Real Option Valuation with Uncertain Costs

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Abstract

In this work we are concerned with valuing the option to invest in a project when the project value and the investment cost are both mean-reverting. Previous works on stochastic project and investment cost concentrate on geometric Brownian motions (GBMs) for driving the factors. However, when the project involved is linked to commodities, mean-reverting assumptions are more meaningful. Here, we introduce a model and prove that the optimal exercise strategy is not a function of the ratio of the project value to the investment V/I – contrary to the GBM case. We also demonstrate that the limiting trigger curve as maturity approaches traces out a non-linear curve in (V, I) space and derive its explicit form. Finally, we numerically investigate the finite-horizon problem using the Fourier space time-stepping algorithm of Jaimungal and Surkov (2009). Numerically, the optimal exercise policies are found to be approximately linear in V/I; however, contrary to the GBM case they are not described by a curve of the form $V^*/I^* = c(t)$. The option price behavior as well as the trigger curve behavior nicely generalize earlier one-factor model results.

Key-words: Real Options; Mean-Reverting; Stochastic Investment; Investment under Uncertainty

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1 Introduction

Quantitative methods to analyze the option to invest in a project enjoy a long and distinguished history that dates back to McDonald and Siegel (1985) and Brennan and Schwartz (1985). These authors pioneered the valuation of projects subject to stochastic cash-flows and developed the real-options methodology in the context of shutting down a plant. Moreover, the value of the option to invest and start producing at a fixed future time T was shown to be given by a European style call option, i.e.,

$$\text{value} = e^{-rT}\mathbb{E}\left[(V_T - I_T)_+\right] \ . \tag{1.1}$$

Here, the expected value is taken under an appropriate risk-adjusted measure, while V_T and I_T represent the project's value and the amount to be invested, respectively, at time T.

Shortly after, McDonald and Siegel (1986) investigated the problem of timing the launch of an investment (see Dixit and Pindyck (1994) for a comprehensive review). If the project can be started at anytime, then (1.1) is modified to its American counterpart. In this case, the maturity date T is replaced by a stopping time τ and the agent chooses τ to maximize the option's value, i.e.,

$$\text{value} = \sup_{\tau \in \mathscr{T}} e^{-r\tau}\mathbb{E}\left[(V_\tau - I_\tau)_+\right] \ , \tag{1.2}$$

where $\mathscr{T}$ is a family of stopping times in $[0, T]$. As such, the problem becomes a free boundary problem for which the optimal investment strategy is computed simultaneously with the option's value.

Traditionally, as in the early work of Tourinho (1979), the project value is assumed to be a geometric Brownian motion (GBM) and the investment cost is constant or deterministic.

Stochastic investment costs have also been studied in the literature. In particular, the perpetual option with project value and investment cost both being GBMs is treated in McDonald and Siegel (1986) (see also Berk, Green, and Naik (1999)). More recently, Elliott, Miao, and Yu (2007) have investigated regime switching investment costs for the option in perpetuity. Not surprisingly, valuing the option with both uncertain investment and project value is similar to valuing exchange options, as in Margrabe (1978), and in uncertain payoffs, as in Fischer (1978).

Assuming GBM may be appropriate in certain circumstances; however, as noticed early on by McDonald and Siegel (1986), in situations where the cash-flows are directly linked to commodities, geometric mean-reversion (GMR) may be more appropriate[1]. Indeed, in an equilibrium framework, when commodity prices are somewhat high, high cost producers are expected to come into the market, thus inducing a downward pressure in prices. Conversely, when prices are somewhat low, high cost producers leave the market, thus inducing an upward pressure in prices. This implies a mean-reversion of commodities prices (cf. Schwartz (1997)) and references therein). Another distinctive feature is that a holder of a commodity may be inclined to hold it unless they are paid an additional premium – a convenience yield – which plays the role of a dividend. When inventories are high, convenience yields should be low and conversely when inventories are low, convenience yields should be high. Such effects can be modeled either through stochastic convenience yield models, as in Gibson and Schwartz (1990) and Miltersen and Schwartz (1998) (who also account for stochastic interest rates), or through two-factor long-term/short-term models, as in Schwartz and Smith (2000). More information on the literature of Real Options in the context of commodities and natural resources, with applications ranging from the valuation of dual-fuel industrial steam boilers to valuing operating flexibility in multinational networks, can be found in Brennan and Trigeorgis (2000) and in Schwartz and Trigeorgis

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[1]The GMR process is also known as the Stochastic Logistic or the Stochastic Verhulst model (cf. Kloeden and Platen (1992)).

(2001).

One case in point where mean-reversion plays a clear role is the option to the invest in an oil field. As most commodities, oil prices tend to mean-revert. Consequently, the value of investing in an oil field is also mean-reverting implying that GBM is a poor modeling choice for such projects. Needless to say, several authors have noticed this and mean-reverting processes in the context of real options have already been considered. Metcalf and Hasset (1995) study the optimal time to invest in a perpetual cash-flow where the cash-flow is driven by either a GBM or GMR. The authors found that, at the aggregate level, there was little difference between GBM and GMR due to two competing effects (i) mean-reversion pulls the trigger levels lower (ii) mean-reversion also prevents extreme events from occurring in realized paths. However, Sarkar (2003) argues that those earlier results are valid only for agents who assign zero market price of risk to the risk factor. Sarkar goes on to demonstrate, in a stochastic cost model rather than a stochastic value model, that when the agent includes a market price of risk there can be significant differences in valuation. As well, Schwartz (1997) states: "... it is very important to consider mean reversion in prices in evaluating projects. The discounted cash flow (CDF) criterion induces investment too early (i.e. when prices are too low), but the real options approach induces investment too late (i.e. when prices are too high) *when it neglects mean reversion in prices*" (pg 972) – emphasis added.

As another important example, consider the classical case of the Antamina mine studied in Moel and Tufano (2000). In 1996, the Peruvian government was privatizing several of its state-owned assets and the Antamina mine was one of the first. This mine contained several metallic ore deposits and the government was selling the right to explore the mine for two years, after which the winning bidder can decide whether to develop it. Such embedded optionality can be regarded as a European option to invest in the mine. Moel and Tufano (2000) utilized stochastic convenience models for the price of the two main metals (copper

and zinc) whose ore are found on the property. To correctly capture the price behavior of these metals, they argue that mean-reversion effects must be taken into account and went on to do so in their model for convenience yield. After calibrating the model to historical data they resorted to a Monte Carlo simulation to value the embedded option. The details of their model itself are not important here, rather the fact that mean-reversion played a critical role is and this ties in well with the earlier discussions.

Considering that several authors model price movements, and hence project values, as mean-reverting while others model investment costs as mean-reverting, it is natural to consider the case when both project value and investment costs are mean-reverting. To date this combination has not been considered in the literature and it is precisely this research gap which we investigate here.

Our main results concern the behavior of the trigger curves for this two factor class of models. Firstly, unlike models in which investment costs I_t and project value V_t are GBMs, we demonstrate that a Bermudan option to invest either immediately or at one fixed future maturity date, traces out a non-linear trigger curve in the (I, V) plane. The reason is that having mean-reversion in both investment cost and project value introduces a "dividend" like effect which becomes stronger as the project value and investment cost move away from their equilibrium point. A linear trigger would result only if the "dividend" was independent of this distance – as it is in the purely GBM case. Secondly, we derive the PDE satisfied by the value of the perpetual option to invest in the project. Unfortunately, due to technical reasons discussed in Section 4, it is not possible to reduce this PDE to a system of ODEs through a separation of variables. Nonetheless, we are able to fully characterize the limiting trigger curve for a finite maturity American option to invest in the project as maturity approaches. Our result generalizes the limiting trigger point $S^* = \max(1, r/\delta) K$ for an American call option on a dividend paying asset with GBM drivers. To the best our knowledge, the analysis of the limiting trigger point for mean-reverting processes – even

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for a single factor model – has not been investigated elsewhere. Finally, we introduce a numerical scheme for valuing the option to invest in the project with a finite maturity and numerically explore some of its features. All of our results are consistent with previous findings, however they generalize those results to the two-factor case.

The remainder of this article is organized as follows. In Section 2, we provide a modeling framework which naturally extends the mean-reverting project value to account for mean-reverting investment costs. The class of risk-neutral measures under which valuation is carried out is also discussed. Based on this modeling assumption, Section 3 investigates the European option to invest in the project and we provide an explicit closed form formula for the value of the real option. Next, in Section 4, we investigate three forms of the early exercise option: (i) the perpetual American option; (ii) the finite time horizon American option; and (iii) the Bermudan option with finite time to maturity. Finally some concluding remarks are made in Section 5.

2   A Mean-Reverting Value and Investment Model

The difficulty with allowing both project value V_t and investment cost I_t to be stochastic lies in the fact that the problem becomes two-dimensional and the optimal policy will, in general, depend on both V_t and I_t. However, since the payoff $(V_T - I_T)_+$ of the option to invest is homogeneous in (V_T, I_T), when the project value and the investment cost are GBMs the optimal policy can be shown to depend only on the ratio V_t/I_t and the option's value inherits the payoff's homogeneity[2]. This was observed quite early in McDonald and Siegel (1986) and it seems that this *trigger ratio* policy has become a paradigm in Real Options pricing. See Dixit and Pindyck (1994) for a review of these triggers for perpetual

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[2]Some early observations on homogeneity were made by Merton (1973) (see Theorem 9, page 149) in the context of Warrants. Specifically, he noticed that the value of a warrant is homogeneous in the share price and strike price if the share price distribution is independent of the share's level. Here, however, we are dealing with two sources of uncertainty.

options with both GBM and mean-reverting project values but constant investment cost. We will see that this very appealing property is not inherited when both processes are mean-reverting.

2.1 The Stochastic Model

We now describe our joint mean-reverting project value and investment cost model.

Firstly, as in earlier works, the project value V_t is assumed to be the exponential of an Ornstein-Uhlenbeck process X_t. Specifically, we write

$$V_t = \exp\{\widehat{\theta} + X_t\}, \tag{2.1a}$$

$$dX_t = -\widehat{\alpha} X_t \, dt + \sigma_X \, d\widehat{W}_t^X. \tag{2.1b}$$

Here, $\widehat{W}_t^X$ is a standard Brownian motion under the real-world measure[3] $\mathbb{P}$. As such, the project value is a mean-reverting diffusion process which reverts to the equilibrium value $\exp\{\widehat{\theta}\}$, $\widehat{\alpha}$ controls the rate at which the project value mean-reverts, and σ_X controls the size of the fluctuations. Similar models have been proposed in the literature for commodity prices as early as Gibson and Schwartz (1990), Cortazar and Schwartz (1994) and Schwartz (1997); jumps were added to these models in Cartea and Figueroa (2005) and more general multi-factor cross-commodity models were introduced in Jaimungal and Surkov (2009). As well, Metcalf and Hasset (1995) consider such models in the context of real options with constant investment costs. Sarkar (2003) investigates stochastic investment cost with mean-reversion while the project value is constant. In this work, we add the effect of stochastic investment cost on top of stochastic project value. It is also possible to incorporate jumps into the project value and the investment cost; however, we opt to

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[3]As usual we work on a complete filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{(\mathscr{F}_t)_{0 \leq t \leq T}\}$, $\mathscr{F}_t = \sigma((\widehat{W}_s^X, \widehat{W}_s^Y)_{0 \leq s \leq t})$ is the natural filtration generated by the driving Brownian motions and $\mathbb{P}$ is the statistical (real-world) probability measure.

leave out such generalizations to maintain simplicity in the exposition.

Next, like in Sarkar (2003), the investment cost I_t is assumed to be the exponential of another, correlated, Ornstein-Uhlenbeck processes Y_t. Specifically, we write

$$I_t = \exp\{\widehat{\phi} + Y_t\}, \tag{2.2a}$$

$$dY_t = -\widehat{\beta} Y_t \, dt + \sigma_Y \, d\widehat{W}_t^Y. \tag{2.2b}$$

Here, $\widehat{W}_t^Y$ is a standard Brownian motion under the real-world measure $\mathbb{P}$ correlated to W_t^X with correlation ρ. The investment cost I_t has an equilibrium level of $\exp\{\widehat{\phi}\}$, $\widehat{\beta}$ controls the rate at which the investment cost mean-reverts, σ_Y controls the size of the fluctuations and ρ controls the strength of the dependence between the project value and the investment costs.

To illustrate the flexibility of the model, in Figure 1 two sample paths for the value and investment are presented. The sample paths are both generated from the same uncorrelated Brownian sample paths to highlight the effect of the correlation. The volatility of the project was assumed to be 80% while the investment cost was assumed to have a volatility of 50%. Panel (a) contains no correlation whereas panel (b) illustrates the behavior when the investment and project value are perfectly correlated. Notice that, as expected, the positive correlation case has a much lower variability in the ratio of value to investment. Naturally, a lower variability will translate into lower option values.

2.2   The Risk-Neutral Measure

In this subsection, we provide the collection of equivalent risk-neutral measures under which the valuation is conducted. As usual, we assume the existence of two tradable assets which at least partially span the project value and investment costs. Rather than going through standard dynamic hedging arguments we simply state our assumptions on the market prices

of risk for the two sources of uncertainty in our model. This assumption imposes a specific structure in the risk-neutral dynamics of the project value and investment cost and we adopt this structure throughout the remainder of the article.

Assumption 2.1 The Market Prices of Risk. *The market prices of risk λ_t^X and λ_t^Y associated with the Brownian factors $\widehat{W}_t^X$ and $\widehat{W}_t^Y$ are assumed to be affine in X_t and Y_t. In particular,*

$$\lambda_t^X = \frac{a_x + b_x\, X_t}{\sigma_X}, \qquad and \qquad \lambda_t^X = \frac{a_y + b_y\, Y_t}{\sigma_Y}, \tag{2.3}$$

where $a_{x,y}$ and $b_{x,y}$ are constants.

Under Assumption 2.1, Girsanov's Theorem implies that the drift adjusted processes

$$W_t^X = \int_0^t \lambda_u^X\, du + \widehat{W}_t^X \qquad and \qquad W_t^Y = \int_0^t \lambda_u^Y\, du + \widehat{W}_t^Y, \tag{2.4}$$

are standard Brownian motions, with correlation ρ, under an equivalent risk-neutral measure $\mathbb{Q}$. This measure will be used for all future valuations. Furthermore, the risk-neutral dynamics of the project value and investment cost become

$$V_t = \exp\{\theta + X_t\}, \tag{2.5a}$$

$$dX_t = -\alpha X_t\, dt + \sigma_X\, dW_t^X, \tag{2.5b}$$

$$I_t = \exp\{\phi + Y_t\}, \tag{2.5c}$$

$$dY_t = -\beta Y_t\, dt + \sigma_Y\, dW_t^Y. \tag{2.5d}$$

Here, several new constants appear: $\alpha = \widehat{\alpha} - b_x$, $\theta = \widehat{\theta} + a_x/\alpha$, $\beta = \widehat{\beta} - b_y$, $\phi = \widehat{\phi} + a_y/\beta$. Notice that under the assumed class of risk-neutral measures, the model is of the same form as the real-world evolution. However, the level at which the processes mean-revert

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to, as well as the rate at which they mean-revert, can differ from their real-world values.

The minimal martingale risk-neutral measure – which is the measure associated with the variance minimizing hedge (cf. Föllmer and Schweizer (1991)) – would have $b_{x,y} = 0$, i.e. constant market prices of risk. Under this measure, the rate of mean-reversion is not altered when moving to the risk-neutral measure. Nonetheless, if $b_{x,y} = 0$ but $a_{x,y} \neq 0$ then the processes will mean-revert to a level distinct from their real-world evolution. We leave the class of measures general and instead allow the agent to decide – possibly through a calibration exercise – on the particular choice of the various constants, or equivalently on α, θ, β and ϕ.

In the remainder of this article, we will refer to the risk-neutral model (2.5) simply as the model. Further, all subsequent Brownian motions are risk-neutral Brownian motions, and expectations $\mathbb{E}[\cdot]$ represent expectations w.r.t. to the risk-neutral measure.

3   The European Option to Invest

We now investigate the option to invest under the modeling assumption (2.5). The European option to invest has value equal to the discounted expectation in (1.1). One of our main results is provided in the following Theorem which is proved in Appendix A.

Theorem 3.1 European Option Price. *The value of the European option to invest in the project at a fixed date T under the modeling assumptions (2.5) is*

$$value = \mathbb{E}[V_T]\Phi\left(d_+\right) - \mathbb{E}[I_T]\Phi\left(d_-\right) \ . \tag{3.1}$$

Here, $\Phi(\cdot)$ denotes the normal cdf, the constants

$$d_\pm = \frac{1}{\widetilde{\sigma}} \ln\left(\frac{\mathbb{E}[V_T]}{\mathbb{E}[I_T]}\right) \pm \tfrac{1}{2}\widetilde{\sigma} \ , \tag{3.2}$$

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and $\widetilde{\sigma}^2$ denotes the effective total variance

$$\widetilde{\sigma}^2 = \sigma_X^2 \frac{1 - e^{-2\alpha T}}{2\alpha} - 2\rho\sigma_X\sigma_Y \frac{1 - e^{-(\alpha+\beta)T}}{\alpha + \beta} + \sigma_Y^2 \frac{1 - e^{-2\beta T}}{2\beta} \; , \tag{3.3}$$

while the expectations are given by

$$\mathbb{E}[V_T] = \exp\left\{ \theta + e^{-\alpha T} X_0 + \frac{\sigma_X^2}{4\alpha}\left(1 - e^{-2\alpha T}\right) \right\} , \quad and \tag{3.4}$$

$$\mathbb{E}[I_T] = \exp\left\{ \phi + e^{-\beta T} Y_0 + \frac{\sigma_Y^2}{4\beta}\left(1 - e^{-2\beta T}\right) \right\} . \tag{3.5}$$

In Figure 2, the value of the European option to invest in the project is shown together with the payoff function. If the optionality was instead a Bermudan option to invest either immediately or at maturity, the option value would be the maximum of the payoff surface (immediate investment) and the European option value (hold to maturity). In the purely GBM case (with no dividends), for the Bermudan option, it would not be optimal to invest immediately. When dividends are included it may be optimal to invest immediately if the project value is large enough, or the investment cost is small enough. In our mean-reverting model, even if the market price of risk is zero (so that there is no analog of a dividend) it is optimal to invest in the project when the project value is large or the investment cost small enough. This feature can be seen graphically in Figure 2 – all points at which the value function intersects the immediate exercise value are optimal exercise points for such a Bermudan option. At a fixed investment cost, once the project value is large enough, the agent should optimally invest in the project.

Another interesting observation is the fact that the trigger curve itself is clearly non-linear. This indicates that the optimal strategy for investing is not provided by monitoring the ratio V/I, and in particular is not provided by a line $V^*/I^* = const.$ as it is in the GBM case. Instead, both processes must be monitored simultaneously. It is the mean-

reverting nature of both processes which causes such distinct results. We will see that similar features flow through to the multiple exercise case.

To further assist in understanding the valuation, in Figure 3 we show how the price varies with the model parameters. In each panel, one model parameter is varied while all other parameters remain constant. The price behavior is easily explained. As the mean-reversion rate α of the project value increases, the option losses value because there is less variability in the project's value. However, as the project's volatility σ_X increases, the variability induces higher option value as usual. Contrastingly, when the mean-reversion rate β of the investment cost rises, the option value increases because this induces less variability in costs and more certainty in the profits once invested and therefore more value in the option. Of course, the opposite effect is seen when the volatility σ_Y of the investment costs increases. As correlation ρ increases the variability of the profits upon investment decreases because project value and investment costs move in tandem with higher correlation. Consequently, the option value decreases as correlation increases.

In addition to the price sensitivities, in Figure 4 we show how the trigger curves evolve as the parameters move. We now explain the observed behavior in turn.

- As the mean-reversion rate α of the project value increases, the trigger curve moves closer to the equilibrium point. This happens because as α increases the effective volatility of the project value decreases; consequently, an agent will invest earlier to gain the potential of the immediate payoff and not wait to maturity.

- As the mean-reversion rate β of the investment cost increases, the trigger curve becomes more and more similar to the trigger curve of a constant investment cost model. In particular when the investment cost is below the equilibrium point, the trigger curve moves upwards as the mean-reversion rate increases, while the trigger curve moves downward when the investment cost is above the trigger level. This

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feature can be explained by observing that the effective volatility of the investment cost increases as β decreases. Therefore an agent will in general wait longer before investing resulting in the trigger curve tilting away from the equilibrium point.

– As the volatility σ_X of the project value increases, the trigger curves move up. This is to be expected because an agent will be willing to take the risk that the project moves deeper in the money when volatility is high and they will delay investing in the project.

– As the volatility σ_Y of the investment cost increases, the trigger curves move down. This is also to be expected because an agent is not willing to take the risk that investment costs go up. They will rather invest immediately at moderate profit levels and forgo the potential of higher future project value in fear of large investment costs.

– As the correlation ρ increases, the trigger curves move downward. An agent is not willing to take the risk of waiting because as correlation increases, large project values are also accompanied by large investment costs and therefore lower profitability.

These results are consistent with and similar in spirit to the results for mean-reverting project value alone with fixed investment costs as described in Chapter 5 §5 of Dixit and Pindyck (1994). Further, the results are also consistent with the behavior when project value is fixed but investment costs are mean-reverting as found in Sarkar (2003).

4   The Early Investment Option

In the previous section, we investigated the European option to invest and the Bermudan option to invest immediately or at maturity. However, agents generally have the ability to invest early in a project, consequently, the value of the option to invest in a project is truly of American type as in (1.2). As is well known, the American option price $f(t, X_t, Y_t)$ is

not known in closed form when the time to maturity is finite. Nonetheless, for infinite time horizon problems – i.e. the perpetual option – with GBM drivers for project value and investment cost, closed form solutions do exist (e.g., Chapter 6 §5 in Dixit and Pindyck (1994)). Furthermore, the infinite time horizon problem for project value with GMR driver and fixed investment cost is also known in closed form (e.g., in Metcalf and Hasset (1995) as well as Chapter 5 §5 in Dixit and Pindyck (1994)).

In the next subsection, we briefly discuss the partial differential equation (PDE) approach for the price, together with appropriate boundary conditions both for the finite-time and perpetual options, and provide a brief discussion about their solutions. In Subsection 4.2.1, we investigate the limiting exercise region for the finite maturity problem and derive an explicit formulae for the trigger curve. In Subsection 4.2.2, we utilize a version of the Fourier space time-stepping algorithm of Jaimungal and Surkov (2009) to numerically solved for the time-dependent trigger surface for the finite maturity problem.

4.1 The PDE approach

A much favored approach to American option valuation is to observe that the value can be recast as a free boundary value problem, and then to solve the resulting PDE either by numerical means for the finite-time horizon, or by writing the solution in terms of special functions for the perpetual case.

Through standard arguments, the value[4] $f(t, x, y)$ of the option to invest under the model (2.5) must solve the PDE

$$\partial_t f + \alpha(\theta - x)\partial_x f + \beta(\phi - y)\partial_y f + \tfrac{1}{2}\left(\sigma_X^2 \partial_{xx} f + 2\rho\sigma_X\sigma_Y \partial_{xy} f + \sigma_Y^2 \partial_{yy} f\right) = rf,$$

(4.1)

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[4]Note that we have written the value in terms of the log-state variables $x = \ln V - \theta$ and $y = \ln I - \phi$ rather than V and I directly as the resulting PDE is simpler in these variables.

together with the value matching and smoothing pasting conditions

$$f(t, x^*(t, y^*), y^*) = e^{\theta + x^*(t, y^*)} - e^{\phi + y^*}, \tag{4.2a}$$

$$\partial_x f(t, x^*(t, y^*), y^*) = e^{\theta + x^*(t, y^*)}, \tag{4.2b}$$

$$\partial_y f(t, x^*(t, y^*), y^*) = -e^{\phi + y^*}, \tag{4.2c}$$

and the limiting conditions

$$\lim_{x \to -\infty} f(t, x, y) = 0, \tag{4.3a}$$

$$\lim_{y \to +\infty} f(t, x, y) = 0, \tag{4.3b}$$

$$\lim_{y \to -\infty} f(t, x, y) = f_0(t, x). \tag{4.3c}$$

Here, $(x^*(t, y^*)$, $y^*)$ represents the trigger curve in the (x, y) plane as a function of time. Condition (4.2a) represents the value matching condition, i.e., the condition that the option value matches the exercise value along the trigger curve. Condition (4.2b) and (4.2c) represent the smooth pasting conditions along the boundary in the direction of the project value and the investment cost, respectively. These last two conditions appear a little different than usual because we are working with the log-variables x and y rather than V and I directly. The limiting conditions (4.3a), (4.3b), and (4.3c) represent the limiting cases of zero project value, infinite and zero investment cost respectively. Also, $f_0(t, x)$ solves the early fixed-investment problem with zero investment.

If the option to invest is perpetual, its value $f(t, x, y)$ is independent of time due to the stationarity of the driving processes. Thus, it solves the equation

$$\alpha(\theta - x)\partial_x f + \beta(\phi - y)\partial_y f + \tfrac{1}{2}\left(\sigma_X^2 \partial_{xx} f + 2\rho\sigma_X\sigma_Y \partial_{xy} f + \sigma_Y^2 \partial_{yy} f\right) = rf \tag{4.4}$$

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together with the contact and smooth pasting conditions (4.2) as well as the limiting conditions (4.3), with the right hand side of (4.3c) replaced by the corresponding perpetual value.

Unlike the case of a fixed investment cost, (4.4) is still a PDE. A tempting approach to solve this problem is to try a separations of variables $f(x, y) = g(x)h(y)$ in an attempt to reduce the problem to an ODE. However, while it is possible to show that such solutions can be written in terms of Kummer functions, they cannot satisfy the conditions (4.3). Due to the subtle issue of analyticity of the Kummer functions at the equilibrium points, together with the limiting conditions, the construction of the analytical solution for the perpetual option price (4.4) is quite difficult and is outside the scope of the current article. We will however report on its solution elsewhere.

4.2 The Finite Time Horizon Problem

4.2.1 The Limiting Trigger Curve

In this section, we analyze the limiting trigger curve as maturity approaches. Indeed, for an American call option on a dividend paying stock, it is well known that the limiting exercise level does not always approach the strike. In fact, if the underlier follows a GBM, then the limiting level is $S^* = \max(1, (r/\delta)) K$ – see e.g. Proposition 33, page 59 in Detemple (2005). Consequently, only when the dividend is large enough would the limiting trigger level equal K. Similarly, here, we show that the limiting trigger curve is not given by the line $V^* = I^*$. The intuition for the result is that the mean-reversion is playing a similar, but distinct role, to a dividend yield.

Proposition 4.1 Limiting Trigger Curves. *The limiting boundary as $t \uparrow T$ of the trigger region traces out a curve in the (V, I) plane satisfying the constraint*

$$\frac{V^*}{I^*} = \max\left(1 \, , \; \frac{\beta\left(\ln I^* - \phi\right) + \left(r - \frac{1}{2}\sigma_Y^2\right)}{\alpha\left(\ln V^* - \theta\right) + \left(r - \frac{1}{2}\sigma_X^2\right)} \right) . \tag{4.5}$$

Proof. See Appendix B. ∎

When $V^* > I^*$ it is possible to characterize V^* as a function of I^* using the Lambert-W function[5] $L(z)$:

$$V^* = \exp\left\{ \widetilde{\theta} + L\left(\left(\beta(\ln I^* - \phi) + r - \tfrac{1}{2}\sigma_Y^2\right) \frac{e^{-\widetilde{\theta}}}{\alpha} \right) \right\}, \tag{4.6}$$

where the constant

$$\widetilde{\theta} = \theta - \frac{r - \frac{1}{2}\sigma_X^2}{\alpha} .$$

Interestingly, the limiting trigger curve (4.6) does not always lie above the maturity trigger curve of $V^* = I^*$. To illustrate this effect, and to gain some intuition on how the trigger curves behave, in Figure 5 we show the limiting trigger curve for several levels of project value mean-reversion α and for several levels of investment mean-reversion β. Notice that certain levels of mean-reversion lead to limiting trigger curves which intersect the maturity trigger curve. Recall that for the American call option, the limiting trigger is $\max(1, r/\delta)K$ and as the dividend yield increases, the limiting trigger curve eventually hits the maturity trigger. Here, the distance from the equilibrium point is acting as an effective dividend and the further one moves away from it the stronger the effective dividend. Moreover, increasing and decreasing the mean-reversion rates alters the effective (invariant) volatilities. As the volatility of the investment decreases and/or the volatility

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[5]The Lambert-W function $L(z)$ solves $L(z)e^{L(z)} = z$.

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of the project decreases, the limiting trigger curve intersects the maturity trigger curve at lower levels. This is intuitive because in both scenarios the amount of optionality embedded in the real option is being reduced and therefore an agent would act to exercise if the investment amount is large and if the project value exceeds the investment amount.

One final observation can be made by comparing Figures 5 and 4: the limiting trigger curves for the American option and the trigger curves for the Bermudan option to invest immediately or at maturity are very similar. We therefore expect that the early trigger curves (and not just the limiting trigger curve) for the American option will inherit a similar structure.

4.2.2 A Recursive Solution

For finite time horizons, no analytical solutions are known, not even for the one dimensional case with GBM project value, therefore we do not attempt to find analytic solutions. Instead we will develop an efficient numerical scheme and investigate the consequences of our model on the trigger curves. Rather than focusing on tree approximations or finite difference schemes of the PDE (4.1) or invoking least squares Monte Carlo (as developed in Carriere (1996) and further developed in Longstaff and Schwartz (2001)), we make use of the mean-reverting Fourier space time-stepping algorithm of Jaimungal and Surkov (2009). More details are to follow.

We will approximate the American option as a limiting sequence of Bermudan options as the time between exercise dates tends to zero. The Bermudan option to invest, where the project can only be invested in at the discrete times $\{t_0, t_1, \ldots, t_n\}$ (e.g. quarterly, monthly, weekly or daily) where $t_n = T$ the maturity date, can priced recursively on the

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exercise dates as follows:

$$\begin{cases} f_{t_n}(x, y) = (e^{\theta + x} - e^{\phi + y})_+ \\[2mm] f_{t_m}(x, y) = \max \left\{ (e^{\theta + x} - e^{\phi + y})_+ \; ; \\[2mm] \qquad\qquad e^{-r\Delta t_m} \; \mathbb{E} \left[f_{t_{m+1}}(X_{t_{m+1}}, Y_{t_{m+1}}) \, | X_{t_m} = x, Y_{t_m} = y \right] \right\}, \end{cases} \quad (4.7)$$

for $m = \{1, 2, \ldots, n - 1\}$. Notice that the second term in the maximization is the holding (or continuation) value of the option to invest on that date. Jaimungal and Surkov (2009) show that this continuation price $f_{t_m}^{cont.}(x, y)$ can be computed via Fourier transforms, resulting in

$$f_{t_m}^{cont.}(x, y) = \mathcal{F}^{-1} \left[\; \mathcal{F} \left[\widetilde{f}_{t_{m+1}}(x, y) \right] (\omega_1, \omega_2) \; e^{\Psi((t_{m+1} - t_m), \omega_1, \omega_2)} \; \right]. \quad (4.8)$$

Here, $\mathcal{F}[.]$ and $\mathcal{F}^{-1}[.]$ represent Fourier and inverse Fourier transforms respectively, Ψ is related to the characteristic function of the generating process

$$\Psi(s, \omega_1, \omega_2) = -\frac{1}{2} \sigma_X^2 \frac{e^{2\alpha s} - 1}{2\alpha} \omega_1^2 - \rho \sigma_X \sigma_Y \frac{e^{(\alpha + \beta)s} - 1}{\alpha + \beta} \omega_1 \omega_2 - \frac{1}{2} \sigma_Y^2 \frac{e^{2\beta s} - 1}{2\beta} \omega_2^2, \quad (4.9)$$

and $\widetilde{f}$ is the option value evaluated on a "mean-reverting grid":

$$\widetilde{f}_{t_{m+1}}(x, y) = f_{t_{m+1}} \left(x e^{-\alpha(t_m - t_{m+1})}, \; y e^{-\beta(t_m - t_{m+1})} \right). \quad (4.10)$$

By comparing the continuation value (4.8) with the value of immediately investing, the optimal strategy can be computed numerically. This requires two fast Fourier transforms to approximately evaluate the Fourier and inverse transforms. Such a procedure is far more efficient than a tree or finite-difference scheme as it requires $O(N \log N)$ computations per exercise date. On the other hand, finite difference schemes will require multiple steps (M)

in between the exercise dates and have computation cost $O(MN)$. For more details see Jaimungal and Surkov (2009).

In Figure 6, we plot the sequence of trigger curves for a ten year option to invest assuming investment can be made only once a year. Naturally, as maturity approaches, the trigger curves move downwards toward the maturity trigger of $V^* = I^*$; however, the early trigger curves lie significantly above the maturity trigger itself. For comparison purposes, the limiting trigger curve (4.6) for the American option to invest is also displayed in the diagram. Not surprisingly the shortest maturity Bermudan trigger curve behaves much like the limiting American trigger curve. Another interesting point is that the trigger curves are approximately described by the straight line $V^* = a(t) + b(t) I^*$ with a non-zero intercept. When the drivers are GBMs the intercept is zero and the trigger curves are exactly linear. As the investment costs move below its equilibrium level the non-linearities become apparent. However, as the investment costs move above its equilibrium level the non-linearities are harder to spot visually, but they do in fact persist.

We are also interested in the limiting case of the Bermudan option tending to an American option. To this end, in Figure 7, we plot the trigger surface for a ten year Bermudan option to invest with daily exercise dates. The solid blue line indicates the equilibrium level of the project value and investment cost, while the black random path is a sample path of the joint investment cost / project value process. Once the path pierces the surface it is optimal to invest in the project. For these specific model parameters and this particular sample path, it was optimal to wait until about year 6. Since the Bermudan option to invest is being monitored on a daily basis it well approximates the American option to invest and these results show that the general features of the Bermudan option with yearly exercise (as shown in Figure 6) are also inherited by the American trigger surface.

5 Conclusions

In this work, we have addressed the problem of the decision of investing, when both the value of the project and the investment cost follows a mean-reverting dynamics. In this case, the optimal policy depends on both the value of the project and the investment level, rather than just on their ratio. The former is known to be the case when the value and the investment are driven by GBMs. This phenomenon precludes the use of a trigger curve for determining the investment frontier, which has been recognized, since the work by McDonald and Siegel (1986), as a specially convenient representation. We have explicitly solve two cases: (i) the European option to invest and (ii) the limiting trigger curve for the American option to invest. Finally, we utilized the Fourier Space Time-Stepping method, developed by Jackson, Jaimungal, and Surkov (2008) and Jaimungal and Surkov (2009), to numerically explore the early trigger levels for the finite-time horizon Bermudan and American options to invest.

There are a few avenues left open for future work.

The first is an analytical one: solving the perpetual American option price. As already pointed in the Section 4.1, the naive separation of variables does not satisfy the boundary conditions for the perpetual option. Although it is possible to solve the PDE, the techniques are outside the scope of the present paper and will be reported elsewhere.

The second avenue left open is related to calibrating this class of models to real data. Calibrating the project value is not terribly difficult – indeed this has been done in several earlier works. However, calibrating the parameters of investment cost process will pose more difficulties. One possible way to proceed would be to utilize a co-integrated model rather than two separate mean-reverting processes. Indeed, one may believe that the costs of a project are somewhat tied into the value that can extracted from it. In fact, even the simple view that the a project may have a fixed cost plus a variable cost which is driven

by production leads to such a picture. One very simple approach to a co-integrated model would be to have the cost I_t satisfy the SDE: $d(\ln I_t) = -\beta(\gamma (\ln V_t - \theta) - (\ln I_t - \phi))dt + \sigma_X dW_t^X$. In this manner, when project values are high, the investment cost is also high, while when project values are low, the investment costs are also low. We have investigated such a co-integrated model, and the results are qualitative similar to those reported here. This model may be easier to calibrate, but at this stage it is still open question.

The third avenue to explore is incorporating technical uncertainty. Here, technical uncertainty is referring to factors that are endogenous to the project – perhaps the amount of reserves that exist in an unexplored oil field. This must be modeled quite distinctly from the mean-reverting project value and investment cost included here. While there has been significant work in this area (cf. Trigeorgis (1999) and e.g. more recently Koussis, Martzoukos, and Trigeorgis (2007)), incorporating mean-reverting project value and investment cost together with technical uncertainty is yet unexplored.

A   Proof: European Option Pricing Formulae

In this appendix we prove Theorem 3.1 – i.e. we derive the value of the European option to invest in a project with stochastic investment and project value. The value is

$$
\begin{aligned}
Opt_0 &= e^{-rT}\mathbb{E}[(V_T - I_T)_+ |\mathscr{F}_0] \\
&= e^{-rT}\,\mathbb{E}^T\!\left[\left.\left(\frac{V_T}{I_T} - 1\right)_+ \right| \mathscr{F}_0\right]\, \mathbb{E}[I_T|\mathscr{F}_0] \\
&= e^{-rT}\mathbb{E}^T[(\xi_T - 1)_+ |\mathscr{F}_0]\, \mathbb{E}[I_T|\mathscr{F}_0]
\end{aligned}
\tag{A.1}
$$

where, $\mathbb{E}^T[\cdot]$ represents expectations with respect to a new measure $\mathbb{P}^T$ defined via the Radon-Nikodym derivative process

$$\eta_t^T \triangleq \left(\frac{d\mathbb{P}^T}{d\mathbb{P}}\right)_t = \frac{\mathbb{E}[I_T|\mathscr{F}_t]}{\mathbb{E}[I_T|\mathscr{F}_0]} , \qquad \text{and} \qquad \xi_t \triangleq \frac{\mathbb{E}[V_T|\mathscr{F}_t]}{\mathbb{E}[I_T|\mathscr{F}_t]} .$$

Note that (i) $\xi_T = V_T/I_T$ and (ii) ξ_t is a $\mathbb{P}^T$-martingale under any modeling assumptions for V_t and I_t (as long as I_t is strictly positive). Property (ii) can be seen from the following simple computation ($0 \le s \le t$):

$$\mathbb{E}^T[\xi_t|\mathscr{F}_s] = \mathbb{E}\left[\frac{\mathbb{E}[V_T|\mathscr{F}_t]}{\mathbb{E}[I_T|\mathscr{F}_t]} \cdot \frac{\mathbb{E}_t[I_T]}{\mathbb{E}_0[I_T]} \middle| \mathscr{F}_s\right] \middle/ \frac{\mathbb{E}[I_T|\mathscr{F}_s]}{\mathbb{E}[I_T|\mathscr{F}_0]} = \frac{\mathbb{E}\left[\mathbb{E}[V_T|\mathscr{F}_t]|\mathscr{F}_s\right]}{\mathbb{E}[I_T|\mathscr{F}_s]}$$
$$= \frac{\mathbb{E}[V_T|\mathscr{F}_s]}{\mathbb{E}[I_T|\mathscr{F}_s]} = \xi_s .$$

For our model (2.1)-(2.2), we have

$$X_T = e^{-\alpha(T-t)} X_t + \sigma_X \int_t^T e^{-\alpha(T-u)} dW_u^X ,$$
$$Y_T = e^{-\beta(T-t)} Y_t + \sigma_Y \int_t^T e^{-\beta(T-u)} dW_u^Y ,$$

so that,

$$\mathbb{E}[V_T|\mathscr{F}_t] = \exp\left\{\theta + e^{-\alpha(T-t)} X_t + \frac{\sigma_X^2}{4\alpha}\left(1 - e^{-2\alpha(T-t)}\right)\right\} ,$$
$$\mathbb{E}[I_T|\mathscr{F}_t] = \exp\left\{\phi + e^{-\beta(T-t)} Y_t + \frac{\sigma_Y^2}{4\beta}\left(1 - e^{-2\beta(T-t)}\right)\right\} .$$

These expressions provide an explicit formula for the ξ_t process. Further, using Ito's lemma and the fact that ξ_t is a $\mathbb{P}^T$-martingale, implies

$$\frac{d\xi_t}{\xi_t} = \sigma_X e^{-\alpha(T-t)} dW_t^{T,X} - \sigma_Y e^{-\beta(T-t)} dW_t^{T,Y} ,$$

where $W_t^{T,X}$ and $W_t^{T,Y}$ are correlated standard $\mathbb{P}^T$-Brownian motions. Consequently,

$$\xi_T \stackrel{d}{=} \xi_0 \exp\left\{-\tfrac{1}{2}\widetilde{\sigma}^2 + \widetilde{\sigma}Z\right\}\ ,$$

where Z is a standard normal random variable and $\widetilde{\sigma}$ is provided in (3.3). Since ξ_T is log-normally distributed, the remaining unknown expectation in (A.1) is

$$\mathbb{E}^T\left[(\xi_T - 1)_+|\mathscr{F}_t\right] = \xi_t \Phi(d_+) - \Phi(d_-)\ ,$$

with $d_\pm$ defined in (3.2). The final pricing result (3.1) is now an easy consequence.

B   Proof: Limiting Trigger Curve

In this appendix, we prove Proposition 4.1.

We will treat an American option as the limiting case of Bermudan options with time between decision dates converging to 0. In particular, if we set $\Delta t = T - t$, then the value of the Bermudan option BO at time $T - \Delta t$ is simply the maximum of the exercise value and the European option value given by Theorem 3.1. So,

$$BO = \max\left(V_{T-\Delta t} - I_{T-\Delta t}\ ,\ e^{-r\Delta t}\left(\mathbb{E}_{T-\Delta t}[V_T]\Phi\left(d_+\right) - \mathbb{E}_{T-\Delta t}[I_T]\Phi\left(d_-\right)\right)\right)\ . \qquad \text{(B.1)}$$

Notice that only the in-the-money exercise value is being compared since the second term in the maximization is always greater than zero. Therefore, we seek (V^*, I^*) such that

$$V^* - I^* = e^{-r\Delta t}\left(\mathbb{E}[V_T \mid V_{T-\Delta t} = V^*]\Phi\left(d_+^*\right) - \mathbb{E}[I_T \mid I_{T-\Delta t} = I^*]\Phi\left(d_-^*\right)\right) \qquad \text{(B.2)}$$

with

$$d^*_\pm = \frac{\ln\left(\mathbb{E}[V_T \mid V_{T-\Delta t} = V^*] \,/\, \mathbb{E}[I_T \mid I_{T-\Delta t} = I^*]\right) \pm \frac{1}{2}\widetilde{\sigma}^2}{\widetilde{\sigma}}$$

Two cases must be treated separately:

- Case I: $V^* > I^*$.

  As $\Delta t \downarrow 0$, $d^*_\pm \uparrow +\infty$ and in particular

  $$\Phi(d^*_\pm) = 1 - \frac{\sqrt{\Delta t}}{2}\frac{e^{-(c^*)^2/2\Delta t}}{\sqrt{\pi/2}\,c^*}\left(1 + o\left(\Delta t\right)\right) , \quad \text{where} \quad c^* = \frac{\ln\left(V^*/I^*\right)}{\sigma} \tag{B.3}$$

  and $\overline{\sigma}^2 = \sigma_X^2 + \sigma_Y^2 - 2\rho\sigma_X\sigma_Y$. Moreover, straightforward computations lead to the limiting behavior of the two expectations

  $$\mathbb{E}[\,V_T \mid V_{T-\Delta t} = V^*\,] = V^*\left(1 - \left(\alpha\left(\ln V^* - \theta\right) - \frac{1}{2}\sigma_X^2\right)\Delta t\right) + o(\Delta t) \tag{B.4a}$$

  $$\mathbb{E}[\,I_T \mid I_{T-\Delta t} = I^*\,] = I^*\left(1 - \left(\beta\left(\ln I^* - \phi\right) - \frac{1}{2}\sigma_Y^2\right)\Delta t\right) + o(\Delta t) \tag{B.4b}$$

  Using Equations (B.3) and (B.4) together in (B.2) we have

  $$V^* - I^* = V^*\left(1 - \left(\alpha\left(\ln V^* - \theta\right) + \left(r - \frac{1}{2}\sigma_X^2\right)\right)\Delta t\right) \tag{B.5}$$

  $$- I^*\left(1 - \left(\beta\left(\ln I^* - \phi\right) + \left(r - \frac{1}{2}\sigma_Y^2\right)\right)\Delta t\right) + o(\Delta t) \tag{B.6}$$

  since $o(\Delta t)$ dominates $\sqrt{\Delta t}e^{-(c^*)^2/2\Delta t}$ from (B.3). Rearranging the above expression and canceling the $o(1)$ terms leads to the result in (4.5) when $V^* > I^*$.

- Case II: $V^* \leq I^*$.

As $\Delta t \downarrow 0$, $d_\pm^* \downarrow -\infty$ and in particular,

$$\Phi(d_\pm^*) = \frac{\sqrt{\Delta t}}{2} \frac{e^{-(c^*)^2/2\Delta t}}{\sqrt{\pi/2}\,(-c^*)} \left(1 + o\left(\Delta t\right)\right) . \tag{B.7}$$

Note that $c^* < 0$ so that this expression is positive. Putting (B.7) together with (B.4) into (B.2) we have

$$V^* - I^* = (V^* - I^*) \frac{\sqrt{\Delta t}}{2} \frac{e^{-(c^*)^2/2\Delta t}}{\sqrt{\pi/2}\,(-c^*)} \left(1 + o\left(\Delta t\right)\right) \tag{B.8}$$

so that as $\Delta t \downarrow 0$ we have $V^* = I^*$.

This completes the proof.

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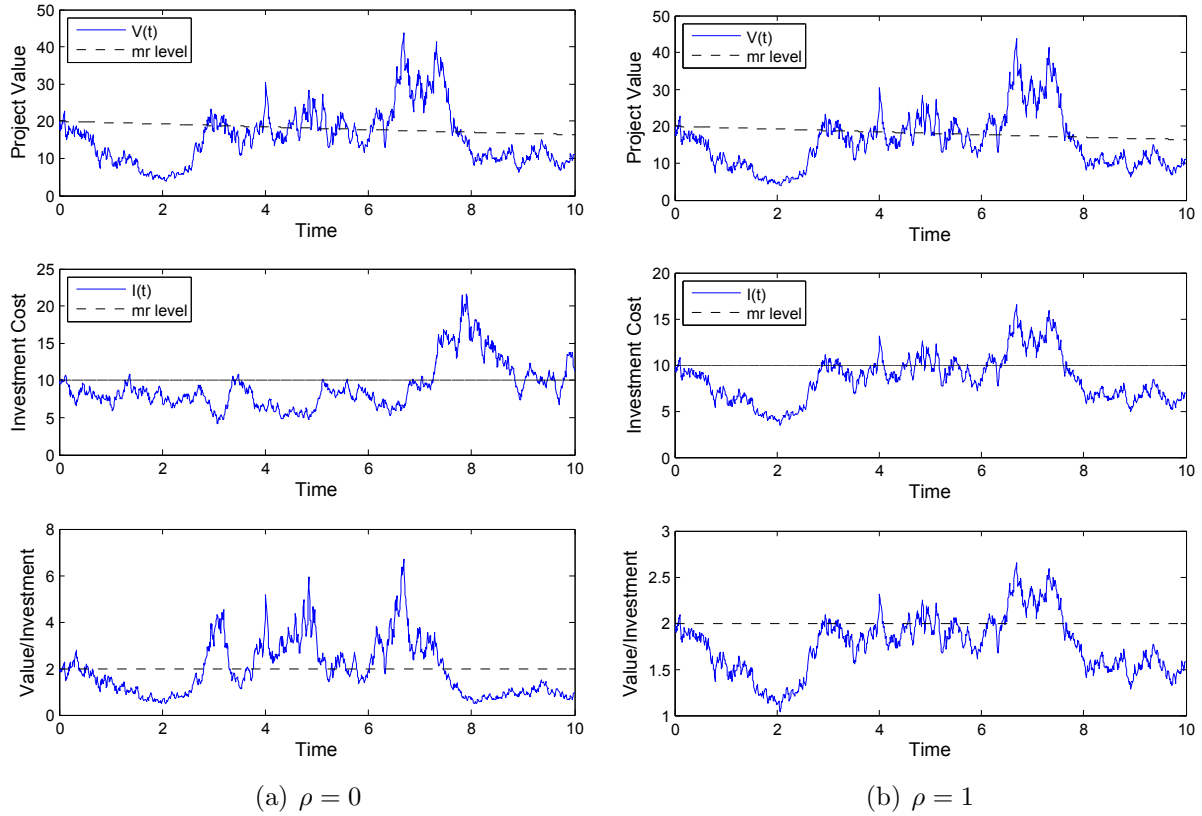Trigeorgis, L. (1999). *Real Options: Managerial Flexibility and Strategy in Resource Allocation.* The MIT Press.

Figure 1: A sample path of project value and investment cost. The lines label *mr level* are the equilibrium mean-reverting levels for the value and investment. The model parameters are: $\alpha = 1$; $\theta = \ln(20)$; $\sigma_X = 0.8$; $\beta = 1$; $\phi = \ln(10)$; $\sigma_Y = 50\%$, and $\lambda = 0$.
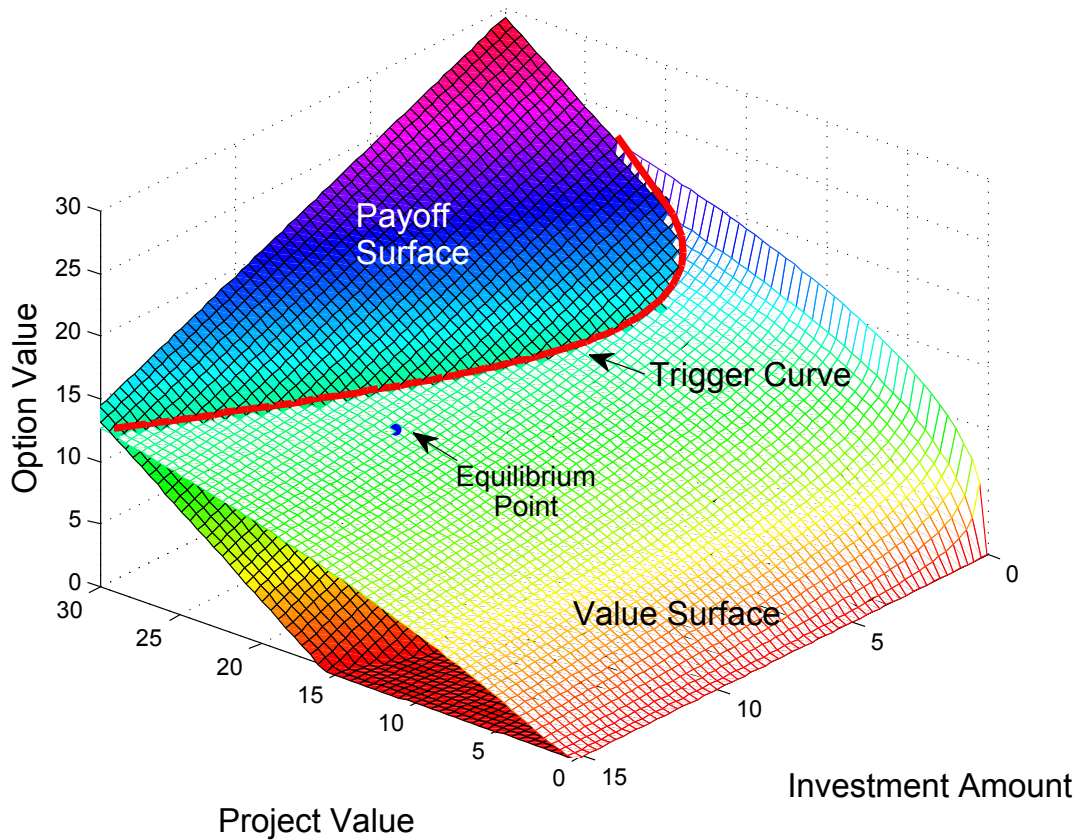
Figure 2: The value of a 1-year European option to invest and the optimal exercise trigger for a 1 year Bermudan option to invest immediately or at maturity. The model parameters are as follows: $\alpha = 1$, $\theta = \ln(20)$, $\sigma_X = 80\%$, $\beta = 1$, $\phi = \ln(10)$, $\sigma_Y = 50\%$, $\rho = 0.5$, and $r = 5\%$.
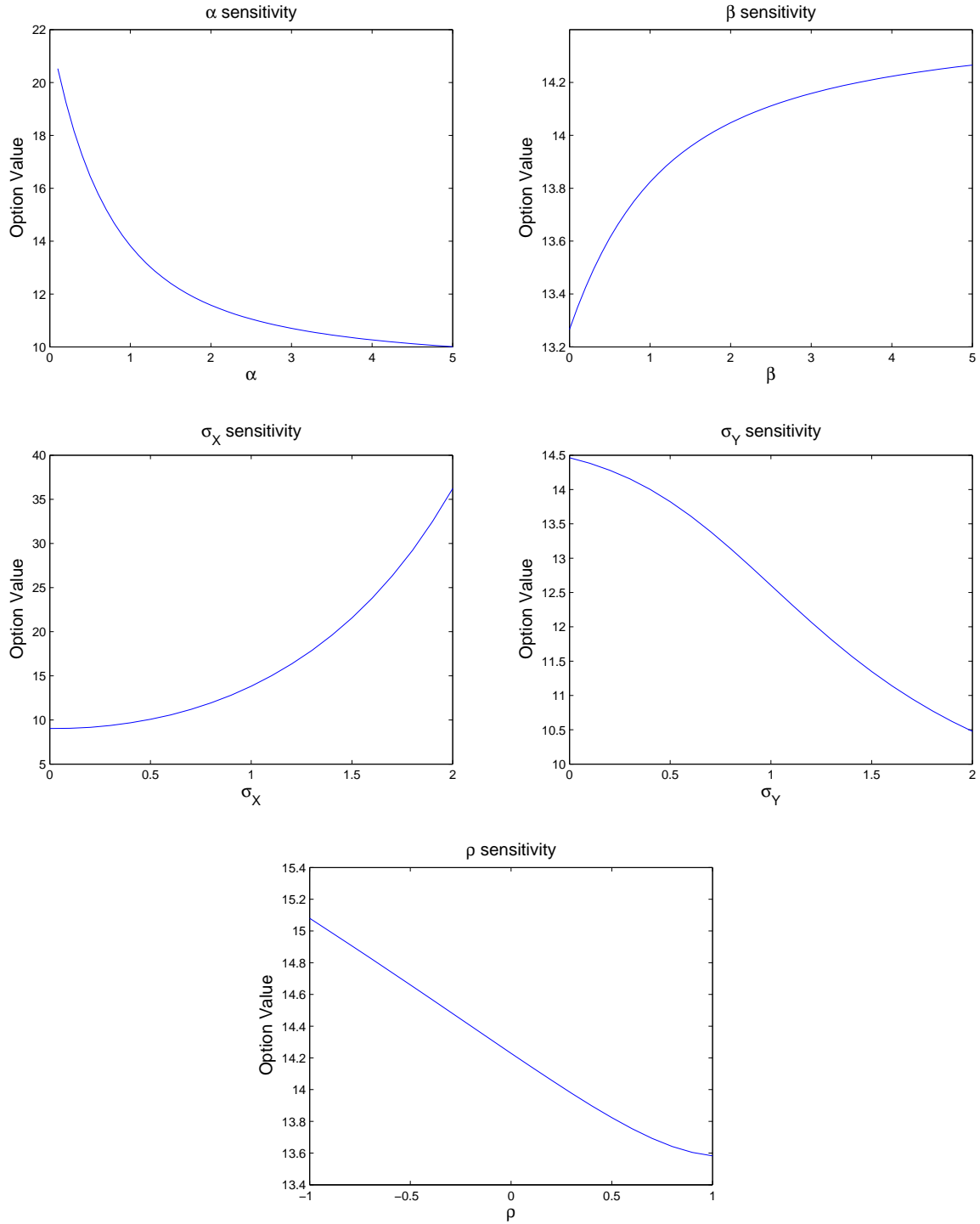
Figure 3: The various sensitivities of the value of the European option to invest in the project at the end of 1 year. The initial project value and investment cost are set at their equilibrium levels: $V = 20$ and $I = 10$. All remaining parameters are kept constant at their levels reported in Figure 2.

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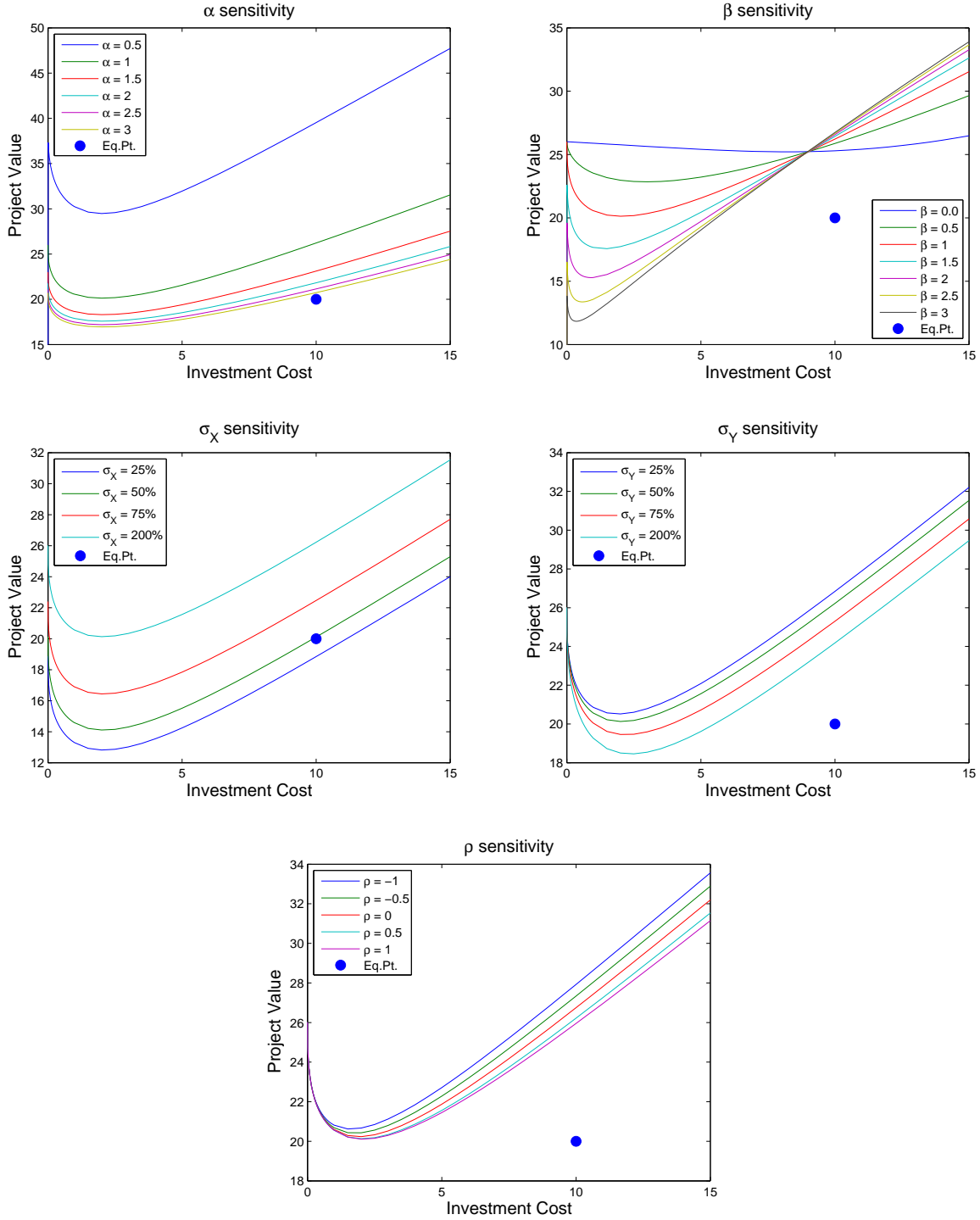
Figure 4: The various sensitivities of the optimal exercise trigger curve for a 1 year Bermudan option to invest immediately or at maturity. The remaining model parameters are as in Figure 2. The point *Eq.Pt.* refers to the long run equilibrium point of the two mean-reverting processes.
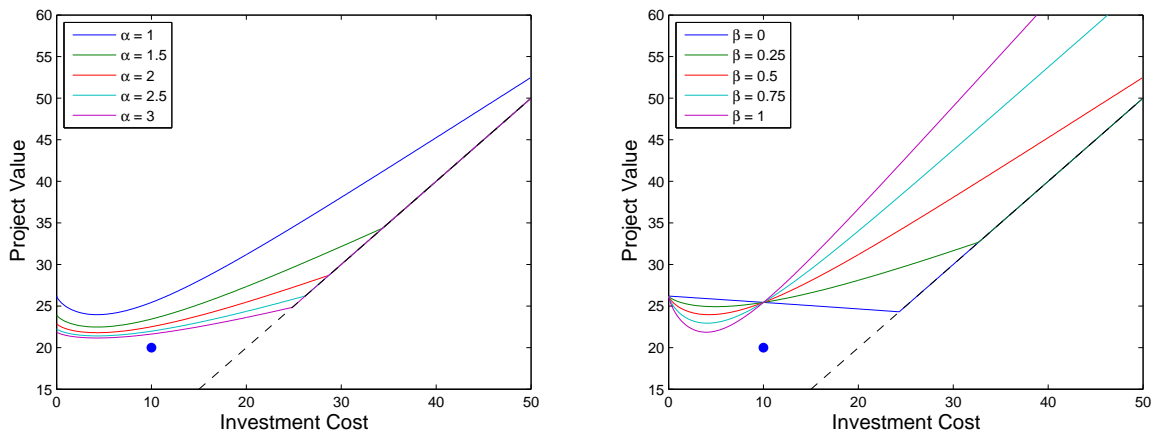
Figure 5: The limiting trigger curves as rates of mean-reversion changes. The solid dot indicates the equilibrium point, while the dashed line is the maturity trigger curve of $V^* = I^*$. The remaining model parameters are as in Figure 2.
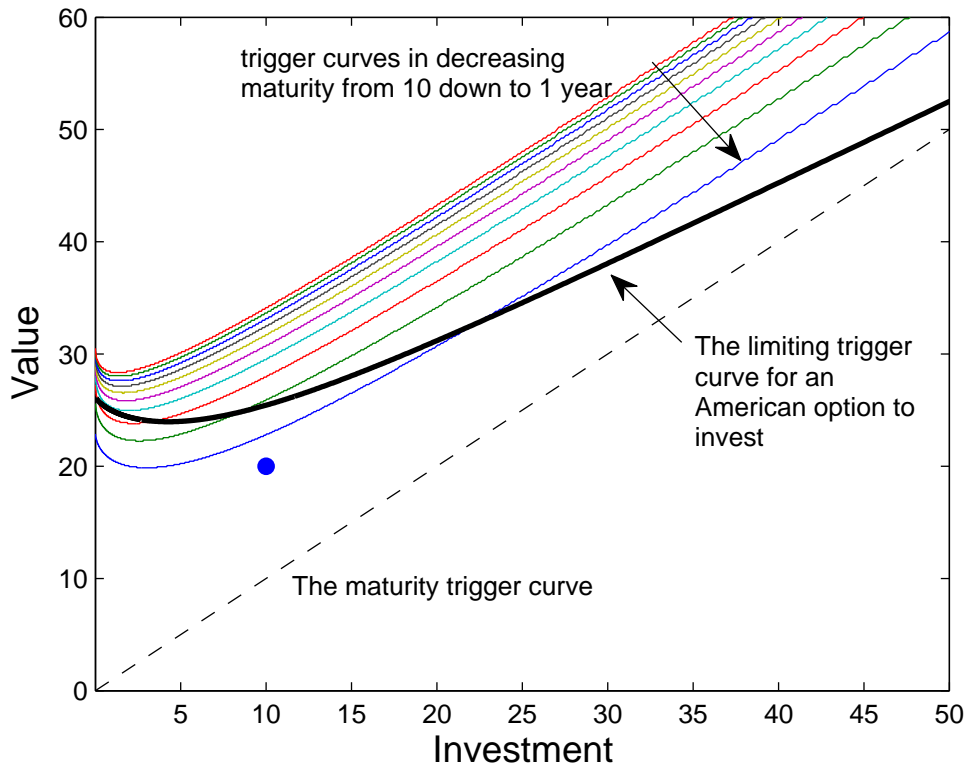
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Figure 6: Trigger curves for the Bermudan option to invest on a yearly basis. The solid dot indicates the equilibrium point, the dashed line is the maturity trigger curve of $V^* = I^*$, and the solid black line is the limiting trigger curve for an American option. Each curve above the dash represents the trigger curve with one more year remaining to maturity. The model parameters are as in Figure 2.
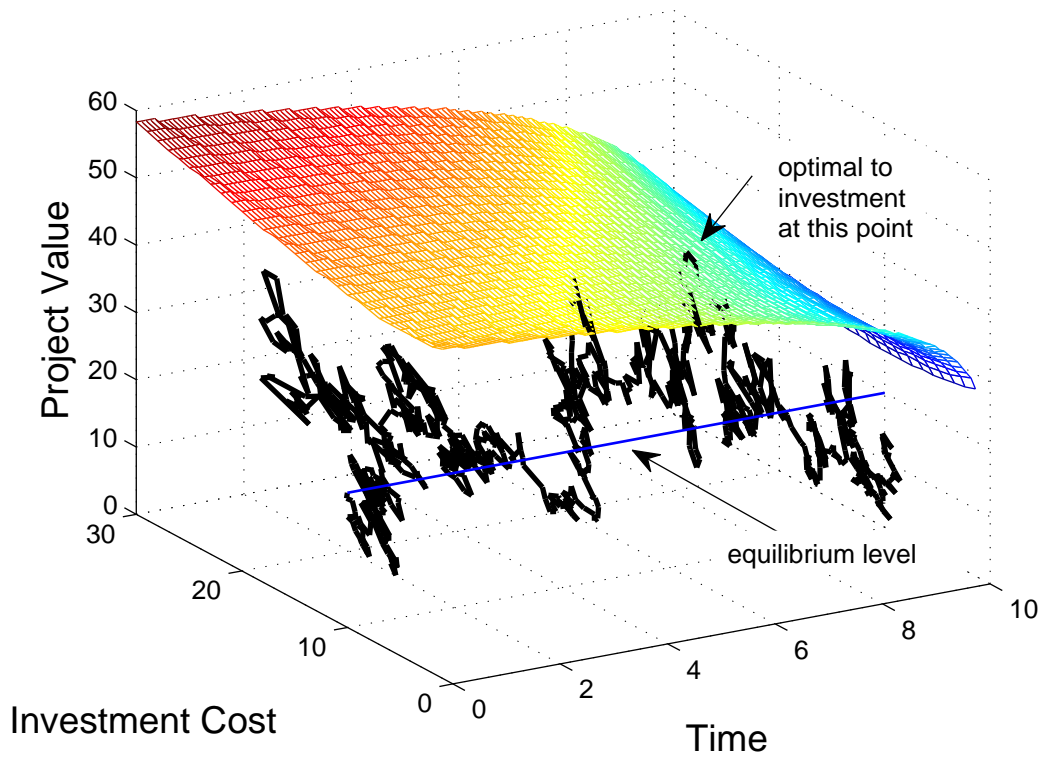
Figure 7: The trigger surface together with a sample path for a ten year option to invest with daily exercise decisions. The straight line is the equilibrium level of the project value/investment cost. When the sample path pierces the surface it is optimal to invest in the project. The model parameters are as in Figure 2.