FIBERS OF THE BAUM-BOTT MAP FOR FOLIATIONS OF DEGREE TWO ON $\mathbb{P}^2$

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Abstract. In this paper we study the fibers of the Baum-Bott map in the space of foliations of degree two on the projective plane $\mathbb{P}^2$. In the main result we prove that its generic fiber contains exactly 240 orbits of the natural action of $Aut(\mathbb{P}^2)$ on the space of foliations.

Contents

1. Introduction

1.1. **The Baum-Bott map.** One of the most basic invariant for singularities of holomorphic foliations of surfaces is the *Baum-Bott index* : if $\mathcal{F}$ is a holomorphic foliation on a neighborhood U of $p \in \mathbb{C}^2$, induced by a holomorphic 1-form $\omega = A(x,y)\,dy - B(x,y)\,dx$, with an unique singularity at p, then the Baum-Bott index of $\mathcal{F}$ at p is defined as

$$BB(\mathcal{F},p) = \frac{1}{(2\pi i)^2}\int_\Gamma \eta \wedge d\eta\,,$$

where η is any $(1,0)$-form, C^∞ on $U \setminus \{p\}$, satisfying $d\omega = \eta \wedge \omega$, and Γ is the boundary of a ball B around p with $p \in B \subset \overline{B} \subset U$ (cf. [Br]). Note that if $f \in \mathcal{O}^*(U)$ and $\omega_1 = f.\omega$ then $d\omega_1 = \eta_1 \wedge \omega_1$, where $\eta_1 = \eta + \frac{df}{f}$, so that

$$\eta_1 \wedge d\eta_1 = \eta \wedge d\eta + d\left(\eta_1 \wedge \frac{df}{f}\right) \implies \int_\Gamma \eta \wedge d\eta = \int_\Gamma \eta_1 \wedge d\eta_1\ .$$

In particular, the Baum-Bott index does not depend on the 1-form representing the foliation.

Another important fact is that it is invariant by biholomorphisms; if $\varphi : (V,q) \to (U,p)$ is a biholomorphism then $BB(\varphi^*(\mathcal{F}),q) = BB(\mathcal{F},p)$ (cf [Br]).

When the dual vector field $X = A(x,y)\partial_x + B(x,y)\partial_y$ has invertible linear part, i.e., $\det DX(p) \neq 0$, a simple computation shows that

$$BB(\mathcal{F},p) = \frac{tr^2(DX(p))}{det(DX(p))}.$$

If the eigenvalues of $DX(p)$ are λ_1 and λ_2 then

$$BB(\mathcal{F},p) = \frac{(\lambda_1+\lambda_2)^2}{\lambda_1\,\lambda_2} = \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2} + 2\ .$$

The numbers λ_2/λ_1 and λ_1/λ_2 will be called the *characteristic values* of the singularity. Note that the charachteristic values satisfy the equation

$$z^2 + (2 - BB(\mathcal{F},p))\,z + 1 = 0\ .$$

Singularities with invertible linear part will be called *non-degenerate singularities.*

In this paper, we will deal with holomorphic foliations on the complex projective plane $\mathbb{P}^2$. A holomorphic foliation on $\mathbb{P}^2$ can be defined in an affine coordinate system $(x,y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ by a holomorphic vector field $X = P(x,y)\,\partial_x + Q(x,y)\,\partial_y$, or by its dual 1-form $\omega = P(x,y)\,dy - Q(x,y)\,dx$, where P and Q are polynomials. We will denote the induced foliation by $\mathcal{F}_X$ or $\mathcal{F}_\omega$. The degree of $\mathcal{F}_X$ is defined as the number of tangencies of the foliation and a generic line $\ell \subset \mathbb{P}^2$. It can be proved that if a vector field $X = P(x,y)\,\partial_x + Q(x,y)\,\partial_y$ induces a degree d foliation then

$$P(x,y) = p(x,y) + x\,g(x,y) \text{ and } Q(x,y) = q(x,y) + y\,g(x,y)\ , \tag{1}$$

where $p,q,g \in \mathbb{C}[x,y]$, $max(dg(p),dg(q)) \leq d$ (dg = degree) and g is homogeneous of degree d. When $g \not\equiv 0$ the set of directions given by $(g(x,y)=0)$, in the line at infinity L_∞ of $\mathbb{C}^2$, defines the set of tangencies of $\mathcal{F}_X$ with L_∞. We will denote the set of foliations of degree d on $\mathbb{P}^2$ by $\mathbb{F}\mathrm{ol}(\mathrm{d},2)$ and the set of singularities of a foliation $\mathcal{F} \in \mathbb{F}\mathrm{ol}(\mathrm{d},2)$ by $sing(\mathcal{F})$. The set of foliations of degree d and with only non-degenerate singularities will be denoted by $\mathbb{F}\mathrm{ol}_{\mathrm{nd}}(\mathrm{d},2)$.

Remark 1.1. We will not assume that P and Q have no common factor, as usual in the theory of complex foliations. With this convention, it follows from (1) that $\mathbb{F}\mathrm{ol}(\mathrm{d},2)$ can be considered as a projective space of dimension $M(d) := (d+3)(d+1)-1$. We would like to remark also that $\mathbb{F}\mathrm{ol}_{\mathrm{nd}}(\mathrm{d},2)$ is a Zariski open subset of $\mathbb{F}\mathrm{ol}(\mathrm{d},2)$.

Remark 1.2. If $\mathcal{F} \in \mathbb{F}\mathrm{ol}(\mathrm{d},2)$ has only isolated singularities then

$$\sum_{p\in sing(\mathcal{F})} mult(\mathcal{F},p) = d^2+d+1 := N(d)\ ,$$

where $mult(\mathcal{F},p)$ denotes the multiplicity of the singularity p. In particular, a foliation $\mathcal{F} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(\mathrm{d},2)$ has exactly $N(d)$ singularities (cf. [Br]).

Given a topological space X we will denote by $\frac{X^m}{S_m}$ the quotient of X^m by the equivalence relation such that the equivalence class of $(x_1,...,x_m) \in X^m$ is

$$[x_1,...,x_m] := \{(x_{\sigma(1)},...,x_{\sigma(m)}) \mid \sigma \in S_m\}\ ,$$

where S_m denotes the set of premutations of $\{1,...,m\}$.

The *Baum-Bott map*,

$$\mathcal{BB}_d : \mathbb{F}\mathrm{ol_{nd}(d,2)} \to \frac{\mathbb{C}^{\mathrm{N(d)}}}{\mathrm{S_{N(d)}}}$$

is defined by

$$\mathcal{BB}_d(\mathcal{G}) = [\mathrm{BB}(\mathcal{G},p_1),\ldots,\mathrm{BB}(\mathcal{G},p_{N(d)})]$$

where $sing(\mathcal{G}) = \{p_1,\ldots,p_{N(d)}\}$. Given $\mathcal{F} \in \mathbb{F}\mathrm{ol_{nd}(d,2)}$ we will denote its fiber $\mathcal{BB}_d^{-1}(\mathcal{BB}_d(\mathcal{F}))$ by $F_d(\mathcal{F})$.

Note that $\mathcal{BB}_d$ extends to a rational map

$$\mathcal{BB}_d : \mathbb{F}\mathrm{ol(d,2)} \dashrightarrow \frac{(\mathbb{P}^1)^{\mathrm{N(d)}}}{\mathrm{S_{N(d)}}} .$$

The well-known Baum-Bott Index Theorem says in the case of foliations of $\mathbb{P}^2$ that (cf. [Br]) :

Theorem 1.1. *If $\mathcal{F} \in \mathbb{F}\mathrm{ol(d,2)}$ has only isolated singularities then*

$$\sum_{p\in sing(\mathcal{F})} \mathrm{BB}(\mathcal{F},\mathrm{p}) = (\mathrm{d}+2)^2 .$$

In particular, $\mathcal{BB}_d$ is not dominant. On the other hand, the following result is known :

Theorem 1.2. *If $d \geq 2$ then the generic rank of $\mathcal{BB}_d$ is d^2+d. In particular, if $d \geq 2$ then the dimension of the generic fiber of $\mathcal{BB}_d$ is $3\,d+2$.*

Theorem 1.2 was proved for $d = 2$ in [AG] and for $d \geq 3$ in [LN-JP].

Remark 1.3. Denote by $Aut(\mathbb{P}^2) \simeq PSL(2,\mathbb{C})$ the group of holomorphic automorphisms of $\mathbb{P}^2$ and consider the natural action Ψ given by

$$(T,\mathcal{F}) \in Aut(\mathbb{P}^2) \times \mathbb{F}\mathrm{ol(d,2)} \overset{\Psi}{\mapsto} \mathrm{T}^*(\mathcal{F}) \in \mathbb{F}\mathrm{ol(d,2)} .$$

We will denote by $\mathcal{O}rb(\mathcal{F})$ the orbit of the foliation $\mathcal{F}$ under this action. Since the Baum-Bott index is invariant by local biholomorphisms, we get

$$\mathcal{O}rb(\mathcal{F}) \subset F_d(\mathcal{F}) \ , \ \forall\, \mathcal{F} \in \mathbb{F}\mathrm{ol_{nd}(d,2)} .$$

In particular, the fiber $F_d(\mathcal{F})$ is foliated by the orbits of Ψ.

When $d = 2$ the dimension of the generic fiber of $\mathcal{BB}_2$ is $8 = dim(Aut(\mathbb{P}^2))$. Therefore, in this case the generic fiber is a finite union of orbits of Ψ.

Other notations that we will use :

- $\mathcal{I}so(\mathcal{F})$ the isotropy group of $\mathcal{F}$:

$$\mathcal{I}so(\mathcal{F}) := \{T \in Aut(\mathbb{P}^2) \mid T^*(\mathcal{F}) = \mathcal{F}\} .$$

- Given $A \subset \mathbb{F}\mathrm{ol(d,2)}$ the saturation of A is by definition

$$\mathcal{S}at(A) = \{T^*(\mathcal{F}) \mid \mathcal{F} \in A \text{ and } T \in Aut(\mathbb{P}^2)\} .$$

Definition 1. When $d \geq 2$, we will say that a fiber $F_d(\mathcal{F})$ is *exceptional* if $dim(F_d(\mathcal{F})) > 3\,d+2$. Otherwise, we will say that the fiber is *non-exceptional.*

1.2. **Statement of the results.** Concerning the generic fiber of the Baum-Bott map on $\mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$, we have the following result :

Theorem 1. *The generic fiber of $\mathcal{BB}_2$ contains exactly 240 orbits of the natural action of $Aut(\mathbb{P}^2)$.*

The proof of Theorem 1 will be done in §2.5. The basic technique will be to reduce the computation of the Baum-Bott indexes of the singularities of a foliation $\mathcal{F} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ to a computation of the residues of a rational form on $\mathbb{P}^1$ (§2.1). For this reduction we will assume that $\mathcal{F}$ has a singularity p satisfying :

(a). *$BB(\mathcal{F},p) \neq 4$, or equivalently 1 is not a characteristic value of $\mathcal{F}$ at p.*
(b). *$\mathcal{F}$ has no invariant straight line through p.*

From now on, we will refer these conditions as *conditions (a) and (b)*, respectively. In lemma 2.1 of §2.1, we will prove that a foliation $\mathcal{F} \in \mathbb{F}\mathrm{ol}(2,2)$ which has a non-degenerate singularity p satisfying (a) and (b) can be represented in some affine coordinate system $(x,y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ by a vector field, which depends of six parameters $\Lambda = (\lambda, A, B, \alpha, \beta, \gamma)$, $X_\Lambda = P_\Lambda(x,x)\,\partial_x + Q_\Lambda(x,y)\,\partial_y$, where

$$(2) \qquad \begin{cases} P_\Lambda(x,y) = \lambda\,x + A\,x^2 + B\,x\,y + (1-\lambda)\,y^2 + x\,(\alpha\,x^2 + \beta\,x\,y + \gamma\,y^2) \\ Q_\Lambda(x,y) = y + (\lambda-1)\,x^2 + A\,x\,y + B\,y^2 + y\,(\alpha\,x^2 + \beta\,x\,y + \gamma\,y^2) \end{cases} .$$

In these coordinates the point p is the origin and $\lambda \neq 1$ is one of the charachteristic values of $\mathcal{F}$ at p.

Notations : Given $\Lambda = (\lambda, A, B, \alpha, \beta, \gamma) \in \mathbb{C}^6$, we will denote by $\mathcal{F}_\Lambda$ the foliation defined by the vector field $X_\Lambda = P_\Lambda \partial_x + Q_\Lambda \partial_y$, where P_Λ and Q_Λ are as in (2). The six dimensional family of foliations $\{\mathcal{F}_\Lambda \,|\, \Lambda \in \mathbb{C}^6\}$, will be denoted by $\mathcal{W}$.

The reduction mentioned above will be done in lemma 2.2 and Corollary 2.1. In §2.2 we will apply the reduction of §3.1 to study foliations $\mathcal{F} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ with $sing(\mathcal{F}) = \{p_1, ..., p_7\}$ and satisfying the following properties :

(c). *p_7 satisfies conditions (a) and (b).*
(d). *If $1 \leq i < j \leq 6$ then $BB(\mathcal{F},p_i) = BB(\mathcal{F},p_j)$.*

The following result will be proved :

Theorem 2. *If $b_o \notin \{0,4,16\}$ then*

(A). *There exists $\mathcal{F} \in \mathcal{W}$ with $sing(\mathcal{F}) = \{p_1, ..., p_6, p_7 = 0\}$, $BB(\mathcal{F},p_7) = b_o$, $BB(\mathcal{F},p_j) = (16 - b_o)/6$, $1 \leq j \leq 6$. In particular, $\mathcal{F}$ satisfies conditions (c) and (d).*
(B). *Assume that the charachteristic values λ and λ^{-1} of $\mathcal{F}$ at p_7 satisfy*

$$\lambda\,,\,\lambda^{-1} \notin \{t \,|\, t = -5 \text{ or } t^3 + 12\,t^2 - 3\,t + 2 = 0\} := \mathcal{A}\ .$$

Then

$$F_2(\mathcal{F}) \cap \mathcal{W} = \mathcal{O}rb(\mathcal{F}) \cap \mathcal{W} = \{\mathcal{F}, \varphi^*(\mathcal{F})\}\ ,$$

where $\varphi(x,y) = (y,x)$.
(C). *If λ or $\lambda^{-1} \in \mathcal{A}$ then $F_2(\mathcal{F})$ is an exceptional fiber of $\mathcal{BB}_2$ and $dim(F_2(\mathcal{F})) = 9$. In particular, there are exactly four exceptional fibers of $\mathcal{BB}_2$ for which the generic element satisfies conditions (c) and (d).*

As a consequence of Theorem 2 we will get the following :

Corollary 1. *Let* $\mathcal{G} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ *with* $sing(\mathcal{G}) = \{q_1, ..., q_7\}$ *and* $BB(\mathcal{G}, q_7) = b_o \notin \{0, 4, 16\}$ *and* $BB(\mathcal{G}, q_j) = (16 - b_o)/6$, $1 \le j \le 6$. *Denote by* λ, λ^{-1} *and* ρ, ρ^{-1} *the characteristic values of* $\mathcal{G}$ *at* q_7 *and* q_j, $1 \le j \le 6$, *respectively. Assume that* $\lambda, \lambda^{-1} \notin \mathcal{A}$ *and*

$$\alpha + \beta + \gamma \neq 1 \ , \ \forall \alpha \in \{\lambda, \lambda^{-1}\} \ , \ \forall \beta \, , \gamma \in \{\rho, \rho^{-1}\} \ . \tag{3}$$

Then $\mathcal{O}rb(\mathcal{F}) = F_2(\mathcal{F})$, *that is the orbit of* $\mathcal{F}$ *coincides with its* $\mathcal{BB}_2$*-fiber.*

An example of foliation satisfying the hypothesis of Corollary 1 is Jouanolou's foliation of degree two, J_2. It has no algebraic invariant curve and satisfies $\mathcal{BB}_2(J_2) = [16/7, ..., 16/7]$, that is it has all Baum-Bott indexes equal (cf. [LN-JP]). Moreover, the charachteristic values at a singularity are the roots λ and λ^{-1} of $7\,z^2 - 2\,z + 7 = 0$, so that $\lambda, \lambda^{-1} \notin \mathcal{A}$ and they also satisfies (3) in Corollary 1. As a consequence we get the following :

Corollary 2. *The Jouanolou foliation of degree two,* J_2, *satisfies* $\mathcal{O}rb(J_2) = F_2(J_2)$.

We would like to remark that $\mathcal{I}so(J_2)$ is a finite subgroup of $Aut(\mathbb{P}^2)$ with 21 transformations (cf. [LN-JP]). The group $\mathcal{I}so(J_2)$ will be used in the proof of Theorem 1 in §2.5.

Remark 1.4. As a consequence of the proof of Theorem 2, we will see that there is no foliation $\mathcal{F} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ with $sing(\mathcal{F}) = \{p_1, ..., p_7\}$, $BB(\mathcal{F}, p_1) = 0$ and $BB(\mathcal{F}, p_j) = 8/3$, $2 \le j \le 7$. In particular, $[0, 8/3, ..., 8/3]$ is not in the immage of $\mathcal{BB}_2$ (see Assertion 2.1 in section 2.2).

1.3. **Examples and related problems.** In this section we will see some examples of exceptional fibers of the Baum-Bott map.

Example 1. *Logarithmic and rational foliations.* A logarithmic 1-form on $\mathbb{P}^2$ is induced by a closed meromorphic 1-form given in homogeneous coordinates by

$$\Omega = \sum_{j=1}^{k} \lambda_j \, \frac{dF_j}{F_j} \not\equiv 0 \ ,$$

where $\lambda_j \in \mathbb{C}^*$ and F_j is a non-constant homogeneous polynomial of degree $d_j \ge 1$, $1 \le j \le k$, with

$$\sum_{j=1}^{k} \lambda_j \, d_j = 0 \ .$$

The above condition implies that $k \ge 2$ and that there exists a closed meromorphic 1-form ω on $\mathbb{P}^2$ such that $\Pi^*(\omega) = \Omega$, where $\Pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$ is the canonical projection. The form ω defines a foliation on $\mathbb{P}^2$, denoted by $\mathcal{F}(\lambda, F)$, $\lambda = (\lambda_1, ..., \lambda_k)$, $F = (F_1, ..., F_k)$, of degree

$$dg(\mathcal{F}(\lambda, F)) = d_1 + ... + d_k - 2 := d(D) \ , \ D = (d_1, ..., d_k) \ .$$

Remark 1.5. When $[\lambda_1 : ... : \lambda_k] = [m_1 : ... : m_k]$, where $(m_1, ..., m_k) \in \mathbb{Z}^k$, then $\mathcal{F}(\lambda, F)$ admits a non-constant rational first integral, which in homogeneous coordinates is expressed as $F_1^{m_1} ... F_k^{m_k}$. This happens when $k = 2$, because $d_1.\lambda_1 + d_2.\lambda_2 = 0$. Conversely, if $\mathcal{F}(\lambda, F)$ admits a non-constant first integral then $[\lambda_1 : ... : \lambda_k] \in \mathbb{P}(\mathbb{Z}^k)$. Foliations with a rational first integral will be called rational foliations.

Let us state some properties of $\mathcal{F}(\lambda, F)$.

1. The algebraic curves $S_j := \Pi(F_j = 0)$, $1 \le j \le k$, are $\mathcal{F}(\lambda, F)$-invariant. Denote $S = \bigcup_j S_j$.
2. If S_j is smooth for all $j = 1, ..., k$ and the singularities of S are nodal then $S_i \cap S_j \subset sing(\mathcal{F}(\lambda, F))$, for all $i \neq j$, and any point $p \in S_i \cap S_j$ is a non-degenerate singularity of $\mathcal{F}(\lambda, F)$ and

$$BB(\mathcal{F}(\lambda, F), p) = -\frac{(\lambda_i - \lambda_j)^2}{\lambda_i\, \lambda_j} \ . \tag{4}$$

3. If $p \in sing(\mathcal{F}(\lambda, F)) \setminus S$ is non-degenerate then

$$BB(\mathcal{F}(\lambda, F), p) = 0 \ . \tag{5}$$

It follows from (4) and (5) that if $\mathcal{F}(\lambda, F)$ has only non-degenerate singularities then the Baum-Bott indexes of its singularities depend only of $[\lambda] = [\lambda_1 : ... : \lambda_k] \in \mathbb{P}^{k-1}$ and not of $F = (F_1, ..., F_k)$. Let us fix some notations.

Let $\mathcal{P}_m$ be the set of homogeneous polynomials of degree m on three variables. Given $D = (d_1, ..., d_k) \in \mathbb{N}^k$ and $\lambda = (\lambda_1, ..., \lambda_k)$, with $\sum_j \lambda_j . d_j = 0$, set

$$\mathcal{P}(D) := \mathcal{P}_{d_1} \times ... \times \mathcal{P}_{d_k} \ ,$$

$$\mathcal{L}og(\lambda, D) = \{\mathcal{F}(\lambda, F) \,|\, F \in \mathcal{P}(D)\} \subset \mathbb{F}\mathrm{ol}(\mathrm{d(D)}, 2) \, .$$

and

$$\mathcal{L}og_{nd}(\lambda, D) = \mathcal{L}og(\lambda, D) \cap \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(\mathrm{d(D)}, 2) \, .$$

Note that (4) and (5) imply that $\mathcal{L}og_{nd}(\lambda, D)$ is contained in a fiber of $\mathcal{BB}_{d(D)}$.

Remark 1.6. It can be proved that $\mathcal{L}og_{nd}(\lambda, D)$ is never empty. In fact, the set

$$\{F \in \mathcal{P}(D) \,|\, \mathcal{F}(\lambda, F) \text{ has a degenerate singularity}\}$$

is a Zariski proper closed set of $\mathcal{P}(D)$. The dimension of $\mathcal{L}og_{nd}(\lambda, D)$ can be calculated and in some cases it is greater than $3\, d(D) + 2$, the dimension of the generic fiber of $\mathcal{BB}_{d(D)}$.

Notation. Given D, F and λ such that $\mathcal{F}(\lambda, F) \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(\mathrm{d(D)}, 2)$ we will denote the fiber $\mathcal{BB}^{-1}_{d(D)}(\mathcal{F}(\lambda, F))$ by $E(\lambda, D)$. Note that $\mathcal{L}og_{nd}(\lambda, D) \subset E(\lambda, D)$.

Remark 1.7. When $D = (1, d + 1)$, $d \ge 2$, then $\lambda_1 + (d + 1)\, \lambda_2 = 0$ and we can take $\lambda = (-(d + 1), 1)$. In this case, $d(D) = d$ and $\mathcal{F}(\lambda, F)$ is defined by

$$\Omega = -(d+1)\frac{dF_1}{F_1} + \frac{dF_2}{F_2} \ .$$

Moreover, F_2/F_1^{d+1} is a rational first integral of $\mathcal{F}(\lambda, F)$. When the curves $\Pi(F_1 = 0)$ and $\Pi(F_2 = 0)$ are transverse and $\Pi(F_2 = 0)$ is smooth then $\mathcal{F}(\lambda, F)$ has $d + 1$ singularities with Baum-Bott index $(d + 2)^2/(d + 1)$ and d^2 with Baum-Bott index 0. Conversely, if $\mathcal{G} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(\mathrm{d}, 2)$ has d^2 singularities with Baum-Bott index 0 then $\mathcal{G} \in \mathcal{L}og_{nd}((-(d+1), 1), (1, d+1))$ (cf. [LN-JP]). In particular, in this case we have

$$\mathcal{L}og_{nd}((-(d + 1), 1), (1, d + 1)) = E((-(d + 1), 1), (1, d + 1)) \ .$$

Moreover,

$$dim(E((-(d{+}1), 1), (1, d{+}1)) = dim(\mathbb{P}(\mathcal{P}_{d+1})) + dim(\mathbb{P}(\mathcal{P}_1)) - 1 = \binom{d+3}{2} > 3\, d{+}2 \ .$$

In particular, if $d = 2$ this fiber has dimension 10. We would like to observe that we don't know any other fiber of $\mathcal{BB}_2$ with dimension 10. This motivates the following :

Problem 1. *Is $E((-3,1),(1,3))$ the unique fiber of $\mathcal{BB}_2$ with dimension 10 ?*

Remark 1.8. Given $D = (d_1, ..., d_k) \in \mathbb{N}^k$ set $N(D) := \sum_{j=1}^k dim(\mathbb{P}(\mathcal{P}_{d_j})) = \sum_{j=1}^k \frac{d_j^2+3\,d_j}{2}$. Note that $dim(\mathcal{L}og_{nd}(\lambda, D)) \leq N(D)$. However, in some cases the equality is true. For instance, when one of the conditions below is fullfilled it can be shown that $dim(\mathcal{L}og_{nd}(\lambda, D)) = N(d)$:

1. $k \geq 3$ and $d_i \neq d_j$ if $i \neq j$.
2. $k = 2$, $d_1 < d_2$ and $d_1 \not| \, d_2$.
3. $k \geq 3$ and if $[\lambda_{\sigma(1)} : ... : \lambda_{\sigma(k)}] = [\lambda_1 : ... : \lambda_k]$ for a permutation $\sigma \in S_k$ then σ is the identity.

We leave the proof to the reader.

For instance, if $D = (1,1,2)$ and λ satisfies condition 3 of remark 1.8, then we have $d(D) = 2$ and $N(D) = 9$. In particular, we obtain that $E(\lambda, (1,1,2))$ is an exceptional fiber of $\mathcal{BB}_2$.

Let us state a related problem.

Problem 2. *When $E(\lambda, D)$ coincides with $\mathcal{L}og_{nd}(\lambda, D)$?*

For instance, if $D = (1, ..., 1) \in \mathbb{N}^k$ and $k \geq 5$ then $d(D) = k-2$ and $N(D) = 2\,k$, so that $N(D) < 3\,d(D) + 2$. In this case, $\mathcal{L}og_{nd}(\lambda, D)$ is always a proper subset of $E(\lambda, D)$.

Example 2. Consider the pencil of foliations $\mathcal{P} := (\mathcal{F}_\alpha)_{\alpha \in \overline{\mathbb{C}}} \subset \mathbb{F}\mathrm{ol}(2,2)$, where $\mathcal{F}_\alpha$ is defined in the affine coordinate system $[x : y : 1] \simeq (x,y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ by $\omega_\alpha = P_\alpha(x,y)\,dy - Q_\alpha(x,y)\,dx$, with

$$\begin{array}{c} P_\alpha(x,y) = 4\,x - 9\,x^2 + y^2 + \alpha\,(2\,y - 4\,x\,y) \\ Q_\alpha(x,y) = 6\,y - 12\,x\,y + 3\,\alpha\,(x^2 - y^2) \end{array} .$$

The following facts can be checked (see also [LN 1]) :

1. The line at infinity $L_\infty = (z = 0)$ and the rational quartic Q defined by $4\,y^2\,(1 - 3\,x) - 4\,x^3 + (3\,x^2 + y^2)^2 = 0$ are $\mathcal{F}_\alpha$-invariant, for every $\alpha \in \overline{\mathbb{C}}$. Set $S := L_\infty \cup Q$.
2. $sing(\mathcal{F}_\alpha) = \{J, K, L, M, N, P_1(\alpha), P_2(\alpha)\}$, where

$$J = (1/2, -1/2) \ , \ K = (0,0) \ , \ L = (1/2, 1/2) \ , \ P_1(\alpha) = \left(\frac{4(\alpha^2+1)}{(\alpha^2+3)^2}, \frac{-8\,\alpha}{(\alpha^2+3)^2}\right) ,$$

$$M = [1 : \sqrt{3}\,i : 0] \ , \ N = [1, -\sqrt{3}\,i : 0] \text{ and } P_2(\alpha) = [1, \alpha, 0] \ .$$

   Note that $J, K, L \in Q$, $M, N \in L_\infty \cap Q$, $P_1(\alpha) \in Q$ and $P_2(\alpha) \in L_\infty$, for all $\alpha \in \overline{\mathbb{C}}$.
3. If $\alpha \notin \{1, -1, \infty, \sqrt{3}\,i, -\sqrt{3}\,i\}$ then the singularities of $\mathcal{F}_\alpha$ are non-degenerate and $BB(\mathcal{F}_\alpha, J) = BB(\mathcal{F}_\alpha, K) = BB(\mathcal{F}_\alpha, L) = 25/6$, $BB(\mathcal{F}_\alpha, P_1(\alpha)) = -25/6$, $BB(\mathcal{F}_\alpha, M) = BB(\mathcal{F}_\alpha, N) = 9/2$ and $BB(\mathcal{F}_\alpha, P_2(\alpha)) = -4/3$.

In particular, $\mathcal{P}$ is contained in the fiber

$$T := \mathcal{BB}_2^{-1}[25/6, 25/6, 25/6, -25/6, 9/2, 9/2, -4/3] \ .$$

On the other hand, it can be checked that $\mathcal{I}so(\mathcal{F}_\alpha)$ is finite for all $\alpha \in \overline{\mathbb{C}}$. This implies that $dim(\mathcal{S}at(\mathcal{P})) = 9$, so that T is an exceptional fiber of $\mathcal{BB}_2$.

Remark 1.9. The pencil $\mathcal{P}$ of this example is flat in the sense of [LN 2]. This means that the unique meromorphic 1-form θ which satisfies $d\omega_\alpha = \theta \wedge \omega_\alpha$, for all $\alpha \in \overline{\mathbb{C}}$, is closed (it can checked that $\theta = \frac{5}{6}\frac{dQ}{Q}$). This also implies that the foliations $\mathcal{F}_\alpha$ admit a common affine transverse structure (cf. [Sc]).

Example 3. Observe that the typical foliation, in the exceptional fibers of examples 1 and 2, have some kind of projective transverse structure. However, as we will see next, it is not true that in every exceptional fiber of the Baum-Bott map the typical foliation has some transverse structure. Let $\mathcal{P}_1 = (\mathcal{F}_\alpha)_{\alpha \in \mathbb{C}}$ be the 1-parameter family of foliations of degree two, where $\mathcal{F}_\alpha$ is defined by the vector field $X_\alpha = P_\alpha(x,y)\,\partial_x + Q_\alpha(x,y)\,\partial_y$, with

$$\begin{aligned} P_\alpha(x,y) &= -5\,x + \alpha\,x^2 + 6\,y^2 + x\,\left(-\tfrac{5}{64}\,\alpha^2\,x^2 - 36\,x\,y - \tfrac{3}{8}\,\alpha\,y^2\right) \\ Q_\alpha(x,y) &= y - 6\,x^2 + \alpha\,x\,y + y\,\left(-\tfrac{5}{64}\,\alpha^2\,x^2 - 36\,x\,y - \tfrac{3}{8}\,\alpha\,y^2\right) \end{aligned}$$

It can be checked that, except for a finite number of parameters, the foliation $\mathcal{F}_\alpha$ has all singularities non-degenerate. Moreover, in the proof of Theorem 2 we will see that if $\mathcal{F}_\alpha \in \mathcal{P}_1 \cap \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2) := \mathcal{P}_{1\,\mathrm{nd}}$ then

$$\mathcal{BB}_2(\mathcal{F}_\alpha) = [-16/5, 16/5, 16/5, 16/5, 16/5, 16/5, 16/5] := M$$

and that $\mathcal{BB}_2^{-1}(M)$ is an exceptional fiber with dimension 9. Concerning the above family of foliations, the following result we will proved in §2.4 :

Proposition 1. *Let $\alpha \in \mathbb{C}$ be such that $\mathcal{F}_\alpha \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$. Then $\mathcal{F}_\alpha$ has no algebraic invariant curve. In particular, $\mathcal{F}_\alpha$ has no projective transverse structure.*

The problem of classification of the exceptional fibers of the Baum-Bott map seems to be very dificult, even in the case of degree two. However, the following one seems to be accessible :

Problem 3. *Classify the exceptional fibers of $\mathcal{BB}_d$, $d \geq 2$, for which the typical foliation has a projective transverse structure.*

2. Foliations of degree two

In this section we will deal only with foliations of degree two. For this reason, we will denote $\mathcal{BB}_2 := \mathcal{BB}$.

2.1. **Reduction of the problem to dimension one.** Let $\mathcal{F} \in \mathbb{F}\mathrm{ol}(2,2)$ be a foliation with a non-degenerate singular point $p \in sing(\mathcal{F})$. If we fix an affine coordinate system $(u,v) \in \mathbb{C}^2 \subset \mathbb{P}^2$ such that $p \in \mathbb{C}^2$ then $\mathcal{F}$ can be represented in this coordinate system by a polynomial vector field Y, where $Y(p) = 0$ and $det(DY(p)) \neq 0$. Let λ_1, λ_2 be the eigenvalues of $DY(p)$ and $\lambda = \lambda_2/\lambda_1$ be a characteristic value of $\mathcal{F}$ at p. We will assume :

(a). $\lambda := \lambda_2/\lambda_1 \neq 1$. This is equivalent to $BB(\mathcal{F}, p) \neq 4$.
(b). There is no $\mathcal{F}$-invariant straight line through $p \in \mathbb{C}^2$.

Lemma 2.1. *In the above situation there exists $\varphi \in Aut(\mathbb{P}^2)$ with $\varphi(0) = p$ and such that $\varphi^*(\mathcal{F})$ can be represented by a polynomial vector $X = P(x,y)\,\partial_x + Q(x,y)\,\partial_y$ with*

$$\begin{cases} P(x,y) = \lambda\,x + A\,x^2 + B\,x\,y + (1-\lambda)\,y^2 + x\,(\alpha\,x^2 + \beta\,x\,y + \gamma\,y^2) \\ Q(x,y) = y + (\lambda-1)\,x^2 + A\,x\,y + B\,y^2 + y\,(\alpha\,x^2 + \beta\,x\,y + \gamma\,y^2) \end{cases} \tag{6}$$

Proof. Since $\lambda \neq 1$ the linear part of $DY(p)$ is semi-simple and is conjugated to $L = \lambda_2\,x\,\partial_x + \lambda_1\,y\,\partial_y$. Therefore, after an affine change of variables, we can suppose that $Y(0) = 0$ and $DY(0) = L$. Set $Y_1 := \lambda_1^{-1}.Y$. Note that $Y_1 = P_1(x,y)\partial_x + Q_1(x,y)\partial_y$ with

$$P_1(x,y) = \lambda\,x + p_2(x,y) + x\,g_2(x,y) \text{ and } Q_1(x,y) = y + q_2(x,y) + y\,g_2(x,y)\ ,$$

where p_2, q_2 and g_2 are homogeneous polynomials of degree two.

Let $R = x\,\partial_x + y\,\partial_y$ be the radial vector field on $\mathbb{C}^2$ and $\mathcal{R}$ the foliation defined by R on $\mathbb{P}^2$. Denote by $Tang(\mathcal{F},\mathcal{R})$ the divisor of tangencies between the foliations $\mathcal{F}$ and $\mathcal{R}$. Observe that $Tang(\mathcal{F},\mathcal{R}) \cap \mathbb{C}^2$ is defined by $(G = 0)$, where

$$G.\,\partial_x \wedge \partial_y = (\lambda-1)^{-1}.\,Y_1 \wedge R \implies$$

$$G(x,y) = x\,y + (\lambda-1)^{-1}.\,(y\,p_2(x,y) - x\,q_2(x,y)) := x\,y - G_3(x,y)\ .$$

We assert that, after a projective change of variables, we can assume that

$$G(x,y) = x\,y - x^3 - y^3\ .$$

Let us prove the assertion. First of all, the cubic $G(x,y) = x\,y - G_3(x,y)$ is irreducible. In fact, since G_3 is homogeneous of degree three, if G was reducible then it would be divisible by x or by y. On the other hand, both curves $(x = 0)$ and $(y = 0)$ are $\mathcal{R}$-invariant. However, this would imply that $\mathcal{F}$ has an invariant straight line through $0 \in \mathbb{C}^2$, which contradicts (b).

In particular, we can write

$$G(x,y) = x\,y - (a\,x^3 + b\,x^2\,y + c\,x\,y^2 + d\,y^3)\ ,$$

where $a, d \neq 0$. Define $T \in Aut(\mathbb{P}^2)$ by

$$T(z,w) = \left(\frac{\rho\,z}{\ell(z,w)}\,,\,\frac{\mu\,w}{\ell(z,w)}\right) = (x,y)\ ,$$

where $\ell(z,w) = 1 + \alpha\,z + \beta\,w$, $\rho^3 = a^{-2}d^{-1}$, $\mu^3 = a^{-1}d^{-2}$, $\alpha = \rho\,b$ and $\beta = \mu\,c$. A straightforward computation shows that

$$T^*(G)(z,w) = G \circ T(z,w) = \rho\,\mu\,\ell^{-3}\,(z\,w - z^3 - w^3) := \rho\,\mu\,\ell^{-3}\,\tilde{G}\ .$$

On the other hand, as the reader can check, we have

$$T^*(R) = \ell.\,(z\,\partial_z + w\,\partial_w) := \ell.\,\tilde{R} \text{ and } T^*(Y_1) = \frac{X}{\ell}\ ,$$

where $X = X_1 + X_2 + \tilde{g}_2.\,\tilde{R}$, with $\tilde{R} = z\,\partial_z + w\,\partial_w$, $X_1 = \lambda\,z\,\partial_z + w\,\partial_w$, $X_2 = \tilde{p}_2\,\partial_z + \tilde{q}_2\,\partial_w$, $\tilde{p}_2$, $\tilde{q}_2$ and $\tilde{g}_2$ homogeneous of degree two. This implies,

$$X \wedge \tilde{R} = T^*(Y_1 \wedge R) = T^*((\lambda-1).\,G.\,\partial_x \wedge \partial_y) = (\lambda-1).\,\tilde{G}.\,\partial_z \wedge \partial_w\ ,$$

which proves the assertion.

In particular, there exists $\varphi \in Aut(\mathbb{P}^2)$ such that $\varphi^*(\mathcal{F})$ can be represented by $X = X_1 + X_2 + \tilde{g}_2\,\tilde{R}$. On the other hand.

$$(\lambda-1).\,\tilde{G}.\,\partial_z \wedge \partial_w = X_1 \wedge \tilde{R} + X_2 \wedge \tilde{R} = [(\lambda-1)\,z\,w + w\,\tilde{p}_2 - z\,\tilde{q}_2]\,\partial_z \wedge \partial_w \implies$$

$$w\,\tilde{p}_2 - z\,\tilde{q}_2 = (1-\lambda)\,(z^3+w^3) \quad \Longrightarrow$$

there exist $A, B \in \mathbb{C}$ such that

$$\begin{aligned}\tilde{p}_2 &= A\,z^2 + B\,z\,w + (1-\lambda)\,w^2\\ \tilde{q}_2 &= (\lambda-1)\,z^2 + A\,z\,w + B\,w^2\end{aligned}$$

This proves the lemma. □

From now on, we fix an affine coordinate system $(x,y) \in \mathbb{C}^2 \subset \mathbb{P}^2$. Set $\Delta = \{(\lambda, A, B, \alpha, \beta, \gamma) \in \mathbb{C}^6 \mid \lambda \neq 0, 1\}$. Given $\Lambda = (\lambda, A, B, \alpha, \beta, \gamma) \in \Delta$, we will denote by $\mathcal{F}_\Lambda$ the foliation defined in the fixed coordinate system by $X_\Gamma := P\,\partial_x + Q\,\partial_y$, where P and Q are as in (6). We will denote also

$$\mathcal{W} = \{\mathcal{F}_\Lambda \mid \Lambda = (\lambda, A, B, \alpha, \beta, \gamma) \in \Delta\}\ .$$

Note that the map $\Lambda \in \Delta \mapsto \mathcal{F}_\Lambda \in \mathcal{W}$ is injective.

Remark 2.1. Let $\mathcal{H} \subset Aut(\mathbb{P}^2)$ be the group of invariance of the divisor defined in the fixed affine coordinate system by $(x\,y - x^3 - y^3 = 0)$. Note that $\mathcal{H}$ is generated by the transformations $\sigma(x,y) = (y,x)$ and $\delta(x,y) = (j\,x, j^2\,y)$, where $j = e^{2\pi i/3}$. In particular, $\mathcal{H}$ is isomorphic to S_3, the group of permutations of three elements. Moreover, as the reader can check, given $\Lambda = (\lambda, A, B, \alpha, \beta, \gamma) \in \Delta$ we have

$$\begin{aligned}\delta^*(X_\Lambda) &= X_{\hat{\delta}(\Lambda)}\ , \text{ where } \hat{\delta}(\Lambda) = (\lambda, j\,A, j^2\,B, j^2\,\alpha, \beta, j\,\gamma)\\ \sigma^*(X_\Lambda) &= \lambda.\,X_{\hat{\sigma}(\Lambda)}\ , \text{ where } \hat{\sigma}(\Lambda) = (\lambda^{-1}, \lambda^{-1}B, \lambda^{-1}A, \lambda^{-1}\gamma, \lambda^{-1}\beta, \lambda^{-1}\alpha)\end{aligned}$$

which implies $\delta^*(\mathcal{F}_\Lambda) = \mathcal{F}_{\hat{\delta}(\Lambda)}$, $\sigma^*(\mathcal{F}_\Lambda) = \mathcal{F}_{\hat{\sigma}(\Lambda)}$ and $\mathcal{H}^*(\mathcal{W}) = \mathcal{W}$.

Remark 2.2. The rational map $\mathcal{BB}|_{\mathcal{W}}$ has generic rank six. In fact, let $\mathbb{F}\mathrm{ol}(\mathrm{a},\mathrm{b})$ be the set of foliations of degree two having a singularity which satisfyies conditions (a) and (b). It follows from [LN] that $\mathbb{F}\mathrm{ol}(\mathrm{a},\mathrm{b})$ contains an open and dense subset of $\mathbb{F}\mathrm{ol}(2,2)$. In particular, $\mathcal{BB}|_{\mathbb{F}\mathrm{ol}(\mathrm{a},\mathrm{b})}$ has generic rank six by Theorem 1.2. On the other hand, lemma 2.1 implies that if $\mathcal{F} \in \mathbb{F}\mathrm{ol}(\mathrm{a},\mathrm{b})$ then $\mathcal{O}rb(\mathcal{F}) \cap \mathcal{W} \neq \emptyset$. The assertion follows now from the fact that $\mathcal{BB}$ is constant along the orbits of $Aut(\mathbb{P}^2)$.

Lemma 2.2. *Let $\Lambda = (\lambda, A, B, \alpha, \beta, \gamma) \in \Delta$ and $\mathcal{F}_\Lambda$ be as before. Then there exists a birrational transformation $\Phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ and an affine coordinate system $(t,v) \in \mathbb{C}^2 \subset \mathbb{P}^2$ such that*

(I). *Φ^{-1} is a biholomorphism in a neighborhood of $sing(\mathcal{F}_\Lambda) \setminus \{0\}$ and*

$$\Phi^{-1}(sing(\mathcal{F}_\Lambda) \setminus \{0\}) \subset (v=0)\ .$$

(II). *$\Phi^*(\mathcal{F}_\Lambda)$ is defined by a vector field*

$$Z_\Lambda = (\lambda-1)\,t^3\,v\,\partial_t + \left(P_\Lambda(t) + t.\,Q_\Lambda(t)\,v + \lambda\,t^3\,v^2\right)\,\partial_v$$

where

$$(7)\qquad \begin{cases} P_\Lambda(t) = t^6 + B\,t^5 + (A+\gamma)\,t^4 + (\lambda+\beta+1)\,t^3 + (B+\alpha)\,t^2 + A\,t + \lambda\\ \qquad\qquad Q_\Lambda(t) = (3-\lambda)\,t^3 + B\,t^2 + A\,t + (3\,\lambda - 1)\end{cases}$$

Proof. The foliation $\mathcal{F}_\Lambda$ is represented in the fixed affine coordinate system by the 1-form

$$\omega_\Lambda = (\lambda\,x + p_2(x,y) + x\,g_2(x,y))\,dy - (y + q_2(x,y) + y\,g_2(x,y))\,dx\ ,$$

where

$$\begin{aligned} p_2(x,y) &= A\,x^2 + B\,x\,y + (1-\lambda)\,y^2 \\ q_2(x,y) &= (\lambda-1)\,x^2 + A\,x\,y + B\,y^2 \\ g_2(x,y) &= \alpha\,x^2 + \beta\,x\,y + \gamma\,y^2 \end{aligned}$$

We begin by a blowing-up $\pi\,:\,(\tilde{\mathbb{P}}^2, D) \to (\mathbb{P}^2, 0)$ at $0 \in \mathbb{C}^2 \subset \mathbb{P}^2$, where $D = \pi^{-1}(0)$. Consider the chart $(t,x) \in \mathbb{C}^2 \subset \tilde{\mathbb{C}}^2$ where $\pi(t,x) = (x, t.\,x)$. A straightforward computation shows that $\pi^*(\omega_\Lambda) = x.\,\theta_\Lambda$, where

$$\theta_\Lambda = x\,\left[\lambda + x\,p_2(1,t) + x^2\,g_2(1,t)\right]\,dt - (1-\lambda)\,\left[t - x\,(t^3+1)\right]\,dx\ .$$

Note that π^{-1} is a biholomorphism in a neighborhood of $sing(\mathcal{F}_\Lambda \setminus \{0\})$ and

$$\pi^{-1}(sing(\mathcal{F}_\Lambda \setminus \{0\})) \subset \big(t - x\,(t^3+1) = 0\big) \setminus (0,0)\ .$$

Define a birrational map Φ_1 by

$$\Phi_1(t,v) = \left(t\,,\frac{1}{v + \frac{1}{t} + t^2}\right) = (t,x)$$

with inverse

$$\Phi_1^{-1}(t,x) = \left(t\,,\frac{1}{x} - \frac{1+t^3}{t}\right)\ .$$

In particular, Φ_1^{-1} is a diffeomorphism in a neighborhood of $\big(t - x\,(t^3+1) = 0\big) \setminus (0,0)$ and

$$\Phi_1^{-1}\left(\big(t - x\,(t^3+1) = 0\big) \setminus (0,0)\right) \subset (v=0)\ .$$

Set $\Phi := \pi \circ \Phi_1$. Note that Φ^{-1} is a biholomorphism in a neighborhood of $sing(\mathcal{F}_\Lambda) \setminus \{0\}$ and

$$\Phi^{-1}(sing(\mathcal{F}_\Lambda) \setminus \{0\}) \subset (v=0)\ .$$

On the other hand, a straightforward computation shows that

$$\Phi_1^*(\theta_\Lambda) = \frac{1}{t^2.\,(v + t^{-1} + t^2)^3}\,.\,\eta_\Lambda\ ,$$

where

$$\eta_\lambda = \big(P_\Lambda(t) + t.\,Q_\Lambda(t)\,v + \lambda\,v^2\big)\,dt - (\lambda-1)\,t^3\,v\,dv\ ,$$

where P_Λ and Q_Λ are as in (7). Since the dual vector field of η_Λ is $Z_\Lambda = (\lambda - 1)\,t^3\,v\,\partial_t + \big(P_\Lambda(t) + t.\,Q_\Lambda(t)\,v + \lambda\,t^3\,v^2\big)\,\partial_v$, we get the lemma. □

Remark 2.3. Note that $\Phi^{-1}(sing(\mathcal{F}_\Lambda)\setminus\{0\}) = (v = P_\Lambda(t) = 0)\setminus(0,0)$. Since $P_\Lambda(t)$ is monic of degree six, we can set $P_\Lambda(t) = \Pi_{j=1}^6(t - \tau_j(\Lambda))$, where $\tau_1(\Lambda), ..., \tau_6(\Lambda)$ are the roots of $P_\Lambda(t) = 0$. When $\mathcal{F}_\Lambda \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ then

(i). $\tau_j(\Lambda) \neq 0$, if $1 \le j \le 6$.
(ii). $\tau_i(\Lambda) \neq \tau_j(\Lambda)$, if $1 \le i < j \le 6$.

We will set $\Phi(\tau_j(\Lambda), 0) = p_j(\Lambda)$, $1 \le j \le 6$. Note that $sing(\mathcal{F}_\Lambda) = \{0, p_1(\Lambda), ..., p_6(\Lambda)\}$.

Corollary 2.1. *Assume that $\mathcal{F}_\Lambda \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$. Let*

$$\omega_\Lambda := \frac{Q_\Lambda^2(t)}{(1-\lambda).\,t.\,P_\Lambda(t)}\,dt\ . \tag{8}$$

With the notations of Remark 2.3 we have

$$BB(\mathcal{F}_\Lambda, p_j(\Lambda)) = Res(\omega_\Lambda, t = \tau_j(\Lambda))\ ,\ 1 \le j \le 6\ . \tag{9}$$

Proof. Let $p_j(\Lambda)$ and $(\tau_j(\Lambda),0) := q_j$, $1 \le j \le 6$, be as in Remark 2.3. Since $\Phi(q_j) = p_j(\Lambda)$ and Φ is a biholomorphism in a neighborhood of q_j, we get

$$BB(\mathcal{F}_\Lambda, p_j(\Lambda)) = BB(Z_\Lambda, q_j)\ ,\ 1 \le j \le 6\ .$$

By assumption the singularities of $\mathcal{F}_\Lambda$ are non-degenerate, and so $det(DZ_\Lambda(q_j)) \neq 0$, which implies

$$BB(Z_\Lambda, q_j) = \frac{tr^2(DZ_\Lambda(q_j))}{det(DZ_\Lambda(q_j))}\ .$$

As the reader can check

$$tr(DZ_\Lambda(q_j)) = \tau_j(\Lambda).\, Q_\Lambda(\tau_j(\Lambda)) \text{ and } det(DZ_\Lambda(q_j)) = (1-\lambda).\, \tau_j(\Lambda)^3.\, P'_\Lambda(\tau_j(\Lambda)) \implies$$

$$BB(Z_\Lambda, q_j) = \frac{Q^2_\Lambda(\tau_j(\Lambda))}{(1-\lambda).\, \tau_j(\Lambda).\, P'_\Lambda(\tau_j(\Lambda))} = Res(\omega_\Lambda, t = \tau_j(\Lambda)) \quad \square$$

Remark 2.4. The Baum-Bott theorem can be proved to a foliation as in Corollary 2.1 by using (9) and the residue theorem. In fact, if we consider the form ω_Λ as a meromorphic 1-form on $\mathbb{P}^1$ then it has eight poles : $\{0, \infty, q_1, ..., q_6\}$. On the other hand, it can be checked that

$$Res(\omega_\Lambda, t = 0) = \frac{(3\,\lambda - 1)^2}{\lambda\,(1-\lambda)} \text{ and } Res(\omega_\Lambda, t = \infty) = \frac{(\lambda - 3)^2}{\lambda - 1}\ .$$

Since $BB(\mathcal{F}_\Lambda, 0) = (\lambda+1)^2/\lambda$, we get from Corollary 2.1 and the residue theorem that

$$BB(\mathcal{F}_\Lambda, 0) + \sum_{j=1}^{6} BB(\mathcal{F}_\Lambda, p_j(\Lambda)) = \frac{(\lambda+1)^2}{\lambda} - \frac{(3\,\lambda-1)^2}{\lambda\,(1-\lambda)} - \frac{(\lambda-3)^2}{\lambda-1} = 16\ .$$

We close this section with the following auxiliary result :

Lemma 2.3. *Let $\mathcal{F} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ and assume that any singularity of $\mathcal{F}$ is contained in a $\mathcal{F}$-invariant straight line. Then $\mathcal{F}$ has a radial singularity.*

Proof. The following fact is well known : let $\mathcal{G} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ and $\ell \subset \mathbb{P}^2$ be a straight line. Then ℓ is $\mathcal{G}$-invariant if, and only if, it contains exactly three singularities of $\mathcal{G}$ (cf. [Br]).

Assume that any singularity of $\mathcal{F} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ is contained in at least an invariant straight line. Set $sing(\mathcal{F}) = \{p_1, ..., p_7\}$. Through p_1 passes an invariant straight line, say ℓ_1. The line ℓ_1 contains two other singularities, say p_2 and p_3, and no other singularity. In particular, the invariant straight line through p_4, say ℓ_2, is distinct from ℓ_1. Since ℓ_1 and ℓ_2 are $\mathcal{F}$-invariant the intersection $\ell_1 \cap \ell_2$ is a singularity of $\mathcal{F}$. We can assume that $\ell_1 \cap \ell_2 = \{p_1\}$. Therefore, ℓ_2 contains the singularities p_1, p_4 and another one, that we can assume to be p_5. Since $p_6, p_7 \notin \ell_1 \cup \ell_2$, then they are contained in other two straight lines, distinct from ℓ_1 and ℓ_2, say ℓ_3 and ℓ_4, respectively. We assert that $\ell_3 = \ell_4$ and it contains the singularities p_1, p_6 and p_7.

In fact, if $\ell_3 \neq \ell_4$, then ℓ_3 contains four singularities of $\mathcal{F}$: p_6, $\ell_3 \cap \ell_1$, $\ell_3 \cap \ell_2$ and $\ell_3 \cap \ell_4$, a contradiction. Therefore, $\ell_3 = \ell_4$ and ℓ_3 contains the singularities p_6, p_7, $\ell_3 \cap \ell_1$ and $\ell_3 \cap \ell_2$. Since it contains exactly three singularities, we must have $\ell_3 \cap \ell_1 = \ell_3 \cap \ell_2 = \{p_1\}$.

In particular, the singularity p_1 is contained in three, two by two, distinct $\mathcal{F}$-invariant straigh lines : ℓ_1, ℓ_2 and ℓ_3. If X is a holomorphic vector field defining

$\mathcal{F}$ in a neighborhood of p_1 then the linear part of X at p_1 must be of the form $\lambda\, R$, where R is the radial vector field and $\lambda \neq 0$. This implies that p_1 is a radial singularity of $\mathcal{F}$. □

Corollary 2.2. *Let $\mathcal{F} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ and assume that $BB(\mathcal{F},p) \neq 4$ for all $p \in sing(\mathcal{F})$. Then $\mathcal{F}$ has at least one singularity satisfying conditions (a) and (b).*

2.2. **Proof of Theorem 2.** With the notations of lemma 2.2 and Remark 2.3, we want to prove that there exists $\Lambda = (\lambda, A, B, \alpha, \beta, \gamma) \in \Delta$ such that $\mathcal{F}_\Lambda \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ and $sing(\mathcal{F}_\Lambda) = \{0, p_1(\Lambda), ..., p_6(\Lambda)\}$ satisfy $BB(\mathcal{F}_\Lambda, 0) = (\lambda+1)^2/\lambda$ and

$$BB(\mathcal{F}_\Lambda, p_i(\Lambda)) = BB(\mathcal{F}_\Lambda, p_j(\Lambda)) := \mu \ , \ 1 \le i < j \le 6 \ .$$

Since $BB(\mathcal{F}_\Lambda, 0) = (\lambda+1)^2/\lambda$, it follows from Baum-Bott theorem that μ must satisfy

$$6\,\mu + \frac{(\lambda+1)^2}{\lambda} = 16 \ . \tag{10}$$

By lemma 2.2 and Corollary 2.1, this is equivalent to prove that there are polynomials $P_\Lambda(t)$ and $Q_\Lambda(t)$ as in (7), such that the form $\omega_\Lambda = \frac{Q_\Lambda^2(t)\,dt}{(1-\lambda)t\,P_\Lambda(t)}$ has all residues at the roots of $P_\Lambda(t) = 0$ equal to μ and $Res(\omega_\Lambda, 0) = (3\,\lambda - 1)^2/\lambda(1-\lambda) := a$. Since $dg(Q_\Lambda^2) < dg(t\,P_\Lambda)$, if we set $P_\Lambda(t) = \Pi_{j=1}^6 (t - \tau_j)$, we must have

$$\omega_\Lambda = \left(\frac{a}{t} + \sum_{j=1}^{6} \frac{Res(\omega_\Lambda, \tau_j)}{t - \tau_j}\right) dt = \left(\frac{a}{t} + \mu \sum_{j=1}^{6} \frac{1}{t-\tau_j}\right) dt = \left(\frac{a}{t} + \mu \frac{P_\Lambda'(t)}{P_\Lambda(t)}\right) dt \, .$$

In other words, we have to find $\Lambda \in \Delta$ such that the identity below is verified :

$$\frac{Q_\Lambda^2(t)}{(1-\lambda).\,t.\,P_\Lambda(t)} \equiv \frac{a}{t} + \mu \frac{P_\Lambda'(t)}{P_\Lambda(t)} \ , \tag{11}$$

where,

$$\begin{array}{c} P_\Lambda(t) = t^6 + B\,t^5 + (A+\gamma)\,t^4 + (\lambda+\beta+1)\,t^3 + (B+\alpha)\,t^2 + A\,t + \lambda \\ Q_\Lambda(t) = (3-\lambda)\,t^3 + B\,t^2 + A\,t + 3\,\lambda - 1 \end{array} .$$

After setting $a = (3\lambda - 1)^2/\lambda(1-\lambda)$ in (11), we obtain the following equivalent identity :

$$\lambda\,(1-\lambda)\,\mu\,t\,P_\Lambda'(t) + (3\lambda-1)^2\,P_\Lambda(t) - \lambda\,Q_\Lambda^2(t) \equiv 0 \ . \tag{12}$$

Identity (12) impose conditions on the coefficients of $t, ..., t^5$ of the right hand side involving the parameters $\lambda, A, B, \alpha, \beta$ and γ. Let us prove that they have a solution, if we assume $(\lambda+1)^2/\lambda \notin \{0, 4, 16\}$. If we substitute (7) and (10) in (12) then we find the following coefficients of t and t^5 :

$$\begin{cases} \text{coeff. of } t \text{ in (12)}: & \frac{1}{6}\,(\lambda+5)\,(\lambda-1)^2.\,A = 0 \\ \text{coeff. of } t^5 \text{ in (12)}: & \frac{1}{6}\,(5\,\lambda+1)\,(\lambda-1)^2.\,B = 0 \end{cases} \tag{13}$$

Since $\lambda \neq 1$ the equations in (13) imply that we have three possible cases :

Case 1. $\lambda = -5 \implies B = 0$ and $A \in \mathbb{C}$.
Case 2. $\lambda = -1/5 \implies A = 0$ and $B \in \mathbb{C}$.
Case 3. $\lambda, \lambda^{-1} \neq -5 \implies A = B = 0$.

Analysis of cases 1 and 2. In case 1, if we set $B = 0$ and $\lambda = -5$ in (7) then we get :

$$\begin{cases} \text{coeff. of } t^3 \text{ in (12)}: & -32\,\beta - 1152 = 0 & \Longrightarrow & \beta = -36 \\ \text{coeff. of } t^2 \text{ in (12)}: & 64\,\alpha + 5\,A^2 = 0 & \Longrightarrow & \alpha = -5\,A^2/64 \\ \text{coeff. of } t^4 \text{ in (12)}: & -128\gamma - 48\,A = 0 & \Longrightarrow & \gamma = -3\,A/8 \end{cases}$$

In particular, $\Lambda = (-5, A, 0, -5\,A^2/64, -36, -3\,A/8)$ and $\mathcal{F}_\Lambda = \mathcal{F}_{X_A}$, where $X_A = P_A\,\partial_x + Q_A\,\partial_y$ and

$$P_A = -5\,x + A\,x^2 + 6\,y^2 + x\,\left(-\frac{5}{64}\,A^2\,x^2 - 36\,x\,y - \frac{3}{8}\,A\,y^2\right)$$

$$Q_A = y - 6\,x^2 + A\,x\,y + y\,\left(-\frac{5}{64}\,A^2\,x^2 - 36\,x\,y - \frac{3}{8}\,A\,y^2\right)$$

Therefore, we get the 1-parameter family of foliations $\mathcal{E}_0 = (\mathcal{F}_{X_A})_{A\in\mathbb{C}}$. As the reader can check, for any $A \in \mathbb{C}$ then $\mathcal{I}so(\mathcal{F}_{X_A})$ is finite. This implies that

$$dim(\mathcal{S}at(\mathcal{E}_0)) = 9\ .$$

On the other hand, $BB(\mathcal{F}_{X_A}, 0) = -5 - 1/5 + 2 = -16/5$, which implies $\mu = 16/5$ and

$$\mathcal{BB}(\mathcal{F}_{X_A}) = [-16/5, 16/5, 16/5, 16/5, 16/5, 16/5, 16/5] := M\ .$$

In particular, the fiber $\mathcal{BB}^{-1}(M)$ is exceptional, because it contains $\mathcal{S}at(\mathcal{E}_0)$.

We would like to observe that Remark 2.1 reduces case 2 to case 1 : let $\sigma(x, y) = (y, x)$. In Remark 2.1 we have seen that $\sigma^*(\mathcal{F}_\Lambda) = \mathcal{F}_{\hat\sigma(\Lambda)}$ where $\hat\sigma(\Lambda) = (\lambda^{-1}, \lambda^{-1}B, \lambda^{-1}A, \lambda^{-1}\gamma, \lambda^{-1}\beta, \lambda^{-1}\alpha)$. On the other hand, we have found in case 1 $\Lambda = (-5, A, 0, -5\,A^2/64, -36, -3\,A/8)$, which implies $\hat\sigma(\Lambda) = (-1/5, 0, -A/5, 3\,A/40, 36/5, A^2/64)$. If we set $B = -A/5$ then we get the solution of case 2 :

$$\hat\sigma(\Lambda) = (-1/5, 0, B, -3\,B/8, 36/5, 25\,B^2/64)\ .$$

Analysis of case 3. If we set $A = B = 0$ then we get :

$$\begin{cases} \text{coeff. of } t^3 \text{ in (12)}: & \frac{1}{2}\,\left[(\lambda+1)^3\,\beta + (\lambda^2 + 18\,\lambda + 1)\,(\lambda - 1)^2\right] = 0 \\ \text{coeff. of } t^2 \text{ in (12)}: & \frac{1}{3}\,\alpha\,(\lambda^3 + 12\,\lambda^2 - 3\,\lambda + 2) = 0 \\ \text{coeff. of } t^4 \text{ in (12)}: & \frac{1}{3}\,\gamma\,(2\,\lambda^3 - 3\,\lambda^2 + 12\,\lambda + 1) = 0 \end{cases} \tag{14}$$

Before going on in the analysis of case 3, let us prove the assertion of Remark 1.4.

Assertion 2.1. We assert that there is no foliation $\mathcal{G} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2, 2)$ with $sing(\mathcal{G}) = \{p_1, ..., p_7\}$, $BB(\mathcal{G}, p_7) = 0$ and $BB(\mathcal{F}, p_j) = 8/3$, $1 \le j \le 6$.

Proof. Assume by contradiction that there exists a foliation $\mathcal{G}$ as above. We will prove first that there is no $\mathcal{G}$-invariant straight line through p_7.

Assume by contradiction that there is a $\mathcal{G}$-invariant straight line ℓ through p_7. Since the singularities of $\mathcal{G}$ are non-degenerate, ℓ contains exactly three singularities of $\mathcal{G}$, p_1 and two others, say p_5 and p_6. In this case, by Camacho-Sad theorem we must have (cf. [Br])

$$CS(\mathcal{G}, \ell, p_5) + CS(\mathcal{G}, \ell, p_6) + CS(\mathcal{G}, \ell, p_7) = C_1(\ell)^2 = 1\ . \tag{15}$$

In our case, $CS(\mathcal{G}, p_j)$ coincides with one of the characteristic values of $\mathcal{G}$ at p_j, $5 \le j \le 7$. The characteristic values of $\mathcal{G}$ at p_5 and p_6 are $\frac{1+2\sqrt{2}i}{3}$ and $\frac{1-2\sqrt{2}i}{3}$, whereas the characteristic value at p_7 is -1. Therefore, (15) is impossible and there is no $\mathcal{G}$-invariant straight line through p_7.

In particular, there exists an affine coordinate system $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ where $\mathcal{G}$ can be represented by a vector field X_Λ, where $\Lambda = (\lambda, A, B, \alpha, \beta, \gamma)$, $\lambda = -1$. Since $\lambda \notin \{5, 1/5\}$ we are in case 3. In particular, we get $A = B = 0$ and α, β, γ satisfy (14). However, the first equation in (14) has no solution if $\lambda = -1$. This finishes the proof of Assertion 2.1. $\square$

Let us continue the analysis of case 3. Since $\lambda \neq -1$ the first relation in (14) implies that

$$\beta = -\frac{(\lambda^2 + 18\,\lambda + 1)\,(\lambda - 1)^2}{(\lambda + 1)^3} := \beta(\lambda)\ .$$

Since the polynomials $\lambda^3 + 12\,\lambda^2 - 3\,\lambda + 2$ and $2\,\lambda^3 - 3\,\lambda^2 + 12\,\lambda + 1$ have no common roots, from the second and third relations in (14) we get three sub-cases :

Case 3.1. $\lambda^3 + 12\,\lambda^2 - 3\,\lambda + 2 \neq 0$ and $2\,\lambda^3 - 3\,\lambda^2 + 12\,\lambda + 1 \neq 0 \implies \alpha = \gamma = 0$.

Case 3.2. $\lambda^3 + 12\,\lambda^2 - 3\,\lambda + 2 = 0$ and $2\,\lambda^3 - 3\,\lambda^2 + 12\,\lambda + 1 \neq 0 \implies \gamma = 0$ and $\alpha \in \mathbb{C}$.

Case 3.3. $\lambda^3 + 12\,\lambda^2 - 3\,\lambda + 2 \neq 0$ and $2\,\lambda^3 - 3\,\lambda^2 + 12\,\lambda + 1 = 0 \implies \alpha = 0$ and $\gamma \in \mathbb{C}$.

Analysis of case 3.1. In this case, we get $\Lambda = (\lambda, 0, 0, 0, \beta(\lambda), 0) := \Lambda(\lambda)$. In particular, if we set

$$X_\lambda = \left(\lambda\, x + (1-\lambda)\, y^2 + \beta(\lambda)\, x^2\, y\right)\, \partial_x + \left(y + (\lambda - 1)\, x^2 + \beta(\lambda)\, x\, y^2\right)\, \partial_y$$

then the foliation $\mathcal{F}_{\Lambda(\lambda)} = \mathcal{F}_{X_\lambda}$ satisfies (A) of Theorem 2.

Let us prove (B) of Theorem 2. From Remark 2.1 we get $\sigma^*(\mathcal{F}_{\Lambda(\lambda)}) = \mathcal{F}_{\hat\sigma(\Lambda(\lambda))}$, where $\hat\sigma(\Lambda(\lambda)) =$

$$\hat\sigma(\lambda, 0, 0, 0, \beta(\lambda), 0) = (\lambda^{-1}, 0, 0, 0, \lambda^{-1}.\, \beta(\lambda), 0) = (\lambda^{-1}, 0, 0, 0, \beta(\lambda^{-1}), 0) = \Lambda(\lambda^{-1})\ .$$

In particular, the unique solutions of identity (12) with $\lambda, \lambda^{-1} \notin \mathcal{A}$ and required Baum-Bott indexes are $\Lambda(\lambda)$ and $\Lambda(\lambda^{-1})$. In particular, $F_2\left(\mathcal{F}_{\Lambda(\lambda)}\right) \cap \mathcal{W} = \{\mathcal{F}_{\Lambda(\lambda)}, \mathcal{F}_{\Lambda(\lambda^{-1})}\}$. Since $\mathcal{F}_{\Lambda(\lambda^{-1})} = \sigma^*(\mathcal{F}_{\Lambda(\lambda)}) \in \mathcal{O}rb\left(\mathcal{F}_{\Lambda(\lambda)}\right)$, we get

$$F_2\left(\mathcal{F}_{\Lambda(\lambda)}\right) \cap \mathcal{W} = \mathcal{O}rb\left(\mathcal{F}_{\Lambda(\lambda)}\right) \cap \mathcal{W}\ . \tag{16}$$

This proves (B) of Theorem 2.

Analysis of cases 3.2 and 3.3. In case 3.2 we get $\Lambda = (\lambda, 0, 0, \alpha, \beta(\lambda), 0)$, where $\alpha \in \mathbb{C}$, $\lambda^3 + 12\,\lambda^2 - 3\,\lambda + 2 = 0$ and $\beta(\lambda) = -(\lambda^2 + 18\,\lambda + 1)(\lambda - 1)^2/(\lambda + 1)^3$.

Let $\lambda_1, \lambda_2, \lambda_3$ be the roots of $s^3 + 12\,s^2 - 3\,s + 2 = 0$. For each $i = 1, 2, 3$ and each $\alpha \in \mathbb{C}$ we get the foliation $\mathcal{F}_{i\alpha} := \mathcal{F}_{X_{i\alpha}}$, where $X_{i\alpha} = P_{i\alpha}\, \partial_x + Q_{i\alpha}\, \partial_y$ and

$$\begin{aligned} P_{i\alpha} &= \lambda_i\, x + (1 - \lambda_i)\, y^2 + x\, \left(\alpha\, x^2 + \beta(\lambda_i)\, x\, y\right) \\ Q_{i\alpha} &= y + (\lambda_i - 1)\, x^2 + y\, \left(\alpha\, x^2 + \beta(\lambda_i)\, x\, y\right) \end{aligned}$$

In this way, we get three one-parameter families of foliations $\mathcal{E}_i := (\mathcal{F}_{i\alpha})_{\alpha \in \mathbb{C}} \subset \mathbb{F}\mathrm{ol}(2, 2)$, $i = 1, 2, 3$. For fixed $i \in \{1, 2, 3\}$ and $\alpha \in \mathbb{C}$ set $sing(\mathcal{F}_{i\alpha}) = \{0, p_1^i(\alpha), ..., p_6^i(\alpha)\}$. If $\mathcal{F}_{i\alpha} \in \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2, 2)$ then :

- $BB(\mathcal{F}_{i\alpha},0)=(\lambda_i+1)^2/\lambda_i$ and
- $BB(\mathcal{F}_{i\alpha},p^i_j(\alpha))=(16-BB(\mathcal{F}_{i\alpha},0))/6=-(\lambda_i^2-14\,\lambda_i+1)/6\,\lambda_i$, if $2\le j\le 7$.

Since λ_i is a root of $s^3+12\,s^2-3\,s+2=0$, we have

$$BB(\mathcal{F}_{i\alpha},p^i_j(\alpha))=\frac{1}{12}(\lambda_i+5)^2\ ,\ 2\le j\le 7\ ,$$

because

$$-\frac{s^2-14\,s+1}{6\,s}=\frac{1}{12}(s+5)^2\quad md(s^3+12\,s^2-3\,s+2)$$

In particular,

$$\mathcal{BB}(\mathcal{F}_{i\alpha})=\left[\frac{(\lambda_i+1)^2}{\lambda_i},\frac{1}{12}\,(\lambda_i+5)^2,...,\frac{1}{12}(\lambda_i+5)^2\right]:=M_i$$

and

$$\mathcal{E}_i\subset\mathcal{BB}^{-1}(M_i)\implies \mathcal{S}at(\mathcal{E}_i)\subset\mathcal{BB}^{-1}(M_i)\ .$$

As the reader can check, for any $\alpha\in\mathbb{C}$ the group $\mathcal{I}so(\mathcal{F}_{i\alpha})$ is finite. This implies that

$$dim(\mathcal{S}at(\mathcal{E}_i))=9\implies$$

$\mathcal{BB}^{-1}(M_i)$ is an exceptional fiber.

Finaly, case 3.3 can be reduced to case 3.2 by using Remark 2.1 as we have done to reduce case 2 to case 1. This finishes the proof of Theorem 2.

2.3. **Proof of Corollary 1.** Let $\mathcal{G}\in\mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ with $sing(\mathcal{G})=\{p_1,...,p_7\}$, where

- The charachteristic values of $\mathcal{G}$ at p_7 are λ, λ^{-1}, where $\lambda\notin\{1,-1\}\cup\mathcal{A}$.
- $BB(\mathcal{G},p_i)=BB(\mathcal{G},p_j)\neq 0$ if $1\le i<j\le 6$.
- If ρ,ρ^{-1} are the characteristic values of $\mathcal{G}$ at any of the points $p_1,...,p_6$ then condition (3) of the hypothesis of Corollary 1 is verified.

We would like to observe that condition (3) and Camacho-Sad theorem imply that there is no $\mathcal{G}$-invariant straight line through p_7. It follows from lemma 2.1 that there exists $\mathcal{F}_\Lambda\in\mathcal{O}rb(\mathcal{G})\cap\mathcal{W}$, with $\Lambda=(\lambda,A,B,\alpha,\beta,\gamma)$. Note that $F_2(\mathcal{G})=F_2(\mathcal{F}_\Lambda)$.

On the other hand, Λ satisfies the conditions of case 3.1 because $\lambda,\lambda^{-1}\notin\mathcal{A}$. In particular, $\Lambda=(\lambda,0,0,0,\beta(\lambda),0)$ and (16) implies

$$F_2(\mathcal{G})\cap\mathcal{W}=F_2(\mathcal{F}_\Lambda)\cap\mathcal{W}=\{\mathcal{F}_\Lambda,\sigma^*(\mathcal{F}_\Lambda)\}\subset\mathcal{O}rb(\mathcal{G})\implies$$

$$F_2(\mathcal{G})=\mathcal{S}at(F_2(\mathcal{G})\cap\mathcal{W})\subset\mathcal{S}at(\mathcal{O}rb(\mathcal{G}))=\mathcal{O}rb(\mathcal{G})\implies$$

$$\mathcal{O}rb(\mathcal{G})=F_2(\mathcal{G})\ .\quad\square$$

2.4. **Proof of Proposition 1.** Consider the family of foliations $(\mathcal{F}_\alpha)_{\alpha\in\mathbb{C}}$ of Proposition 1. Recal that $\mathcal{F}_\alpha$ is defined by the vector field $X_\alpha=P_\alpha\,\partial_x+Q_\alpha\,\partial_y$, where

$$\begin{array}{c}P_\alpha(x,y)=-5\,x+\alpha\,x^2+6\,y^2+x\,\left(-\frac{5}{64}\,\alpha^2\,x^2-36\,x\,y-\frac{3}{8}\,\alpha\,y^2\right)\\ Q_\alpha(x,y)=y-6\,x^2+\alpha\,x\,y+y\,\left(-\frac{5}{64}\,\alpha^2\,x^2-36\,x\,y-\frac{3}{8}\,\alpha\,y^2\right)\end{array}\ .$$

Fix $\alpha\in\mathbb{C}$ such that $\mathcal{F}:=\mathcal{F}_\alpha\in\mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ and set $sing(\mathcal{F})=\{p_1,...,p_6,p_7=0\}$. Since $BB(\mathcal{F},0)=-16/5$ and $BB(\mathcal{F},p_j)=16/5$ if $1\le j\le 6$, its charachteristic values are :

(i). $\lambda_{71}=-5$, $\lambda_{72}=-1/5$ at p_7.
(ii). $\lambda_{j1}=(3+4\,i)/5$ and $\lambda_{j2}=(3-4\,i)/5$ at p_j, $1\le j\le 6$.

Since $\lambda_{ji}\notin\mathbb{Q}_+$, $1\le j\le 7$, $i=1,2$, we get (iii). There are exactly two analytic separatrices, say S_{ji}, $i = 1, 2$, of $\mathcal{F}$ through p_j, which are smooth, $1 \le j \le 7$. Moreover, we can choose the charachteristic values at p_j in such a way that $CS(\mathcal{F}, S_{ji}) = \lambda_{ji}$, $1 \le j \le 7$, $1 = 1, 2$, where CS denotes the Camacho-Sad index (cf. [C-S]).

Now, suppose by contradiction that $\mathcal{F}$ has an irreducible invariant algebraic curve, say Z. Let

$$A(Z) = \{(j, i) \mid 1 \le j \le 7 \;,\; 1 \le i \le 2 \text{ and } Z \supset S_{ji}\} \;.$$

It follows from a version of Camacho-Sad theorem in [LN] that

(iv). $A(Z) \neq \emptyset$ and $A(Z)$ is a proper subset of $\{(j, i) \,|\, 1 \le j \le 7 \;,\; i = 1, 2\}$.

(v). $\sum_{(j,i) \in A(Z)} \lambda_{ji} = 3\, dg(Z) - \mathcal{X}(Z^*) \in \mathbb{Z}_+$, where $\mathcal{X}(Z^*)$ denotes the Euler charachteristic of the normalization of Z.

As the reader can check, there are only three possibilities for the above sum to be a positive integer :

$$\begin{array}{rr} 1^{st}: & -\frac{1}{5} + \left(\frac{3}{5} + \frac{4}{5}\, i\right) + \left(\frac{3}{5} - \frac{4}{5}\, i\right) = 1 \\ 2^{nd}: & -5 + 5 \times \left[\left(\frac{3}{5} + \frac{4}{5}\, i\right) + \left(\frac{3}{5} - \frac{4}{5}\, i\right)\right] = 1 \\ 3^{rd}: & 5 \times \left[\left(\frac{3}{5} + \frac{4}{5}\, i\right) + \left(\frac{3}{5} - \frac{4}{5}\, i\right)\right] = 6 \end{array}$$

In the first two cases we get

$$3\, dg(Z) - \mathcal{X}(Z^*) = 1 \quad \Longrightarrow$$

Z is a $\mathcal{F}$-invariant straight line. since -5 and $-1/5$ are the charachteristic values of $\mathcal{F}$ at 0, we get $0 \in Z$. But, $\mathcal{F}$ has no invariant straight line through 0, and so the first and second cases cannot happen for the curve Z.

In the third case, the curve Z contains five separatrixes with Camacho-Sad index $(3 + 4\, i)/5$ and five with index $(3 - 4\, i)/5$. Since $\mathcal{F}$ has six singularities with these charachteristic numbers, the curve Z must contain $k \in \{4, 5\}$ pairs of separatrices through the same singularity. These points are nodal singularities of Z. If we set $dg(Z) = d$ then the genus formula for nodal curves implies that

$$g(Z^*) = \frac{(d-1)(d-2)}{2} - k \ge 0 \quad \Longrightarrow \quad d^2 - 3\, d + 2 \ge 2\, k \ge 8 \quad \Longrightarrow \quad d \ge 5 \;.$$

This also implies that

$$\mathcal{X}(Z^*) = 2 - 2\, g(Z^*) = -d^2 + 3\, d + 2\, k \;.$$

Therefore, from $3\, d - \mathcal{X}(Z^*) = 6$, we get

$$3\, d - (-d^2 + 3\, d + 2\, k) = 6 \quad \Longrightarrow \quad d^2 = 6 + 2\, k \quad \Longrightarrow \quad d = 4 \text{ and } k = 5 \;,$$

a contradiction. This finishes the proof of Proposition 1.

2.5. **Proof of Theorem 1.** The idea is to use Corollary 2 and the fact that the isotropy group of the Jouanolou's foliation of degree two has 21 elements.

Jouanolou's foliation of degree two, J_2, can be defined in some affine coordinate system $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$ by the vector field

$$X_J := \left(1 - x\, y^2\right)\, \partial_x + \left(x^2 - y^3\right)\, \partial_y \;.$$

It can be shown that $\mathcal{I}so(J_2) = \langle\, S, T\,\rangle$, where

$$S(x,y) = \left(\frac{1}{y}, \frac{x}{y}\right)$$
$$T(x,y) = (\zeta^{-2}\, x, \zeta\, y) \quad , \; \zeta = e^{2\pi i/7} \; ,$$

in the above affine coordinates (cf. [LN-JP] pg. 1566). For instance, the reader can check that $T^*(X_J) = \zeta^2\, X_J$ and $S^*(X_J) = y^{-1}.\, X_J$.

The singular set of J_2 is the orbit of $p_1 = (1,1)$ by T :

$$sing(J_2) = \{p_1, ..., p_7\} \;, \text{ where } p_j := T^{j-1}(1,1) \;, \; 1 \le j \le 7 \;.$$

This implies that

$$BB(J_2, p_i) = BB(J_2, p_j) \;, \; \forall\, 1 \le i < j \le 7 \;,$$

and so,

$$BB(J_2, p_j) = \frac{16}{7} \;, \; \forall\; j = 1, ..., 7 \;,$$

by the Baum-Bott theorem.

Remark 2.5. The charachteristic values of the Jouanolou foliation J_2 at any of its singularities are λ_o and $\lambda_o^{-1} = \overline{\lambda_o}$, where $\lambda_o = \frac{1+4\sqrt{3}\,i}{7}$. Since J_2 satisfies the hypothesis of Corollary 1 we have :

- $\mathcal{O}rb(J_2) = F_2(J_2)$.
- $\mathcal{O}rb(J_2) \cap \mathcal{W} = \{\mathcal{F}_{\Lambda_1}, \mathcal{F}_{\Lambda_2}\}$, where $\Lambda_1 = (\lambda_o, 0, 0, 0, \beta(\lambda_o), 0)$ and $\Lambda_2 = \big(\overline{\lambda_o}, 0, 0, 0, \beta(\overline{\lambda_o}), 0\big)$.

Since all singularities of J_2 are non-degenerate we can find a neighborhood $\mathcal{U}_1$ of J_2 in $\mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ and holomorphic functions $P_j\colon \mathcal{U}_1 \to \mathbb{P}^2$, $1 \le j \le 7$, with the following properties :

(i). $P_j(J_2) = p_j$, $1 \le j \le 7$.
(ii). For all $\mathcal{F} \in \mathcal{U}_1$ we have $sing(\mathcal{F}) = \{P_1(\mathcal{F}), ..., P_7(\mathcal{F})\}$.

For each $j \in \{1, ..., 7\}$ fix a small ball W_j in the Fubini-Study metric, centered at p_j, $1 \le j \le 7$, in such a way that $W_i \cap W_j = \emptyset$ if $i \ne j$. By taking $\mathcal{U}_1$ small we can assume that

(iii). $P_j(\mathcal{U}_1) \subset W_j$, $1 \le j \le 7$.

Define $B \,:\, \mathcal{U}_1 \to \mathbb{C}^7$ by

$$B(\mathcal{F}) = (BB(\mathcal{F}, P_1(\mathcal{F})), ..., BB(\mathcal{F}, P_7(\mathcal{F})) := (B_1(\mathcal{F}), ..., B_7(\mathcal{F})) \;.$$

By Baum-Bott theorem we have $B(\mathcal{U}_1) \subset \Sigma$, where

$$\Sigma = \left\{ (b_1, ..., b_7) \in \mathbb{C}^7 \;\Big|\; \sum_{j=1}^{7} b_j = 16 \right\} \;.$$

On the other hand, in Theorem 2 of [LN-JP] it is proved that the map B has rank six at J_2. As a consequence, there exist neighborhoods $\mathcal{U}_2 \subset \mathcal{U}_1$ of J_2 in $\mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ and V_2 of $(16/7, ..., 16/7)$ in Σ such that $B|_{\mathcal{U}_2} \,:\, \mathcal{U}_2 \to V_2$ has rank six.

Recall that $\mathcal{O}rb(J_2)$ is a smooth submanifold of $\mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ of dimension eight. A *transverse section* to $\mathcal{O}rb(J_2)$ through J_2 is, by definition, the immage Γ of an embedding $f \,:\, U \to \mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$, where $U \subset \mathbb{C}^6$ is a neighborhood of $0 \in \mathbb{C}^6$, such that $f(0) = J_2$ and Γ is transverse to $\mathcal{O}rb(J_2)$ at J_2.

Let us fix some notations :

(I). $b_0 := B(J_2) = (16/7, ..., 16/7)$.

(II). Given $b = (b_1, ..., b_7) \in \mathbb{C}^7$ we will denote $[b] = [b_1, ..., b_7]$, the immage of b by the symmetrization map $\mathbb{C}^7 \to \mathbb{C}^7/S_7$.

(III). Given $\sigma \in S_7$, we will denote by $\hat{\sigma} : \mathbb{C}^7 \to \mathbb{C}^7$ the map defined by

$$\hat{\sigma}(b_1, ..., b_7) = (b_{\sigma(1)}, ..., b_{\sigma(7)}) \ .$$

Note that $\hat{\sigma}(\Sigma) = \Sigma$ for all $\sigma \in S_7$.

Lemma 2.4. *There exists a transverse section Γ to $\mathcal{O}rb(J_2)$ with the following properties :*

(A). *$\Gamma \subset \mathcal{U}_2$. Moreover, if we set $V := B(\Gamma) \subset \Sigma$ then $B|_\Gamma : \Gamma \to V$ is a biholomorphism.*

(B). *Γ is $\mathcal{I}so(J_2)$-invariant. In other words, $\varphi^*(\Gamma) = \Gamma$ for all $\varphi \in \mathcal{I}so(J_2)$.*

(C). *V is invariant by the action $(\sigma, b) \in S_7 \times \mathbb{C}^7 \mapsto \hat{\sigma}(b) \in \mathbb{C}^7$. In other words, $\hat{\sigma}(V) = V$ for all $\sigma \in S_7$.*

(D). *If $\mathcal{F}_1, \mathcal{F}_2 \in \Gamma$ are in the same orbit then there is $\varphi \in \mathcal{I}so(J_2)$ such that $\mathcal{F}_2 = \varphi^*(\mathcal{F}_1)$.*

The proof of lemma 2.4 will be done at the end of the section.

Corollary 2.3. *Let Γ and $V = B(\Gamma)$ be as in lemma 2.4. Given $b \in V$ set $N(b) =$ "the number of orbits contained in $\mathcal{BB}^{-1}[b]$ cutting Γ". Then :*

(A). *If $b \in V$ then $N(b)$ divides 240.*

(B). *If $b = (b_1, ..., b_7) \in V$ is such that $b_i \neq b_j$ for all $i \neq j$ then $N(b) = 240$.*

Proof. By (A) of lemma 2.4 the map $B|_\Gamma : \Gamma \to V$ is a biholomorphism. Given $b \in V$ we will denote $\mathcal{F}_b := (B|_\Gamma)^{-1}(b) \in \Gamma$. With the above notations we have $\mathcal{F}_{b_0} = J_2$, $B(\mathcal{F}_b) = b$ and $\mathcal{BB}(\mathcal{F}_{\hat{\sigma}(b)}) = [b]$ for all $\sigma \in S_7$.

Let us introduce a group homomorphism $\Phi : \mathcal{I}so(J_2) \to S_7$. Given $\mathcal{F} \in \Gamma$ and $\varphi \in \mathcal{I}so(J_2)$ we have $\varphi^*(\mathcal{F}) \in \Gamma$, by (B) of lemma 2.4. If Y is a vector field defining $\mathcal{F}$ then the vector field $\varphi^*(Y) = (d\varphi)^{-1}.Y \circ \varphi$ defines $\varphi^*(\mathcal{F})$. In particular,

$$p \in sing(\varphi^*(Y)) \iff \varphi(p) \in sing(Y) \implies$$

$$p \in sing(\varphi^*(\mathcal{F})) \iff \varphi(p) \in sing(\mathcal{F}) \implies sing(\mathcal{F}) = \varphi(sing(\varphi^*(\mathcal{F}))) \ .$$

Since

$$sing(\mathcal{F}) = \{P_1(\mathcal{F}), ..., P_7(\mathcal{F})\} \text{ and } sing(\varphi^*(\mathcal{F})) = \{P_1(\varphi^*(\mathcal{F})), ..., P_7(\varphi^*(\mathcal{F}))\} \ ,$$

there exists an unique permutation $\Phi(\varphi) \in S_7$ such that

$$P_{\Phi(\varphi)(j)}(\mathcal{F}) = \varphi(P_j(\varphi^*(\mathcal{F}))) \ , \ 1 \le j \le 7 \ . \tag{17}$$

It can be checked by using (17) that Φ is a group homomorphism and that

$$\Phi(T)(1, ..., 7) = (7, 1, 2, 3, 4, 5, 6) \text{ and } \Phi(S)(1, ..., 7) = (1, 5, 2, 6, 3, 7, 4) \ .$$

This implies that $\Phi : J_2 \to \Phi(J_2) = \langle \Phi(T), \Phi(S) \rangle$ is a group isomorphism. In particular,

$$\# \, \Phi(J_2) = 21 \ .$$

Remark 2.6. If $\mathcal{F} \in \Gamma$ and $\varphi \in \mathcal{I}so(J_2)$ then, with the notation of (III), we have

$$B(\varphi^*(\mathcal{F})) = \widehat{\Phi(\varphi)}(B(\mathcal{F})) \ .$$

Proof. If $\mathcal{F} \in \Gamma$ and $\varphi \in \mathcal{I}so(J_2)$ then it follows from (17) and the definitions that,

$$B(\varphi^*(\mathcal{F})) = (BB(\varphi^*(\mathcal{F}), P_1(\varphi^*(\mathcal{F}))), ..., BB(\varphi^*(\mathcal{F}), P_7(\varphi^*(\mathcal{F})))) =$$

$$= (BB(\varphi^*(\mathcal{F}), \varphi^{-1}(P_{\Phi(\varphi)(1)}(\varphi^*(\mathcal{F})))), ..., BB(\varphi^*(\mathcal{F}), \varphi^{-1}(P_{\Phi(\varphi)(7)}(\varphi^*(\mathcal{F}))))) =$$

$$= (BB(\mathcal{F}, P_{\Phi(\varphi)(1)}(\mathcal{F})), ..., BB(\mathcal{F}, P_{\Phi(\varphi)(7)}(\mathcal{F}))) = \widehat{\Phi(\varphi)}(B(\mathcal{F}) \quad \square$$

Now, fix $b = (b_1, ..., b_7) \in V$. Clearly,

$$\mathcal{BB}^{-1}[b] \cap \Gamma = \{\mathcal{F}_{\hat{\sigma}(b)} \mid \sigma \in S_7\} \implies \#(\mathcal{BB}^{-1}[b] \cap \Gamma) = 7!\ .$$

On the other hand, if $\sigma_1, \sigma_2 \in S_7$ are such that $\mathcal{F}_{\hat{\sigma}_1(b)}$ and $\mathcal{F}_{\hat{\sigma}_2(b)}$ are in the same orbit then there exists $\varphi \in \mathcal{I}so(J_2)$ such that

$$\varphi^*(\mathcal{F}_{\hat{\sigma}_1(b)}) = \mathcal{F}_{\hat{\sigma}_2(b)}\ ,$$

by (D) of lemma 2.4. It follows from Remark 2.6 that

$$\hat{\sigma}_2(b) = B(\mathcal{F}_{\hat{\sigma}_2(b)}) = B(\varphi^*(\mathcal{F}_{\hat{\sigma}_1(b)})) = \widehat{\Phi(\varphi)}(B(\mathcal{F}_{\hat{\sigma}_1(b)})) = \widehat{\Phi(\varphi)}(\hat{\sigma}_1(b)) = \Phi(\widehat{\varphi) \circ \sigma_1}(b)\ .$$

If we assume that $b_i \neq b_j$ for $i \neq j$, then the above relation implies that $\mathcal{F}_{\hat{\sigma}_1(b)}$ and $\mathcal{F}_{\hat{\sigma}_2(b)}$ are in the same orbit if, and only if, $\sigma_2 = \Phi(\varphi) \circ \sigma_1$, for some $\varphi \in \mathcal{I}so(J_2)$. In particular, we obtain that $N(b) =$ "the number of lateral classes of the subgroup $\Phi(\mathcal{I}so(J_2))$ in S_7" : $\frac{7!}{21} = 240$.

In the general case, we obtain that $\mathcal{F}_{\hat{\sigma}_1(b)}$ and $\mathcal{F}_{\hat{\sigma}_2(b)}$ are in the same orbit if, and only if, the permutation $\sigma := \sigma_2^{-1} \circ \Phi(\varphi) \circ \sigma_1$ satisfies $\hat{\sigma}(b) = b$. This implies that $N(b) =$ "the number of lateral classes of $\Phi(J_2)$ in some subgroup of S_7 that contains $\Phi(J_2)$", which is a divisor of 240. $\square$

The proof of Theorem 1 will be achieved if we show that there exists a neighborhood $V_1 \subset V$ of b_0 in Σ such that for any $b \in V_1$ then any orbit contained in $\mathcal{BB}^{-1}[b]$ cuts the transverse section Γ. In fact, if this is true then :

- for any $b \in V_1$ the number of orbits contained in $\mathcal{BB}^{-1}[b]$ is at most 240.
- for any $b = (b_1, ..., b_7) \in V_1$, with $b_i \neq b_j$ for all $i \neq j$, the number of orbits contained in $\mathcal{BB}^{-1}[b]$ is exactly 240.

By using the above facts, Theorem 1 will be a consequence of the following :

- if the fiber of $b \in \Sigma$ is not exceptional then each orbit contained in $\mathcal{BB}^{-1}[b]$ is an irreducible component of $\mathcal{BB}^{-1}[b]$.
- if the fibers of $b, b' \in \Sigma$ are not exceptional and the fiber of b is generic then the number of irreducible components of $\mathcal{BB}^{-1}[b']$ is $\leq$ the number of irreducible components of $\mathcal{BB}^{-1}[b]$.
- the set $\{b \in \Sigma \,|\, \mathcal{BB}^{-1}[b]$ is a generic fiber$\}$ is open and dense in Σ.
- the set $\{b \in V_1 \,|\, b = (b_1, ..., b_7)$ and $b_i \neq b_j$ if $i \neq j\}$ is open and dense in V_1.

The existence of a neighborhood V_1 as above will be a consequence of the next result.

Lemma 2.5. *Let $(\mathcal{F}_n)_{n\geq 1}$ be a sequence in $\mathbb{F}\mathrm{ol}_{\mathrm{nd}}(2,2)$ such that $\underset{n\to\infty}{\ell im}\, \mathcal{BB}(\mathcal{F}_n) = [b_0]$. Then there exists $n_o \in \mathbb{N}$ such that $\mathcal{O}rb(\mathcal{F}_n) \cap \Gamma \neq \emptyset$ for all $n \geq n_o$.*

Proof. Since Γ is a transverse to $\mathcal{O}rb(J_2)$ at J_2, it is sufficient to prove that J_2 is in the adherence of $\bigcup_n \mathcal{O}rb(\mathcal{F}_n)$. We will prove first that $\mathcal{F}_n$ has at least one singularity satisfying conditions (a) and (b) of the hypothesis of lemma 2.1, for n large. Set $sing(\mathcal{F}_n) = \{p_1^n, ..., p_7^n\}$.

Since $\lim\limits_{n\to\infty} \mathcal{BB}(\mathcal{F}_n) = [16/7, ..., 16/7]$ we get

$$\lim_{n\to\infty} BB(\mathcal{F}_n, p_j^n) = \frac{16}{7} \ , \ \forall j \in \{1, ..., 7\} \ .$$

In particular, there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$ then $BB(\mathcal{F}_n, p_j^n) \neq 4$, $1 \leq j \leq 7$. Therefore, Corollary 2.2 implies that $\mathcal{F}_n$ has at least one singularity satisfying conditions (a) and (b) and we can apply lemma 2.1 to $\mathcal{F}_n$ for large n.

Without lost of generality, we can assume that the singularity p_7^n of $\mathcal{F}_n$ satisfies conditions (a) and (b) for all $n \in \mathbb{N}$. Let $\lambda_n, \lambda_n^{-1}$ be the charachteristic values of $\mathcal{F}_n$ at p_7^n. Since $\lim\limits_{n\to\infty} BB(\mathcal{F}_n, p_7^n) = 16/7$, we get $\lim\limits_{n\to\infty} \{\lambda_n, \lambda_n^{-1}\} = \{\lambda_o, \overline{\lambda}_o\}$, where $\lambda_o = (1 + 4\sqrt{3}i)/7$. Without lost of generality, we can assume that $\lim\limits_{n\to\infty} \lambda_n = \lambda_o$. In particular, if we fix an affine coordinate system $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$, then there exists $\Lambda_n = (\lambda_n, A_n, B_n, \alpha_n, \beta_n, \gamma_n) \in \mathbb{C}^6$ such that $\mathcal{G}_n := \mathcal{F}_{\Lambda_n} \in \mathcal{O}rb(\mathcal{F}_n)$. In this coordinate syatem $\mathcal{G}_n$ is represented by the vector field $X_n = P_n\partial_x + Q_n\partial_y$, where

$$\begin{cases} P_n = \lambda_n\, x + A_n\, x^2 + B_n\, x\, y + (1 - \lambda_n)y^2 + x\,(\alpha_n\, x^2 + \beta_n\, x\, y + \gamma_n\, y^2) \\ Q_n = y + (\lambda_n - 1)x^2 + A_n\, x\, y + B_n\, y^2 + y\,(\alpha_n\, x^2 + \beta_n\, x\, y + \gamma_n\, y^2) \end{cases} .$$

It is enough to prove the following :

Assertion 2.2. $\lim\limits_{n\to\infty} \Lambda_n = (\lambda_o, 0, 0, 0, \beta(\lambda_o), 0) := \Lambda_o$. *In particular,* $\lim\limits_{n\to\infty} \mathcal{G}_n = \mathcal{F}_{\Lambda_o} \in \mathcal{O}rb(J_2)$.

Proof. Assume first that the sequence $(\Lambda_n)_{n\geq 1}$ is bounded in $\mathbb{C}^6$. In this case, it is enough to prove that any convergent subsequence of $(\Lambda_n)_{n\geq 1}$ converges to Λ_o. Without lost of generality we will suppose that $\lim\limits_{n\to\infty} \Lambda_n = (\lambda_o, A_o, B_o, \alpha_o, \beta_o, \gamma_o) = \tilde{\Lambda}_o \in \mathbb{C}^6$. Let $\omega_n := \omega_{\Lambda_n}$ be as in Corollary 2.1,

$$\omega_n = \frac{q_n^2(t)}{(1 - \lambda_n)\, t\, p_n(t)}\, dt \ ,$$

where

$$\begin{cases} p_n(t) = t^6 + B_n t^5 + C_n t^4 + D_n t^3 + E_n t^2 + A_n t + \lambda_n \\ q_n(t) = (3 - \lambda_n)t^3 + B_n t^2 + A_n t + 3\lambda_n - 1 \end{cases} , \tag{18}$$

with $C_n = A_n + \gamma_n$, $D_n = \lambda_n + \beta_n + 1$ and $E_n = B_n + \alpha_n$. In particular,

$$\lim_{n\to\infty} \omega_n = \omega_{\tilde{\Lambda}_o} = \frac{q_o^2(t)}{(1 - \lambda_o)\, t\, p_o(t)}\, dt := \omega_o \ ,$$

where $p_o(t)$ and $q_o(t)$ are as in (18) with $n = o$. Denote by $\tau_1^o, ..., \tau_6^o$ the roots of $p_o(t) = 0$. Let $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the birrational transformation of lemma 2.2. Choose the roots $\tau_1^n, ..., \tau_6^n$ of $p_n(t) = 0$ and the singularities $q_1^n, ..., q_6^n$ of $\mathcal{G}_n$ in such a way that $\Phi(\tau_j^n) = q_j^n$, $1 \leq j \leq 6$, $n \in \mathbb{N}$. Since $Res(\omega_n, 0) = (3 - \lambda_n)^2/(\lambda_n.(1 - \lambda_n) := \mu_n$ and $Res(\omega_n, \tau_j^n) = BB(\mathcal{G}_n, q_j^n)$, $1 \leq j \leq 6$, $n \in \mathbb{N}$, we can write

$$\omega_n = \left(\frac{\mu_n}{t} + \sum_{j=1}^{6} \frac{BB(\mathcal{G}_n, q_j^n)}{t - \tau_j^n}\right) dt \ . \tag{19}$$

Since $\underset{n\to\infty}{\ell im}\, BB(\mathcal{G}_n, q_j^n) = 16/7$ for all $j \in \{1, ..., 6\}$ and the roots of $p_n(t) = 0$ converge to the roots of $p_o(t) = 0$ we obtain from (19) :

$$\underset{n\to\infty}{\ell im}\, \omega_n = \left(\frac{\mu_o}{t} + \frac{16}{7}\sum_{j=1}^{6} \frac{1}{t - \tau_j^o}\right) dt \ , \ \mu_o = \frac{(3-\lambda_o)^2}{\lambda_o(1-\lambda_o)} \ .$$

This implies

$$\frac{\mu_o}{t} + \frac{16}{7}\frac{p_o'(t)}{p_o(t)} \equiv \frac{q_o^2(t)}{(1-\lambda_o)\, p_o(t)} \ .$$

On the other hand, since $\lambda_o \notin \mathcal{A}$, as we have seen in the proof of Theorem 2, the above equation in p_o and q_o implies that $\tilde{\Lambda}_o = (\lambda_o, 0, 0, 0, \beta(\lambda_o), 0) = \Lambda_o$. Therefore, $\underset{n\to\infty}{\ell im}\, \Lambda_n = \Lambda_o$.

Let us assume, by contradiction, that the sequence $(\Lambda_n)_{n\geq 1}$ is unbounded. It follows from (18) that the components of Λ_n are symmetric polynomial of $\tau^n := (\tau_1^n, ..., \tau_6^n)$, $n \in \mathbb{N}$. In particular, the sequence $(\tau^n)_{n\geq 1}$ is unbounded. By taking subsequences and reordering the roots, if necessary, we can assume that :

I. $\underset{n\to\infty}{\ell im}\, \tau_1^n = \infty$.

II. The sequences $(\tau_j^n)_{n\geq 1}$ converge in $\mathbb{P}^1$, $2 \leq j \leq 6$. In other words, $\underset{n\to\infty}{\ell im}\, \tau_j^n = \tau_j^o \in \mathbb{C} \cup \{\infty\}$, $2 \leq j \leq 6$.

Since $p_n(t)$ is monic and its constant coefficient is λ_n, we get

$$\Pi_{j=1}^6 \tau_j^n = \lambda_n \implies \underset{n\to\infty}{\ell im}\, \Pi_{j=1}^6 \tau_j^n = \lambda_o \notin \{0, \infty\} \implies$$

there exists $j \in \{2, ..., 6\}$ such that $\underset{n\to\infty}{\ell im}\, \tau_j^n = 0$. Set $k := \#\{j \,|\, \underset{n\to\infty}{\ell im}\, \tau_j^n = \infty\}$ and $\ell := \#\{j \,|\, \underset{n\to\infty}{\ell im}\, \tau_j^n = 0\} \geq 1$.

By reordering again the indexes $j = 2, ..., 6$, we can assume that

(III). $\underset{n\to\infty}{\ell im}\, \tau_j^n = \infty$, $1 \leq j \leq k$.

(IV). $\underset{n\to\infty}{\ell im}\, \tau_j^n = 0$, $k+1 \leq j \leq k+\ell$.

(V). If $k + \ell < 6$ then $\underset{n\to\infty}{\ell im}\, \tau_j^n = \tau_j^o \in \mathbb{C} \setminus \{0\}$, $k+\ell+1 \leq j \leq 6$.

Now, we use that $BB(\mathcal{G}_n, q_j^n) = Res(\omega_n, \tau_j^n)$ and $\underset{n\to\infty}{\ell im}\, BB(\mathcal{G}_n, q_j^n) = 16/7$, $1 \leq j \leq k$. Note that

$$BB(\mathcal{G}_n, q_j^n) = Res(\omega_n, \tau_j^n) = \frac{q_n^2(\tau_j^n)}{(1-\lambda_n)\, \tau_j^n\, p_n'(\tau_j^n)} =$$

$$= \frac{\left((3-\lambda_n)(\tau_j^n)^3 + B_n\, (\tau_j^n)^2 + A_n\, \tau_j^n + 3\lambda_n - 1\right)^2}{(1-\lambda_n)\tau_j^n \Pi_{i\neq j}(\tau_j^n - \tau_i^n)} \implies$$

$$BB(\mathcal{G}_n, q_j^n) = \frac{\left((3-\lambda_n) + B_n/\tau_j^n + A_n/(\tau_1^n)^2 + (3\lambda_n - 1)/(\tau_j^n)^3\right)^2}{(1-\lambda_n)\Pi_{i\neq j}(1 - \tau_i^n/\tau_j^n)} \ . \tag{20}$$

We will use also the relations below, that follow from (18),

$$B_n = -(\tau_1^n + ... + \tau_6^n) \text{ and } A_n = -\sum_{j_1<...<j_5} \tau_{j_1}^n ... \tau_{j_5}^n \ . \tag{21}$$

Let us prove that $k \geq 2$ and $\ell \geq 2$. Assume by contradiction that $k = 1$. In this case, $\underset{n\to\infty}{\ell im}\, \tau_1^n = \infty$ and $\underset{n\to\infty}{\ell im}\, \tau_j^n$ is finite for $j > 1$. Therefore, (21) implies $\underset{n\to\infty}{\ell im}\, \frac{A_n}{(\tau_1^n)^2} = 0$ and $\underset{n\to\infty}{\ell im}\, \frac{B_n}{\tau_1^n} = -1$. It follows from (20) and $\underset{n\to\infty}{\ell im}\, BB(\mathcal{G}_n, q_1^n) = 16/7$ that

$$\frac{16}{7} = \underset{n\to\infty}{\ell im} \left(\frac{\left((3-\lambda_n) + B_n/\tau_1^n + A_n/(\tau_1^n)^2 + (3\lambda_n - 1)/(\tau_1^n)^3\right)^2}{(1-\lambda_n)\Pi_{i\neq 1}(1 - \tau_i^n/\tau_1^n)} \right) = \frac{(2-\lambda_o)^2}{1-\lambda_o}$$

which contradicts $\lambda_o = (1 + 4\sqrt{3}\, i)/7$.

By the same reason, $\ell \geq 2$. In fact, by considering the change of variables $t = 1/s$ we get the expression of the form ω_n in a neighborhood of $t = \infty$:

$$\omega_n = \frac{\tilde{q}_n^2(s)}{(1 - 1/\lambda_n)\, s\, \tilde{p}_n(s)}\, ds \ ,$$

where

$$\tilde{p}_n(s) = s^6 + \tfrac{A_n}{\lambda_n} s^5 + \tfrac{C_n}{\lambda_n} s^4 + \tfrac{D_n}{\lambda_n} s^3 + \tfrac{E_n}{\lambda_n} s^2 + \tfrac{B_n}{\lambda_n} s + \tfrac{1}{\lambda_n}$$
$$\tilde{q}_n(s) = \left(3 - \tfrac{1}{\lambda_n}\right) s^3 + \tfrac{A_n}{\lambda_n} s^2 + \tfrac{B_n}{\lambda_n} s + \tfrac{3}{\lambda_n} - 1$$

Since the roots of $\tilde{p}_n(s) = 0$ are $1/\tau_j^n := \zeta_j^n$, $1 \leq j \leq 6$, we have $\ell = \#\{j \mid \underset{n\to\infty}{\ell im}\, \zeta_j^n = \infty\}$. Hence, by the same argument as before we get $\ell \geq 2$.

Let us prove that $k \geq 3$ and $\ell \geq 3$. Suppose by contradiction that $k = 2$. Since $\ell \geq 2$, we have $\underset{n\to\infty}{\ell im}\, \tau_1^n = \underset{n\to\infty}{\ell im}\, \tau_2^n = \infty$, $\underset{n\to\infty}{\ell im}\, \tau_3^n = \underset{n\to\infty}{\ell im}\, \tau_4^n = 0$ and $\underset{n\to\infty}{\ell im}\, \tau_j^n \in \mathbb{C}$ if $j > 4$. By taking subsequences and reordering again, if necessary, we can assume that $\underset{n\to\infty}{\ell im}\, \frac{\tau_2^n}{\tau_1^n} = x \in \mathbb{C}$. In this case, (21) implies that $\underset{n\to\infty}{\ell im}\, \frac{A_n}{(\tau_1^n)^2} = 0$ and $\underset{n\to\infty}{\ell im}\, \frac{B_n}{\tau_1^n} = -1 - x$. Therefore,

$$\underset{n\to\infty}{\ell im} \left(\frac{\left((3-\lambda_n) + B_n/\tau_1^n + A_n/(\tau_1^n)^2 + (3\lambda_n - 1)/(\tau_1^n)^3\right)^2}{(1-\lambda_n)\Pi_{i\neq 1}(1 - \tau_i^n/\tau_1^n)} \right) = \frac{(2-\lambda_o - x)^2}{(1-\lambda_o)(1-x)} \implies$$

$$\frac{16}{7} = \frac{(\lambda_o + 1)^2}{\lambda_o} = \frac{(2-\lambda_o-x)^2}{(1-\lambda_o)(1-x)} \ .$$

If $x = 0$ we get the same contradiction of the precedent case. On the other hand, if $x \neq 0$ then $\underset{n\to\infty}{\ell im}\, \frac{A_n}{(\tau_2^n)^2} = 0$ and $\underset{n\to\infty}{\ell im}\, \frac{B_n}{\tau_2^n} = -1 - 1/x$, which implies

$$\underset{n\to\infty}{\ell im} \left(\frac{\left((3-\lambda_n) + B_n/\tau_2^n + A_n/(\tau_2^n)^2 + (3\lambda_n - 1)/(\tau_2^n)^3\right)^2}{(1-\lambda_n)\Pi_{i\neq 2}(1 - \tau_i^n/\tau_2^n)} \right) = \frac{(2-\lambda_o - 1/x)^2}{(1-\lambda_o)(1-1/x)}$$

and so

$$\frac{(\lambda_o+1)^2}{\lambda_o} = \frac{(2-\lambda_o-x)^2}{(1-\lambda_o)(1-x)} = \frac{(2-\lambda_o-1/x)^2}{(1-\lambda_o)(1-1/x)} = \frac{16}{7} \ .$$

It can be checked that there is no $x \in \mathbb{C}$ satisfying the above relations. This contradiction proves that $k \geq 3$.

As in the preceding case, by considering the expression of ω_n after the change of variables $t = 1/s$, it can be proved that $\ell \geq 3$. We leave the details of this proof to the reader.

Therefore, we must have $k = \ell = 3$, so that $\lim\limits_{n\to\infty} \tau_j^n = \infty$, if $1 \leq j \leq 3$, and $\lim\limits_{n\to\infty} \tau_j^n = 0$, if $4 \leq j \leq 6$. In particular, from (19) we get $\lim\limits_{n\to\infty} \omega_n = \omega_o$, where

$$\omega_o = \frac{\mu_1}{t}\,dt \not\equiv 0 \ , \ \mu_1 = \mu_o + 3 \times \frac{16}{7} = \frac{64}{7} - \frac{80\sqrt{3}}{21}\,i \neq 0 \ .$$

The contradiction will be provided by a more general result, in which we don't assume $\lim\limits_{n\to\infty} BB(\mathcal{G}_n, q_j^n) = 16/7$, $1 \leq j \leq 6$.

Lemma 2.6. *Let* $\omega_n = \frac{q_n^2(t)}{(1-\lambda_n)\,t\,p_n(t)}\,dt$ *be as in (19) and* $\tau_1^n, ..., \tau_6^n$ *be the roots of* $p_n(t) = 0$. *Assume that :*

(1). $\lim\limits_{n\to\infty} \lambda_n = \lambda \in \mathbb{C} \setminus \{1\}$.
(2). $\lim\limits_{n\to\infty} \omega_n = \frac{\mu_1}{t}\,dt$, *where* $\mu_1 \in \mathbb{C}$.
(3). $\lim\limits_{n\to\infty} \tau_j^n = \infty$, *if* $j = 1, 2, 3$, *and* $\lim\limits_{n\to\infty} \tau_j^n = 0$, *if* $j = 4, 5, 6$.

Then $\mu_1 = 0$.

Proof. Given a polynomial $\varphi(t) = a_0\,t^k + a_1\,t^{k-1} + ... + a_k$ of degree k ($a_0 \neq 0$), set $[\varphi] := [a_0 : a_1 : ... : a_k] \in \mathbb{P}^k$. For instance,

$$[q_n] = [3 - \lambda_n : B_n : A_n : 3\,\lambda_n - 1] \in \mathbb{P}^3 \ ,$$

and

$$[p_n] = [1 : B_n : C_n : D_n : E_n : A_n : \lambda_n] \in \mathbb{P}^6 \ .$$

It follows from (1) and (3) of the hypothesis of lemma 2.6 that the sequence $(B_n, C_n, D_n, E_n, A_n) \in \mathbb{C}^5$ is unbounded and

$$\lim_{n\to\infty} [p_n] = [0 : 0 : 0 : 1 : 0 : 0 : 0] \in \mathbb{P}^6 \ ,$$

which means that there exists a sequence $(c_n)_n$ in $\mathbb{C}$ with $\lim\limits_{n\to\infty} c_n = 0$ and $\lim\limits_{n\to\infty} c_n.p_n(t) = t^3$.

We have two possibilities :

1^{st}. The sequence $(A_n, B_n) \in \mathbb{C}^2$ is bounded. In this case $\lim\limits_{n\to\infty} \omega_n = 0$, because $(B_n, C_n, D_n, E_n, A_n)$ is unbounded.

2^{nd}. (A_n, B_n) is unbounded. In this case, we get

$$\lim_{n\to\infty} [q_n] = [0 : B_o : A_o : 0] \in \mathbb{P}^3 \ .$$

This means that there exists a sequence $(d_n)_n$ in $\mathbb{C}$ with $\lim\limits_{n\to\infty} d_n = 0$ and $\lim\limits_{n\to\infty} d_n.q_n(t) = B_o\,t^2 + A_o\,t \not\equiv 0$. In particular, we get

$$\lim_{n\to\infty} \frac{d_n^2}{c_n}\,\omega_n = \frac{(B_o\,t^2 + A_o\,t)^2}{(1-\lambda)\,t^4}\,dt \ .$$

Suppose by contradiction that $\mu_1 \neq 0$. In this case, the above limit and the hypothesis (2) of the lemma imply that

$$\ell im_{n\to\infty} \frac{d_n^2}{c_n} = \frac{(B_o\, t^2 + A_o\, t)^2}{\mu_1\,(1-\lambda)\, t^3}\ ,\ \forall t\ .$$

However, this is impossible, because $\ell im_{n\to\infty} \frac{d_n^2}{c_n}$ is a constant and the right hand side is not. □

This finishes the proof of lemma 2.5. □

Proof of lemma 2.4. Let S, T and X_J be as before. As we have mentioned, we have $T^*(X_J) = \zeta^2\, X_J$ and $S^*(X_J) = y^{-1}.\, X_J$. Denote by $\mathcal{X}$ the set of polynomial vector fields of the form $p(x,y)\,\partial_x + q(x,y)\,\partial_y + g(x,y)\,R$, where $max(dg(p), dg(q)) \leq 2$, $g(x,y)$ is homogeneous of degree two and $R = x\,\partial_x + y\,\partial_y$. Note that

$$\mathcal{X} = \left\langle x^i\, y^j\, \partial_x\,,\ x^i\, y^j\, \partial_y\,,\ x^k\, y^\ell\, R \mid 0 \leq i,j,k,\ell \leq 2\ ,\ k+\ell = 2 \text{ and } i+j \leq 2 \right\rangle_{\mathbb{C}}$$

and $\mathbb{F}\mathrm{ol}(2,2) \simeq \mathbb{P}(\mathcal{X})$. We will consider $\mathbb{F}\mathrm{ol}(2,2)$ parametrized in homogeneous coordinates by $\mathcal{X}$.

A vector field in the set

$$\mathcal{P} = \{x^i\, y^j\, \partial_x\,,\ x^i\, y^j\, \partial_y\,,\ x^k\, y^\ell\, R \mid 0 \leq i,j,k,\ell \leq 2\ ,\ k+\ell = 2 \text{ and } i+j \leq 2\}$$

will be called a *monomial*. Note that $\mathcal{P}$ is base of $\mathcal{X}$. Given $X \in \mathcal{X}$ and $W \in \mathcal{P}$ we will denote by $Cf_W(X)$ the coefficient of W when we write X in the base $\mathcal{P}$.

Remark 2.7. Given a polynomial vector field Y on $\mathbb{C}^2$ define $\tilde{S}(Y) := y.\, S^*(Y)$ and $\tilde{T}(Y) = \zeta^{-2}\, T^*(Y)$. It can be checked that $\tilde{S}(\mathcal{X}) = \mathcal{X}$ and $\tilde{S}\colon \mathcal{X} \to \mathcal{X}$ is a linear isomorphism. We leave this computation to the reader. Since $T^*(X_J) = \zeta^2\, X_J$ and $S^*(X_J) = y^{-1}.\, X_J$, we get $\tilde{S}(X_J) = X_J$ and $\tilde{T}(X_J) = X_J$. Observe also that

$$\tilde{S} \circ \tilde{T} = \tilde{T}^2 \circ \tilde{S}\ ,\ \tilde{S}^3 = I \text{ and } \tilde{T}^7 = I\ , \tag{22}$$

where I is the identity of $\mathcal{X}$.

Proof. We will prove the first relation and leave the others to the reader. By a direct computation, it can be checked that $T \circ S = S \circ T^2$. This implies $S^* \circ T^* = (T^*)^2 \circ S^*$. On the other hand, for any $Y \in \mathcal{X}$ we have

$$\tilde{T}^2(\tilde{S}(Y)) = \tilde{T}^2(y\, S^*(Y)) = \zeta^{-4}\,(T^*)^2(y\, S^*(Y)) =$$
$$= \zeta^{-4}\,(y \circ T^2)\,.\,(T^*)^2 \circ S^*(Y) = \zeta^{-2}\, y\, S^*(T^*(Y)) = \tilde{S}(\tilde{T}(Y)) \qquad \square$$

Let $G := \left\langle \tilde{T}, \tilde{S} \right\rangle$ be the group generated by the linear isomorphisms $\tilde{S}$ and $\tilde{T}$ of $\mathcal{X}$.

Observe that :

- $\varphi(X_J) = X_J$ for any $\varphi \in G$.
- (22) implies that G is isomorphic to $\mathcal{I}so(J_2)$.

In particular, we can consider the action of $\mathcal{I}so(J_2)$ on $\mathbb{F}\mathrm{ol}(2,2)$, parametrized in the homogeneous coordinates by $\mathcal{X}$, by the action of the group G on $\mathcal{X}$.

The first part of the proof of lemma 2.4 will consist in finding a G-invariant 6-dimensional subspace $E \subset \mathcal{X}$ such that $X_J + E$ is transverse to $T_{X_J}\mathcal{O}rb(J_2)$, the tangent space of $\mathcal{O}rb(J_2)$ at X_J.

Since $\tilde{T}^7 = I$ the eigenvalues of $\tilde{T}$ are ζ^j, $0 \le j \le 6$, $\zeta = e^{2\pi i/7}$. In particular, if we denote by E_j the eigenspace of the eigenvalue ζ^j, then the canonical decomposition of the operator $\tilde{T} : \mathcal{X} \to \mathcal{X}$ can be written as

$$\mathcal{X} = \bigoplus_{j=0}^{6} E_j \ .$$

Remark also that (22) implies :

$$\tilde{S}(E_r) = E_{4r\ md(7)} \ , \ \forall \ r \quad \Longrightarrow$$

$$\tilde{S}(E_0) = E_0 \ , \ E_1 \overset{\tilde{S}}{\to} E_4 \overset{\tilde{S}}{\to} E_2 \overset{\tilde{S}}{\to} E_1 \text{ and } E_3 \overset{\tilde{S}}{\to} E_5 \overset{\tilde{S}}{\to} E_6 \overset{\tilde{S}}{\to} E_3 \ . \tag{23}$$

We leave the proof of (23) to the reader.

On the other hand, it can be checked directly that (see also [LN-JP]) :

$$E_0 = \left\langle \partial_x \, , \, x^2\,\partial_y, y^2\,R \right\rangle , \, E_1 = \left\langle y\,\partial_x \, , \, x^2\,R \right\rangle , \, E_2 = \left\langle y^2\,\partial_x \, , \, x\,\partial_y \right\rangle ,$$

$$E_3 = \left\langle x^2\,\partial_x \, , \, x\,y\,\partial_y \right\rangle , \, E_4 = \langle \partial_y \, , \, x\,y\,R \rangle , \, E_5 = \langle x\,\partial_x \, , \, y\,\partial_y \rangle , \, E_6 = \left\langle x\,y\,\partial_x \, , \, y^2\,\partial_y \right\rangle \ .$$

Now, we use some results proved in [LN-JP]. If we consider $B : \mathcal{U}_1 \to \mathbb{C}^7$ as a map defined in a neighborhood of $X_J \in \mathcal{X}$, then Theorem 2 of [LN-JP] implies that $dim(ker(DB(X_J))) = 9$ and its projection on $T_{J_2}\mathbb{F}\text{ol}(2,2)$ coincides with $T_{J_2}\mathcal{O}rb(J_2)$. In fact, in lemma 3.5 of §3.4 of [LN-JP] it is proved that a monomial $W \in \mathcal{P}$ is in $ker(DB(X_J))$ if, and only if, $W \in E_0$. Therefore, it follows from (23) that if we choose two monomials, say $W_1 = y\,\partial_x \in E_1$ and $W_3 = x^2\,\partial_x \in E_3$, then the subspace

$$E = \left\langle W_1, \tilde{S}(W_1), \tilde{S}^2(W_1), W_3, \tilde{S}(W_3), \tilde{S}^2(W_3) \right\rangle$$

has dimension 6, is G-invariant and transverse to $ker(DB(X_J))$. Moreover, $DB(X_J) : E \to T_{b_0}\Sigma$ is an isomorphism. In particular, we get :

(i). The projection of $X_J + E$ in $\mathbb{F}\text{ol}(2,2)$, say $\tilde{E}$, is transverse to $\mathcal{O}rb(J_2)$ at J_2.

(ii). There are neighborhoods Γ_1 of J_2 in $\tilde{E}$ and V_1 of b_0 in Σ such that $B_1 := B|_{\Gamma_1} : \Gamma_1 \to V_1$ is a biholomorphism.

If we set

$$V := \bigcap_{\sigma \in S_7} \hat{\sigma}(V_1) \text{ and } \Gamma := B_1^{-1}(V)$$

then Γ and V satisfy (A), (B) and (C) of lemma 2.4.

Assertion 2.3. *We assert that, if V and Γ are sufficiently small then Γ satisfies (D) of lemma 2.4. In other words, if $\mathcal{F}_1, \mathcal{F}_2 \in \Gamma$ are in the same orbit then there exists $\varphi \in \mathcal{I}so(J_2)$ such that $\mathcal{F}_2 = \varphi^*(\mathcal{F}_1)$.*

Proof. Since $dim(\mathcal{O}rb(J_2)) = 8$ and Γ is a transverse section to $\mathcal{O}rb(J_2)$, if Γ is small enough then there exists a neighborhood $\mathcal{U}$ of the identity $I \in Aut(\mathbb{P}^2)$ with the following property :

($*$) if $\mathcal{F} \in \Gamma$, $\varphi \in \mathcal{U}$ and $\varphi^*(\mathcal{F}) \in \Gamma$ then $\varphi = I$.

From now on, we will assume that Γ satisfies property $(*)$.

Let us assume, by contradiction, that Assertion 2.3 is not true. This implies that there are sequences $(\mathcal{F}_{1n})_n$, $(\mathcal{F}_{2n})_n$, of foliations in Γ, and $(\varphi_n)_n$ in $Aut(\mathbb{P}^2)\backslash\mathcal{I}so(J_2)$ such that $\lim\limits_{n\to\infty} \mathcal{F}_{jn} = J_2$, $j = 1, 2$, and $\varphi_n^*(\mathcal{F}_{1n}) = \mathcal{F}_{2n}$ for all $n \geq 1$. Note that $(*)$ implies that the sequence $(\varphi_n)_n$ is discrete in $Aut(\mathbb{P}^2)$.

The idea is to prove that there exists $\lim\limits_{n\to\infty} \varphi_n = \varphi \in Aut(\mathbb{P}^2)$. If we assume this fact, then we will have

$$J_2 = \lim_{n\to\infty} \mathcal{F}_{2n} = \lim_{n\to\infty} \varphi_n^*(\mathcal{F}_{1n}) = \varphi^*(J_2) \implies \varphi \in \mathcal{I}so(J_2) .$$

On the other hand, since the sequence is discrete in $Aut(\mathbb{P}^2)$, we must have $\varphi_n = \varphi \in \mathcal{I}so(J_2)$ for n large, a contradiction.

Remark 2.8. We say that four points in $\mathbb{P}^2$ are in general position if they are two by two distinct and any three of them, distinct two by two, are not in the same straight line. We would like to observe that any four points, distinct two by two, in $sing(J_2)$ are in general position. This fact, can be checked directly by using that $sing(J_2) = \{p_1, T(p_1), ..., T^6(p_1)\}$, where $p_1 = (1, 1)$ and $T(x, y) = (\zeta^{-2}\, x, \zeta\, y)$, $\zeta = e^{2\pi i/7}$.

The following result will be used :

Lemma 2.7. *Let $(\psi_n)_n$ be a sequence in $Aut(\mathbb{P}^2)$. Assume that there are sequences $(x_{jn})_n$ and $(y_{jn})_n$ in $\mathbb{P}^2$, $j \in \{1, 2, 3, 4\}$, such that*

(A). *$\psi_n(x_{jn}) = y_{jn}$, for all $n \in \mathbb{N}$ and $j = 1, 2, 3, 4$.*
(B). *$\lim\limits_{n\to\infty} x_{jn} = x_j \in \mathbb{P}^2$ and $\lim\limits_{n\to\infty} y_{jn} = y_j \in \mathbb{P}^2$, $j = 1, 2, 3, 4$.*
(C). *The four points in both sets $\{x_1, ..., x_4\}$ and $\{y_1, ..., y_4\}$, are in general position.*

Then there exists $\lim\limits_{n\to\infty} \psi_n = \psi \in Aut(\mathbb{P}^2)$.

The proof of lemma 2.7 can be done by using the following facts :

- given two sets of four points in $\mathbb{P}^2$, say $\{z_1, z_2, z_3, z_4\}$ and $\{w_1, w_2, w_3, w_4\}$, whose points are in general position, then there exists an unique $\phi \in Aut(\mathbb{P}^2)$ such that $\phi(z_j) = w_j$, $1 \leq j \leq 4$.
- if n is big enough then the points in both sets $\{x_{1n}, ..., x_{4n}\}$ and $\{y_{1n}, ..., y_{4n}\}$ are in general position.

We leave the details to the reader.

Let $P_1, ..., P_7$ be the local holomorphic maps, defined before, such that $sing(\mathcal{F}) = \{P_1(\mathcal{F}), ..., P_7(\mathcal{F})\}$ and $P_j(J_2) = p_j$, $1 \leq j \leq 7$. We have seen in the proof of Corollary 2.3 that

$$sing(\mathcal{F}_{1n}) = \varphi_n(sing(\varphi_n^*(\mathcal{F}_{1n}))) = \varphi_n(sing(\mathcal{F}_{2n})) .$$

Since $sing(\mathcal{F}_{jn}) = \{P_1(\mathcal{F}_{jn}), ..., P_7(\mathcal{F}_{jn})\}$, $j = 1, 2$, for all $n \in \mathbb{N}$ there exists a permutation $\sigma_n \in S_7$ such that

$$\varphi_n(P_i(\mathcal{F}_{2n})) = P_{\sigma_n(i)}(\mathcal{F}_{1n}) \ , \ \forall n \in \mathbb{N} \ , \ \forall i = 1, ..., 7 .$$

By taking a subsequence, if necessary, we can assume that $\sigma_n = \sigma \in S_7$ for all $n \in \mathbb{N}$, because S_7 is finite. In particular,

$$\varphi_n(P_i(\mathcal{F}_{2n})) = P_{\sigma(i)}(\mathcal{F}_{1n}) \ , \ \forall n \in \mathbb{N} \ , \ \forall i = 1, ..., 7 . \tag{24}$$

If we set $x_{jn} = P_j(\mathcal{F}_{2n})$ and $y_{jn} = P_{\sigma(j)}(\mathcal{F}_{1n})$, $j = 1, ..., 4$, then

- $\varphi_n(x_{jn}) = y_{jn}$ for all $n \in \mathbb{N}$ and $j = 1, 2, 3, 4$.
- $\underset{n\to\infty}{\ell im}\, x_{jn} = p_j$ and $\underset{n\to\infty}{\ell im}\, y_{jn} = p_{\sigma(j)}$, $j = 1, 2, 3, 4$.
- the points in both sets $\{p_1, ..., p_4\}$ and $\{p_{\sigma(1)}, ..., p_{\sigma(4)}\}$ are in general position.

Therefore, lemma 2.7 implies that there exists $\underset{n\to\infty}{\ell im}\, \varphi_n = \varphi \in Aut(\mathbb{P}^2)$. This finishes the proof of lemma 2.4 and of Theorem 1. $\square$

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