu_i)$. Since either basin consists of entire stable sets, this proves that $B(\mu_1)$ and $B(\mu_2)$ intersect each other, and so $\mu_1 = \mu_2$. This completes the proof of Proposition 3.2.

3.2. **Local product structure.** Here we prove Proposition 3.3. Let $m$ be any Gibbs $u$-state and $\ell_0$ be any center leaf. Since the center leaves form a fiber bundle, we may find a neighborhood $V \subset N/\mathcal{W}^c$ and a homeomorphism

$$\phi : V \times \ell_0 \mapsto \pi_c^{-1}(V), \quad (\ell, \zeta) \mapsto \phi(\theta, \zeta)$$

that maps each vertical $\{\ell\} \times \ell_0$ to the corresponding center leaf $\ell$. Clearly, we may choose $V$ to be the image of the bracket (recall Section 2.2)

$$W^u_\varepsilon(\ell_0) \times W^s_\varepsilon(\ell_0) \to V, \quad (\xi, \eta) \mapsto [\xi, \eta]$$

for some small $\varepsilon > 0$. Then, by dynamical coherence, the homeomorphism

$$W^u_\varepsilon(\ell_0) \times W^s_\varepsilon(\ell_0) \times \ell_0 \to \pi_c^{-1}(V), \quad (\xi, \eta, \zeta) \mapsto \phi([\xi, \eta], \zeta) \tag{8}$$

maps each $\{\xi\} \times W^s_\varepsilon(\ell_0) \times \ell_0$ onto a center stable leaf and each $W^u_\varepsilon(\ell_0) \times \{\eta\} \times \ell_0$ onto a center unstable leaf. For each $x \in \pi_c^{-1}(V)$, let $\mathcal{W}^u_{loc}(x)$ denote the local strong unstable leaf over $V$, that is, the connected component of $\mathcal{W}^u(x) \cap \pi_c^{-1}(V)$ that contains $x$. Each $\mathcal{W}^u_{loc}(x)$ is a graph over the unstable set $W^u(\pi_c(x))$ and the center stable holonomy defines a homeomorphism

$$h^{cs}_{x,y} : \mathcal{W}^u_{loc}(x) \to \mathcal{W}^u_{loc}(y)$$

between any two local strong unstable leaves. By assumption, all these homeomorphisms are absolutely continuous. Now let

$$m \mid \pi_c^{-1}(V) = \int m_x \, d\hat{m}$$

be the disintegration of $m$ relative to the partition of $\pi_c^{-1}(V)$ into local strong unstable leaves. By definition of Gibbs $u$-states, each $m_x$ is equivalent to the Lebesgue measure along $\mathcal{W}^u_{loc}(x)$. It follows that the center stable holonomies are absolutely continuous relative to the conditional probabilities of $m$ along local strong unstable leaves:

$$m_x(E) = 0 \text{ if and only if } m_y(h^{cs}_{x.y}(E)) = 0 \tag{9}$$

for $x$ and $y$ in some full $m$-measure subset of $\pi_c^{-1}(V)$ and for any measurable set $E \subset \mathcal{W}^u_{loc}(x)$. By the construction of (8), center stable holonomies preserve the coordinate $\xi$. Thus, identifying $\pi_c^{-1}(V)$ with the space $W_\varepsilon(\ell_0) \times W^s_\varepsilon(\ell_0) \times \ell_0$ through the homeomorphism (8), property (9) becomes

$$m_x(A \times W^s_\varepsilon(\ell_0) \times \ell_0) = 0 \text{ if and only if } m_y(A \times W^s_\varepsilon(\ell_0) \times \ell_0) = 0 \tag{10}$$

for any measurable set $A \subset W^u_\varepsilon(\ell_0)$ and for $m$-almost every $x$ and $y$ in $\pi_c^{-1}(V)$. Let $\mu \mid V = \int \mu^u_\eta \, d\mu^s(\eta)$ be the disintegration of $\mu$ relative to the partition of $V$ into unstable slices $W^u(\ell_0) \times \{\eta\}$; notice that $\mu^s$ is just the projection of $\mu \mid V$ to $W^s_\varepsilon(\ell_0)$. Projecting $m \mid \pi_c^{-1}(V)$ down to $V \approx W^u_\varepsilon(\ell_0) \times W^s_\varepsilon(\ell_0)$, property (10) yields

$$\mu_\eta(A \times W^s_\varepsilon(\ell_0)) = 0 \text{ if and only if } \mu_{\eta'}(A \times W^s_\varepsilon(\ell_0)) = 0 \tag{11}$$

for any measurable set $A \subset W^u_\varepsilon(\ell_0)$ and for $\mu$-almost every $\eta$ and $\eta'$ in $V$. This means that the conditional probabilities $\mu^u_\eta$ are (almost) all equivalent. Consequently, there is $\rho : W^u_\varepsilon(\ell_0) \times W^s_\varepsilon(\ell_0) \to (0, \infty)$ such that $\mu^u_\eta = \rho(\cdot, \eta)\mu^u$ at $\mu$-almost every point, where $\mu^u$ denotes the projection of $\mu \mid V$ to $W^u_\varepsilon(\ell_0)$. Replacing in the disintegration of $\mu \mid V$, we get that $\mu \mid V = \rho \, \mu^u \times \mu^s$. This proves that $\mu$ has local product structure, as claimed.

3.3. **Positive Gibbs $u$-states.** Here we prove Proposition 3.5. We begin by proving the following fact, which is interesting in itself:

**Proposition 3.6.** *For $f \in \mathcal{P}^1_*(N)$, given $c > 0$ and $l \geq 1$ there is $n_0$ such that $\#\big(S \cap \Gamma_{c,l}\big) < n_0$ for every center leaf $S$, where*

$$\Gamma_{c,l} = \{x \in N : \liminf \frac{1}{n} \sum_{i=1}^{n} \log \|Df^{-l} \mid E^c(f^{il}(x))\|^{-1} \geq c\}.$$

*Proof.* Recall that $\mathrm{vol}^c$ denotes the Riemannian volume on center leaves. The main ingredient is

**Lemma 3.7.** *Given $c > 0$ and $l \geq 1$ there exists $\delta > 0$ such that for any $x \in S \cap \Gamma_{c,l}$ and any neighborhood $U$ of $x$ inside the center leaf $S$ that contains $x$, one has*

$$\liminf \frac{1}{n} \sum_{i=0}^{n-1} \mathrm{vol}^c(f^{il}(U)) \geq \delta.$$

*Proof.* Let $x \in S \cap \Gamma_{c,l}$ be fixed. Fix $0 < c_1 < c_2 < c$ and define $H(c_2)$ to be the set of $c_2$-hyperbolic times for $x$, that is, the set of times $m \geq 1$ such that

$$\frac{1}{k} \sum_{i=m-k+1}^{m} \log \|Df^{-l} \mid E^c_{f^{il}(x)}\|^{-1} \geq c_2 \quad \text{for all } 1 \leq k \leq m. \tag{12}$$

By the Pliss Lemma (see [2, 5]) , there exist $n_1 \ge 1$ and $\delta_1 > 0$ such that

$$\#\big(H(c_2) \cap [1, n)\big) \ge n\delta_1 \quad \text{for all } n \ge n_1.$$

Notice that (12) implies $Df^{-kl}$ is an exponential contraction on $E^c_{f^{ml}(x)}$:

$$\|Df^{-kl} \mid E^c_{f^{ml}(x)}\| \le \prod_{i=m-k+1}^{m} \|Df^{-l} \mid E^c_{f^{il}(x)}\| \le e^{-c_2 k} \quad \text{for all } 1 \le k \le m.$$

It also follows from [5] that the points $f^{ml}(x)$ with $m \in H(c_2)$ admit backward-contracting center disks with size uniformly bounded from below: there is $r > 0$ depending only on $f$ and the constants $c_1$ and $c_2$ such that

$$f^{-kl}(B^c_r(f^{ml}(x))) \subset B^c_{e^{-c_1 k} r}(f^{(m-k)l}(x)) \quad \text{for all } 1 \le k \le m.$$

where $B^c_\rho(y)$ denotes the ball inside $\mathcal{W}^c_y$ of radius $\rho$ around any point $y$. Let $a_1 > 0$ be a lower bound for $m^c(B^c_r(y))$ over all $y \in N$. Fix $n_2$ such that the ball of radius $e^{-c_1 k} r$ around $x$ is contained in $U$ for every $k \ge n_2$. Then, in particular,

$$f^{ml}(U) \supset B^c_r(f^{ml}(x)) \quad \text{and so} \quad m^c(f^{ml}(U)) \ge a_1$$

for every $m \in H(c_2)$ with $m \ge n_2$. So, for $n \gg \max\{n_1, n_2\}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} m^c(f^{il}(U)) \ge \frac{1}{n} a_1 \big[\#(H(c_2) \cap [1, n)) - n_2\big] \ge \frac{1}{n} a_1 \big[n\delta_1 - n_2\big] \ge \frac{\delta_1}{2} a_1 \,.$$

To finish the proof of Lemma 3.7 it suffices to take $\delta = a_1\delta_2/2$. $\square$

To deduce Proposition 3.6 from Lemma 3.7, take any $n_0 \ge V/\delta$ where $V$ is an upper bound for the volume of center leaves. Suppose $S \cap \Gamma_{c,l}$ contains $n_0$ distinct points $x_j$, $j = 1, \ldots, n_0$. Let $U_j$, $j = 1, \ldots, n_0$ be pairwise disjoint neighborhoods of the $x_j$ inside $S$. Take $n$ large enough that

$$\frac{1}{n}\sum_{i=0}^{n-1} m^c(f^i(U_j)) > \delta \quad \text{for } 1 \le j \le n_0.$$

Then

$$V \ge \frac{1}{n}\sum_{i=0}^{n-1} m^c(f^i(S)) \ge \sum_{j=1}^{n_0} \frac{1}{n}\sum_{i=0}^{n-1} m^c(f^i(U_j)) > n_0\delta > V.$$

This contradiction proves Proposition 3.6. $\square$

*Proof of Proposition 3.5.* We argue by contradiction. Suppose there exists some ergodic Gibbs $u$-state $\nu$ supported in $\mathcal{W}^c(\Lambda)$ whose center Lyapunov exponents are all positive.

**Lemma 3.8.** *There is $k_0 \ge 1$ and some ergodic Gibbs $u$-state $\nu_*$ of $f^{k_0}$ supported in $\mathcal{W}^c(\Lambda)$ such that*

$$\int \log \|Df^{-k_0} \mid E^c_x\|^{-1} d\nu_*(x) > 0. \tag{13}$$

*Proof.* Arguing as in [48, Section 2.1] one can find $k_0 \ge 1$ such that

$$\int \log \|Df^{-k_0} \mid E^c_x\|^{-1} d\nu(x) > 0$$

The measure $\nu$ needs not be ergodic for $f^{k_0}$ but, since it is ergodic for $f$, it has a finite number $k$ of ergodic components $\nu_i$ ($k$ divides $k_0$). Moreover,

$$\int \log \|Df^{-k_0} \mid E^c_x\|^{-1} d\nu_i(x) > 0$$

for some ergodic component $\nu_i$. Since, by Proposition 3.1, each ergodic component $\nu_i$ is a Gibbs $u$-state, this completes the proof of the lemma. $\square$

Let $k_0 \geq 1$ be fixed from now on and $\lambda > 0$ denote the expression on the left hand side of (13). Let $g = f^{k_0}$ and

$$\Gamma = \{x \in N : \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Dg^{-1} \mid E^c_{g^j(x)}\|^{-1} = \lambda\}$$

be the set of regular points of $\log \|Dg^{-1} \mid E^c\|$ for the transformation $g$. By ergodicity, $\nu_*(\Gamma) = 1$. A statement similar to the next corollary was proved by Ruelle, Wilkinson [41] when the diffeomorphism is $C^{1+\varepsilon}$ and the center is 1-dimensional.

**Corollary 3.9.** *There is $n_0 \geq 1$ such that $\#\big(\mathcal{W}^c(w) \cap \Gamma\big) < n_0$ for every $w \in N$.*

*Proof.* Just use Proposition 3.6 with $c = \lambda/2$ and $l = k_0$. Clearly, $\Gamma \subset \Gamma_{c,l}$. $\square$

Let $\ell_0$ be any periodic center leaf intersecting $\operatorname{supp} \nu_*$ (periodic center leaves are dense in the support, by Proposition 3.2) and $\kappa \geq 1$ be minimal such that $g^\kappa(\ell_0) = \ell_0$. Since $\nu_*$ is a Gibbs $u$-state and $\Gamma$ has full measure, $\operatorname{vol}^u(\mathcal{W}^u(x) \setminus \Gamma) = 0$ for $\nu_*$-almost every $x$, where $\operatorname{vol}^u$ denotes the Riemannian volume along strong unstable manifolds. In particular, the stable set $\mathcal{W}^s(\ell_0) = \cup_{z\in\ell_0} \mathcal{W}^s(z)$ must intersect some strong unstable disk $D^u$ such that $\operatorname{vol}^u(D^u \setminus \Gamma) = 0$. See Figure 1.

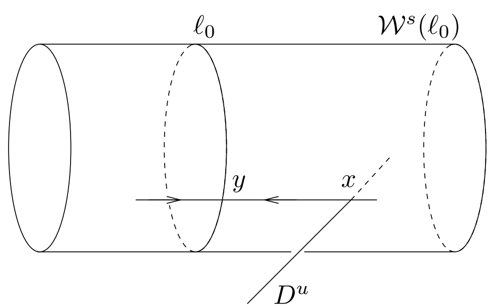


FIGURE 1.

**Lemma 3.10.** *Every point $x \in D^u \cap \mathcal{W}^s(\ell_0)$ belongs to the strong stable manifold of some periodic point $y \in \ell_0$ of $f$ with period bounded by $k_0\kappa n_0$.*

*Proof.* Let $y \in \ell_0$ be such that $x \in \mathcal{W}^s(y)$ and let $g_0 = g^\kappa \mid \ell_0$. Suppose first that the orbit of $y$ under $g_0$ is infinite. We refer the reader to Figure 2. Fix $y^* \in \omega(y)$ and let $(y_j)_j$ be an injective sequence of iterates of $y$ converging to $y^*$. Let $(x_j)_j$ be a sequence of iterates of $x$ with $x_j \in \mathcal{W}^s(y_j)$ and $d(x_j, y_j) \to 0$. Choose disks $D^u_j$ around the $x_j$ inside the forward iterates of $D^u$, small but with *uniform size.* Since $\Gamma$ is an invariant set, $m^u(D^u_j \setminus \Gamma) = 0$ for every $j$. For every large $j$, the center leaves $\mathcal{W}^c(x_j)$ is close to $\ell_0$ and so one can define a $cs$-holonomy map $\pi^{cs}$ from $D^u_j$ to the local strong unstable leaf through $y^*$. Since $\mathcal{W}^{cs}$ is absolutely continuous, the image of every $D^u_j \cap \Gamma$ is a full volume measure subset of a neighborhood of $y^*$ inside $\mathcal{W}^u(y^*)$, where these neighborhoods also have uniform size for all large $j$. Let $J = \{j_0, j_0+1, \ldots, j_0+n_0\}$ where $j_0$ is some large integer and $n_0$ is as in Corollary 3.9. On the one hand, it follows from the previous considerations that

$$\Gamma^* = \bigcap_{j\in J} \pi^{cs}(D^u_j \cap \Gamma)$$

is a full volume measure subset of some neighborhood of $y^*$ inside $\mathcal{W}^u(y^*)$. Fix some $w \in \Gamma^*$ close to $y^*$. For each $j \in J$, let $w_j \in D^u_j \cap \Gamma$ be such that $\pi^{cs}(w_j) = w$. Moreover, let $z_j$ be the point where the local strong stable manifold of $w_j$ intersects $\mathcal{W}^{cu}(y^*) = \mathcal{W}^{cu}(w)$. It is clear from the definition that $w_j \in \mathcal{W}^{cs}(w)$ and so $z_j \in \mathcal{W}^c(w)$ for all $j \in J$. Moreover, by choosing $w$ close enough to $y^*$ we can ensure that $w_j$ is close to $x_j$ for every $j \in J$ and so $z_j$ is close to $y_j$ for all $j \in J$. The latter implies that the $z_j$ are all distinct. Observe also that $z_j \in \Gamma$ for all $j \in J$, because $\Gamma$ is (clearly) saturated by strong stable leaves. This proves that $\#(\mathcal{W}^c(w) \cap \Gamma) \geq \#J > n_0$, in contradiction with Corollary 3.9. This contradiction proves that the $g_0$-orbit of $y$ can not be infinite.

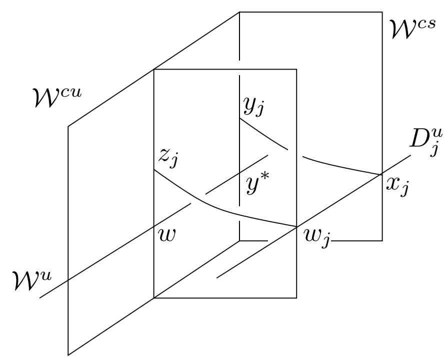


Figure 2.

Similar arguments handle the case when $y$ is a periodic point for $g_0$. Let $k \geq 1$ be the (minimal) period of $y$ for $g_0$. Forward iterates of $D^u$ accumulate on the strong unstable manifolds of the iterates of $y$. Using, in much the same way as before, that the center stable foliation is absolutely continuous and $\Gamma$ is saturated by strong stable leaves, we find $w \in \mathcal{W}^{cu}(y)$ arbitrarily close to $y$ whose center leaf $\mathcal{W}^c_w$ intersects $\Gamma$ at points close to each of the $k$ iterates of $y$. In view of Corollary 3.9 this implies that $k < n_0$. This means that the period of $y$ for $f$ is less than $k_0\kappa n_0$ as stated. The proof of Lemma 3.10 is complete. $\square$

**Lemma 3.11.** *Every point $z \in \ell_0$ is periodic for $f$, with period bounded by $k_0\kappa n_0$.*

*Proof.* Let $y \in \ell_0$ be a periodic point as in Lemma 3.10 and let $z \in \ell_0$ be arbitrary. Choose $y' \in \mathcal{W}^u(y) \cap \mathcal{W}^c(\Lambda_i) \setminus \ell_0$ and $z' \in \mathcal{W}^s(z) \cap \mathcal{W}^c(\Lambda_i) \setminus \ell_0$. By accessibility, there exists some $su$-path connecting $y'$ to $z'$ or, in other words, there exist points

$$b_0 = y, a_1 = y', b_1, \dots, a_i, b_i, \dots, a_s = z', b_s = z$$

which belong to $\mathcal{W}^c(\Lambda_i)$ such that $a_j$ and $b_j$ belong to the same strong stable manifold and $b_j$ and $a_{j+1}$ belong to the same strong unstable manifold. We are going to find an (arbitrarily) nearby $su$-path

$$\tilde{b}_0 = y, \tilde{a}_1, \tilde{b}_1, \dots, a_i, b_i, \dots, \tilde{a}_s, \tilde{b}_s \tag{14}$$

with $\tilde{b}_s \in \ell_0$ and such that every $\tilde{b}_i$ belongs to some periodic center leaf in $\mathcal{W}^c(\Lambda_i)$. The first step is to observe that, since periodic leaves are dense, one may always find periodic leaves $\ell_1, \dots, \ell_{s-1}$ arbitrarily close to $\mathcal{W}^c(b_1), \dots, \mathcal{W}^c(b_{s-1})$, respectively. Let $\ell_s = \ell_0$. Assume $\tilde{b}_0, \tilde{a}_1, \dots, \tilde{b}_k$ have been defined, for some $0 \leq k < s$. Since $\mathcal{W}^u(b_k)$ intersects the stable set of $\mathcal{W}^c(b_{k+1})$ transversely at $a_{k+1}$, and stable and unstable sets vary continuously with the base point, we can find $\tilde{b}_{k+1} \in \ell_{k+1}$ close to $b_{k+1}$ such that $\mathcal{W}^u(\tilde{b}_k)$ intersects $\mathcal{W}^s(\tilde{b}_{k+1})$ at some point $\tilde{a}_{k+1}$ close to $a_k$. Repeating this procedure $s$ times, we obtain an $su$-path as in (14).

The next step is to prove that the points $\tilde{b}_i$ themselves are periodic. Recall that $\tilde{b}_0 = y$ is taken to be periodic and $D^u$ intersects $\mathcal{W}^s(\tilde{b}_0)$. So, the iterates accumulate on $\mathcal{W}^u(\tilde{b}_0)$ and, in particular, on $\tilde{a}_1$. This implies there exist points $w \in \ell_1$ arbitrarily close to $\tilde{b}_1$ whose strong stable manifold intersects $f^n(D^u)$ for some $n$. Since $\Gamma$ has full volume inside every $f^n(D^u)$, we may use Lemma 3.10 to conclude that $w$ is periodic, with period uniformly bounded. Consequently, $\tilde{b}_1$ itself is periodic. It also follows that the iterates of $D^u$ accumulate on $\mathcal{W}^u(\tilde{b}_1)$. This means we may now repeat the construction with $\tilde{b}_1$ in the place of $\tilde{b}_0$ and conclude that $\tilde{b}_2$ is periodic. After $s$ steps we conclude that $\tilde{z} = \tilde{b}_s$ is periodic. Since $\tilde{z}$ is arbitrarily close to $z$, and all the periods are bounded, we get that $z$ itself is periodic. This completes the proof of the lemma. □

In particular, Lemma 3.11 implies that no periodic point on the support of $\nu_*$ is hyperbolic. This is a contradiction since, by a classical result of Katok [29], the support of any hyperbolic measure contains hyperbolic periodic points. This completes the proof of Proposition 3.5. □

## 4. Mostly contracting center

In this section we prove some useful facts about partially hyperbolic diffeomorphisms with mostly contracting center direction. We call $\mathcal{W}^u$-*disk* any image of a ball in $E^u$ embedded inside some strong unstable leaf.

**Lemma 4.1.** *The center direction of $f$ is mostly contracting if and only if the center Lyapunov exponents of all ergodic Gibbs $u$-states are negative.*

*If $f \in \mathcal{P}^k_1(N)$, $k > 1$ and $\Lambda$ is an attractor of $f_c$, then the center direction of $f \mid \mathcal{W}^c(\Lambda)$ is mostly contracting if and only if the center Lyapunov exponent is negative for every ergodic Gibbs $u$-state supported in $\mathcal{W}^c(\Lambda)$.*

*Proof.* Bonatti, Viana [19] show that if the center direction is mostly contracting then the center exponents of every ergodic Gibbs $u$-state are negative. To prove the converse, let $D$ be any disk inside a strong unstable leaf. By [17, Lemma 11.12] every Cesaro accumulation point of the iterates of Lebesgue measure on $D$ is a Gibbs $u$-state. By [17, Lemma 11.13] every ergodic component of a Gibbs $u$-state is again a Gibbs $u$-state. This implies that the iterates $f^n(D)$ accumulate on the support of some ergodic Gibbs $u$-state $\nu$. The hypothesis implies that $\nu$-almost every point has a Pesin (local) stable manifold which is an embedded disk of dimension $d_{cs}$. Using also the absolute continuity of the Pesin stable foliation (Pesin [35]), we conclude that a positive Lebesgue measure subset of points in some $f^n(D)$ belong to the union of these $d_s$-disks. This implies that (5) holds on a positive Lebesgue measure subset of $D$, as we wanted to show.

The second part of the lemma follows from similar arguments. □

### 4.1. Supports of Gibbs $u$-states.

**Lemma 4.2.** *If the center direction of $f$ is mostly contracting then the supports of the ergodic Gibbs $u$-states of $f$ are pairwise disjoint.*

*Proof.* Let $m_1$ and $m_2$ be ergodic Gibbs $u$-states of $f$ and suppose $\operatorname{supp} m_1 \cap \operatorname{supp} m_2$ contains some point $z$. Let $D$ be any $\mathcal{W}^u$-disk around $z$. Then $D \subset \operatorname{supp} m_1 \cap \operatorname{supp} m_2$, since the supports are $\mathcal{W}^u$-saturated (Proposition 3.1). By Lemmas 11.12 and 11.13 in [17], every ergodic component $\nu$ of every Cesaro accumulation point of the iterates of Lebesgue measure on $D$ is an ergodic Gibbs $u$-state. Clearly, the support of $\nu$ is contained in $\operatorname{supp} m_1 \cap \operatorname{supp} m_2$. By Pesin theory (see [19] for this particular setting) $\nu$-almost every point has a local stable manifold which is an embedded $d_{cs}$-disk. Recall (Proposition 3.1) that the density of Gibbs $u$-states along strong unstable leaves is positive and finite. Thus, we may find a $\mathcal{W}^u$-disk $D_\nu \subset \operatorname{supp}\nu$ such that every point $x$ in a full Lebesgue measure subset $D_\nu^*$ has a Pesin stable manifold and belongs to the basin of $\nu$. Moreover, $D_\nu$ is accumulated by $\mathcal{W}^u$-disks $D_i \subset \operatorname{supp} m_1$ such that Lebesgue almost every point is in the basin of $m_1$. Assuming $D_i$ is close enough to $D_\nu$, it must intersect the union of the local stable manifolds through the points of $D_\nu^*$ on some positive Lebesgue measure subset $D_i^*$ (because the Pesin local stable lamination is absolutely continuous [35]). Then $D_i^*$ is contained in the basin of $\nu$, and some full Lebesgue measure subset is contained in the basin of $m_1$. That implies $m_1 = \nu$. Analogously, $m_2 = \nu$, and so the ergodic Gibbs $u$-states $m_1$ and $m_2$ coincide. That completes the proof of the lemma. $\square$

*Remark* 4.3. It follows from Proposition 3.1 and Lemma 4.2 that if $f$ has mostly contracting center direction and minimal strong unstable foliation then it has a unique Gibbs $u$-state. This was first observed in [19].

**Proposition 4.4.** *Suppose the center direction of $f$ is mostly contracting, and let $m$ be an ergodic Gibbs $u$-state of $f$. Then the support of $m$ has a finite number of connected components. Moreover, each connected component $S$ is $\mathcal{W}^u$-saturated and $\mathcal{W}^u(x)$ is dense in $S$ for any $x \in S$.*

*Proof.* Let $p$ be any periodic point in the support of $m$ with stable index equal to $d_{cs}$ (such periodic points do exist, by Katok [29]) and let $\kappa$ be its period. By Proposition 3.1, the unstable manifold of every $f^j(p)$ is contained in $\operatorname{supp} m$. We claim that $\cup_{j=1}^{\kappa}\mathcal{W}^u(f^j(p))$ is dense in $\operatorname{supp} m$. To see this, let $D$ be any disk inside $\mathcal{W}^u(p)$. Consider the forward iterates of Lebesgue measure on $D$. Using Lemmas 11.12 and Lemma 11.13 in [17], one gets that any ergodic component of any Cesaro accumulation point of these iterates is an ergodic Gibbs $u$-state $\nu$ supported inside the closure of $\cup_{j=1}^{\kappa}\mathcal{W}^u(f^j(p))$. By Lemma 4.2, the Gibbs $u$-states $m$ and $\nu$ must coincide. In particular, $\operatorname{supp} m$ is contained in the closure of $\cup_{j=1}^{\kappa}\mathcal{W}^u(f^j(p))$. That proves our claim.

Since $m$ is ergodic for $f$, its ergodic decomposition relative to $f^\kappa$ has the form $m = l^{-1}\sum_{i=1}^{l} f_*^i\tilde{m}$ where $l$ divides $\kappa$ and $\tilde{m}$ is $f^\kappa$-invariant and ergodic. Then

$$\operatorname{supp} m = \bigcup_{i=1}^{l} f^i(\operatorname{supp}\tilde{m}).$$

We claim that the $f^i(\operatorname{supp}\tilde{m})$, $i = 1, \dots, l$ are precisely the connected components of $\operatorname{supp} m$. On the one hand, the previous paragraph gives that $p \in f^s(\operatorname{supp}\tilde{m})$ for some $s$. Replacing either $p$ or $\tilde{m}$ by an iterate, we may suppose $s = 0$. Then, by the argument in the previous paragraph applied to $f^\kappa$ (it is clear from the definition (5) that if $f$ has mostly contracting then so does any positive iterate), $\operatorname{supp}\tilde{m}$ coincides with the closure of $\mathcal{W}^u(p)$ and, in particular, it is connected. On the other hand, Lemma 4.2 gives that the $f^i(\operatorname{supp}\tilde{m})$, $i = 1, \dots, l$ are pairwise disjoint. Since they are closed, it follows that they are also open in $\operatorname{supp} m$. This proves our claim.

We are left to prove that the strong unstable foliation is minimal in each connected component $S_i = f^i(\operatorname{supp}\tilde{m})$. This will follow from an argument of Bonatti, Díaz, Ures [16]:

**Lemma 4.5.** *There is a neighborhood $U_i^s$ of $f^i(p)$ inside $W^s(f^i(p))$ such that every unstable leaf in $S_i$ has some transverse intersection with $U_i^s$.*

*Proof.* For any $x \in S_i$, let $D_x$ be a small $\mathcal{W}^u$-disk around $x$. Since $\tilde{m}_j$ is the unique ergodic $u$-state of $f^\kappa$ with support contained in $S_j$. It is also the unique Cesaro accumulation point of the iterates of $\mathrm{vol}_{D_x}$ under $f^\kappa$. In particular, there is $n_x \geq 1$ such that $f^{n_x\kappa}(D_x)$ intersects the local stable manifold of $f^i(p)$ transversely. This implies that $D_x$ intersects the global stable manifold of $f^i(p)$ transversely. Then, by continuity of the strong unstable foliation, there is a neighborhood $V_x$ of $x$ and a bounded open set $U_x \subset W^s_{f^\kappa}(f^i(p))$ such that $\mathcal{W}^u(y)$ intersects $U_x$ transversely for every $y \in V_x$. The family $\{V_x : x \in S_i\}$ is an open cover of the compact set $S_i$. Let $\{V_{x_1}, \cdots, V_{x_m}\}$ be a finite subcover. Choose $U^s_j$ a bounded neighborhood of $f^i(p)$ inside $W^s_{f^\kappa}(f^i(p))$ containing $U_{x_j}$ for all $j = 1, \dots, m$. It follows from the construction that every strong unstable leaf contained in $S_i$ intersects $U^s_i$ transversely. This finishes the proof of the lemma. $\square$

Let us go back to proving Proposition 4.4. The lemma gives that $\mathcal{W}^u(f^{-n\kappa}(x))$ intersects $U^s_i$ transversely, and so $\mathcal{W}^u(x)$ intersects $f^{n\kappa}(U^s_i)$ transversely, for every $x \in S_i$ and every $n \geq 0$. Since $f^{n\kappa}(U^s_i)$ converges to $f^i(p)$ when $n \to \infty$, it follows that $W^u(f^i(p))$ is contained in the closure of $W^u(x)$. Hence, $W^u(x)$ is dense in $S_j$, as claimed. The proof of the proposition is complete. $\square$

4.2. **Bernoulli property.** An invariant ergodic measure $\eta$ of a transformation $g$ is called *Bernoulli* if $(g, \eta)$ is ergodically conjugate to a Bernoulli shift.

**Theorem 4.6.** *Suppose $f$ is a $C^k$, $k > 1$ partially hyperbolic diffeomorphism with mostly contracting center direction. Then there is $l \geq 1$ and a $C^k$ neighborhood $\mathcal{U}$ of $f$ such that for any $g \in \mathcal{U}$, every ergodic $u$-state of $g^l$ is Bernoulli.*

*Proof.* Let $m_1, \dots, m_u$ be the ergodic Gibbs $u$-states of $f$. Proposition 4.4 gives that for each $j = 1, \dots, u$ there exists $l_j \geq 1$ such that the support of $m_j$ has $l_j$ connected components $S_{j,i}$, $i = 1, \dots, l_j$. Moreover, each connected component $S_{j,i}$ carries an ergodic component $m_{j,i} = f^i_* \tilde{m}_j$ of the Gibbs $u$-state $m_j$ for the iterate $f^{l_j}$. Let $l$ be any common multiple of $l_1, \dots, l_u$. Then every $S_{j,i}$ is fixed under $f^l$. Moreover, every Gibbs $u$-state $m_{j,i}$ is $f^l$-invariant and $f^{nl}$-ergodic for every $n \geq 1$: otherwise $S_i$ would break into more than one connected component (cf. the proof of Lemma 4.2). Then, by Ornstein, Weiss [33], every $m_{i,j}$ is a Bernoulli measure for $f^l$. We claim that $\{m_{j,i} : 1 \leq j \leq u \text{ and } 1 \leq i \leq l_j\}$ contains all the ergodic $u$-states of $f^{nl}$ for every $n \geq 1$. Indeed, let $m_*$ be any ergodic $u$-state for $f^{nl}$. Then

$$m = \frac{1}{nl} \sum_{k=1}^{nl} f^k_* m_*$$

is a $u$-state for $f$. Let $m = a_1 m_1 + \cdots + a_u m_u$ be its ergodic decomposition for $f$ and let $s$ be such that $a_s > 0$. Then $\operatorname{supp} m_s \subset \operatorname{supp} m$. Since $\operatorname{supp} m_s$ is $f$-invariant, it must intersect $\operatorname{supp} m_*$. Using Lemma 4.2 for $f^{nl}$ we conclude that $m_*$ must coincide with some ergodic component of $m_s$ for the iterate $f^{nl}$. In other words, it must coincide with $m_{s,i}$ for some $i = 1, \dots, l_s$, and this proves our claim.

Now we extend these conclusions to any diffeomorphism $g$ in a $C^k$, $k > 1$ neighborhood of $f$. By Andersson [9], any such $g$ has mostly contracting center direction, and so the previous argument applies to it. However, we must also prove that the integer $l$ can be taken uniform on a whole neighborhood of $f$. Notice that the only constraint on $l$ was that it should be a multiple of the periods $l_j$ of the ergodic components $m_j$. Observe that [9] also gives that the number of ergodic Gibbs $u$-states does not exceed the number of ergodic Gibbs $u$-states of $f$. So, we only need to check that the periods $l_j$ remain uniformly bounded for any $g$ in a neighborhood. We do this by arguing with periodic points, as follows. Let us fix, once and for all, $f$-periodic points $p_j$ with stable index $d_{cs}$ in the support of each $m_j$, $j = 1, \dots, u$. The period of each $p_j$ is a (fixed) multiple of $l_j$. Let $p_j(g)$ be the continuation of these periodic points for some nearby diffeomorphism $g$, and let $\{m_1(g), \dots, m_s(g)\}$, with $s \le u$ be the ergodic Gibbs $u$-states of $g$. We claim that every $\operatorname{supp} m_j(g)$, $1 \le j \le s$ contains some $p_i(g)$, $1 \le i \le u$. This can be seen as follows. As observed before, any accumulation point of Gibbs $u$-states of $g$ when $g \to f$ is a Gibbs $u$-state for $f$. We fix some small $\varepsilon > 0$ and consider the $\varepsilon$-neighborhoods $B(p_j, \varepsilon)$ of the periodic points $p_j$. Then, for any $g$ close enough to $f$ every ergodic Gibbs $u$-state $m_j(g)$ must give positive weight to some $B(p_i, \varepsilon)$ and, consequently, also to $B(p_i(g), 2\varepsilon)$. By continuous dependence of stable manifolds of periodic points on the dynamics, and the fact that the supports of Gibbs $u$-states are $u$-saturated, it follows that $\operatorname{supp} m_j(g)$ contains some $\mathcal{W}^u$-disk that intersects $W^s(p_i(g))$ transversely. Then, the support of $m_j(g)$ must contain $p_i(g)$. This proves our claim. It follows that the period $l_j(g)$ of each ergodic Gibbs $u$-state of $g$ divides the period of some $p_i(g)$ which, of course, coincides with the period of $p_i$. Since the latter have been fixed once and for all, this proves that the $l_j(g)$ are indeed uniformly bounded on a neighborhood of $f$. The proof of the theorem is complete. □

4.3. **Abundance of mostly contracting center.** We also give a family of new examples of diffeomorphisms with mostly contracting center.

**Theorem 4.7.** *Suppose* $\dim M = 3$. *The set of ergodic diffeomorphisms such that either $f$ or $f^{-1}$ has mostly contracting center direction is $C^1$ open and dense in the space of $C^k$, $k > 1$ partially hyperbolic volume preserving diffeomorphisms with 1-dimensional center and some fixed compact center leaf.*

*Proof.* Denote by $\mathcal{V}_m^k$ the set of $C^k$ volume preserving partially hyperbolic diffeomorphisms with 1-dimensional center and some fixed compact center leaf. This is a $C^1$ open set, cf. [28, Theorem 4.1]. Moreover, the diffeomorphisms such that both the strong stable foliation and the strong unstable foliation is minimal fill an open and dense subset $U_1$ of $\mathcal{V}_m^1$. This follows from a conservative version of the results of [16]: one only has to observe that blenders, that they use for the proof in the dissipative context, can be constructed also in the conservative setting, as shown by [26]. By [15], there is an open and dense subset $U_2$ for which the center Lyapunov exponent

$$\int \log |Df \mid E^c(x)| dm(x) \neq 0.$$

Furthermore, by [24], there is an open and dense subset $U_3$ of $\mathcal{V}_m^1$ consisting of accessible diffeomorphisms. Let $U = U_1 \cap U_2 \cap U_3$. Before proceeding, let us recall that $C^\infty$ are $C^1$ dense in the space of volume preserving diffeomorphisms, by [11]. In particular, the $C^1$ open and dense subset $U$ has non-trivial intersection with the space of $C^k$ diffeomorphisms, for any $k > 1$.

We claim that for every $C^k$, $k > 1$ diffeomorphism $f$ in $U$, either $f$ or its inverse has a unique ergodic Gibbs $u$-state and the corresponding center Lyapunov exponent is negative. In particular, by Lemma 4.1, either $f$ or its inverse has mostly contracting center direction. The first step is to note that $f$ is ergodic, since it is accessible (see [22, 27, 38]). Then the Lebesgue measure vol is an ergodic Gibbs $u$-state for both $f$ and $f^{-1}$. Since the strong stable and strong unstable foliations are minimal, the Gibbs $u$-state is unique; see Remark 4.3. This completes the proof of Theorem 4.7. □

## 5. Finiteness and stability of physical measures

In this section we prove Theorem D. As remarked before, Theorem A is a particular case. We begin by recalling certain ideas from Bonatti, Gomez-Mont, Viana [18] and Avila, Viana [13] that we use for handling the case when the center Lyapunov exponent vanishes.

5.1. **Smooth cocycles.** By assumption, the center leaves of $f$ define a fiber bundle $\pi_c : N \to N/\mathcal{W}^c$ over the leaf space. Then $f$ may be seen as a smooth cocycle (as defined in [13]) over $f_c$:

$$\begin{array}{rccc} f: & N & \to & N \\ & \downarrow & & \downarrow \\ f_c: & N/\mathcal{W}^c & \to & N/\mathcal{W}^c \end{array}$$

It follows from the form of our maps that the strong stable manifold $\mathcal{W}^s(x)$ of every point $x \in M$ is a graph over the stable set $W^s_{\pi_c(x)}$ of $\pi_c(x) \in N/\mathcal{W}^c$. For each $\eta \in W^s(\xi)$, the strong stable holonomy defines a homeomorphism $h^s_{\xi,\eta} : \xi \to \eta$ between the two center leaves. In fact (see [13, Proposition 4.1]),

$$h^s_{\xi,\eta}(\theta) = \lim_{n\to\infty} (f^n \mid \eta)^{-1} \circ (f^n \mid \xi)(\theta), \tag{15}$$

(for large $n$ one can identify $f^n(\xi) \approx f^n(\eta)$ via the fiber bundle structure) for each $\theta \in \xi$ and the limit is uniform on the set of all $(\xi, \eta, \theta)$ with $\theta \in \xi$ and $\xi$ and $\eta$ in the same local stable set. These *s-holonomy* maps satisfy

- $h^s_{\eta,\zeta} \circ h^s_{\xi,\eta} = h^s_{\xi,\zeta}$ and $h^s_{\xi,\xi} = \mathrm{id}$
- $f \circ h^s_{\xi,\eta} = h^s_{f_c(\xi),f_c(\eta)} \circ f$
- $(\xi, \eta, \theta) \mapsto h^s_{\xi,\eta}(\theta)$ is continuous on the set of triples $(\xi, \eta, \theta)$ with $\xi$ and $\eta$ in the same local stable set and $\theta \in \mathcal{W}^c(\xi)$.

Let $m$ be any $f$-invariant probability measure and $\mu = (\pi_c)_*(m)$. A *disintegration* of $m$ into conditional probabilities along the center leaves is a measurable family $\{m_\xi : \xi \in \operatorname{supp}\mu\}$ of probability measures with $m_\xi(\xi) = 1$ for $\mu$-almost every $\xi$ and

$$m(E) = \int m_\xi(E)\, d\mu(\xi) \tag{16}$$

for every measurable set $E \subset M$. By Rokhlin [40], such a family exists and is essentially unique. A disintegration is called *s-invariant* if

$$(h^s_{\xi,\eta})_* m_\xi = m_\eta \quad \text{for every } \xi,\, \eta \in \operatorname{supp}\mu \text{ in the same stable set.}$$

In a dual way one defines *u-holonomy* maps and *u-invariance*. We call a disintegration *bi-invariant* if it is both $s$-invariant and $u$-invariant, and we call it *continuous* if $m_\xi$ varies continuously with $\xi$ on the support of $\mu$, relative to the weak* topology.

**Proposition 5.1.** *Let $f \in \mathcal{P}^k_*(N)$, $k > 1$ be such that the center stable foliation is absolutely continuous. Let $m$ be an ergodic Gibbs u-state with vanishing center Lyapunov exponents. Then $m$ admits a disintegration $\{m_\xi : \xi \in \operatorname{supp}\mu\}$ into conditional probabilities along the center leaves which is continuous and bi-invariant.*

*Proof.* Proposition 3.3 gives that $(\pi_c)_* m$ has local product structure. Thus, we are in a position to use Theorem D of Avila, Viana [13] to obtain the conclusion of the present proposition. □

5.2. **Zero Lyapunov exponent case.** The following result provides a characterization of the systems exhibiting ergodic Gibbs $u$-states with vanishing central exponent.

**Proposition 5.2.** *Let $f \in \mathcal{P}^k_1(N)$, $k > 1$ be such that the center stable foliation is absolutely continuous. Let $\Lambda$ be an attractor of $f_c$ such that $f$ is accessible on $\Lambda$, and let $m$ be an ergodic Gibbs u-state with vanishing center Lyapunov exponent. Then*

(1) *the conditional probabilities $\{m_x : x \in \Lambda\}$ along the center leaves are equivalent to the Lebesgue measure on the leaves, with densities uniformly bounded from zero and infinity;*

(2) $\operatorname{supp} m = W^c(\Lambda)$ *and this is the unique Gibbs u-state supported in* $\mathcal{W}^c(\Lambda)$*;*
(3) *the basin of $m$ covers a full Lebesgue measure subset of a neighborhood of* $\mathcal{W}^c(\Lambda)$.

*Proof.* By Proposition 5.1, there is a disintegration $\{m_x : x \in \Lambda\}$ of $m$ along the center foliation which is continuous, $s$-invariant, and $u$-invariant. Let $\xi$ and $\eta$ be any two points in $\mathcal{W}^c(\Lambda)$. By accessibility on $\Lambda$, one can find an $su$-path $b_0 = \xi, b_1, \ldots, b_{s-1}, b_s = \eta$ connecting $\xi$ to $\eta$. This $su$-path induces a holonomy map $h : \mathcal{W}^c(\xi) \to \mathcal{W}^c(\eta)$, defined as the composition of all strong stable/unstable holonomy maps $h_i : \mathcal{W}^c(b_{i-1}) \to \mathcal{W}^c(b_i)$. The fact that the disintegration is bi-invariant gives, in particular, that

$$m_\eta(h(B^c_\varepsilon(\xi)) = m_\xi(B^c_\varepsilon(\xi)) \tag{17}$$

It is a classical fact that the strong stable and strong unstable foliations are absolutely continuous in a strong sense: their holonomy maps have bounded Jacobians. See [20, 31, 1]. Those arguments extend directly to their restrictions to each center stable or center unstable leaf, respectively: the restricted strong stable and strong unstable foliations are also absolutely continuous with bounded Jacobians. By compactness, the $su$-path may be chosen such that the number $s$ of legs and the length of each leg are uniformly bounded, independent of $\xi$ and $\eta$ [1]. Then, we may fix a uniform upper bound constant $K > 1$ on the Jacobians of all associated strong stable and strong unstable holonomies. Notice $\operatorname{vol}^c(B_r(\zeta)) = 2r$, since the center leaves are one-dimensional. Then

$$K^{-1} \operatorname{vol}^c(B^c_\varepsilon(\xi)) \le \operatorname{vol}^c(h(B^c_\varepsilon(\xi))) \le K \operatorname{vol}^c(B^c_\varepsilon(\xi)). \tag{18}$$

From (17) and (18) we obtain

$$\frac{1}{K}\frac{m_\xi(B^c_\varepsilon(\xi))}{\operatorname{vol}^c(B^c_\varepsilon(\xi))} \le \frac{m_\eta(h(B^c_\varepsilon(\xi)))}{\operatorname{vol}^c(h(B^c_\varepsilon(\xi)))} \le K\frac{m_\xi(B^c_\varepsilon(\xi))}{\operatorname{vol}^c(B^c_\varepsilon(\xi))}.$$

and, taking the limit as $\varepsilon \to 0$,

$$\frac{1}{K}\frac{dm_\xi}{d\operatorname{vol}^c}(\xi) \le \frac{dm_\eta}{d\operatorname{vol}^c}(\eta) \le K\frac{dm_\xi}{d\operatorname{vol}^c}(\xi).$$

Since we can always find $\eta$ where the density is less or equal than 1 (respectively, greater or equal than 1), this implies that

$$\frac{dm_\xi}{d\operatorname{vol}^c}(\xi) \in \left[K^{-1}, K\right] \tag{19}$$

for every $\xi$, and that proves claim (1).

Now let $m'$ be any other ergodic Gibbs $u$-state supported in $\mathcal{W}^c(\Lambda)$. The center Lyapunov exponent of $m'$ must vanish: otherwise, by [29], there would be some hyperbolic periodic point in $\mathcal{W}^c(\Lambda)$, and that is incompatible with the conclusion in part (1) that there exist invariant conditional probabilities equivalent to Lebesgue measure along the center leaves. So, all the previous considerations apply to $m'$ as well. In particular, it has a continuous disintegration $\{m'_x : x \in \Lambda\}$ along the center foliation such that each $m'_x$ is equivalent with $\operatorname{vol}^c$. Moreover, by Proposition 3.2, $(\pi_c)_*(m) = (\pi_c)_*(m')$. Then, $\operatorname{vol}^c$-almost every point in almost every center leaf, relative to $(\pi_c)_*(m) = (\pi_c)_*(m')$, belongs to the basin of both $m$ and $m'$. In

[1]This may be deduced from [12] as follows. By Proposition 8.3 in [12], given any $x_0 \in M$ there exists $w \in M$ such that $x_0$ is connected to every point in a neighborhood of $w$ by a uniformly bounded $su$-path. Then the same is true if one replaces $w$ by an arbitrary point $z \in M$: connect $w$ to $z$ by some $su$-path; the "same" $su$-path determines a bijection between neighborhoods of $w$ and $z$; concatenating with $su$-paths from $x_0$ to the neighborhood of $w$ one obtains uniformly bounded $su$-paths from $x_0$ to any point near $z$. The claim now follows by compactness of the ambient manifold.

particular, the two basins intersect, and that implies $m = m'$. That completes the proof of claim (2).

From conclusion (1) we get that there exists a full $m$-measure set $\Delta$ consisting of leaves such that $\mathrm{vol}^c$-almost every point in the leaf belongs to the basin of $m$. Then, since $m$ is a Gibbs $u$-state, we can find a $\mathcal{W}^u$-disk $D_0^u$ such that $\Delta \cap D_0^u$ has full measure in $D_0^u$. Consider the $cu$-disk

$$D_0^{cu} = \bigcup_{\xi \in D_0^u} \mathcal{W}^c(\xi)$$

Observe that the center foliation $\mathcal{W}^c$ is absolutely continuous on each center unstable leaf, because the corresponding holonomy maps between unstable leaves coincide with the corresponding holonomy maps for the center stable foliation, which we assume to be absolutely continuous. Using this fact, and a Fubini argument, we get that Lebesgue almost every point in $D_0^{cu}$ belongs to the basin of $m$. Next, consider the open set

$$D_0 = \bigcup_{\zeta \in D_0^{cu}} \mathcal{W}^s_{loc}(\zeta).$$

Since the strong stable foliation is absolutely continuous and the basin is $\mathcal{W}^s$-saturated, it follows that Lebesgue almost every point in $D_0$ belongs to the basin of $m$. It is clear that we can cover $\mathcal{W}^c(\Lambda)$ by finitely many such open sets $D_0$. This proves (3), and so the proof of the lemma is complete. $\square$

5.3. **Construction of physical measures.** We are nearly done with the proof of Theorem D. By Proposition 3.5, all ergodic Gibbs $u$-states have non-negative center Lyapunov exponent. The case when the exponent vanishes for some Gibbs $u$-state is handled by Proposition 5.2: we get alternative (a) of the theorem in this case. Finally, if the center Lyapunov exponent is negative for all Gibbs $u$-states over some attractor $\Lambda_i$ of $f_c$ then, by Lemma 4.1, the center direction of $f$ is mostly contracting on that attractor $\Lambda_i$. Then, by Bonatti, Viana [19], there are finitely many ergodic Gibbs $u$-states supported in $\mathcal{W}^c(\Lambda_i)$, these $u$-states are the physical measures of $f$, and the union of their basins covers a full volume measure subset of a neighborhood of $\mathcal{W}^c(\Lambda_i)$. By Theorem 4.6, all these physical measures are Bernoulli for some iterate of $f$. Thus, we get alternative (b) of the theorem in this case.

From now on, let $\{m_{i,j}\}_{j=1}^{J(i)}$ be the physical measures supported on each attractor $\Lambda_i$. As we have just seen, their basins cover a full Lebesgue measure subset of a neighborhood $U_i$ of $\mathcal{W}^c(\Lambda_i)$. We want to prove that the union of all these basins contains a full Lebesgue measure subset of the ambient manifold. Suppose otherwise, that is, suppose the complement $C$ of this union has positive Lebesgue measure. Let $C_0 \subset C$ be the set of Lebesgue density points of $C$. Notice that $C_0$ is $f$-invariant and $\mathrm{vol}(C_0) = \mathrm{vol}(C)$. Since the unstable foliation is absolutely continuous, there is a $\mathcal{W}^u$-disk $D^u$ such that $\mathrm{vol}_{D^u}(D^u \cap C_0) > 0$. Denote $I^u = D^u \cap C_0$. Then every Cesaro accumulation point of the iterates of Lebesgue measure on $I^u$ is a Gibbs $u$-state (see [17], section 11.2), and so its ergodic components are ergodic Gibbs $u$-states. Let $m^*$ be any such accumulation point and $m_{i,j}$ be an ergodic component of $m^*$. The support of $m_{i,j}$ is contained in $U_i$, and so there is $n_0 \geq 1$ such that $f^{n_0}(I^u)$ intersects $U_i$. Recalling that $C_0$ is invariant, we get that $\mathrm{vol}(C_0 \cap U_i) > 0$. This contradicts the definition of $C_0$, since Lebesgue almost every point in $U_i$ belongs to the basin of $m_{i,l}$ for some $l = 1, \ldots, J(i)$. This contradiction proves that the union of the basins does have full Lebesgue measure in $N$. That completes the proof of Theorem D.

5.4. **Number of physical measures.** In this section, we give explicit upper bounds on the number of physical measures for some diffeomorphisms with mostly contracting center direction:

**Theorem 5.3.** *Let $f \in \mathcal{P}_1^k(N)$, $k > 1$ be accessible on some attractor $\Lambda$ and have absolutely continuous center stable foliation. Assume there exists some center leaf $\ell \subset \mathcal{W}^c(\Lambda)$ such that $f^\kappa(\ell) = \ell$ for some $\kappa \geq 1$ and $f^\kappa \mid \ell$ is Morse-Smale with periodic points $p_1, \cdots, p_s$.*

*Then the center direction is mostly contracting over $\Lambda$ and $f$ has at most $s$ physical measures supported in $\mathcal{W}^c(\Lambda)$. If $\mathcal{W}^u(p_i)$ intersects $W^s(\ell) \setminus \cup_{j=1}^s \mathcal{W}^s(p_j)$ for every $i$ then $f$ has at most $s/2$ physical measures supported in $\mathcal{W}^c(\Lambda)$.*

*Proof.* Since $f$ has hyperbolic periodic point in $\mathcal{W}^c(\Lambda)$ the restriction of $f$ to $\mathcal{W}^c(\Lambda)$ can not be conjugate to a rotation extension over $\Lambda$. Thus, by Theorem D, $f$ has mostly contracting center direction over $\Lambda$.

**Lemma 5.4.** *Suppose $f \in \mathcal{P}_1^k(N)$, $k > 1$ has mostly contracting center direction on an attractor $\Lambda$ and let $p$ be any periodic point in $\mathcal{W}^c(\Lambda)$. Then any disk $D^u$ in unstable manifold of $p$ contains a positive measure subset $I^u$ such that any $\xi \in I^u$ belongs to the basin of some physical measure and has local stable manifold $W^s_{loc}(\xi)$.*

*Proof.* As in the proof of Lemma 4.1, there is a positive measure subset $I^u$ of $D^u$ belonging to the basin of some physical measure $m$, and for $\xi \in I^u$, there is $n_0$ such that $f^{n_0}(\xi)$ belongs to the Pesin stable manifold of some point $\zeta$. Iterating backward we obtain a local stable manifold for $\xi$. □

Suppose $f$ has physical measures $\{m_j\}_{j=1}^J$ on $\mathcal{W}^c(\Lambda)$. Let $p_t$, $t = 1, \ldots, s$ be fixed as in Theorem 5.3. Since the support of each physical measure is a $u$-saturated compact set, the following fact is an immediate consequence of Lemma 4.2:

**Corollary 5.5.** *For each $1 \leq t \leq s$ there is at most one physical measure whose support intersects $W^s(p_t)$.*

As observed before, the unstable foliation is minimal in every attractor in the quotient. So, the orbit of every strong unstable leaf intersects $W^s(\ell) = \cup_{t=1}^s W^s(p_t)$. Since the supports of physical measures are $\mathcal{W}^u$-saturated and invariant, it follows that for every $1 \leq j \leq J$ there exists some $1 \leq t \leq s$ such that $\operatorname{supp} m_j$ intersects $W^s(p_t)$. So, by Corollary 5.5, $J \leq s$.

Let $\{p_{s_i}\}_{i=1}^{s/2}$ be periodic points in $\ell$ with stable index $d_s$ (i.e. repellers for $f \mid \ell$) and let $\{p_{s_i}\}_{i=s/2+1}^{s}$ be periodic points in $\ell$ with stable index $d_{cs}$ (i.e. attractors for $f \mid \ell$). We claim that if $\mathcal{W}^u(p_i)$ intersects $\mathcal{W}^s(\ell) \setminus \cup_{j=1}^s \mathcal{W}^s(p_j)$ for every $i$, then the support of every physical measure contains some $p_i$, $s/2+1 \leq i \leq s$. Indeed, by the previous observations the support must intersect $W^s(p_i)$ for some $i$, corresponding to either an attractor or a repeller of $f \mid \ell$. In the former case, the claim is proved; in the latter case, our assumption on $\ell$ implies that the support intersects the stable set of some other periodic point $p_j$ which is an attractor, and so the claim follows in just the same way. So, by the previous argument, the number of physical measures can not exceed $s/2$ in this case. The proof of Theorem 5.3 is complete. □

5.5. **Statistical stability.** We also want to analyze the dependence of the physical measures on the dynamics. For this, we assume $N = M \times S^1$ and restrict ourselves to the set $\mathcal{S}^k(N) \subset \mathcal{P}_1^k(N)$ of skew-product maps. Notice that every $f \in \mathcal{S}^k(N)$ is dynamically coherent, has compact one-dimensional center leaves, and absolutely continuous center stable foliation. As pointed out before, partially hyperbolicity is an open property and accessibility holds on an open and dense subset of $\mathcal{S}^k(N)$.

**Theorem 5.6.** *For any $k > 1$ there exists a $C^1$ open and $C^k$ dense subset $\mathcal{B}^k(N)$ of $\mathcal{S}^k(N)$ such that every $f \in \mathcal{B}^k(N)$ has mostly contracting center direction. Moreover, on a $C^k$ open and dense subset of $\mathcal{B}^k(N)$ the number of physical measures is locally constant and the physical measures depend continuously on the diffeomorphisms.*

*Proof.* Notice that every $f \in \mathcal{S}^k(N)$ is dynamically coherent, has compact one-dimensional center leaves, and absolutely continuous center stable foliation. By a variation of an argument of Niţică, Török [32], one gets that the set of diffeomorphisms in $\mathcal{S}^k(N)$ which are accessible on all attractors are $C^1$ open and $C^r$ dense. Let us comment a bit on this, since our setting is not exactly the same. The heart of the proof is to show that the accessibility class of any point contains the corresponding center leaf. This is done by considering 4-leg $su$-paths linking the point to every nearby point in the center leaf; in this way one gets that every accessibility class is open in the center leaf; then, connectivity gives the conclusion. The only difference in our case is that we deal with accessibility over each of the attractors, not the whole ambient manifold. However, the arguments remains unchanged, just with the additional caution to choose the corners of the 4-leg $su$-path to be points over the attractor. It is easy to see that the set of diffeomorphisms in $\mathcal{S}^k(N)$ which have a center leaf containing some hyperbolic periodic point is $C^1$ open and $C^r$ dense. Take $\mathcal{B}^k$ be the intersection of above two sets. Then by Theorem D, any $f \in \mathcal{B}^k$ has mostly contracting center bundle. By Andersson [9], for any partially hyperbolic diffeomorphism $f$ with mostly contracting center direction there is a $C^k$, $k > 1$ neighborhood $\mathcal{U}$ of $f$ such that any $g \in \mathcal{U}$ has mostly contracting center direction also, and on a $C^k$ open and dense subset of $\mathcal{U}$, the number of physical measures is locally constant and the physical measures depend continuously on the diffeomorphism. This ends the proof of Theorem 5.6. □

## 6. Absolute continuity for mostly contracting center

Throughout this section $f : N \to N$ is a partially hyperbolic, dynamically coherent, $C^k$, $k > 1$ diffeomorphism with mostly contracting center direction. Recall the later is a robust (open) condition, by Andersson [9]. We develop certain criteria for proving absolute continuity of the center stable, center unstable, and center foliations and we apply these tools to exhibit several robust examples of absolute continuity. In particular, this yields a proof of Theorem E.

The starting point for our criteria is the observation that for maps with mostly contracting center the Pesin stable manifolds are contained in, and have the same dimension as the center stable leaves. Since the Pesin stable lamination is absolutely continuous ([35, 37]), in this way one can get a local property of absolute continuity for the center stable foliation. This initial step of the construction is carried out in Section 6.2. Then one would like to propagate this behavior to the whole ambient manifold, in order to obtain actual absolute continuity. It is important to point out that this can not possibly work without additional conditions. Example 6.1 below illustrates some issues one encounters. A more detailed analysis, including explicit robust counter-examples will appear in [49]. Suitable assumptions are introduced in Section 6.1, where we also give the precise statements of our criteria. In Section 6.3 we present the main tool for propagating local to global behavior. The criteria are proved in Sections 6.4 through 6.6.

Before proceeding, let us give a simple example of a map whose center foliation is leafwise absolutely continuous and locally absolutely continuous, but not globally absolutely continuous. This kind of construction explains why Pesin theory alone can not give (global) absolute continuity of center foliations, even when the center direction is mostly contracting.

*Example* 6.1. Let us start with $f_0 : S^1 \times [0,1] \to S^1 \times [0,1]$, $f_0(x,t) = (2x, g(t))$ where $g : [0,1] \to [0,1]$ is a $C^2$ diffeomorphism such that $g(0) = 0$, $g(1) = 1$, $g(t) < t$ for all $t \in (0,1)$, and $0 < g'(t) < 2$ for every $t \in [0,1]$. Then $f_0$ is a partially hyperbolic endomorphism of the cylinder, with the vertical segments as center leaves. Next, let $f : S^1 \times [0,1] \to S^1 \times [0,1]$ be a $C^2$-small perturbation, preserving the two boundary circles $\mathcal{C}_i = S^1 \times \{i\}$, $i = 0, 1$ and the vertical line $\{0\} \times [0,1]$ through the fixed point $(0,0)$. Moreover, the horizontal derivatives of $f$ at the endpoints of this vertical line should be different:

$$\frac{\partial f}{\partial x}(0,0) \neq \frac{\partial f}{\partial x}(0,1). \tag{20}$$

By the stability of center foliations ([28], the new map $f$ has a center foliation whose leaves are curve segments with endpoints in the two boundary circles. Thus, they induce a holonomy map $h : \mathcal{C}_0 \to \mathcal{C}_1$ that conjugates the two expanding maps $f \mid \mathcal{C}_0$ and $f \mid \mathcal{C}_1$. Condition (20) implies that the conjugacy can not be absolutely continuous (see [43]). This shows that the center foliation is not absolutely continuous. Yet, it is absolutely continuous restricted to $S^1 \times [0,1)$, as we are going to explain. Notice that our assumptions imply that $g'(0) < 1 < g'(1)$ and so the lower boundary component $\mathcal{C}_0$ is an attractor for $f_0$, with $S^1 \times [0,1)$ as its basin of attraction. Then the same is true for the perturbation $f$. Moreover, restricted to this basin, the center leaves coincide with the Pesin stable manifolds of the points in the attractor, and so they do form an absolutely continuous foliation. In particular, this also shows that the center foliation is leafwise absolutely continuous.

6.1. **Criteria for absolute continuity.** We assume that some small cone field around the strong unstable bundle has been fixed. We call *$u$-disk* any embedded disk of dimension $d_u$ whose tangent space is contained in that unstable cone field at every point. Previously, we introduced the special case of $\mathcal{W}^u$-disks, which are contained in strong unstable leaves. To begin with, in Section 6.4 we prove that upper leafwise absolute continuity always holds in the present context:

**Proposition 6.2.** *The center stable foliation of $f$ is upper leafwise absolutely continuous, if it exists.*

For the next criterion we assume the diffeomorphism is non-expanding along the center direction. This notion is defined as follows. Assume also $f$ is dynamically coherent. Given $r > 0$ and $* \in \{s, cs, c, cu, u\}$, we denote by $\mathcal{W}^*_r(x) \subset \mathcal{W}^*(x)$ the ball of radius $r$ around $x$, relative to the distance induced by the Riemannian metric of $N$ on the leaf $\mathcal{W}^*(x)$. In what follows we always suppose $r$ is small enough so that $\mathcal{W}^*_r(x)$ is an embedded disk of dimension $d_*$ for all $x \in M$ and every choice of $*$. We use $\widehat{W}^s(p)$ and $\widehat{W}^u(p)$ to denote the stable and unstable sets of a periodic point $p$. We say that $f$ is *non-expanding along the center direction* if there exist $\rho > 0$ and $\varepsilon > 0$ such that

- $f^n(\mathcal{W}^{cs}_\varepsilon(x)) \subset \mathcal{W}^{cs}_\rho(f^n(x))$ for every $n \geq 0$ and almost any $x$ in any $u$-disk.
- the support of every ergodic Gibbs $u$-state $m$ contains some periodic point $p$ such that $\widehat{W}^s(p) \supset \mathcal{W}^{cs}_{2\rho}(p)$

**Proposition 6.3.** *If $f$ is non-expanding along the center direction then the center stable foliation is absolutely continuous.*

The proof of this proposition is given in Sections 6.2 through 6.5. We will see that the hypothesis holds for a classical construction of partially hyperbolic, robustly transitive diffeomorphisms due to Mañé [30] (Section 7.1). It also holds for a more recent class of examples introduced by Bonatti, Viana [19], which are not even partially hyperbolic (though they do admit a dominated invariant splitting of the tangent bundle), but this fact will not be proved here.

Let $f \in \mathcal{P}_1^k(N)$. Let $\ell$ be a periodic center leaf $\ell$, with period $\kappa \geq 1$. For $* \in \{s, u\}$, we denote $\mathcal{W}^*(\ell) = \cup_{\zeta \in \ell} \mathcal{W}^*(\zeta)$. We call *homoclinic leaf* associated to $\ell$ any center leaf $\ell'$ contained in $\mathcal{W}^s(\ell) \cap \mathcal{W}^u(\ell)$. Then there exist strong stable and strong unstable holonomy maps

$$h^s : \ell \to \ell' \quad \text{and} \quad h^u : \ell \to \ell' \tag{21}$$

We say that $\ell$ is *in general position* if

(a) $f^\kappa \mid \ell$ is Morse-Smale with a single periodic attractor $a$ and a single periodic repeller $r$;
(b) $h^s(a \cup r)$ is disjoint from $h^u(a \cup r)$, for some homoclinic leaf associated to the center leaf $\ell$.

Notice that $\mathcal{W}^s(\ell') \setminus \mathcal{W}^s(h^s(r))$ is contained in the stable manifold $\widehat{W}^s(a)$ of the attractor. Thus, condition (b) implies that $\mathcal{W}^u(a)$ and $\mathcal{W}^u(r)$ intersect $\widehat{W}^s(a)$ transversely. Analogously, $\mathcal{W}^s(a)$ and $\mathcal{W}^s(r)$ intersect $\widehat{W}^u(r)$ transversely.

**Proposition 6.4.** *Suppose $f \in \mathcal{P}_1^k(N)$ has some center leaf $\ell$ in general position and such that every strong unstable leaf intersects $\mathcal{W}^s(\ell)$. Then the center stable foliation of $f$ is absolutely continuous.*

This proposition is proved in Section 6.6. In Section 7.2 we use it to prove Theorems B and E, and in Section 7.3 we give an application to volume preserving systems. Noticing that, apart from dynamical coherence, all the hypotheses of Proposition 6.4 are robust, we get the following immediate consequence:

**Corollary 6.5.** *Suppose $f \in \mathcal{P}_1^k(N)$ is robustly dynamically coherent and has some periodic center leaf $\ell$ in general position and such that every strong unstable leaf intersects $\mathcal{W}^s(\ell)$. Then the center stable foliation is robustly absolutely continuous.*

6.2. **Local absolute continuity.** The following lemma will allow us to obtain some property of local absolute continuity:

**Lemma 6.6.** *For any ergodic u-state $m$ of $f$ and any disk $D$ contained in an unstable leaf inside* $\operatorname{supp} m$*, there is a positive measure subset $\Gamma$ such that the points in $\Gamma$ have (Pesin) stable manifolds with uniform size. Moreover, these stable manifolds form an absolutely continuous lamination, in the following sense: there is $K > 0$ such that for any two u-disks $D_1$, $D_2$ sufficiently close to $D$, the stable manifolds of points in $\Gamma$ define a holonomy map between subsets of $D_1$ and $D_2$, and this is absolutely continuous, with Jacobian between $1/K$ and $K$.*

*Proof.* Because $f$ has mostly contracting center direction, $m$ is a hyperbolic ergodic measure of $f$, by Pesin theory, there is a Pesin block $\Lambda$ with positive $m$ measure such that every point $x \in \Lambda$ has uniform size of stable manifold, and these stable manifolds on $\Lambda$ is uniformly absolutely continuous. Notice that the stable manifolds are contained in the center stable leaves. Since $m$ is a $u$-state, there is a disk $D_0$ contained in an unstable leaf inside the support and intersecting $\Lambda$ on a $m^u$-positive measure subset $D_0^*$. Then the points in $D_0^*$ have stable manifolds of size bounded below by some $\delta_0 > 0$. Denote $B_0 = \cup_{x \in D_0^*} W^s_{\delta_0}(x)$. Since $m$ is a $u$-state, $m(B_0) = a_0 > 0$. We claim that there is $n_0 > 0$ such that $(f^{n_0})_* \operatorname{vol}_D(B_0) \neq 0$.

Let us prove this claim. Let $D_\varepsilon^*$ be the $\varepsilon$-neighborhood of $D_0^*$ inside the corresponding unstable leaf. Denote by $B_\varepsilon = \cup_{x \in D_\varepsilon^*} W^{cs}_{\delta_0}(x)$, it is an open set, and $m(B_\varepsilon) \geq a_0 > 0$. Because every Cesaro accumulation point of the iterates of Lebesgue measure on $D$ is a Gibbs $u$-state with support contained in $\operatorname{supp} m$, and there is a unique ergodic $u$-state with support contained in $\operatorname{supp} m$, then $m$ is the unique Cesaro accumulation of the iterates of Lebesgue measure on $D$. Since $B_\varepsilon$ is open, one has $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (f^i)_* \operatorname{vol}_D(B_\varepsilon) \geq m(B_\varepsilon) = a_0$, so there is arbitrarily big $n$ such that $f_*^n(\operatorname{vol}_D)(B_\varepsilon) > a_0/2$. For $\delta > 0$ sufficiently small, denote by

$D_\delta = \{x \in D, d^u(x, \partial(D)) \geq \delta\}$, one has $m^u(D \setminus D_\delta) < a_0/4$. Then there is $y \in D_\delta$ such that $f^n(y) \in B_\varepsilon$ and $f^n(D)$ contains a disk $D_y$ around $y$ and for any $x \in D^*_\varepsilon$ one has $W^{cs}_{\delta_0}(x) \cap D_y \neq \emptyset$. Then the stable manifolds of $D^*_0$ define a holonomy map between $D^*_0$ and $B_0 \cap D_y$, by the uniform absolute continuity of these stable manifolds, $\mathrm{vol}_{D_y}(D_y \cap B_0) > 0$, then $f^{n_0}_*(\mathrm{vol}_D)(B_0) > 0$. This proves the claim.

This claim implies $\mathrm{vol}_{f^n(D)}(f^n(D) \cap B^*_0) > 0$, let $\Gamma = D \cap f^{-n_0}(B^*_0)$, then $\mathrm{vol}_D(\Gamma) > 0$, every point in $\Gamma$ has uniform size of stable manifold, and these stable manifolds are uniformly absolutely continuous. □

Suppose $f \in \mathrm{Diff}^k(N)$, $k > 1$ admits dominated splitting $E^{cu} \oplus E^{cs}$, and it is dynamically coherent, that is, it has center stable and center unstable foliation. We call *cs-block* for $f$ the image $\mathcal{B} = h(\Sigma \times I^{d_{cs}})$ of any embedding $h : \Sigma \times I^{d_{cs}} \to N$, with $\Sigma \subset I^{d_{cu}}$, satisfying the following properties:

1. $h(\{a\} \times I^{d_{cs}})$ is contained in $\mathcal{W}^{cs}(h(a, 0))$, for every $a \in \Sigma$
2. $h(\{a\} \times I^{d_{cs}})$ is contained in the stable set of $h(a, 0)$, for every $a \in \Sigma$
3. $h(\Sigma \times \{0\})$ is a positive measure subset of some disk $D$ transverse to $\mathcal{W}^{cs}$;
4. there is $K > 0$ such that for any $u$-disks $D_1, D_2 \subset N$ which cross $h(\Sigma \times I^u)$, that is, $D_i$ intersects $h(a \times I^{cs})$ for every $a \in \Sigma$, there is a holonomy map $h^{cs}$ induced by $\mathcal{W}^{cs}$ from $D_1 \cap h(a \times I^{cs})$ to $D_2 \cap h(a \times I^{cs})$, the Jacobian of the holonomy map between $\mathrm{vol}_{D_1}$ and $\mathrm{vol}_{D_2}$ is bounded by $K$ from above and $1/K$ from below.

We also say that $\mathcal{B}$ is a $cs$-block over the disk $D$ in (3). If $D$ is contained in the unstable manifold of an index $d_{cs}$ periodic point $p$, then we say the $cs$-block is associated with p.

*Remark* 6.7. If $D$ is in the support of some ergodic Gibbs $u$-state $m$ then $m(\mathcal{B}) > 0$: this is a consequence of the absolute continuity property (4) and the fact that Gibbs $u$-states have positive densities along strong unstable leaves (Proposition 3.1).

We say that the $cs$-block has size $r > 0$ if the *plaque* $h(\{a\} \times I^{d_{cs}})$ contains $\mathcal{W}^{cs}_r(h(a, 0))$ for every $a \in \Sigma$. If a map $\tilde{h} : \Sigma \times I^{d_{cs}} \to N$ satisfies

$$\tilde{h} \mid \Sigma \times \{0\} \equiv h \mid \Sigma \times \{0\} \quad \text{and} \quad \tilde{h}(a \times I^{d_{cs}}) \subset h(a \times I^{d_{cs}})$$

for every $a \in \Sigma$ then $\tilde{\mathcal{B}} = \tilde{h}(\Sigma \times I^{d_{cs}})$ is called a sub-block of $\mathcal{B}$.

**Lemma 6.8.** *Let $m$ be an ergodic $u$-state of $f$ and $p \in \operatorname{supp} m$ be a periodic point of stable index $d_{cs}$ whose stable manifold $\widehat{W}^s(p)$ has size $r$. Then there is a cs-block associated with $p$ with size $r$.*

*Proof.* By Lemma 6.6, there is a $cs$-block over any $u$-disk $D \subset \mathcal{W}^u(p)$. Let $\kappa$ be the period of $p$. For every large $n$, the backward image $f^{-n\kappa}(\mathcal{B})$ is a $cs$-block of size $r$ over the $u$-disk $f^{-n}(D)$. □

6.3. **Recurrence to $cs$-blocks.** The next proposition is a key ingredient in the proof of our criteria for absolute continuity.

**Proposition 6.9.** *Let $m_i$, $i = 1, \dots, s$ be the ergodic Gibbs $u$-states of $f$ and $B_i$, $i = 1, \dots, n$ be cs-blocks over $\mathcal{W}^u$-disks $D_i \subset \operatorname{supp} m_i$. Then for any positive Lebesgue measure subset $D^*$ of any $\mathcal{W}^u$-disk $D$, there exists $n > 0$ arbitrarily large and there exists $1 \leq i \leq s$ such that* $\mathrm{vol}_D(D^* \cap f^{-n}(B_i)) > 0$.

*Proof.* (For notational simplicity, we use $m^u$ to denote $\mathrm{vol}_{f^n(\Gamma)}$ for any $u$-disk $\Gamma$ and any $n > 0$.) Let $D^*_\varepsilon = \cup_{x \in D^*} B_\varepsilon(x, D)$ where $B_\varepsilon(x, D)$ is the ball in $D$ with radius $\varepsilon$ and center in $x$, and $m, m_\varepsilon$ be Cesaro accumulation points of the iterates of

Lebesgue measure on $D^*$ and $D^*_\varepsilon$ respectively, such that there is $\{n_j\}_{j=1}^\infty$ satisfying

$$\lim_{j\to\infty}\frac{1}{n_j m^u(D^*)}\sum_{i=0}^{n_j-1}(f^i)_*m^u \mid D^* = m;$$

$$\lim_{j\to\infty}\frac{1}{n_j m^u(D^*_\varepsilon)}\sum_{i=0}^{n_j-1}(f^i)_*m^u \mid D^*_\varepsilon = m_\varepsilon.$$

Then they are $u$-states, denote by $m = a_1m_1+\cdots+a_m m_s$ the ergodic decomposition of $m$, suppose $a_1 \neq 0$. For $\varepsilon$ sufficiently small, $m^u(D^*_\varepsilon) \approx m^u(D^*)$, and then $m_\varepsilon \approx m$, denote by $m_\varepsilon = a_{1,\varepsilon}m_1 + \cdots + a_{s,\varepsilon}m_m$ the ergodic decomposition of $m_\varepsilon$, one has $a_{1,\varepsilon} \approx a_1$. Denote $D^*_1 = D_1 \cap B_1$ and $D^*_{1,\delta} = \cup_{x\in D^*_1} B^u_\delta(x)$ and

$$B^*_{1,\delta} = \{z : z \in \mathcal{W}^{cs}_{loc}(x) \cap \mathcal{W}^u_{loc}(y) \text{ for } x \in D^*_{1,\delta} \text{ and } y \in B_1\}.$$

Then there is $K_1 > 0$ such that for any $u$-disk $\Gamma$ which crosses $B^*_{1,\delta}$, one has

$$\frac{m^u(\Gamma\cap B_1)}{m^u(\Gamma\cap B^*_{1,\delta})} > K_1,$$

that is because $m^u(\Gamma \cap B_1) > \frac{1}{K}m^u(D_1 \cap B_1) > 0$ and $m^u(\Gamma \cap B^*_{1,\delta})$ is bounded above, where $K$ is the bound for the Jacobian of the center stable foliation in $B_1$. We can choose $\varepsilon$ properly such that $m^u(\partial(D^*_\varepsilon)) = 0$, by Remark 6.7, suppose $m_1(B_1) = b_0 > 0$. Because $B^*_{1,\delta}$ is open,

$$\lim_{j\to\infty}\frac{1}{n_j m^u(D^*_\varepsilon)}\sum_{i=0}^{n_j-1}\big(f^i_*m^u \mid D\big)\big(f^i(D^*_\varepsilon)\cap B^*_{1,\delta}\big) \geq m_\varepsilon(B_{1,\delta}) \gtrsim b_0a_\varepsilon > \frac{a_1b_0}{2}.$$

So there is $n_j$ arbitrarily big such that

$$\big(f^{n_j}_*m^u \mid D\big)\big(f^{n_j}(D^*_\varepsilon)\cap B^*_{1,\delta}\big) \geq \frac{a_1b_0}{4}m^u(D^*_\varepsilon).$$

We claim that there is $b_1 > 0$ such that, for every $\varepsilon > 0$ sufficiently small,

$$\big(f^{n_j}_*m^u \mid D\big)\big(f^{n_j}(D^*_\varepsilon)\cap B_1\big) \geq 2b_1m^u(D^*_\varepsilon).$$

Let us prove the claim. For $\varepsilon_1 < \varepsilon$, denote $D^*_{\varepsilon,\varepsilon_1} = \{x \in D^*_\varepsilon; d_D(x,\partial(D^*_\varepsilon)) > \varepsilon_1\}$. Then, for $\varepsilon_1$ sufficiently small, one has $m^u(D^*_{\varepsilon,\varepsilon_1}) > m^u(D^*_\varepsilon) - a_1b_0/8$. It follows that

$$\big(f^{n_j}_*m^u \mid D\big)\big(B^*_{1,\delta}\big) \geq \frac{a_1b_0}{8}m^u(D^*_\varepsilon)$$

and for any $x \in f^{n_j}(D^*_{\varepsilon,\varepsilon_1}) \cap B^*_{1,\delta}$, there is a $u$-disk $D_x \subset f^{n_j}(D^*_\varepsilon)$ containing $x$ such that $D_x \cap \mathcal{W}^{cs}_{loc}(y) \neq \emptyset$ for any $y \in D^*_{1,\delta}$. Since

$$\frac{m^u(D_x\cap B_1)}{m^u(D_x\cap B^*_{1,\delta})} > K_1,$$

and the distortion property on $\mathcal{W}^u$, there is $K_2 > 0$ such that

$$\frac{m^u(f^{-n_j}(D_x\cap B_1))}{m^u(f^{-n_j}(D_x\cap B^*_{1,\delta}))} > K_2.$$

Then, taking $b_1 = K_2a_1b_0/16$, we get our claim:

$$m^u(D^*_\varepsilon\cap f^{-n_j}(B_1)) > K_2m^u(D^*_{\varepsilon,\varepsilon_1}\cap f^{-n_j}(B^*_{1,\delta})) > 2b_1m^u(D^*_\varepsilon),$$

Since $\lim_{\varepsilon\to 0} m^u(D^*_\varepsilon \setminus D^*) = 0$, this proves that $m^u(D^* \cap f^{-n_j}(B_1)) > b_1m^u(D^*)$. This completes the proof of the proposition. $\square$

*Remark* 6.10. Assuming there exists a unique Gibbs $u$-state, the arguments in the proof of Proposition 6.9 yield a slightly stronger conclusion that will be useful in the sequel: there exists $b_1 > 0$ such that for any positive Lebesgue measure subset $D^*$ of any $\mathcal{W}^u$-disk $D$ and for any $1 \le i \le s$ there exist arbitrarily large values of $n > 0$ such that $\operatorname{vol}_D(D^* \cap f^{-n}(B_i)) \ge b_1 \operatorname{vol}_D(D^*)$.

6.4. **Upper leafwise absolute continuity.** Here we prove Proposition 6.2. Suppose there exists some measurable set $Y$ with $\operatorname{vol}(Y) > 0$ that meets almost every center stable leaf $\mathcal{W}^{cs}(z)$ on a zero $\operatorname{vol}^{cs}$-measure subset. Up to replacing $Y$ by some full measure subset, we may suppose that every $x \in f^n(Y)$ is a Lebesgue density point of $f^n(Y)$ for every $n \ge 0$:

$$\lim_{\rho\to 0} \frac{\operatorname{vol}(B_\rho(x) \cap f^n(Y))}{\operatorname{vol}(B_\rho(x))} = 1. \tag{22}$$

Since $f$ has finitely many ergodic $u$-states and their basins cover a full measure subset of $N$ (see [19]), it is no restriction to suppose that $Y$ is contained in the basin of some ergodic Gibbs $u$-state $m$. Let $\mathcal{B}$ be a $cs$-block over some $u$-disk contained in the support of $m$ (recall Proposition 3.1 and Section 6.2). Since the strong unstable foliation is absolutely continuous (see [20]), we can find a $u$-disk $D$ such that $D^* = D \cap Y$ has positive $\operatorname{vol}_D$-measure. By Proposition 6.9, there exists $n > 0$ such that $\operatorname{vol}_{f^n(D)}(f^n(D^*) \cap \mathcal{B}) > 0$. Take $y \in D^*$ such that $f^n(y) \in \mathcal{B}$ and $f^n(y)$ is a Lebesgue density point for $f^n(D^*) \cap \mathcal{B}$ inside $f^n(D)$. Then, for every small $\rho > 0$,

$$\frac{\operatorname{vol}_{f^n(D)}\big(B^u_\rho(f^n(y)) \cap \mathcal{B}\big)}{\operatorname{vol}_{f^n(D)}\big(B^u_\rho(f^n(y))\big)} \ge \frac{\operatorname{vol}_{f^n(D)}\big(B^u_\rho(f^n(y)) \cap f^n(D^*) \cap \mathcal{B}\big)}{\operatorname{vol}_{f^n(D)}\big(B^u_\rho(f^n(y))\big)} \approx 1,$$

where $B^u_\rho(x)$ denotes the connected component of $B_\rho(x) \cap \mathcal{W}^u(x)$ that contains $x$. Then, since the center stable foliation is uniformly absolutely continuous on the $cs$-block, there exists $c > 0$ such that

$$\frac{\operatorname{vol}\big(B_\rho(f^n(y)) \cap \mathcal{B}\big)}{\operatorname{vol}\big(B_\rho(f^n(y))\big)} \ge c \quad \text{for all small } \rho > 0.$$

Together with (22), this implies that $\operatorname{vol}(f^n(Y) \cap \mathcal{B}) > 0$. On the other hand, the hypothesis implies that $f^n(Y)$ intersects almost every center stable leaf on a zero Lebesgue measure subset. Using, once more, that the center stable leaf is absolutely continuous on the $cs$-block, we get that $\operatorname{vol}(f^n(Y) \cap \mathcal{B}) = 0$. This contradicts the previous conclusion, and that contradiction completes the proof of Proposition 6.2.

6.5. **Non-expansion along the center.** Now we prove Proposition 6.3. Let the ergodic $u$-states $\{m_i\}_{i=1}^m$, periodic points $\{p_i\}_{i=1}^m$, and $\rho, \varepsilon$ given in the definition of non-expansion along the center. By Lemma 6.8, we can choose $cs$-blocks $\{\mathcal{B}_i\}_{i=1}^m$ associated with $m_i$ with size $\rho$.

In order to prove the center stable foliation is absolutely continuous, we just need show that for any two $u$-disks $D_1, D_2$ which are $\varepsilon$ near, the holonomy map induced by $\mathcal{W}^{cs}$ between $D_1$ and $D_2$ maps Lebesgue positive measure subset to a Lebesgue positive measure subset, where two $u$-disks $D_1, D_2$ are $\varepsilon$ near if for any $x \in D_1$, there is $y \in D_2$ belonging to $\mathcal{W}^{cs}_\varepsilon(x)$.

Suppose $D_1^* \subset D_1$ is a positive measure subset, denote by $D_2^* \subset D_2$ the image of $D_1^*$ under cs-holonomy map. Since $f$ is non-expanding along the center, we can assume that for any $x \in D_1^*$, one has $f^n(\mathcal{W}^{cs}_\varepsilon(x)) \subset \mathcal{W}^{cs}_\rho(f^n(x))$ for $n > 0$. Choose $\tilde{B}_i$ a sub-block of $B_i$ with arbitrarily small size, then by Proposition 6.9, there is $n$ and $j$ such that $m^u(f^n(D_1^*) \cap \tilde{B}_j) > 0$. Because $f^n(D_1^*)$ and $f^n(D_2^*)$ are $\rho$ near, and the cs-holonomy map in $B_i$ is absolutely continuous, one has that $m^u(f^n(D_2^*) \cap B_j) > 0$, this implies $m^u(D_2^*) > 0$, so $\mathcal{W}^{cs}$ is absolutely continuous.

6.6. **Center leaves in general position.** We are going to prove Proposition 6.4. Let us start by giving an overview of the argument. We need to compare a set on any $u$-disk with its projection to another $u$-disk under $cs$-holonomy. The idea is to consider appropriate iterates of both $u$-disks intersecting a given $cs$-block, and then take advantage of the uniform structure on the $cs$-block. The problem is that, because $cs$-blocks have gaps along the center direction, one can not immediately ensure that iterates of both disks intersect the same $cs$-block. To this end, we use the twisting property in the assumption of general position to find a pair of $cs$-blocks whose union covers the whole center direction, in the sense that it intersects any large iterate of any $u$-disk. Then, we show that some iterate of any of the disks intersects both $cs$-blocks, which gives the required property.

Now we fill in the details in the proof. Let $f$ and $\ell$ be as in the statement of the proposition. For simplicity, consider the center leaf $\ell$ to be fixed (in other words, $\kappa = 1$) and we also take the attractor $a$ and repeller $r$ of $f \mid \ell$ to be fixed. Extension to the general case is straightforward.

**Lemma 6.11.** *The diffeomorphism $f$ has a unique ergodic $u$-state and its support contains the attractor $a$.*

*Proof.* By Lemma 4.2, the supports of all ergodic Gibbs $u$-states are pairwise disjoint. Thus, it suffices to show that the support of any ergodic $u$-state contains $a$. By Proposition 3.1, the support of $m$ consists of entire unstable leaves. So, it suffices to prove that every strong unstable leaf intersects the stable manifold $\widehat{W}^s(a)$ of the attractor. By hypothesis, every strong unstable leaf intersects $\mathcal{W}^s(\ell)$. Moreover, $\mathcal{W}^s(\ell)$ is the union of $\widehat{W}^s(a)$ with the strong stable leaf through the repeller $r$. If a strong unstable leaf $L$ intersects $\mathcal{W}^s(r)$ then its forward orbit accumulates on $\mathcal{W}^u(r)$ and, in particular, on $h^u(r)$. Since $\ell$ is in general position, $h^u(r) \neq h^s(r)$ and so $h^u(r)$ belongs to the stable manifold of $a$. Hence, in any case, $L$ does intersect $\widehat{W}^s(a)$. This completes the argument. $\square$

Consider the four points $a_s = h^s(a)$, $a_u = h^u(a)$, $r_s = h^s(r)$, $r_u = h^u(r)$ in $\ell'$. For $\rho$, $\varepsilon > 0$ small, and $\zeta \in \ell'$, denote

$$\mathcal{W}^s_\rho(\ell') = \cup_{\xi \in \ell'} \mathcal{W}^s_\rho(\xi) \quad \text{and} \quad V^{cs}_\varepsilon(\zeta) = \cup_{\xi \in B^c_\varepsilon(\zeta)} \mathcal{W}^s_\rho(\xi).$$

Let $\tilde{\mathcal{B}}$ be a $cs$-block over $\mathcal{W}^u_{loc}(a)$ (Lemma 6.8). Then for $n$ large, $f^{-n}(\tilde{\mathcal{B}})$ intersects $\mathcal{W}^u_{loc}(a_s)$ in a set $\tilde{D}^*_1$ with positive Lebesgue measure, and we may choose a $cs$-block $\mathcal{B}_1 \subset f^{-n}(\tilde{\mathcal{B}})$ over a $u$-disk $\tilde{D}_1 \supset \tilde{D}^*_1$ such that

$$\mathcal{W}^u_{2\tau}(\zeta) \cap \mathcal{B}_1 \neq \emptyset \quad \text{for all } \zeta \in \mathcal{W}^s_\rho(\ell') \setminus V^{cs}_\varepsilon(r_s).$$

We think of the union $W^u_{2\tau}(V^{cs}_\varepsilon(r_s))$ of the local unstable manifolds through the local center stable manifold of $r_s$ as the gap of $\mathcal{B}_1$ along the center direction. See Figure 3.

Dually, consider a $cs$-block $\mathcal{B}_2 \subset f^{-n}(\tilde{\mathcal{B}})$ over a $u$-disk $\tilde{D}_2 \subset \mathcal{W}^u(a)$ such that

$$\mathcal{W}^u_{2\tau}(\zeta) \cap \mathcal{B}_2 \neq \emptyset \quad \text{for all } \zeta \in \mathcal{W}^s_\rho(\ell') \setminus V^{cs}_\varepsilon(r_u).$$

Again, the union $W^u_{2\tau}(V^{cs}_\varepsilon(r_u))$ of the local unstable manifolds through the local center stable manifold of $r_u$ is the gap of $\mathcal{B}_2$ along the center direction. Moreover, we may fix $\delta_0 > 0$ such that, for any $\zeta \in \mathcal{W}^s_\rho(\ell')$, either

$$m^u(\mathcal{W}^u_{2\tau}(\zeta) \cap \mathcal{B}_1) > \delta_0 \quad \text{or} \quad m^u(\mathcal{W}^u_{2\tau}(\zeta) \cap \mathcal{B}_2) > \delta_0.$$

This is, in precise terms, what we meant when we announced that the union $\mathcal{B}_1 \cup \mathcal{B}_2$ of the two $cs$-blocks would cover the whole center direction.

Now consider a new $cs$-block $\mathcal{B}$ defined as the union of

$$(\mathcal{W}^u_{2\tau}(\xi) \cap \mathcal{B}_2) \cup (\mathcal{W}^u_{2\tau}(\xi) \cap \mathcal{B}_1)$$

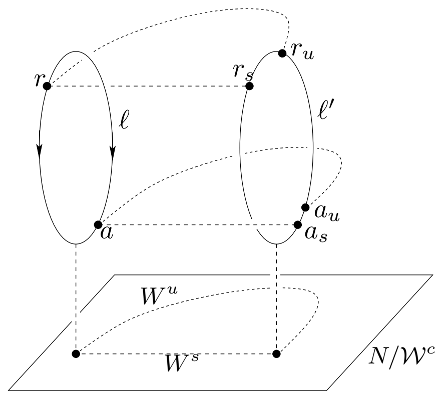


FIGURE 3.

over all $\xi \in W^s_\rho(\ell') \setminus \big(V^{cs}_\varepsilon(r_s) \cup V^{cs}_\varepsilon(r_u)\big)$. In other words, $\mathcal{B}$ is obtained from $\mathcal{B}_1 \cup \mathcal{B}_2$ by removing the two gaps. Thus, $\mathcal{B} = \mathcal{B}^1 \cup \mathcal{B}^2$ with $\mathcal{B}^1 \subset \mathcal{B}_1$ and $\mathcal{B}^2 \subset \mathcal{B}_2$. We are going to show that arbitrarily large iterates of any $u$-disk intersect both connected components of $\mathcal{B}$ on positive measure subsets.

**Lemma 6.12.** *Given any $u$-disk $D$ and any positive* $\operatorname{vol}_D$*-measure subset $D^*$ there exists $\zeta \in D^*$ and $k$ arbitrarily large such that*

$$\operatorname{vol}_{f^k(D)}\big(\mathcal{W}^u_{2\tau}(f^n(\zeta)) \cap f^k(D^*) \cap \mathcal{B}^i\big) > 0 \quad \textit{for both } i = 1, 2.$$

*Proof.* It is no restriction to suppose every point of $D^*$ is a Lebesgue density point. Fix $\varepsilon > 0$ small (the precise choice will be given later). Take any point $x \in D^*$ and let $r > 0$ small enough so that $\operatorname{vol}_D(D^*_r) > (1-\varepsilon)\operatorname{vol}_D(D_r)$, where $D_r$ is the disk of radius $r$ around $x$ and $D^*_r = D_r \cap D^*$. By Proposition 6.9 and Remark 4.3 there exists $b_1 > 0$, independent of $x$ and $r$, such that

$$\operatorname{vol}_D(D^*_r \cap f^{-n_i}(\mathcal{B}^1)) \geq b_1 \operatorname{vol}_D(D^*_r) \geq b_1(1-\varepsilon)\operatorname{vol}_D(D_r)$$

for a sequence $n_i \to \infty$. Let $\rho > 0$ be slightly smaller than $r$, so that

$$\operatorname{vol}_D(D_\rho) > (1-\varepsilon)\operatorname{vol}_D(D_r).$$

Then, for any $n_i$ sufficiently large and any $y \in D_\rho$, we have $f^{-n_i}(W^u_{loc}(f^{n_i}(y))) \subset D_r$. Since the local unstable manifold of $f^{n_i}(y)$ cuts across both $\mathcal{B}^1$ and $\mathcal{B}^2$, this means that we can associate to $y \in D^*_\rho \cap f^{-n_i}(\mathcal{B}_1)$ the following subsets of $D_r$:

$$D^1_i(y) = f^{-n_i}(W^u_{loc}(f^{n_i}(y)) \cap \mathcal{B}^1) \quad \text{and} \quad D^2_i(y) = f^{-n_i}(W^u_{loc}(f^{n_i}(y)) \cap \mathcal{B}^2).$$

By bounded distortion, there exists $\kappa = \kappa(f) > 0$ such that

$$\operatorname{vol}_D(D^2_i(y)) \geq \kappa \operatorname{vol}_D(D^1_i(y)) \quad \text{for every } y \text{ and every } i.$$

We also denote by $D^1_i$ and $D^2_i$ the (disjoint) unions of $D^1_i(y)$ and $D^2_i(y)$, respectively, over all $y \in D^*_\rho \cap f^{-n_i}(\mathcal{B}_1)$. Then, the previous inequality gives

$$\operatorname{vol}_D(D^2_i) \geq \kappa \operatorname{vol}_D(D^1_i) \quad \text{for every } i.$$

By Proposition 6.9 and Remark 6.10, there exists a sequence $(n_i)_i$ of positive integers and there exists $b_1 > 0$ such that

$$\operatorname{vol}_D(D^*_\rho \cap f^{-n_i}(\mathcal{B}^1)) \geq b_1 \operatorname{vol}_{D_\rho}(D^*_\rho) \geq b_1(1-\varepsilon)^2 \operatorname{vol}_D(D_\rho).$$

Consequently,

$$\operatorname{vol}_D(D^1_i) \geq b_1(1-\varepsilon)^2 \operatorname{vol}_D(D_\rho) \geq b_1(1-\varepsilon)^3 \operatorname{vol}_D(D_r).$$

This implies that $\mathrm{vol}_D(D_i^2) \geq b_2 \,\mathrm{vol}_D(D_r)$, where the constant $b_2 > 0$ is independent of $i$ and the choice of $r$. Now, suppose the lemma is false. Then $D_i^2(y) \cap D^*$ is empty, for every $y \in D_\rho^* \cap f^{-n_i}(\mathcal{B}^1)$, that is, $D_i^2 \cap D^* = \emptyset$. It follows that $\mathrm{vol}_D(D_r^*) \leq (1 - b_2)\,\mathrm{vol}_D(D_r)$. This contradicts the choice of $D_r^*$ at the beginning of the proof, as long as we fix $\varepsilon < b_2$. The proof of the lemma is complete. □

*Proof of Proposition 6.4.* Let $h^{cs} : D^1 \to D^2$ be a $cs$-holonomy between $u$-disks $D_1$ and $D_2$. Let $D_1^* \subset D_1$ be a positive $\mathrm{vol}_{D_1}$-measure subset and $D_2^* = h^{cs}(D_1^*)$. We want to prove that $\mathrm{vol}_{D_2}(D_2^*)$ is also positive. By Lemma 6.12, there exists $\zeta \in D_1^*$ and $k \geq 1$ such that

$$\mathrm{vol}_{f^k(D)}\left(\mathcal{W}^u_{2\tau}(f^k(\zeta)) \cap f^k(D_1^*) \cap \mathcal{B}^i\right) > 0 \quad \text{for both } i = 1, 2. \tag{23}$$

Notice that for $k$ big enough, $\mathcal{W}^u_{2\tau}(f^k(\zeta))$ and $\mathcal{W}^u_{2\tau}(f^k(h^{cs}(\zeta)))$ are contained in nearby $cu$-disks. That is because the stable foliation is uniformly contracting. Then $\mathcal{W}^u_{2\tau}(h^{cs}(\zeta)) \cap \mathcal{W}^s_\rho(\ell') \neq \emptyset$. This implies $\mathcal{W}^u_{2\tau}(h^{cs}(\zeta)) \cap \tilde{\mathcal{B}}_1 \neq \emptyset$ or $\mathcal{W}^u_{2\tau}(h^{cs}(\zeta)) \cap \mathcal{B}_2 \neq \emptyset$. Since $\tilde{\mathcal{B}}_1, \mathcal{B}_2$ are $cs$-blocks, whose $cs$-foliations are uniformly absolutely continuous, from (23) one gets that

$$\mathrm{vol}_{f^k(D_2)}\left(\mathcal{W}^u_{2\tau}(f^k(h^{cs}(\zeta))) \cap f^k(D_2^*) \cap \mathcal{B}^i\right) > 0$$

for either $i = 1$ or $i = 2$. This implies that $\mathrm{vol}_{D_2}(D_2^*) > 0$. Thus, the center stable foliation is absolutely continuous, as claimed. □

## 7. Robust absolute continuity

Here we use the results in the previous section to give examples of open sets of diffeomorphisms with absolutely continuous center stable/unstable foliations.

7.1. **Mañé's example.** Mañé [30] constructed a $C^1$ open set of diffeomorphisms $\mathcal{U}$ such that every $f \in \mathcal{U}$ is partially hyperbolic (but not hyperbolic), dynamically coherent, and transitive. From Proposition 6.3 one gets that every $C^k$, $k > 1$ diffeomorphism $f$ in some non-empty $C^1$ open subset $\mathcal{U}'$ has absolutely continuous center stable foliation. To explain this, let us recall some main features in Mañé's construction.

One starts from a convenient linear Anosov map $A : \mathbb{T}^3 \to \mathbb{T}^3$ with eigenvalues $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3$. Let $p$ be a fixed point of $A$ and $\rho > 0$ be small. One deforms $A$ inside the $\rho$-neighborhood of $p$, so as to create some fixed point with stable index 1, while keeping the diffeomorphism unchanged outside $B_\rho(p)$. Mañé [30] shows that this can be done in such a way that the diffeomorphism $f_0 : \mathbb{T}^3 \to \mathbb{T}^3$ thus obtained is partially hyperbolic, with splitting $E^s \oplus E^c \oplus E^s$ where all factors have dimension 1, and every diffeomorphism in some $C^1$ neighborhood $\mathcal{U}$ is dynamically coherent and transitive. The presence of periodic points with both stable indices 1 and 2 ensures that $f_0$ is not Anosov. Bonatti, Viana [19] observed that every $C^k$, $k > 1$ diffeomorphism $f \in \mathcal{U}$ has mostly contracting center direction. Here, as well as in the steps that follow, one may have to reduce the neighborhood $\mathcal{U}$. Then Bonatti, Díaz, Ures [16] showed that the unstable foliation of every $f \in \mathcal{U}$ is minimal. According to [19], this implies that every $C^k$, $k > 1$ diffeomorphism $f \in \mathcal{U}$ admits a unique physical measure, whose basin contains Lebesgue almost every point. The non-expansion condition in Proposition 6.3 can be checked as follows.

A crucial observation is that the center stable bundle $E^c \oplus E^s$ is uniformly contracting outside $B_\rho(p)$, for all diffeomorphisms in a neighborhood, because $f_0 = A$ outside $B_\rho(p)$. Let $q$ be another fixed or periodic point of $A$ and assume $\rho$ was chosen much smaller than the distance from $p$ to the orbit of $q$. Then $q$ remains a periodic point for $f_0$, with stable index 2 and stable manifold of size $\geq 5\rho$.

Let $q_f$ denote the hyperbolic continuation of $q$ for every $f$ in a neighborhood of $f_0$: $q_f$ is a periodic point with stable index 2 and stable manifold of size $\geq 4\rho$. The fact that $E^c \oplus E^s$ is uniformly contracting outside $B_\rho(p)$ also implies that $f^n(\mathcal{W}^{cs}_\rho(x)) \subset \mathcal{W}^{cs}_{2\rho}(f^n(x))$ for all $x \in \mathbb{T}^3$ and $n \geq 0$. This proves that $f$ is non-expanding along the center direction, and so we may apply Proposition 6.3 to conclude that the center stable foliation of every $f$ near $f_0$ is absolutely continuous.

We ignore whether the center unstable foliation and the center foliation are absolutely continuous or not in this case. However, in the next section, a different construction allows us to give examples where all three invariant foliations are robustly absolutely continuous.

7.2. **Robust absolute continuity for all invariant foliations.** Here we prove Theorem E and use it to deduce Theorem B. We begin with an intermediate result:

**Proposition 7.1.** *Let $f_0 : N \to N$ be a $C^k$, $k > 1$ skew-product $f_0(x,\theta) = (g_0(x), h_0(x,\theta))$, where $g_0$ is a transitive Anosov diffeomorphism. Assume that $f_0$ is accessible and has some periodic center leaf in general position. Then there exists a $C^k$ neighborhood $\mathcal{V}$ of $f_0$ such that for every $f \in \mathcal{V}$, the center stable, center unstable, and center foliation are absolutely continuous.*

*Proof.* Every skew-product has absolutely continuous center stable and center unstable foliation and is robustly dynamically coherent (by [28]; the center foliation of a partially hyperbolic skew-product is always plaque expansive). In particular, $f_0$ satisfies all the hypotheses of Theorem D. The presence of a Morse-Smale center leaf prevents $f_0$ from being conjugate to a rotation extension. Thus, the center direction is mostly contracting in a whole neighborhood of $f_0$. The assumption that $g_0$ is transitive also ensures that every strong unstable leaf intersects $\mathcal{W}^s(\ell)$. So, we are in a position to apply Corollary 6.5 to conclude that the center stable foliation is robustly absolutely continuous. The same reasoning applied to the inverse of $f_0$ gives that the center unstable foliation is also robustly absolutely continuous. From the following general fact we get that the center foliation is also robustly absolutely continuous:

**Lemma 7.2** (Pugh, Viana, Wilkinson [39])**.** *Let $\mathcal{F}^1$, $\mathcal{F}^2$, $\mathcal{F}^3$ be foliation in some smooth manifold $N$ such that $\mathcal{F}^1$ and $\mathcal{F}^2$ are transverse at every point and the leaves of $\mathcal{F}^3$ are coincide with the intersections of leaves of $\mathcal{F}^1$ and $\mathcal{F}^2$: for every point $x \in N$, $\mathcal{F}^3(x) = \mathcal{F}^1(x) \cap \mathcal{F}^2(x)$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are absolutely continuous then so is $\mathcal{F}_3$.*

*Proof.* Suppose $D_1, D_2$ are two disks transverse with $\mathcal{F}^3$, and $h^3 : D_1 \to D_2$ is the holonomy map induced by $\mathcal{F}^3$. Then $\mathcal{F}^1$ and $\mathcal{F}^2$ induce two foliations $\hat{\mathcal{F}}^1_i$ and $\hat{\mathcal{F}}^2_i$ on $D_i$, $i = 1, 2$, and these two foliations absolutely continuous in $D_i$. Fix $l_1 \subset D_1$ a leaf of $\hat{\mathcal{F}}^2_1$, and denote by $l_2 = h^3(l_1)$, then $l_2$ is a leaf of $\hat{\mathcal{F}}^2_2$. Since the foliations $\hat{\mathcal{F}}^1_i$, $i = 1, 2$ are absolutely continuous, one has that the disintegration of the Lebesgue measure $\mathrm{vol}_{D_1}$ along the foliation $\hat{\mathcal{F}}^1_1$ is

$$\mathrm{vol}_{D_1} = \varphi_x(y) d\,\mathrm{vol}_{\hat{\mathcal{F}}^1_1(x)}(y) d\,\mathrm{vol}_{l_1}(x), \quad \text{where } \varphi_x(y) > 0$$

and the disintegration of the Lebesgue measure of $D_2$ along the foliation $\hat{\mathcal{F}}^1_2$ is

$$\mathrm{vol}_{D_2} = \phi_x(y) d\,\mathrm{vol}_{\hat{\mathcal{F}}^1_2(x)}(y) d\,\mathrm{vol}_{l_2}(x), \quad \text{where } \phi_x(y) > 0.$$

Now for any set $\Delta_1 \subset D_1$ with $\mathrm{vol}_{D_1}(\Delta_1) > 0$, denote its image for $h^3$ by $\Delta_2$. By the above formulas for the disintegration, there is a positive $\mathrm{vol}_{l_1}$ measure subset $\Gamma_1 \subset l_1$ such that for any $x \in \Gamma_1$, one has

$$\mathrm{vol}_{\hat{F}^1_1(x)}(\Delta_1 \cap \hat{F}^1_1(x)) > 0.$$

Denote $\Gamma_2 = h^3(\Gamma_1) \subset l_2$. By the absolute continuity of $\mathcal{F}^1$ and $\mathcal{F}^2$, $\mathrm{vol}_{l_2}(\Delta_2) > 0$ and $\mathrm{vol}_{\hat{F}_2^1(x)}(\Delta_2 \cap \hat{F}_2^1(x)) > 0$ for any $x \in \Gamma_2$. This implies $\mathrm{vol}_{D_2}(\Delta_2) > 0$, and so the proof is complete. □

This completes the proof of Proposition 7.1. □

To complete the proof of Theorem E it suffices to note that any skew-product $f_0$ with a Morse-Smale center leaf, as in the statement of the theorem, is approximated by skew-products with center leaves in general position: all that is missing is property (b) in the definition of general position, and this can be achieved by a $C^k$ small perturbation inside the space of skew-products. Then Theorem E follows from Proposition 7.1.

Now Theorem B is deduced as follows. For any skew-product $f_0$ as in the statement, Let $\mathcal{U}$ be an open set that accumulates on $f_0$ as given by Theorem E: for any $f \in \mathcal{U}$ the center stable, center unstable, and center foliations are absolutely continuous. Then let $\mathcal{V} \subset \mathcal{U}$ be an open subset such that every $f \in \mathcal{V}$ is accessible ([32]). Then every $f \in \mathcal{V}$ has finitely many physical measures, with basins containing almost every point. The Morse-Smale behavior on the center leaf $\ell$ prevents $f$ from being conjugate to a rotation extension. Thus, we are in case (b) of Theorem D. From the fact that $\ell$ contains a unique periodic attractor we also get that the physical measure is unique (see Theorem 5.6). The same argument applies for $f^{-1}$. This finishes the proof of Theorem B.

7.3. **Volume preserving systems.** Here we prove Theorem C and a pair of related results. Based on these, we also describe a, partially conjectural, scenario for absolute continuity of foliations of conservative and dissipative systems.

Part (1) of Theorem C is a direct consequence of the main result of Baraviera, Bonatti [15]. Part (2) is given by the following result:

**Lemma 7.3.** *For any $f \in \mathcal{W}_0$ with $\lambda^c(f) > 0$, the center foliation and the center stable foliation are not upper leafwise absolutely continuous.*

*Proof.* Fix $c \in (0, \lambda^c(f))$. Then, by the Birkhoff ergodic theorem, the set

$$\Gamma_{c,1} = \{x \in N : \lim \frac{1}{n} \sum_{i=1}^{n} \log \|Df^{-1} \mid E^c(f^i(x))\|^{-1} \geq c\}$$

has positive volume. Then, by Proposition 3.6, there is $n_0 \geq 1$ such that the intersection of any center leaf with $\Gamma_{c,1}$ has at most $n_0$ points. In particular, the intersection has zero volume inside the center leaf. So, the center foliation of $f$ is not upper leafwise absolutely continuous. Next, observe that the set $\Gamma_{c,1}$ consists of entire strong stable leaves. So, the intersection of $\Gamma_{c,1}$ with any center stable leaf consists of no more than $n_0$ strong stable leaves. This implies that the intersection has zero volume inside the center stable leaf. Consequently, the center stable foliation is not upper leafwise absolutely continuous. In particular, we get that the center foliation and the center stable foliation are not absolutely continuous, as claimed. □

Now we prove part (3) of the theorem. Let $p \in M$ be a periodic point of $g_0$ and $a \in M$ be a homoclinic point associated to $p$. For simplicity, we take the periodic point to be fixed. Let us begin by constructing $\mathcal{W}_1$. The first step is to approximate $f_0$ by some diffeomorphism $f_1$ such that $\lambda^c(g) > 0$ for any $g$ in a $C^1$ neighborhood. This can be done by the perturbation method in [15]; the perturbation may be chosen such that $f_1 = f_0$ on a neighborhood of $\{p\} \times S^1$, and we assume that this is the case in what follows. The second step is to find $f_2$ arbitrarily close to $f_1$ such that, denoting by $\ell_p$ and $\ell_a$ the center leaves associated to the continuation of $p$ and $a$,

- every strong unstable leaf of $f_2$ intersects $W^s(\ell_p)$;
- the restriction of $f_2$ to $\ell_p$ is a Morse-Smale diffeomorphism, with a single attractor $\xi$ and a single repeller $\eta$
- and $\mathcal{W}^u(\eta)$ and $W^s(\xi)$ are in general position (we call this non-strong connection).

These properties remain valid in a small neighborhood of $f_2$. As a final step, we use [16, 27, 26] to find a diffeomorphism $f_3$ arbitrarily close to $f_2$ and such that the strong stable and the strong unstable foliations are minimal in a whole $C^1$ neighborhood of $f_3$. We take $\mathcal{W}_1$ to be such a neighborhood. By [19], for every diffeomorphism $f \in \mathcal{W}$ the inverse $f^{-1}$ has mostly center direction. Then, by [9], the same is true in a whole $C^k$ neighborhood $\mathcal{W}_f$ in the space of all (possibly dissipative). diffeomorphisms. Hence, we are in a position to apply Corollary 6.5 to conclude that the center unstable foliation is absolutely continuous for every diffeomorphism in $\mathcal{W}_f$.

This finishes the proof of Theorem C. The next proposition is a variation of results in [14] where center foliations are replaced by center stable or center unstable foliations.

**Proposition 7.4.** *Let $f_0$ be as in Theorem A, where $M$ is a surface, and let $f$ be any $C^1$ nearby accessible, volume preserving diffeomorphism with $\lambda^c(f) = 0$. If either the center stable foliation or the center unstable foliation is absolutely then $f$ is smoothly conjugate to a rotation extension and the center foliation is a smooth foliation.*

*Proof.* Suppose $\mathcal{W}^{cs}$ is absolutely continuous. Then we may apply Theorem D. In this case Lebesgue measure is a Gibbs $u$-state with zero center exponent, and so we are in the elliptic case (a) of the theorem. In particular, the center foliation is leafwise absolutely continuous. Then we can apply [14] to conclude that the center foliation is smooth and $f$ is smoothly conjugate to a rigid model. In present case, where the center fiber bundle is trivial, we get that $f$ is topologically conjugate to a rotation extension (cf. Remark 4.3). □

*Remark* 7.5. Suppose $f$ is partially hyperbolic, dynamically coherent, volume preserving, and all the center exponents are negative at almost every point. Then the center stable foliation of $f$ is upper leafwise absolutely continuous. This is a fairly direct consequence of Pesin theory. Indeed, if all the Lyapunov exponents are negative then the Pesin local stable manifold of almost every point is a neighborhood of the point inside its center stable leaf. Then the absolute continuity of Pesin laminations [35] implies that the center stable foliation is upper leafwise absolutely continuous.

We close with a couple of conjectures on the issue of absolute continuity. The first one deals with dissipative systems.

*Conjecture* 7.6. Let $k > 1$ and $\mathcal{C}_k$ be the space of partially hyperbolic, dynamically coherent $C^k$ diffeomorphisms with mostly contracting center direction. Then, for an open and dense subset,

- if there is a unique physical measure then the center stable foliation is absolutely continuous;
- if there is more than one physical measure then the center stable foliation is not upper leafwise absolutely continuous.

Examples of the second situation will appear in a forthcoming paper [49].

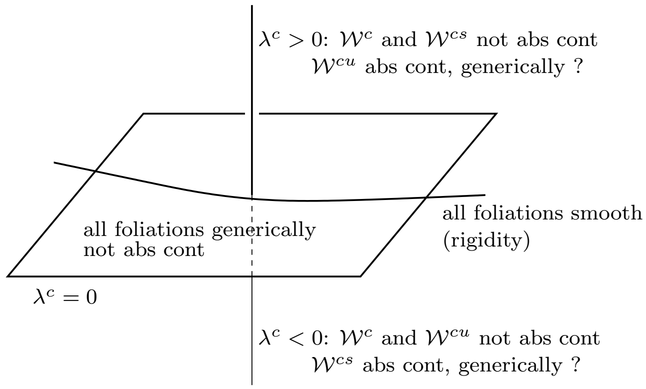


FIGURE 4.

*Conjecture* 7.7. Let $k > 1$ and $\mathcal{V}_k$ be the space of partially hyperbolic, dynamically coherent, volume preserving $C^k$ diffeomorphisms whose center Lyapunov exponents are negative at almost every point. Then, for an open and dense subset, the center stable foliation is absolutely continuous.

Figure 4 outlines a scenario for these issues in a relevant special case, namely near the map $f_0 = g_0 \times \mathrm{id}$ as in Theorem A. Accessibility is assumed throughout (but is not needed for the negative results in $\lambda^c \neq 0$). Generically means for open and dense in $C^k$ topology, $k \geq 1$. Upper leafwise absolute continuity of the center unstable is known for $\lambda^c > 0$, as we have seen, and we have also found an open subset with (full) absolute continuity of the center unstable.

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