HOMOGENIZATION OF DEGENERATE POROUS MEDIUM TYPE EQUATIONS IN ERGODIC ALGEBRAS

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Abstract. We consider the homogenization problem for general porous medium type equations of the form $u_t = \Delta f(x, \frac{x}{\varepsilon}, u)$, where $f(x, y, \cdot)$ is increasing but may degenerate in the sense that $f_u(x, y, \cdot)$ may vanish on a set with empty interior. We address both, the Cauchy problem and the initial-boundary value problem, with null boundary condition. The homogenization is carried out in the general context of ergodic algebras.

1. Introduction

In this paper we consider the homogenization of a porous medium type equation of the general form

$$u_t = \Delta f(x, \frac{x}{\varepsilon}, u),$$

where $f(x, y, \cdot)$ is increasing and locally Lipschitz, uniformly in (x, y), and may degenerate, in the sense that $f_u(x, y, \cdot)$ may vanish in a set with empty interior. We consider both, the Cauchy problem and the initial-boundary value problem with null boundary condition. In the case of the Cauchy problem, the discussion here largely extends the corresponding one in [3] concerning the homogenization of the particular type of such equations where $f(x, y, u) = f(u) + V(y)$, in the nondegenerate case. As in [3], we assume that for each (x, u), $f(x, \cdot, u)$ belongs to a given general ergodic algebra, but we restrict the initial data to "well-prepared" ones, that is, functions of the form $g(x, \frac{x}{\varepsilon}, \varphi_0(x))$, where, for each (x, y), $g(x, y, \cdot)$ is the inverse of $f(x, y, \cdot)$, and $\varphi_0 \in L^\infty(\mathbb{R}^n)$. Actually, in this case, as in [3], we just consider $f = f(y, u)$, since the general case $f = f(x, y, u)$ follows easily from this simpler case where the notation is less cumbersome. In the case of the initial-boundary value problem, the discussion in this paper largely extends the corresponding one in [15], where we consider the special case $f(x, y, u) = f(u) + V(x, y)$ with f nondegenerate. As in [15], the method applied in this case allow initial data which do not need to be "well-prepared". However, again as in [15], we have to restrict the homogenization analysis to regular algebras with mean value. The latter is a concept introduced here which includes the Fourier-Stieltjes algebras introduced in [15]. We prove that such an algebra with mean value (algebra w.m.v., in short) is ergodic. We recall that the theory of algebras w.m.v. and ergodic algebras was first developed by Zhikov and Krivenko in [28] (see also [17]).

As in [2, 3, 15], one of the main tools used here in the study of the homogenization problem is the parametrized family of two-scale of Young measures. We recall that two-scale Young measures were first introduced by W. E in [12] as a nonlinear extension of the concept of two-scale convergence introduced by Nguetseng in [23] and further developed by Allaire in [1].

This paper is organized as follows. In Section 2 we recall the concepts of algebra w.m.v., generalized Besicovitch space and ergodic algebra. We also recall a general result established in [3] which relates such algebras and the translation operators acting on them with the continuous functions defined on certain compact spaces and certain groups of homeomorphisms of these compact spaces. In Section 3, we introduce the concept of regular algebra w.m.v., prove that these are ergodic algebras, and that this concept includes the Fourier-Stieltjes spaces $\mathrm{FS}(\mathbb{R}^n)$. In Section 4, we briefly recall the general result of [3] on the existence

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of two-scale Young measures associated with a given algebra w.m.v. In Section 5, we provided a self-contained discussion about the well-posedness of the Cauchy problem and the initial-boundary value problem with null boundary condition for degenerate porous medium type equations. In Section 6, we consider the homogenization problem for porous medium type equations defined in all $\mathbb{R}^n$ and we analize the case of the Cauchy problem. Finally, in Section 7, we consider the homogenization problem for porous medium type equations defined in a bounded domain and we analize the case of the initial-boundary value problem with null boundary condition.

2. Ergodic Algebras

In this section we recall some basic facts about algebras with mean values and ergodic algebras that will be needed for the purposes of this paper. To begin with, we recall the notion of mean value for functions defined in $\mathbb{R}^n$.

Definition 2.1. Let $g \in L^1_{\text{loc}}(\mathbb{R}^n)$. A number $M(g)$ is called the *mean value of g* if

$$\lim_{\varepsilon\to 0}\int_A g(\varepsilon^{-1}x)\,dx = |A|M(g) \tag{2.1}$$

for any Lebesgue measurable bounded set $A \subseteq \mathbb{R}^n$, where $|A|$ stands for the Lebesgue measure of A. This is the same as saying that $g(\varepsilon^{-1}x)$ converges, in the duality with L^∞ and compactly supported functions, to the constant $M(g)$. Also, if $A_t := \{x \in \mathbb{R}^n : t^{-1}x \in A\}$ for $t > 0$ and $|A| \neq 0$, (2.1) may be written as

$$\lim_{t\to\infty}\frac{1}{t^n|A|}\int_{A_t} g(x)\,dx = M(g). \tag{2.2}$$

We will also use the notation $⨍_{\mathbb{R}^n} g\,dx$ for $M(g)$.

As usual, we denote by $\text{BUC}(\mathbb{R}^n)$ the space of the bounded uniformly continuous real-valued functions in $\mathbb{R}^n$.

We recall now the definition of algebra with mean value introduced in [28].

Definition 2.2. Let $\mathcal{A}$ be a linear subspace of $\text{BUC}(\mathbb{R}^n)$. We say that $\mathcal{A}$ is an *algebra with mean value* (or *algebra w.m.v.*, in short), if the following conditions are satisfied:

(A) If f and g belong to $\mathcal{A}$, then the product fg belongs to $\mathcal{A}$.
(B) $\mathcal{A}$ is invariant under the translations $\tau_y : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto x + y$, $y \in \mathbb{R}^n$, that is, if $f \in \mathcal{A}$, then $\tau_y f \in \mathcal{A}$, for all $y \in \mathbb{R}^n$, where $\tau_y f := f \circ \tau_y$, $f \in \mathcal{A}$.
(C) Any $f \in \mathcal{A}$ possesses a mean value.
(D) $\mathcal{A}$ is closed in $\text{BUC}(\mathbb{R}^n)$ and contains the unity, i.e., the function $e(x) := 1$ for $x \in \mathbb{R}^n$.

Remark 2.1. Condition (D) was not originally in [28] but its inclusion does not change matters since any algebra satisfying (A), (B) and (C) can be extended to an algebra satisfying (A)–(D) in an unique way modulo isomorphisms.

For the development of the homogenization theory in algebras with mean value, as it is done in [28, 17] (see also [7, 3]), in similarity with the case of almost periodic functions, one introduces, for $1 \le p < \infty$, the space $\mathcal{B}^p$ as the abstract completion of the algebra $\mathcal{A}$ with respect to the Besicovitch seminorm

$$|f|_p := \left(⨍_{\mathbb{R}^n} |f|^p\,dx\right)^{1/p}$$

Both the action of translations and the mean value extend by continuity to $\mathcal{B}^p$, and we will keep using the notation $\tau_y f$ and $M(f)$ even when $f \in \mathcal{B}^p$. Furthermore, for $p > 1$ the product in the algebra extends to a bilinear operator from $\mathcal{B}^p \times \mathcal{B}^q$ into $\mathcal{B}^1$, with q equal to the dual exponent of p, satisfying

$$|fg|_1 \le |f|_p|g|_q.$$

In particular, the operator $M(fg)$ provides a nonnegative definite bilinear form on $\mathcal{B}^2$.

Since there is an obvious inclusion between elements of this family of spaces, we may define the space $\mathcal{B}^\infty$ as follows:

$$\mathcal{B}^\infty = \{f \in \bigcap_{1\le p<\infty} \mathcal{B}^p \; : \; \sup_{1\le p<\infty} |f|_p < \infty\},$$

We endow $\mathcal{B}^\infty$ with the (semi)norm

$$|f|_\infty := \sup_{1\le p<\infty} |f|_p.$$

Obviously the corresponding quotient spaces for all these spaces (with respect to the null space of the seminorms) are Banach spaces, and in the case $p = 2$ we obtain a Hilbert space. We denote by $\stackrel{\mathcal{B}^p}{=}$, the equivalence relation given by the equality in the sense of the $\mathcal{B}^p$ semi-norm. We will keep the notation $\mathcal{B}^p$ also for the corresponding quotient spaces.

Remark 2.2. A classical argument going back to Besicovitch [4] (see also [17], p.239) shows that the elements of $\mathcal{B}^p$ can be represented by functions in $L^p_{\text{loc}}(\mathbb{R}^n)$, $1 \le p < \infty$.

We next recall a result established in [3] which provides a connection between algebras with mean value and the algebra $C(\mathcal{K})$ of continuous functions on a certain compact (Hausdorff) topological space. We state here only the parts of the corresponding result in [3] that will be used in this paper.

Theorem 2.1 (cf. [3])**.** *For an algebra $\mathcal{A}$, we have:*

(i) *There exist a compact space $\mathcal{K}$ and an isometric isomorphism i identifying $\mathcal{A}$ with the algebra $C(\mathcal{K})$ of continuous functions on $\mathcal{K}$.*

(ii) *The translations $\tau_y : \mathbb{R}^n \to \mathbb{R}^n$, $\tau_y x = x + y$, induce a family of homeomorphisms $T(y) : \mathcal{K} \to \mathcal{K}$, $y \in \mathbb{R}^n$, satisfying the group properties $T(0) = I$, $T(x + y) = T(x) \circ T(y)$, such that the mapping $T : \mathbb{R}^n \times \mathcal{K} \to \mathcal{K}$, $T(y, z) := T(y)z$, is continuous.*

(iii) *The mean value on $\mathcal{A}$ extends to a Radon probability measure $\mathfrak{m}$ on $\mathcal{K}$ defined by*

$$\int_{\mathcal{K}} i(f)\, d\mathfrak{m} := \fint_{\mathbb{R}^n} f\, dx, \qquad f \in \mathcal{A},$$

which is invariant by the group of homeomorphisms $T(y) : \mathcal{K} \to \mathcal{K}$, $y \in \mathbb{R}^n$, that is, $\mathfrak{m}(T(y)E) = \mathfrak{m}(E)$ for all Borel sets $E \subseteq \mathcal{K}$.

(iv) *For $1 \le p \le \infty$, the Besicovitch space $\mathcal{B}^p / \stackrel{\mathcal{B}^p}{=}$ is isometrically isomorphic to $L^p(\mathcal{K}, \mathfrak{m})$.*

A function $f \in \mathcal{B}^2$ is said to be *invariant* if $\tau_y f \stackrel{\mathcal{B}^2}{=} f$, for all $y \in \mathbb{R}^n$. More clearly, $f \in \mathcal{B}^2$ is invariant if

$$M\big(|\tau_y f - f|^2\big) = 0, \qquad \text{for all } y \in \mathbb{R}^n. \tag{2.3}$$

The concept of ergodic algebra is then introduced as follows.

Definition 2.3. An algebra w.m.v. $\mathcal{A}$ is called an *ergodic algebra* if any invariant function f belonging to the corresponding space $\mathcal{B}^2$ is equivalent (in $\mathcal{B}^2$) to a constant.

A very useful alternative definition of ergodic algebra is also given in [17], p. 247, and shown therein to be equivalent to Definition 2.3. We state that as the following lemma.

Lemma 2.1 (cf. [17])**.** *Let $\mathcal{A} \subseteq \mathrm{BUC}(\mathbb{R}^n)$ be an algebra w.m.v.. Then $\mathcal{A}$ is ergodic if and only if*

$$\lim_{t\to\infty} M_y \left(\Big| \frac{1}{|B(0;t)|} \int_{B(0;t)} f(x+y)\, dx - M(f) \Big|^2 \right) = 0 \qquad \forall f \in \mathcal{A}. \tag{2.4}$$

3. Regular algebras w.m.v. and the Fourier-Stieltjes space $\mathrm{FS}(\mathbb{R}^n)$.

In this section we introduce the concept of regular algebra w.m.v. and recall the definition and some basic properties of the Fourier-Stieltjes space introduced by the authors in [15], which is, to the best of our knowledge, the largest known example of a regular algebra w.m.v..

For any $f \in L^\infty(\mathbb{R}^n)$, let us denote by $\hat{f}$ the Fourier transform of f defined as the following distribution

$$\langle \hat{f}, \phi \rangle := \int f(x)\check{\phi}(x)\,dx, \qquad \text{for all } \phi \in C_c^\infty(\mathbb{R}^n),$$

where $\check{\phi}$ denotes the usual inverse Fourier transform of ϕ, i.e.,

$$\check{\phi}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int \phi(y) e^{iy\cdot x}\,dx.$$

Given an algebra w.m.v. $\mathcal{A}$, let us denote by $V(\mathcal{A})$ the subspace formed by the elements $f \in \mathcal{A}$ such that $M(f) = 0$, namely,

$$V(\mathcal{A}) := \{f \in \mathcal{A} : M(f) = 0\}.$$

Also, let us denote by $Z(\mathcal{A})$ the subset of those $f \in \mathcal{A}$ such that the distribution $\hat{f}$ has compact support not containing the origin 0, that is,

$$Z(\mathcal{A}) := \{f \in \mathcal{A} : \operatorname{supp}(\hat{f}) \text{ is compact and } 0 \notin \operatorname{supp}(\hat{f})\}.$$

We collect in the following lemma some useful properties of the functions in $Z(\mathcal{A})$, whose proof is found in [17], p. 246.

Lemma 3.1 (cf. [17])**.** *Let $\mathcal{A}$ be an algebra w.m.v. in $\mathbb{R}^n$ and $f \in Z(\mathcal{A})$. Then:*

(i) *There exists $u \in C^\infty(\mathbb{R}^n) \cap Z(\mathcal{A})$ such that $\Delta u = f$, where Δ is the usual Laplace operator in $\mathbb{R}^n$; $u = f * \zeta$ for certain smooth function ζ, fast decaying together with all its derivatives, satisfying $\hat{\zeta} \in C_c^\infty(\mathbb{R}^n)$ and $0 \notin \operatorname{supp}(\hat{\zeta})$.*

(ii) *For any Borelian $Q \subseteq \mathbb{R}^n$, with $|Q| > 0$, we have*

$$\lim_{t\to\infty} \frac{1}{t^n|Q|} \int_{Q_t} f(x+y)\,dx = 0, \qquad \textit{uniformly in } y \in \mathbb{R}^n. \tag{3.1}$$

The fundamental result about ergodic algebras, proved by Zhikov and Krivenko [28], is the following.

Theorem 3.1 (cf. [28])**.** *If $\mathcal{A}$ is an ergodic algebra, then $Z(\mathcal{A})$ is dense in $V(\mathcal{A})$ in the topology of the corresponding space $\mathcal{B}^2$.*

The following immediate corollary of Theorem 3.1, established in [3], will be used in Section 6 concerning the homogenization of a porous medium type equation.

Lemma 3.2 (cf. [3])**.** *Let $\mathcal{A}$ be an ergodic algebra in $\mathrm{BUC}(\mathbb{R}^n)$ and $h \in \mathcal{B}^2$ such that $M(h\Delta f) = 0$ for all $f \in \mathcal{A}$ such that $\Delta f \in \mathcal{A}$. Then h is $\mathcal{B}^2$-equivalent to a constant.*

Theorem 3.1 also motivates the following definition.

Definition 3.1. An algebra w.m.v. $\mathcal{A}$ is said to be *regular* if $Z(\mathcal{A})$ is dense in $V(\mathcal{A})$ in the topology of the sup-norm.

We have the following important fact about regular algebras w.m.v..

Proposition 3.1. *If $\mathcal{A}$ is a regular algebra w.m.v., then $\mathcal{A}$ is ergodic.*

Proof. We are going to use the characterization of ergodic algebras provided by Lemma 2.1. Let $f \in \mathcal{A}$. Clearly, to prove (2.4), we may assume $M(f) = 0$. Now, since $\mathcal{A}$ is regular, given $\varepsilon > 0$, we may find $g \in Z(\mathcal{A})$ such that $\|f - g\|_\infty < \varepsilon$. Hence,

$$\limsup_{t\to\infty} M_y\left(\Big|\frac{1}{|B(0;t)|}\int_{B(0;t)} f(x+y)\,dx\Big|^2\right) \le 2\lim_{t\to\infty} M_y\left(\Big|\frac{1}{|B(0;t)|}\int_{B(0;t)} g(x+y)\,dx\Big|^2\right) + 2\varepsilon^2 = 2\varepsilon^2,$$

where we used Lemma 3.1(ii) for the last equality. This implies (2.4). □

We next state a property of regular algebras w.m.v. which will be used in our application to homogenization of porous medium type equations on bounded domains in the final part of this paper.

Lemma 3.3. *Let $\mathcal{A}$ be a regular algebra w.m.v. If $f \in V(\mathcal{A})$, then for any $\varepsilon > 0$ there exists a function $u_\varepsilon \in Z(\mathcal{A})$ satisfying the inequalities*

$$f - \varepsilon \le \Delta u_\varepsilon \le f + \varepsilon. \tag{3.2}$$

Proof. This follows immediately from Lemma 3.1(i) and Definition 3.1. □

The space $\mathrm{FS}(\mathbb{R}^n)$ introduced in [15] provides a very encompassing example of a regular algebra w.m.v..

Definition 3.2. The Fourier-Stieltjes space, denoted by $\mathrm{FS}(\mathbb{R}^n)$, is the completion relatively to the sup-norm of the space of functions $\mathrm{FS}_*(\mathbb{R}^n)$ defined by

$$\mathrm{FS}_*(\mathbb{R}^n) := \left\{ f : \mathbb{R}^n \to \mathbb{R} \,:\, f(x) = \int_{\mathbb{R}^n} e^{ix\cdot y}\,d\nu(y) \text{ for some } \nu \in \mathcal{M}_*(\mathbb{R}^n) \right\}, \tag{3.3}$$

where by $\mathcal{M}_*(\mathbb{R}^n)$ we denote the space of complex-valued measures μ with finite total variation, i.e., $|\mu|(\mathbb{R}^n) < \infty$.

Recall that a subalgebra $\mathcal{B} \subseteq \mathcal{A}$ is called an ideal of $\mathcal{A}$ if for any $f \in \mathcal{A}$ and $g \in \mathcal{B}$ we have $fg \in \mathcal{B}$. Let $C_0(\mathbb{R}^n)$ denote the closure of $C_c^\infty(\mathbb{R}^n)$ with respect to the sup norm. The following result was established in [15].

Proposition 3.2 (cf. [15])**.** $\mathrm{FS}(\mathbb{R}^n) \subseteq \mathrm{BUC}(\mathbb{R}^n)$ *and it is an algebra w.m.v. containing* $C_0(\mathbb{R}^n)$ *as an ideal. Moreover,* $\mathrm{FS}(\mathbb{R}^n)$ *is a regular algebra w.m.v. and the space* $\mathrm{PAP}(\mathbb{R}^n)$ *of the perturbed almost periodic functions, defined as*

$$\mathrm{PAP}(\mathbb{R}^n) := \{f \in \mathrm{BUC}(\mathbb{R}^n) \,:\, f = g + \psi,\ g \in \mathrm{AP}(\mathbb{R}^n),\ \psi \in C_0(\mathbb{R}^n)\},$$

is a closed strict subalgebra of $\mathrm{FS}(\mathbb{R}^n)$.

4. Two-scale Young Measures

In this section we recall the theorem giving the existence of two-scale Young measures established in [3]. We begin by recalling the concept of vector-valued algebra with mean value.

Given a Banach space E and an algebra w.m.v. $\mathcal{A}$, we denote by $\mathcal{A}(\mathbb{R}^n;E)$ the space of functions $f \in \mathrm{BUC}(\mathbb{R}^n;E)$ satisfying the following:

(i) $L_f := \langle L, f\rangle$ belongs to $\mathcal{A}$ for all $L \in E^*$;
(ii) The family $\{L_f \,:\, L \in E^*,\ \|L\| \le 1\}$ is relatively compact in $\mathcal{A}$.

Theorem 4.1 (cf. [3])**.** *Let E be a Banach space, $\mathcal{A}$ an algebra w.m.v. and $\mathcal{K}$ be the compact associated with $\mathcal{A}$. There is an isometric isomorphism between $\mathcal{A}(\mathbb{R}^n;E)$ and $C(\mathcal{K};E)$. Denoting by $g \mapsto \underline{g}$ the canonical map from $\mathcal{A}$ to $C(\mathcal{K})$, the isomorphism associates to $f \in \mathcal{A}(\mathbb{R}^n;E)$ the map $\tilde f \in C(\mathcal{K};E)$ satisfying*

$$\underline{\langle L, f\rangle} = \langle L, \tilde f\rangle \in C(\mathcal{K}) \qquad \forall L \in E^*. \tag{4.1}$$

In particular, for each $f \in \mathcal{A}(\mathbb{R}^n;E)$, $\|f\|_E \in \mathcal{A}$.

For $1 \le p < \infty$, we define the space $L^p(\mathcal{K}; E)$ as the completion of $C(\mathcal{K}; E)$ with respect to the norm $\|\cdot\|_p$, defined as usual,

$$\|f\|_p := \left(\int_{\mathcal{K}} \|f\|_E^p \, d\mathfrak{m} \right)^{1/p}.$$

As a standard procedure, we identify functions in L^p that coincide $\mathfrak{m}$-a.e. in $\mathcal{K}$.

Similarly, we define the space $\mathcal{B}^p(\mathbb{R}^n; E)$ as the completion of $\mathcal{A}(\mathbb{R}^n; E)$ with respect to the seminorm

$$|f|_p := \left(\fint_{\mathbb{R}^n} \|f\|_E^p \, dx \right)^{1/p},$$

identifying functions in the same equivalence class determined by the seminorm $|\cdot|_p$. Clearly, the isometric isomorphism given by Theorem 4.1 extends to an isometric isomorphism between $\mathcal{B}^p(\mathbb{R}^n; E)$ and $L^p(\mathcal{K}; E)$.

The next theorem gives the existence of two-scale Young measures associated with an algebra $\mathcal{A}$. For the proof, we again refer to [3].

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and $\{u_\varepsilon(x)\}_{\varepsilon>0}$ be a family of functions in $L^\infty(\Omega; K)$, for some compact metric space K.

Theorem 4.2. *Given any infinitesimal sequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ there exist a subnet $\{u_{\varepsilon_{i(d)}}\}_{d\in D}$, indexed by a certain directed set D, and a family of probability measures on K, $\{\nu_{z,x}\}_{z\in\mathcal{K}, x\in\Omega}$, weakly measurable with respect to the product of the Borel σ-algebras in $\mathcal{K}$ and $\mathbb{R}^n$, such that*

$$\lim_D \int_\Omega \Phi\Big(\frac{x}{\varepsilon_{i(d)}}, x, u_{\varepsilon_{i(d)}}(x)\Big)\, dx = \int_\Omega \int_{\mathcal{K}} \langle \nu_{z,x}, \underline{\Phi}(z,x,\cdot)\rangle \, d\mathfrak{m}(z)\, dx \qquad \forall \Phi \in \mathcal{A}\left(\mathbb{R}^n; C_0(\Omega\times K)\right). \tag{4.2}$$

Here $\underline{\Phi} \in C\left(\mathcal{K}; C_0(\Omega\times K)\right)$ denotes the unique extension of Φ. Moreover, equality (4.2) still holds for functions Φ in the following function spaces:

(1) $\mathcal{B}^1(\mathbb{R}^n; C_0(\Omega\times K))$;
(2) $\mathcal{B}^p(\mathbb{R}^n; C(\bar\Omega\times K))$ *with* $p>1$;
(3) $L^1(\Omega; \mathcal{A}(\mathbb{R}^n; C(K)))$.

As in the classical theory of Young measures we have the following consequence of Theorem 4.2.

Theorem 4.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $\{u_\varepsilon\} \subseteq L^\infty(\Omega; \mathbb{R}^m)$ be uniformly bounded and let $\nu_{z,x}$ be a two-scale Young measure generated by a subnet $\{u_{\varepsilon(d)}\}_{d\in D}$, according to Theorem 4.2. Assume that U belongs either to $L^1(\Omega; \mathcal{A}(\mathbb{R}^n; \mathbb{R}^m)))$ or to $\mathcal{B}^p(\mathbb{R}^n; C(\bar\Omega; \mathbb{R}^m))$ for some $p>1$. Then*

$$\nu_{z,x} = \delta_{\underline{U}(z,x)} \quad \textit{if and only if} \quad \lim_D \|u_{\varepsilon(d)}(x) - U(\frac{x}{\varepsilon(d)}, x)\|_{L^1(\Omega)} = 0. \tag{4.3}$$

5. Some results about a porous medium type equation

In this section, we review some results about the Cauchy problem and an initial-boundary value problem for a porous medium type equation which will be used later. More specifically, we consider the Cauchy problem

$$\partial_t u - \Delta f(x,u) = 0, \qquad (x,t) \in \mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0,+\infty), \tag{5.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{5.2}$$

and, for $\Omega \subseteq \mathbb{R}^n$ open bounded with smooth boundary, we consider the initial-boundary value problem

$$\partial_t u - \Delta f(x,u) = 0, \qquad (x,t) \in Q := \Omega \times (0,+\infty), \tag{5.3}$$

$$u(x,0) = u_0(x),\ x\in\Omega, \qquad u(x,t) = 0,\ (x,t) \in \partial\Omega\times(0,\infty). \tag{5.4}$$

For the purposes of this paper, we assume that $f(x,u)$ satisfies the following hypotheses, where I is an arbitrary compact interval of $\mathbb{R}$:

(f1) $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ ($f : \Omega \times \mathbb{R} \to \mathbb{R}$, resp.) is continuous and, for each $x \in \mathbb{R}^n$ ($x \in \Omega$, resp.), $f(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is increasing and locally Lipschitz continuous uniformly in x.

(f2) $D_x^\alpha f(x, u)$ and $D_x^\alpha f_u(x, u)$, $|\alpha| \le 2$, are uniformly bounded for $(x, u) \in \mathbb{R}^n \times I$ (resp., $\Omega \times I$).

(f3) There exists a constant $\theta_0 > 0$ such that

$$-f_u(x, u) + \theta_0 \sum_{i=1}^n |f_{u,x_i}(x, u)| \le 0, \tag{5.5}$$

for all $(x, u) \in \mathbb{R}^n \times I$ (resp., $(x, u) \in \Omega \times I$).

Observe that assumption **(f3)** is trivially satisfied by functions of the form $f(x, u) = a(x)u|u|^{\gamma(x)} + b(x)$, with γ, a, b smooth, $\gamma(x) > \gamma_0 > 0$ and $a(x) > a_0 > 0$, $x \in \mathbb{R}^n$ ($x \in \Omega$).

Specifically for the problem (5.3),(5.4) we also assume:

(f4) $f(x, 0) = 0$ for $x \in \partial\Omega$.

Concerning the initial data, we assume

$$u_0 \in L^\infty(\mathbb{R}^n) \qquad (u_0 \in L^\infty(\Omega), \text{ resp.}).$$

Because of our assumptions on $f(x, u)$, namely the fact that $f(x, \cdot)$ is increasing, for each x, equation (5.1) is only mildly degenerate, in other words, it still belongs to the "non-degenerate" class, in the classification of [5]. Nevertheless, it is degenerate in the sense that $f_u(x, \cdot)$ can vanish on a set $\mathcal{N} \subseteq \mathbb{R}$, provided $\mathcal{N}$ does not contain a non-empty open interval. The simplest and prototypical example is the classical porous medium equation, for which $f(x, u) = u|u|^\gamma$, $\gamma \ge 1$. We remark that for the latter, due to a comparison principle, we can always guarantee that $u(x, t) \ge 0$ if $u_0(x) \ge 0$, which is physically desirable. For this reason, we can view $f(u) = u^{\gamma+1}$, $u \ge 0$, as defined in $\mathbb{R}$, trivially extended as $u|u|^\gamma$. This motivates our choice of taking $f(x, \cdot)$ as defined in the whole $\mathbb{R}$, which is a matter of convenience.

The study of the well-posedness of the Cauchy problem for general quasilinear degenerate parabolic equations starts with Volpert and Hudjaev [26], for initial data in BV, where the L^1-stability was achieved completely only in the isotropic case, that is, for a diagonal viscosity matrix. The results in [26] were extended to the initial boundary value problem in [27]. Well-posedness in the isotropic case with initial data in L^∞ was established by Carrillo [5] in the homogeneous case where the coefficients do not explicitly depend on (x, t). A purely L^1 well-posedness theory for the homogeneous anisotropic case was established by Chen and Perthame in [9]. The latter was extended to the non-homogeneous anisotropic case in [8]. We refer to the bibliography in the cited papers for a more complete list of references on the subject.

Equation (5.1) is a particular case of a degenerate non-homegeneous isotropic equation and, as we said above, its degeneration is of a mild type which makes its study a bit simpler than that of the general degenerate equation. Here we will briefly sketch its analysis in order to introduce some notations and some particular results that will be needed in our study of the homogenization of porous medium type equations in Section 6. For the stability results, we follow closely the analysis in [5] and show which adaptations of the results in [5] need to be made in order to handle the explicit dependence on x of f. For the existence of solutions, which follows from the compactness of the sequence of solutions of regularized (nondegenerate) problems, we use a method motivated by Kruzhkov [19]. We remark that recently Panov [24] has obtained a very general compactness result that, in particular, would imply the one proved here. However the techniques used in [24] are out of the scope of the present paper and we think it is appropriate here to provide a simple and direct proof of this compactness result.

Definition 5.1. A function $u \in L^\infty(\mathbb{R}_+^{n+1})$ is said to be a weak solution of the problem (5.1),(5.2) if the following hold:

(1) $f(x, u(x, t)) \in L^2_{\text{loc}}((0, \infty); H^1_{\text{loc}}(\mathbb{R}^n))$;

(2) For any $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$, we have

$$\int_{\mathbb{R}_+^{n+1}} u\varphi_t - \nabla f(x,u)\cdot\nabla\varphi\,dx\,dt + \int_{\mathbb{R}^n} u_0\varphi(x,0)\,dx = 0. \tag{5.6}$$

Similarly, a function $u \in L^\infty(Q)$ is said to be a weak solution of the problem (5.3),(5.4) if the following hold:

(3) $f(x,u(x,t)) \in L^2_{\text{loc}}((0,+\infty); H^1_0(\Omega))$.

(4) Given $\varphi \in C_c^\infty(\Omega\times\mathbb{R})$, we have

$$\int_Q u\,\partial_t\varphi - \nabla f(x,u)\cdot\nabla\varphi\,dx\,dt + \int_\Omega u_0(x)\varphi(x,0)\,dx = 0. \tag{5.7}$$

Let u be a weak solution of either (5.1),(5.2) or (5.3),(5.4). Denoting by $\langle\cdot,\cdot\rangle$ the usual pairing between $H^{-1}(U)$ and $H_0^1(U)$ when $U\subseteq\mathbb{R}^n$ is open, we can conclude from (5.6) (resp., from (5.7)) that

$$\partial_t u \in L^2_{\text{loc}}(\mathbb{R}_+; H^{-1}_{\text{loc}}(\mathbb{R}^n)), \qquad (\text{resp., } \partial_t u \in L^2_{\text{loc}}(\mathbb{R}_+; H^{-1}_{\text{loc}}(\Omega)))$$

so that the equality (5.6) is equivalent to

$$\int_0^\infty \langle\partial_t u,\varphi\rangle\,dt + \int_{\mathbb{R}_+^{n+1}} \nabla f(x,u)\cdot\nabla\varphi\,dx\,dt - \int_{\mathbb{R}^n} u_0\varphi(x,0)\,dx = 0, \tag{5.8}$$

for all $\varphi\in C_c^\infty(\mathbb{R}^{n+1})$, while (5.7) is equivalent to

$$\int_0^\infty \langle\partial_t u,\varphi\rangle\,dt + \int_Q \nabla f(x,u)\cdot\nabla\varphi\,dx\,dt - \int_\Omega u_0\varphi(x,0)\,dx = 0. \tag{5.9}$$

Let $H_\delta:\mathbb{R}\to\mathbb{R}$ be the approximation of the function sgn given by

$$H_\delta(s) := \begin{cases} 1, & \text{for } s>\delta,\\ \dfrac{s}{\delta}, & \text{for } |s|\le\delta\\ -1, & \text{for } s<-\delta \end{cases}.$$

Given a nondecreasing Lipschitz continuous function $\vartheta:\mathbb{R}\to\mathbb{R}$ and $k\in\mathbb{R}$, we define

$$B_\vartheta^k(x,\lambda) := \int_k^\lambda \vartheta(f(x,r))dr.$$

Concerning the function B_ϑ^k, we will make use of the following lemma which is a version of a lemma in [5], whose proof remains essentially the same and for which, therefore, we refer to [5].

Lemma 5.1. *Let $u\in L^\infty(\mathbb{R}_+^{n+1})$ be a weak solution of (5.1),(5.2). Then, for a.e. $t\in(0,+\infty)$, we have*

$$\int_0^t\int_{\mathbb{R}^n} B_\vartheta^k(x,u)\varphi_s\,ds\,dx + \int_{\mathbb{R}^n} B_\vartheta^k(x,u_0)\varphi(x,0)\,dx - \int_{\mathbb{R}^n} B_\vartheta^k(x,u(t))\varphi(x,t)\,dx$$
$$= -\int_0^t \langle\partial_s u, \vartheta(f(x,u))\varphi\rangle\,ds$$

$\forall k\in\mathbb{R}$ and for all $0\le\varphi\in C_c^\infty(\mathbb{R}^{n+1})$.

Similarly, let $u\in L^\infty(Q)$ be a weak solution of (5.3),(5.4). Then, for a.e. $t\in(0,+\infty)$, we have

$$\int_0^t\int_\Omega B_\vartheta^k(x,u)\varphi_s\,ds\,dx + \int_\Omega B_\vartheta^k(x,u(x,0))\varphi(x,0)\,dx - \int_\Omega B_\vartheta^k(x,u(t))\varphi(x,t)\,dx$$
$$= -\int_0^t \langle\partial_s u, \vartheta(f(x,u))\varphi\rangle\,ds,$$

$\forall k\in\mathbb{R}$ and $\forall\, 0\le\varphi\in C_c^\infty(\Omega\times\mathbb{R})$.

Let us denote

$$\vartheta_\delta(\lambda; y) := H_\delta(\lambda - f(y,k)) \quad \text{and} \quad B_{\vartheta_\delta}(x,\lambda;y) := B_{\vartheta_\delta(\cdot;y)}(x,\lambda).$$

Next we state and prove a lemma which is also an adaptation of a similar result in [5].

Lemma 5.2 (Entropy production term)**.** *Let u be a weak solution of the Cauchy problem* (5.1),(5.2)*, with $u_0 \in L^\infty(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^{n+1}_+} B^k_{\vartheta_\delta}(x,u;y)\varphi_t - H_\delta(f(x,u) - f(y,k))\nabla f(x,u)\cdot\nabla\varphi\,dx\,dt \\ = \int_{\mathbb{R}^{n+1}_+} |\nabla f(x,u)|^2 H'_\delta(f(x,u)-f(y,k))\varphi\,dx\,dt, \tag{5.10}$$

for all $k \in \mathbb{R}$ and all $0 \le \varphi \in C^\infty_c(\mathbb{R}^{n+1}_+)$.

Similarly, let $u \in L^\infty(Q)$ be a weak solution of (5.3),(5.4)*. Then,*

$$\int_Q B^k_{\vartheta_\delta}(x,u;y)\varphi_t - H_\delta(f(x,u) - f(y,k))\nabla f(x,u)\cdot\nabla\varphi\,dx\,dt \\ = \int_Q |\nabla f(x,u)|^2 H'_\delta(f(x,u)-f(y,k))\varphi\,dx\,dt \tag{5.11}$$

for all $k \in \mathbb{R}$ and all $0 \le \varphi \in C^\infty_c(Q)$.

Proof. By the Lemma 5.1, we have

$$-\int_0^{+\infty}\langle \partial_t u, H_\delta(f(x,u)-f(y,k))\varphi\rangle\,dt = \int_{\mathbb{R}^{n+1}_+} B^k_{\vartheta_\delta}(x,u;y)\varphi_t\,dx\,dt.$$

Since u is a weak solution and $H_\delta(f(x,u)-f(y,k))\varphi$ is a test function for each fixed y and k, we get

$$-\int_0^{+\infty}\langle \partial_t u, H_\delta(f(x,u)-f(y,k))\varphi\rangle\,dt - \int_{\mathbb{R}^{n+1}_+}\{\nabla f(x,u)\cdot\nabla(H_\delta(f(x,u)-f(y,k))\varphi)\}\,dx\,dt = 0.$$

This equality with the previous one gives

$$\int_{\mathbb{R}^{n+1}_+}\{B^k_{\vartheta_\delta}(x,u;y)\varphi_t - \nabla f(x,u)\cdot\nabla(H_\delta(f(x,u)-f(y,k))\varphi)\}\,dx\,dt = 0,$$

and this equality yields (5.10).

The proof of (5.11) follows similarly with obvious adaptations.

□

The following theorem, which follows from (5.10) (resp., (5.11)), by using doubling of variables, and the trick of completing the square in [5], theorem 13, p. 339, will be used in our analysis of the homogenization problem in Section 6. We give its proof here for the reader's convenience.

Theorem 5.1. *Let u_1, u_2 be weak solutions of the Cauchy problem* (5.1),(5.2) *with initial data $u_{01}, u_{02} \in L^\infty(\mathbb{R}^n)$. Then we have the following:*

(i) *For all $0 \le \varphi \in C^\infty_c(\mathbb{R}^{n+1}_+)$, we have*

$$\int_{\mathbb{R}^{n+1}_+} |u_1(x,t) - u_2(x,t)|\varphi_t - \nabla|f(x,u_1(x,t)) - f(x,u_2(x,t))|\cdot\nabla\varphi\,dx\,dt \ge 0. \tag{5.12}$$

(ii) *If u_2 is a stationary solution, then*

$$\int_{\mathbb{R}^{n+1}_+} B^{u_2(x)}_{\vartheta_\delta}(x,u_1(x,t))\varphi_t\,dx\,dt - \int_{\mathbb{R}^{n+1}_+} H_\delta(f(x,u_1(x,t))-f(x,u_2(x)))\nabla[f(x,u_1(x,t))-f(x,u_2(x))]\cdot\nabla\varphi\,dx\,dt = \int_{\mathbb{R}^{n+1}_+} |\nabla[f(x,u_1(x,t))-f(x,u_2(x))]|^2 H'_\delta(f(x,u_1(x,t))-f(x,u_2(x)))\varphi\,dx\,dt, \tag{5.13}$$

for all $0\le\varphi\in C_c^\infty(\mathbb{R}^{n+1}_+)$.

Similarly, if u_1,u_2 are weak solutions of the initial-boundary value problem (5.3),(5.4)*, with initial data $u_{01},u_{02}\in L^\infty(\Omega)$, then we have the following:*

(iii) *For all $0\le\varphi\in C_c^\infty(Q)$, we have*

$$\int_Q |u_1(x,t)-u_2(x,t)|\varphi_t - \nabla|f(x,u_1(x,t))-f(x,u_2(x,t))|\cdot\nabla\varphi\,dx\,dt\ge 0. \tag{5.14}$$

(iv) *If u_2 is a stationary solution, then*

$$-\int_Q B^{u_2(x)}_{\vartheta_\delta}(x,u_1(x,t))\varphi_t\,dx\,dt + \int_Q H_\delta(f(x,u_1(x,t))-f(x,u_2(x)))\nabla[f(x,u_1(x,t))-f(x,u_2(x))]\cdot\nabla\varphi\,dx\,dt = -\int_Q |\nabla[f(x,u_1(x,t))-f(x,u_2(x))]|^2 H'_\delta(f(x,u_1(x,t))-f(x,u_2(x)))\varphi\,dx\,dt, \tag{5.15}$$

for all $0\le\varphi\in C_c^\infty(Q)$.

Proof. We begin by proving the assertions concerning the Cauchy problem (5.1),(5.2). Let $u_1=u(x,t)$ and $u_2=u_2(y,s)$. By (5.10) applied to u_1, we have

$$\int_{\mathbb{R}^{n+1}_+}\{B^k_{\vartheta_\delta}(x,u_1;y)\phi_t - H_\delta(f(x,u_1)-f(y,k))\nabla_x f(x,u_1)\cdot\nabla_x\phi\}\,dx\,dt = \int_{\mathbb{R}^{n+1}_+}|\nabla_x f(x,u_1)|^2 H'_\delta(f(x,u_1)-f(y,k))\phi\,dx\,dt,$$

for all $k\in\mathbb{R}$ and for all $0\le\phi\in C_c^\infty((\mathbb{R}^{n+1}_+)^2)$. Setting $k=u_2$ and integrating in y,s, we obtain

$$\int_{(\mathbb{R}^{n+1}_+)^2}\{B^{u_2}_{\vartheta_\delta}(x,u_1;y)\phi_t - H_\delta(f(x,u_1)-f(y,u_2))\nabla_x f(x,u_1)\cdot\nabla_x\phi\,dx\,dt\,dy\,ds = \int_{(\mathbb{R}^{n+1}_+)^2}|\nabla_x f(x,u_1)|^2 H'_\delta(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds. \tag{5.16}$$

Now, applying (5.10) to u_2, taking $k=u_1$ and integrating in x,t, we obtain

$$\int_{(\mathbb{R}^{n+1}_+)^2}\Big\{B^{u_1}_{\vartheta_\delta}(y,u_2;x)\phi_s + H_\delta(f(x,u_1)-f(y,u_2))\nabla_y f(y,u_2)\cdot\nabla_y\phi\Big\}\,dx\,dt\,dy\,ds = \int_{(\mathbb{R}^{n+1}_+)^2}|\nabla_y f(y,u_2)|^2 H'_\delta(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds \tag{5.17}$$

Now, note that

$$0=\int_{\mathbb{R}_+^{n+1}} \nabla_y f(y,u_2)\cdot\nabla_x[H_\delta(f(x,u_1)-f(y,u_2))\phi]\,dx\,dt$$
$$=\int_{\mathbb{R}_+^{n+1}} \Big\{\nabla_y f(y,u_2)\cdot\nabla_x f(x,u_1)H_\delta'(f(x,u_1)-f(y,u_2))\phi+H_\delta(f(x,u_1)-f(y,u_2))\nabla_y f(y,u_2)\cdot\nabla_x\phi\Big\}\,dx\,dt$$

and so we have

$$\int_{(\mathbb{R}_+^{n+1})^2} H_\delta(f(x,u_1)-f(y,u_2))\nabla_y f(y,u_2)\cdot\nabla_x\phi\,dx\,dt\,dy\,ds$$
$$=-\int_{(\mathbb{R}_+^{n+1})^2} \nabla_y f(y,u_2)\cdot\nabla_x f(x,u_1)H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds \tag{5.18}$$

Analogously,

$$\int_{(\mathbb{R}_+^{n+1})^2} H_\delta(f(x,u_1)-f(y,u_2))\nabla_x f(x,u_1)\cdot\nabla_y\phi\,dx\,dt\,dy\,ds$$
$$=\int_{(\mathbb{R}_+^{n+1})^2} \nabla_y f(y,u_2)\cdot\nabla_x f(x,u_1)H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds \tag{5.19}$$

Adding (5.16) and (5.19) yields

$$\int_{(\mathbb{R}_+^{n+1})^2} \Big\{B_{\vartheta_\delta}^{u_2}(x,u_1;y)\phi_t+H_\delta(f(x,u_1)-f(y,u_2))\nabla_x f(x,u_1)\cdot(\nabla_x+\nabla_y)\phi\Big\}\,dx\,dt\,dy\,ds$$
$$=\int_{(\mathbb{R}_+^{n+1})^2} \Big\{|\nabla_x f(x,u_1)|^2+\nabla_x f(x,u_1)\cdot\nabla_y f(y,u_2)\Big\}H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds \tag{5.20}$$

Further, multiplying (5.17) by -1 and adding to (5.18) gives

$$\int_{(\mathbb{R}_+^{n+1})^2} \Big\{B_{\vartheta_\delta}^{u_1}(y,u_2;x)\phi_s+H_\delta(f(x,u_1)-f(y,u_2))\nabla_y f(y,u_2)\cdot(\nabla_x+\nabla_y)\phi\Big\}\,dx\,dt\,dy\,ds$$
$$=\int_{(\mathbb{R}_+^{n+1})^2} \Big\{|\nabla_y f(y,u_2)|^2-\nabla_x f(x,u_1)\cdot\nabla_y f(y,u_2)\Big\}H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds. \tag{5.21}$$

Now, adding (5.20) and (5.21) we obtain

$$\int_{(\mathbb{R}_+^{n+1})^2} \Big\{B_{\vartheta_\delta}^{u_2}(x,u_1;y)\phi_t+B_{\vartheta_\delta}^{u_1}(y,u_2;x)\phi_s$$
$$-H_\delta(f(x,u_1)-f(y,u_2))(\nabla_x+\nabla_y)(f(x,u_1)-f(y,u_2))\cdot(\nabla_x+\nabla_y)\phi\Big\}\,dx\,dt\,dy\,ds$$
$$=+\int_{(\mathbb{R}_+^{n+1})^2} |(\nabla_x+\nabla_y)(f(x,u_1)-f(y,u_2))|^2H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds. \tag{5.22}$$

We then use test functions as $\phi(x,t,y,s):=\varphi(\frac{x+y}{2},\frac{t+s}{2})\rho_k(\frac{x-y}{2})\theta_l(\frac{t-s}{2})$, where $0\le\varphi\in C_c^\infty(\mathbb{R}_+^{n+1})$, and $\rho_k,\ \theta_l$ are classical approximations of the identity in $\mathbb{R}^n$ and $\mathbb{R}$, respectively, as in the doubling of variables method. Hence, letting $k\to\infty$ first, later $\delta\to0$ and then letting $l\to\infty$, we obtain (5.12).

To obtain (5.13), we observe that if u_2 is stationary solution then $B_{\vartheta_\delta}^{u_1}(y,u_2;x)$ and $B_{\vartheta_\delta}^{u_2}(x,u_1;y)$ are independent of s and so, we can write the trivial equality where both members are null

$$\int_{(\mathbb{R}_+^{n+1})^2} B_{\vartheta_\delta}^{u_1}(y,u_2;x)\phi_s\,dx\,dt\,dy\,ds=\int_{(\mathbb{R}_+^{n+1})^2} B_{\vartheta_\delta}^{u_2}(x,u_1;y)\phi_s\,dx\,dt\,dy\,ds$$

Combining the previous equality in (5.22), we have

$$\int_{(\mathbb{R}_+^{n+1})^2}\Big\{B_{\vartheta_\delta}^{u_2}(x,u_1;y)(\phi_t+\phi_s)$$

$$-H_\delta(f(x,u_1)-f(y,u_2))(\nabla_x+\nabla_y)(f(x,u_1)-f(y,u_2))\cdot(\nabla_x+\nabla_y)\phi\Big\}\,dx\,dy\,dt\,ds$$

$$=\int_{(\mathbb{R}_+^{n+1})^2}|(\nabla_x+\nabla_y)(f(x,u_1)-f(y,u_2))|^2H_\delta'(f(x,u_1)-f(y,u_2))\phi\,dx\,dt\,dy\,ds.$$

Now, using test functions as above and letting $k,l\to\infty$, we get (5.13).

The relations (5.14) and (5.15) concerning problem (5.3),(5.4) are proved in an entirely similar way. □

Remark 5.1. As usual, we denote $(s)_\pm:=\{\pm s,0\}$. The same arguments in the above proof lead to an inequality similar to (5.12) (resp., (5.14)) with $|u_1-u_2|$, $|f(x,u_1)-f(x,u_2)|$ replaced by $(u_1-u_2)_\pm$, $(f(x,u_1)-f(x,u_2))_\pm$, respectively, just by using $B_{(\vartheta_\delta)_\pm}^k$, $(H_\delta)_\pm$, instead of $B_{\vartheta_\delta}^k$, H_δ, respectively. We thus obtain

$$\int_{\mathbb{R}_+^{n+1}}(u_1(x,t)-u_2(x,t))_\pm\varphi_t-\nabla(f(x,u_1(x,t))-f(x,u_2(x,t)))_\pm\cdot\nabla\varphi\,dx\,dt\ge 0. \tag{5.23}$$

in the case of problem (5.1),(5.2), and

$$\int_Q(u_1(x,t)-u_2(x,t))_\pm\varphi_t-\nabla(f(x,u_1(x,t))-f(x,u_2(x,t)))_\pm\cdot\nabla\varphi\,dx\,dt\ge 0, \tag{5.24}$$

in the case of problem (5.3),(5.4), where we mean one inequality holding with $(\cdot)_+$ and another holding for $(\cdot)_-$. Moreover, in the latter case, to obtain (5.14) and (5.24) we only need that $u_i\in L^\infty(Q)$ satisfies (5.7) and $f(x,u_i(x,t))\in L^2_{\text{loc}}((0,\infty);H^1(\Omega))$ instead of $f(x,u_i(x,t))\in L^2_{\text{loc}}((0,\infty);H^1_0(\Omega))$, $i=1,2$, as can be easily checked.

Concerning the Cauchy problem (5.1),(5.2), we now consider the following weight function $\Lambda:\mathbb{R}^n\to\mathbb{R}$ defined by

$$\Lambda(x):=e^{-\sqrt{1+|x|^2}}. \tag{5.25}$$

The relevance of the weight function Λ for our purposes is that

$$|D_i\Lambda(x)|\le\Lambda(x),\quad\text{for } i=1,\dots,n,\quad\text{and}\quad|\Delta\Lambda(x)|\le(n+1)\Lambda(x),\quad\text{for } x\in\mathbb{R}^n. \tag{5.26}$$

Concerning the initial-boundary value problem (5.3),(5.4), we let $\xi\in H^1_0(\Omega)$ be the eigenfunction of $-\Delta$ associated with the eigenvalue $\lambda_1>0$ such that $\xi>0$ in Ω (see, e.g., [14]).

Theorem 5.2 (Uniqueness)**.** *Let u_1,u_2 be weak solutions of the Cauchy problem* (5.1),(5.2) *with initial data $u_{01},u_{02}\in L^\infty(\mathbb{R}^n)$. Then, there exists $C>0$ such that for a.e. $t>0$, we have*

$$\int_{\mathbb{R}^n}|u_1(t)-u_2(t)|\Lambda(x)\,dx\le e^{Ct}\int_{\mathbb{R}^n}|u_{01}(x)-u_{02}(x)|\Lambda(x)\,dx. \tag{5.27}$$

Similarly, let u_1,u_2 be weak solutions of the initial-boundary value problem (5.3),(5.4) *with initial data $u_{01},u_{02}\in L^\infty(\Omega)$. Then, there exists $C>0$ such that for a.e. $t>0$, we have*

$$\int_\Omega|u_1(t)-u_2(t)|\xi(x)\,dx\le e^{Ct}\int_\Omega|u_{01}(x)-u_{02}(x)|\xi(x)\,dx. \tag{5.28}$$

Proof. Taking $\varphi(x,t)=\delta_h(t)\Lambda(x)$, with $0\le\delta_h\in C_c^\infty((0,+\infty))$ in (i) of Theorem (5.1), we obtain

$$\int_{\mathbb{R}_+^{n+1}}\Big\{-|u_1-u_2|\delta_h'(t)\Lambda(x)-|f(x,u_1)-f(x,u_2)|\delta_h(t)\Delta\Lambda\Big\}\,dx\,dt\le 0.$$

Observe that

$$-\int_{\mathbb{R}_+^{n+1}}|u_1-u_2|\delta_h'(t)\Lambda(x)\,dx\,dt\le\int_{\mathbb{R}_+^{n+1}}\Big\{|f(x,u_1)-f(x,u_2)|\delta_h(t)|\Delta\Lambda|\Big\}\,dx\,dt$$
$$\le C\int_{\mathbb{R}_+^{n+1}}|u_1-u_2|\delta_h(t)\Lambda(x)\,dx\,dt,$$

where we use that $|\Delta\Lambda|\le(n+1)\Lambda$ and the Lipschitz condition on $f(x,u)$. We define

$$\beta(s):=\int_{\mathbb{R}^n}|u_1(x,s)-u_2(x,s)|\Lambda(x)\,dx.$$

Then, using a suitable sequence of functions δ_h and letting $h\to0$, we arrive at

$$\beta(t)\le\int_{\mathbb{R}^n}|u_{01}(x)-u_{02}(x)|\Lambda(x)\,dx+C\int_0^t\beta(s)\,ds.$$

Hence, we may apply Gronwall's lemma to conclude the proof of (5.27).

The proof of (5.28) is entirely similar starting now by taking $\varphi(x,t)=\delta_h(t)\xi(x)$ in (iii) of Theorem 5.1. □

Remark 5.2. Noting that $(f(x,u_1)-f(x,u_2))_\pm\le C(u_1-u_2)_\pm$, respectively, and using Remark 5.1 we see that the same arguments show that

$$\int_{\mathbb{R}^n}(u_1(t)-u_2(t))_\pm\Lambda(x)\,dx\le e^{Ct}\int_{\mathbb{R}^n}(u_{01}(x)-u_{02}(x))_\pm\Lambda\,dx \tag{5.29}$$

and

$$\int_\Omega(u_1(t)-u_2(t))_\pm\xi(x)\,dx\le e^{Ct}\int_\Omega(u_{01}(x)-u_{02}(x))_\pm\xi(x)\,dx \tag{5.30}$$

for a.e. $t>0$ for weak solutions of problems (5.1),(5.2) and (5.3),(5.4), respectively. Moreover, as a consequence of Remark 5.1, for the problem (5.3),(5.4), to obtain (5.30) we only need that $u_i\in L^\infty(Q)$ satisfies (5.7) and $f(x,u_i(x,t))\in L^2_{\rm loc}((0,\infty);H^1(\Omega))$ instead of $f(x,u_i(x,t))\in L^2_{\rm loc}((0,\infty);H^1_0(\Omega))$, $i=1,2$, provided

$$(f(x,u_1(x,t))-f(x,u_2(x,t)))_\pm|\partial\Omega\equiv0,\qquad a.e.\ t\in(0,\infty),\quad\text{respectively,}$$

the latter meaning the trace for functions in $H^1(\Omega)$.

The above remark immediately implies the following result.

Corollary 5.1 (Monotonicity)**.** *Let u_1,u_2 to be as in the Theorem 5.2. Suppose that $u_{01}\le u_{02}$ a.e. in $\mathbb{R}^n$ (resp., in Ω). Then,*

$$u_1\le u_2,\qquad a.e.\ in\ \mathbb{R}^n\ (resp.,\ a.e.\ in\ \Omega).$$

Remark 5.3. We remark that so far we have only used that $f(x,u)$ satisfies **(f1)**. In particular, for the stability and monotonicity results it sufficies **(f1)**. The assumptions **(f2)**, **(f3)** and **(f4)** will be only needed for the subsequent discussion on the existence of solutions.

Our next goal is to prove the existence of a weak solution for (5.1),(5.2) and for (5.3),(5.4).

Before we begin properly the discussion about the existence question, we state a well known result on the compactness in the space L^1, which will be needed. The proof, which we ommit here, follows in a standard way by mollification and application of Arzela-Ascoli theorem.

Lemma 5.3. *Let $U \subseteq \mathbb{R}^n$ be an open set and $\mathbb{F} \subseteq L^1_{loc}(U)$ be a family uniformly bounded in $L^1(B)$, for any closed ball $B \subseteq U$. Suppose that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for $|y| < \delta$ we have*

$$\int_B |u(x+y) - u(x)|\, dx < \varepsilon, \quad \forall\, u \in \mathbb{F}.$$

Then, $\mathbb{F}$ is relatively compact in $L^1_{loc}(U)$.

We consider the following regularized version of (5.1),(5.2),

$$\partial_t u - \Delta f^\varepsilon(x,u) = 0, \qquad (x,t) \in \mathbb{R}^{n+1}_+, \tag{5.31}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{5.32}$$

where $f^\varepsilon(x,\lambda) := f(x,\lambda) + \varepsilon\lambda$ and, for the moment, we assume

$$u_0 \in W^{2,\infty}(\mathbb{R}^n). \tag{5.33}$$

Similarly, we also consider the regularized version of (5.3),(5.4),

$$\partial_t u - \Delta f^\varepsilon(x,u) = 0, \qquad (x,t) \in Q, \tag{5.34}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \qquad u|\partial\Omega \times (0,\infty) = 0, \tag{5.35}$$

where f^ε is as above and, again, for the moment, we assume

$$u_0 \in W^{2,\infty}_0(\Omega). \tag{5.36}$$

The existence and uniqueness of a classical solution of (5.31),(5.32) (resp., (5.34),(5.35)) for $\varepsilon > 0$ having bounded derivatives is proved, for example, in [21].

Motivated by [19] we now establish the following result.

Theorem 5.3. *Let u_ε be the solution of the regularized problem (5.31),(5.32). Then, for $|y| < \delta < 1$ and $t \in [0,T]$, we have*

$$\int_{\mathbb{R}^n} |u_\varepsilon(x+y,t) - u_\varepsilon(x,t)|\Lambda(x)\, dx \le c_0\delta, \tag{5.37}$$

where the constant c_0 is independent of ε. Moreover, for some constant $M > 0$ independent of ε, we have

$$\int_{\mathbb{R}^n} |u_\varepsilon(x,t+s) - u_\varepsilon(x,t)|\Lambda(x)\, dx \le \min_{0<\delta<1} \left\{ (2c_0 + \|\nabla\Lambda\|_1)\delta + s\, M\left(\frac{1}{\delta^2} + \frac{2}{\delta} + 1\right)\|\Lambda\|_1 \right\} \overset{s\to 0}{\longrightarrow} 0. \tag{5.38}$$

Similarly, if u_ε is the solution of (5.34),(5.35), then for any $0 \le \varphi \in C^\infty_c(\Omega)$ and $|y| < \delta$, with δ sufficiently small,

$$\int_\Omega |u_\varepsilon(x+y,t) - u_\varepsilon(x,t)|\varphi(x)\, dx \le c_1\delta, \tag{5.39}$$

where the constant c_1 is independent of ε. Moreover, for some constant $M > 0$ independent of ε, we have

$$\int_\Omega |u_\varepsilon(x,t+s) - u_\varepsilon(x,t)|\varphi(x)\, dx \le \min_{0<\delta<1} \left\{ (2c_1 + \|\nabla\Lambda\|_1)\delta + s\, M\left(\frac{1}{\delta^2} + \frac{2}{\delta} + 1\right)\|\varphi\|_1 \right\} \overset{s\to 0}{\longrightarrow} 0. \tag{5.40}$$

Proof. 1. To deduce (5.37), for each $k = 1, \cdots, n$ define $v^k := \partial_{x_k} u_\varepsilon$ and observe that

$$\partial_t v^k - \Delta(f^\varepsilon_u(x,u)v^k) - \nabla\cdot(f^\varepsilon_{x_k u}(x,u)\nabla u) - \big(\nabla f^\varepsilon_{x_k u}\big)(x,u)\cdot\nabla u = -\big(\Delta f^\varepsilon\big)(x,u), \tag{5.41}$$

where, for simplicity of notation, we denote u_ε by u, $\big(f^\varepsilon_{x_1 x_k u}(x,u), \cdots, f^\varepsilon_{x_n x_k u}(x,u)\big)$ by $\big(\nabla f^\varepsilon_{x_k u}\big)(x,u)$ and $\sum_{i=1}^n f^\varepsilon_{x_i x_i}(x,u)$ by $\big(\Delta f^\varepsilon\big)(x,u)$.

We fix a number $T > 0$ and let $g^k \in C^\infty([0,T]\times\mathbb{R}^n)$ be such that $g^k(t) \in C^\infty_c(\mathbb{R}^n)$ for all $t \in [0,T]$. Now, taking $0 < t_0 \le T$, multiplying the equation (5.41) by g^k, integrating by parts and summing over k from 1 to n, we get

$$
\begin{aligned}
&\int_0^{t_0}\int_{\mathbb{R}^n} -\sum_{k=1}^n \Big\{\partial_t g^k + f_u^\varepsilon(x,u)\Delta g^k - \sum_{i=1}^n \big(f^\varepsilon_{x_i u}(x,u) g^i_{x_k} - f^\varepsilon_{x_i x_k u}(x,u) g^i\big)\Big\} v^k\,dx\,dt \\
&\qquad + \int_{\mathbb{R}^n}\sum_{k=1}^n v^k(t_0) g^k(t_0)\,dx = \int_{\mathbb{R}^n}\sum_{k=1}^n \Big\{ v^k(0) g^k(0) - (\Delta f)(x,u) g^k(t_0)\Big\}\,dx.
\end{aligned} \tag{5.42}
$$

For $k = 1, \cdots, n$ and $g = (g^1, \cdots, g^n)$, we define

$$
\mathcal{L}_k(g) := \partial_t g^k + f_u^\varepsilon(x,u)\Delta g^k - \sum_{i=1}^n \big(g^i_{x_k} f^\varepsilon_{x_i u}(x,u) - f^\varepsilon_{x_i x_k u}(x,u) g^i\big). \tag{5.43}
$$

Let φ_h^k, $k = 1, \cdots, n$ be the solution of the Cauchy problem

$$
\begin{cases}
\mathcal{L}_k(\varphi_h) = 0, & (x,t) \in \mathbb{R}^n \times (0,t_0),\\
\varphi_h^k(t_0) = \operatorname{sgn}(v^k(t_0)) * \rho_h\, e^{-|x|}, & x \in \mathbb{R}^n,
\end{cases} \tag{5.44}
$$

where $\rho_h = h^{-n}\rho(h^{-1}x)$, and $0 \le \rho \in C_c(\mathbb{R}^n)$ is a standard symmetric mollifier satisfying $\operatorname{supp}\rho \subseteq \{x : |x| \le 1\}$ and $\int_{\mathbb{R}^n}\rho\,dx = 1$.

Now, observe that

$$
\begin{aligned}
0 = 2\mathcal{L}_k(\varphi_h)\varphi_h^k &= \partial_t(\varphi_h^k)^2 + f_u^\varepsilon(x,u)\Delta(\varphi_h^k)^2 - 2 f_u^\varepsilon(x,u)|\nabla\varphi_h^k|^2 \\
&\quad - 2\sum_{i=1}^n f^\varepsilon_{x_i u}(x,u)\varphi^i_{h,x_k}\varphi_h^k + 2\sum_{i=i}^n f^\varepsilon_{x_i x_k u}(x,u)\varphi_h^i\varphi_h^k
\end{aligned}
$$

Summing over k, using the Cauchy inequality with δ, the fact that $f^\varepsilon_{x_j u}(x,u) = f_{x_j u}(x,u)$ and (5.5), we have

$$
\begin{aligned}
0 &\le \partial_t|\varphi_h|^2 + f_u^\varepsilon(x,u)\Delta|\varphi_h|^2 + 2\Big(-f_u(x,u) + \theta_0\sum_{i=1}^n |f_{x_i u}(x,u)|\Big)\sum_{k=1}^n |\nabla\varphi_h^k|^2 + c(\theta_0)|\varphi_h|^2 \\
&\le \partial_t|\varphi_h|^2 + f_u^\varepsilon(x,u)\Delta|\varphi_h|^2 + c|\varphi_h|^2.
\end{aligned} \tag{5.45}
$$

2. In this step, we prove that

$$
|\varphi_h|^2 \le c(\theta_0, T)\, e^{-\frac{|x|}{M}},
$$

for all $(x,t) \in \mathbb{R}^n \times [0,t_0]$.

We begin by defining $\mathcal{L}(v) := \partial_t v + f_u^\varepsilon(x,u)\Delta v$, $w := e^{ct}|\varphi_h|^2$, and observing that (5.45) implies $\mathcal{L}(w) \ge 0$. From the latter, it follows by the maximum principle that $|\varphi_h(x,t)| \le 1$ for all $(x,t) \in \mathbb{R}^n \times [0,t_0]$. In particular, given $q_0 > n\, e^{2cT}$, we obtain that $|w| \le q_0$ for all $(x,t) \in \mathbb{R}^n \times [0,t_0]$.

Now, set

$$
q(x,t) := q_0\, e^{\frac{1}{M}\left(t_0 - t - |x|\right)},
$$

with $M > \sup_{\mathbb{R}^n\times I} f_u(x,u)$, $I \supset [-\|u_\varepsilon\|_\infty, \|u_\varepsilon\|_\infty]$ for $0 < \varepsilon < 1$, and note that

$$
\mathcal{L}(q) = -q\Big\{\frac{1}{M}\Big(1 - \frac{f_u^\varepsilon(x,u)}{M}\Big) + \frac{f_u^\varepsilon(x,u)}{M}\frac{n-1}{|x|}\Big\} \le 0,
$$

which yields $\mathcal{L}(w - q) \ge 0$. It is easily seen that

$$
w - q|_{\{0 \le t \le t_0;\, |x| = t_0 - t\}} = w - q_0 \le 0, \qquad w(x,t_0) - q(x,t_0) \le 0.
$$

Then, the claim follows by the maximum principle (cf., e.g., [25]).

3. Let $0 \le \rho \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \rho \subseteq [-1,1]$ and $\int_{\mathbb{R}} \rho\, dx = 1$. Set

$$\eta_m(\lambda) := 1 - \int_{-\infty}^{\lambda} \rho(s-m)\, ds,$$

for $m \in \mathbb{N}$, and take

$$g^k(x,t) := \varphi_h^k(x,t)\, \eta_m(|x|)$$

as a test function in (5.42). Hence

$$\begin{aligned}\int_{\mathbb{R}^n} \sum_{k=1}^n v^k(t_0)\operatorname{sgn}(v^k(t_0)) * \rho_h\, e^{-|x|}\eta_m(|x|)\, dx = \sum_{k=1}^n \int_0^{t_0}\int_{\mathbb{R}^n} \Big\{ 2f^\varepsilon(x,u)\nabla\varphi_h^k \cdot \nabla\eta_m(|x|) \\ + f^\varepsilon(x,u)\varphi_h^k \Delta\eta_m(|x|) - \sum_{i=1}^n f^\varepsilon_{x_i u}(x,u)\partial_{x_k}\eta_m(|x|)\varphi_h^k \Big\}\, dx\, dt \\ + \int_{\mathbb{R}^n}\sum_{k=1}^n \Big\{ v^k(0)\varphi_h^k(x,0) + -(\Delta f)(x,u)\varphi_h^k(x,t_0)\Big\}\eta_m(|x|)\, dx.\end{aligned} \tag{5.46}$$

Thus, letting $m \to \infty$ first and then letting $h \to 0$, we obtain an estimate of the form

$$\int_{\mathbb{R}^n} \sum_{k=1}^n |v^k(t_0)|\, e^{-|x|}\, dx \le c(T, \theta_0, \|\nabla u_0\|_\infty, \|\nabla h\|_\infty) < \infty,$$

for all $t_0 \in [0,T]$, where, in particular, the right-hand side does not depend on ε. Consequently

$$\int_{\mathbb{R}^n} |u_\varepsilon(x+y,t) - u_\varepsilon(x,t)|\Lambda(x)\, dx \le c_0|y|,$$

for some c_0 independent of ε, which gives (5.37).

4. To deduce (5.38), we first note that from the maximum principle and the hypotheses on f, we know that there exists $M > 0$ such that $|f^\varepsilon(x, u_\varepsilon(x,t))| \le M$ for all $(x,t) \in \mathbb{R}_+^{n+1}$ and for all $\varepsilon > 0$. Now, fix t, s, ε and set $w(x) := u_\varepsilon(x,t+s) - u_\varepsilon(x,t)$. Given $\varphi \in W^{2,\infty}(\mathbb{R}^n)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} w(x)\varphi(x)\Lambda(x)\, dx &= \int_{\mathbb{R}^n}\int_t^{t+s} \partial_t u_\varepsilon(x,\tau)\varphi\Lambda\, d\tau\, dx = \int_{\mathbb{R}^n}\int_t^{t+s} \Delta f^\varepsilon(x,u_\varepsilon)\varphi\Lambda\, d\tau\, dx \\ &= \int_{\mathbb{R}^n}\int_t^{t+s} f^\varepsilon(x,u_\varepsilon)\Delta(\varphi\Lambda)\, d\tau\, dx \\ &= \int_{\mathbb{R}^n}\int_t^{t+s} \Big\{ f^\varepsilon(x,u_\varepsilon)\Delta\varphi\Lambda + 2f^\varepsilon(x,u_\varepsilon)\nabla\varphi\cdot\nabla\Lambda + f^\varepsilon(x,u_\varepsilon)\varphi\Delta\Lambda\Big\}\, d\tau\, dx,\end{aligned}$$

and this implies

$$\Big|\int_{\mathbb{R}^n} w(x)\varphi(x)\Lambda(x)\, dx\Big| \le M\Big\{ \|\Delta\varphi\|_\infty + 2\|\nabla\varphi\|_\infty + \|\varphi\|_\infty\Big\}\|\Lambda\|_1 s. \tag{5.47}$$

Taking $\varphi = (\operatorname{sgn} w) * \rho_\delta$ and observing that $\|\nabla\varphi\|_\infty \le \frac{c}{\delta}$, $\|\Delta\varphi\|_\infty \le \frac{c}{\delta^2}$ and $\|\varphi\|_\infty \le 1$, where c only depends on the dimension, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |w(x)|\Lambda(x)\,dx &= \Big(\int_{\mathbb{R}^n} w(x)\ \mathrm{sgn}(w(x))\,\Lambda(x)\,dx\Big)\int_{\mathbb{R}^n}\rho(y)\,dy\\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} w(x-\delta y)\ \mathrm{sgn}(w(x-\delta y))\,\Lambda(x-\delta y)\rho(y)\,dx\,dy\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} w(x)\varphi(x)\Lambda(x)\,dx &= \int_{\mathbb{R}^n} w(x)\Lambda(x)\Big(\int_{\mathbb{R}^n}\mathrm{sgn}(w(y))\,\rho_\delta(x-y)\,dy\Big)\,dx\\ &= \int_{\mathbb{R}^n} w(x)\Lambda(x)\Big(\int_{\mathbb{R}^n}\mathrm{sgn}(w(x-\delta y))\rho(y)\,dy\Big)\,dx\\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n} w(x)\,\Lambda(x)\ \mathrm{sgn}(w(x-\delta y))\,\rho(y)\,dx\,dy.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{\mathbb{R}^n} |w(x)|\Lambda(x)\,dx - \int_{\mathbb{R}^n} w(x)\varphi(x)\Lambda(x)\,dx\\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n}\Big\{w(x-\delta y)\ \mathrm{sgn}(w(x-\delta y))\,\Lambda(x-\delta y) - w(x)\,\Lambda(x)\ \mathrm{sgn}(w(x-\delta y))\Big\}\rho(y)\,dx\,dy\\ &= \int_{\mathbb{R}^n\times\mathbb{R}^n}\Big\{\big[w(x-\delta y)-w(x)\big]\ \mathrm{sgn}(w(x-\delta y))\Lambda(x) + \big[\Lambda(x-\delta y)-\Lambda(x)\big]\ \mathrm{sgn}(w(x-\delta y))\,w(x-\delta y)\Big\}\rho(y)\,dx\,dy.\end{aligned}$$

Therefore,

$$\Big|\int_{\mathbb{R}^n} |w(x)|\Lambda(x)\,dx - \int_{\mathbb{R}^n} w(x)\varphi(x)\Lambda(x)\,dx\Big| \le (2c_0 + \|\nabla\Lambda\|_1)\delta. \tag{5.48}$$

Thus, we conclude from (5.48) and from (5.47) that

$$\int_{\mathbb{R}^n} |w(x)|\Lambda(x)\,dx \le\ (2c_0+\|\nabla\Lambda\|_1)\delta + s\,M\Big\{\frac{1}{\delta^2}+\frac{2}{\delta}+1\Big\}\|\Lambda\|_1,$$

for all $0<\delta<1$, which completes the proof of the assertions concerning the Cauchy problem (5.31),(5.32).

5. As to the proof of the assertions concerning the initial-boundary value problem (5.34),(5.35) we have the following. The proof of (5.39) is achieved following the same lines as the proof of (5.37) with the following small adaptations. We first get the analogue of equation (5.42),

$$\begin{aligned}&\int_0^{t_0}\int_\Omega -\sum_{k=1}^n\Big\{\partial_t g^k + f_u^\varepsilon(x,u)\Delta g^k - \sum_{i=1}^n\big(f_{x_iu}^\varepsilon(x,u)g_{x_k}^i - f_{x_ix_ku}^\varepsilon(x,u)g^i\big)\Big\}v^k\,dx\,dt\\ &\quad+\int_{\mathbb{R}^n}\sum_{k=1}^n v^k(t_0)g^k(t_0)\,dx = \int_\Omega\sum_{k=1}^n\Big\{v^k(0)g^k(0) - (\Delta f)(x,u)g^k(t_0)\Big\}\,dx.\end{aligned}\tag{5.49}$$

Now we define φ_h^k, $k=1,\cdots,n$, as the solution of the initial-boundary value problem

$$\begin{cases}\mathcal{L}_k(\varphi_h)=0, & (x,t)\in\Omega\times(0,t_0),\\ \varphi_h^k(t_0)=\big(\mathrm{sgn}(v^k(t_0))\chi_{\Omega_{2h}}\big)*\rho_h, & x\in\Omega,\\ \varphi_h^k(x,t)=0, & (x,t)\in\partial\Omega\times(0,t_0),\end{cases}\tag{5.50}$$

where χ_A denotes, as usual, the indicator function of the set A, and $\Omega_{2h} := \{ x \in \Omega \,:\, \text{dist}(x, \partial\Omega) > 2h \}$. Again we have inequality (5.45) from which we deduce by the maximum principle that $|\varphi_h| \le 1$. Since now we are in a bounded domain Ω, the latter inequality for $|\varphi_h|$ is enough and we may skip step 2.

In (5.49), we now take $g^k(x,t) = \eta_\delta(x)\varphi_h^k(x,t)$ where $\eta_\delta \in C_c^\infty(\Omega)$, $\|\eta_\delta\|_\infty \le 1$ and $\eta_\delta \to \chi_\Omega$ pointwise as $\delta \to 0$. We obtain the analogue of equation (5.46)

$$\int_\Omega \sum_{k=1}^n v^k(t_0)\big(\operatorname{sgn}(v^k(t_0))\chi_{\Omega_h}\big) * \rho_h \eta_\delta(x)\,dx = \sum_{k=1}^n \int_0^{t_0}\int_\Omega \Big\{ 2 f^\varepsilon(x,u)\nabla\varphi_h^k\cdot\nabla\eta_\delta(x) + f^\varepsilon(x,u)\varphi_h^k\Delta\eta_\delta(x) - \sum_{i=1}^n f^\varepsilon_{x_i u}(x,u)\partial_{x_k}\eta_\delta(x)\varphi_h^k \Big\}\,dx\,dt + \int_\Omega \sum_{k=1}^n \Big\{ v^k(0)\varphi_h^k(x,0) - (\Delta f)(x,u)\varphi_h^k(x,t_0)\Big\}\eta_m(|x|)\,dx. \tag{5.51}$$

We use integration by parts to move the derivatives from η_δ to the product of remaining functions in the integrals of the first three terms inside the integral sign on the right-hand member of (5.51). We then make $\delta \to 0$, use Gauss-Green (divergence) theorem and the fact that $\varphi_h(x,t)$ and $f^\varepsilon(x, u_\varepsilon(x,t))$ vanish for $x \in \partial\Omega$ to conclude that those three integrals converge to 0 as $\delta \to 0$. We then make $h \to 0$, and the remaining of the proof of (5.39) is entirely similar to the corresponding part of the proof of (5.37).

The proof of (5.40) is totally similar to the one of (5.38) given above.

□

Theorem 5.4 (Existence)**.** *Let u_ε be the unique solution of* (5.31),(5.32)*. There exists $u \in L^\infty(\mathbb{R}_+^{n+1})$ such that, passing to a suitable subsequence if necessary, $u_\varepsilon \to u$ a.e. in $\mathbb{R}_+^{n+1}$ as $\varepsilon \to 0$. Moreover, u is the unique weak solution of* (5.1),(5.2)*. Finally, we may relax* (5.33)*, take $u_0 \in L^\infty(\mathbb{R}^n)$, and still obtain a weak solution for* (5.1),(5.2)*, which is unique.*

Similarly, if u_ε be the unique solution of (5.34),(5.35)*, there exists $u \in L^\infty(Q)$ such that, passing to a suitable subsequence if necessary, $u_\varepsilon \to u$ a.e. in Q as $\varepsilon \to 0$. Moreover, u is the unique weak solution of* (5.3),(5.4)*. Finally, we may relax* (5.36)*, take $u_0 \in L^\infty(\Omega)$, and still obtain a weak solution for* (5.3),(5.4)*, which is unique.*

Proof. We only prove the part concerning the Cauchy problem (5.1),(5.2). The assertions concerning the problem (5.3),(5.4) are proved in an entirely similar (even easier) way.

1. Let us first prove the case where (5.33) holds. By Theorem 5.3, $\{u_\varepsilon\}_{\varepsilon>0}$ satisfies the hypotheses of Lemma 5.3. Therefore, there exists $u \in L^\infty(\mathbb{R}_+^{n+1})$ such that, passing to a subsequence if necessary, $u_\varepsilon \to u$ in $L^1_{loc}(\mathbb{R}_+^{n+1})$.

2. Multiplying (5.31) by $f^\varepsilon(x,u_\varepsilon)\Lambda$ and integrating by parts, we get

$$\int_0^T\int_{\mathbb{R}^n}\Big\{\partial_t u_\varepsilon f^\varepsilon(x,u_\varepsilon)\Lambda + \nabla f^\varepsilon(x,u_\varepsilon)\cdot\nabla(f^\varepsilon(x,u_\varepsilon)\Lambda)\Big\}\,dx\,dt = 0.$$

This yields the equality

$$\int_0^T\int_{\mathbb{R}^n}\Lambda\,\partial_t\Big[\int_0^{u_\varepsilon} f^\varepsilon(x,s)ds\Big]\,dx\,dt + \int_0^T\int_{\mathbb{R}^n}|\nabla f^\varepsilon(x,u_\varepsilon)|^2\Lambda + \nabla f^\varepsilon(x,u_\varepsilon)\cdot\nabla\Lambda f^\varepsilon(x,u_\varepsilon)\Big\}\,dx\,dt = 0,$$

which gives

$$\int_0^T\int_{\mathbb{R}^n}|\nabla f^\varepsilon(x,u_\varepsilon)|^2\Lambda\,dx\,dt = -\int_0^T\int_{\mathbb{R}^n}\nabla f^\varepsilon(x,u_\varepsilon)\cdot\nabla\Lambda f^\varepsilon(x,u_\varepsilon)\,dt\,dx + \int_{\mathbb{R}^n}\Lambda(x)\int_{u_0}^{u_\varepsilon(T)} f^\varepsilon(x,s)\,ds\,dx$$
$$\le n\int_0^T\int_{\mathbb{R}^n}|\nabla f^\varepsilon(x,u_\varepsilon)||f^\varepsilon(x,u_\varepsilon)|\Lambda\,dx\,dt + \int_{\mathbb{R}^n}\Lambda(x)\Big|\int_{u_0}^{u_\varepsilon(T)} f^\varepsilon(x,s)\,ds\Big|\,dx,$$

where we have used that $|\nabla\Lambda| \le n\Lambda$. The Cauchy inequality with δ gives

$$\int_0^T \int_{\mathbb{R}^n} |\nabla f^\varepsilon(x,u_\varepsilon)|^2 \Lambda\, dx\, dt \le \int_0^T \int_{\mathbb{R}^n} \left\{ 2n\delta |\nabla f^\varepsilon(x,u_\varepsilon)|^2\Lambda + \frac{n}{4\delta}(f^\varepsilon)^2(x,u_\varepsilon)\Lambda \right\} dx\, dt + \int_{\mathbb{R}^n} \Lambda(x) \left| \int_{u_0(x)}^{u_\varepsilon(T)} f^\varepsilon(x,s)\, ds \right| dx.$$

taking $\delta = \dfrac{1}{4n}$ in the previous inequality, we obtain

$$\frac{1}{2}\int_0^T \int_{\mathbb{R}^n} |\nabla f^\varepsilon(x,u_\varepsilon)|^2 \Lambda\, dx\, dt \le n^2 \int_0^T \int_{\mathbb{R}^n} (f^\varepsilon)^2(x,u_\varepsilon)\Lambda\, dx\, dt + \int_{\mathbb{R}^n} \Lambda(x) \left| \int_{u_0(x)}^{u_\varepsilon(T)} f^\varepsilon(x,s)\, ds \right| dx.$$

Therefore,

$$\int_0^T \int_{\mathbb{R}^n} |\nabla f^\varepsilon(x,u_\varepsilon)|^2 \Lambda\, dx\, dt \le c(\|u_0\|_\infty, T) \int_{\mathbb{R}^n} \Lambda(x)\, dx,$$

for all $0 < \varepsilon < 1$. Given $R > 0$, it follows that

$$\Lambda(R) \int_0^T \int_{B_R} |\nabla f^\varepsilon(x,u_\varepsilon)|^2\, dx\, dt \le \int_0^T \int_{B_R} |\nabla f^\varepsilon(x,u_\varepsilon)|^2 \Lambda(x)\, dx\, dt \le c \int_{\mathbb{R}^n} \Lambda(x)\, dx$$

and so

$$\int_0^T \int_{B_R} |\nabla f^\varepsilon(x,u_\varepsilon)|^2\, dx\, dt \le \frac{c}{\Lambda(R)} \int_{\mathbb{R}^n} \Lambda(x)\, dx,$$

for any $0 < \varepsilon < 1$.

Thus,

$$\|f^\varepsilon(x,u_\varepsilon)\|_{L^2(0,T;H^1(B_R))} \le c(R,T,\|u_0\|_\infty),$$

uniformly in ε. Hence, there exists $v \in L^2_{\text{loc}}(\mathbb{R}_+; H^1_{\text{loc}}(\mathbb{R}^n))$ such that $f^\varepsilon(x,u_\varepsilon) \to v$ weakly. Since $f^\varepsilon(x,u_\varepsilon) \to f(x,u)$ a.e., then $v = f(x,u)$ and for this reason we conclude that $f(x,u) \in L^2_{\text{loc}}(\mathbb{R}_+; H^1_{\text{loc}}(\mathbb{R}^n))$.

3. Finally, when $u_0 \in L^\infty(\mathbb{R}^n)$, we may approximate u_0 in $L^1_{\text{loc}}(\mathbb{R}^n)$ by a sequence $u_{0k} \in W^{2,\infty}(\mathbb{R}^n)$ obtaining a sequence u_k of weak solutions of (5.1),(5.2), with initial data $u_0 = u_{0k}$, and then use the stability Theorem 5.2 to deduce that u_k is a Cauchy sequence in $L^1_{\text{loc}}(\mathbb{R}^{n+1}_+)$. We then easily conclude that the limit $u \in L^\infty(\mathbb{R}^{n+1}_+)$ of the sequence u_k is a weak solution of (5.1),(5.2).

□

6. Homogenization of Porous Medium Type Equations: The Cauchy problem

In this section, we consider the following homogenization problem

$$\begin{cases} \partial_t u = \Delta f(\frac{x}{\varepsilon}, u), & (x,t) \in \mathbb{R}^{n+1}_+, \\ u(x,0) = u_0(\frac{x}{\varepsilon}, x), & x \in \mathbb{R}^n, \end{cases} \tag{6.1}$$

where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies **(f1)**, **(f2)**, **(f3)** of Section 5, and is such that for each $u \in \mathbb{R}$, $f(\cdot,u) : \mathbb{R}^n \to \mathbb{R}$ belongs to a given ergodic algebra $\mathcal{A}(\mathbb{R}^n)$. Further, there exists a continuous function $g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that, for each $z \in \mathbb{R}^n$, $g(z, f(z,u)) = u$ and $f(z, g(z,v)) = v$, for all $u, v \in \mathbb{R}$, and, for each $v \in \mathbb{R}$, $g(\cdot, v) \in \mathcal{A}(\mathbb{R}^n)$.

In (6.1) we assume that u_0 satisfies

$$u_0 \in L^\infty(\mathbb{R}^n; \mathcal{A}(\mathbb{R}^n)). \tag{6.2}$$

Let $\bar{f}:\mathbb{R}\to\mathbb{R}$ be given by

$$p = \fint g(z,\bar{f}(p))\,dz. \tag{6.3}$$

Also, let

$$\bar{u}_0(x) = \fint u_0(x,z)\,dz, \tag{6.4}$$

and let $\mathcal{K}$ be the compact space given by Theorem 2.1 such that $\mathcal{A}(\mathbb{R}^n)\sim C(\mathcal{K})$, and $\mathfrak{m}$ be the associated invariant probability measure on $\mathcal{K}$. For each $\alpha\in\mathbb{R}$, define

$$\psi_\alpha(\cdot) := g(\cdot,\alpha),$$

and notice that ψ_α is a steady solution of (6.1).

Theorem 6.1. *Let $u_\varepsilon(x,t)$ be the weak solution of* (6.1)*. Suppose that*

$$u_0(x,z) = g(z,\varphi_0(x)) \tag{6.5}$$

for some $\varphi_0\in L^\infty(\mathbb{R}^n)$. Then u_ε weak star converge in $L^\infty(\mathbb{R}_+^{n+1})$ to $\bar{u}(x,t)$, where the latter is the weak solution to the Cauchy problem

$$\begin{cases}\partial_t\bar{u} = \Delta\bar{f}(\bar{u}), & (x,t)\in\mathbb{R}_+^{n+1},\\ \bar{u}(x,0)=\bar{u}_0(x), & x\in\mathbb{R}^n.\end{cases} \tag{6.6}$$

Moreover, we have

$$u_\varepsilon(x,t) - g\big(\frac{x}{\varepsilon},\bar{f}(\bar{u}(x,t))\big)\to 0, \qquad \text{as } \varepsilon\to 0 \text{ in } L^1_{loc}(\mathbb{R}_+^{n+1}). \tag{6.7}$$

Proof. 1. First, we observe that the weak solutions u_ε, $\varepsilon>0$, of (6.1) are bounded uniformly with respect to ε in $L^\infty(\mathbb{R}_+^{n+1})$. For this, we note that if α_1,α_2 are such that $\alpha_1\le\varphi_0(x)\le\alpha_2$ for $x\in\mathbb{R}$, we have

$$g(\frac{x}{\varepsilon},\alpha_1)\le u_0(\frac{x}{\varepsilon},x)\le g(\frac{x}{\varepsilon},\alpha_2) \qquad \text{for all } x\in\mathbb{R}^n.$$

By the monotonicity of the solution operator of (6.1) (see Corollary 5.1), we get

$$g(\frac{x}{\varepsilon},\alpha_1)\le u_\varepsilon(x,t)\le g(\frac{x}{\varepsilon},\alpha_2) \qquad \text{for all } (x,t)\in\mathbb{R}_+^{n+1}.$$

Thus, in the sequel, we denote by K a compact interval containing the image of all the functions u_ε, $\varepsilon>0$.

Let $\nu_{z,x,t}\in\mathcal{M}(K)$, with $(z,x,t)\in\mathcal{K}\times\mathbb{R}_+^{n+1}$, be the two-scale space time Young measures associated with a subnet of $\{u_\varepsilon\}_{\varepsilon>0}$ with test functions oscillating only on the space variable. Following [13], [2] and [3], the theorem will be proved by adapting DiPerna's method in [11], that is, by showing that $\nu_{z,x,t}$ is a Dirac measure for almost all $(z,x,t)\in\mathcal{K}\times\mathbb{R}_+^{n+1}$. Since we are going to show that $\nu_{z,x,t}$ does not depend on the chosen subnet (so that, a posteriori, a full limit as $\varepsilon\to0$ occurs), in order to simplify our notation we will use the notation $\lim_{\varepsilon\to0}$, with no reference to the subnet.

Observe that, for every $\alpha\in\mathbb{R}$, the weak solutions u_ε and $\psi_\alpha(\frac{x}{\varepsilon})$ satisfy (see Theorem 5.1)

$$\int_{\mathbb{R}_+^{n+1}}|u_\varepsilon(x,t)-\psi_\alpha(\frac{x}{\varepsilon})|\phi_t+|f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-f(\frac{x}{\varepsilon},\psi_\alpha(\frac{x}{\varepsilon}))|\Delta\phi\,dx\,dt+\int_{\mathbb{R}^n}|u_0(\frac{x}{\varepsilon},x)-\psi_\alpha(\frac{x}{\varepsilon})|\phi(x,0)\,dx\ge0, \tag{6.8}$$

for all $0\le\phi\in C_c^\infty(\mathbb{R}^{n+1})$. In (6.8), we take $\phi(x,t)=\varepsilon^2\varphi(\frac{x}{\varepsilon})\psi(x,t)$ with $0\le\psi\in C_c^\infty(\mathbb{R}_+^{n+1})$, φ, $\Delta\varphi\in\mathcal{A}(\mathbb{R}^n)$ and $\varphi\ge0$. Observe that

$$\Delta\phi=\Delta\varphi(\frac{x}{\varepsilon})\psi(x,t)+2\varepsilon\nabla\varphi(\frac{x}{\varepsilon})\cdot\nabla\psi(x,t)+\varepsilon^2\varphi(\frac{x}{\varepsilon})\Delta\psi(x,t).$$

Letting $\varepsilon \to 0$ and using Theorem 4.2, we get

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \psi(x,t)\langle \nu_{z,x,t}, |f(z,\cdot)-f(z,\psi_\alpha(z)|\rangle \underline{\Delta\varphi}(z)\, d\mathfrak{m}(z)\, dx\, dt \ge 0.$$

Now apply the inequality above to $\|\varphi\|_\infty \pm \varphi$ to obtain

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \psi(x,t)\langle \nu_{z,x,t}, |f(z,\cdot)-\alpha|\rangle \underline{\Delta\varphi}(z)\, d\mathfrak{m}(z)\, dx\, dt = 0, \tag{6.9}$$

for all φ such that $\varphi, \Delta\varphi \in \mathcal{A}(\mathbb{R}^n)$ and all $0 \le \psi \in C_c^\infty(\mathbb{R}^{n+1}_+)$.

2. As in [13], we define a new family of parametrized measures $\mu_{z,x,t}$ supported on a compact set $K' \supset \{f(z,\lambda) : (\lambda, z) \in K \times \mathbb{R}^n\}$ by

$$\langle \mu_{z,x,t}, \theta\rangle := \langle \nu_{z,x,t}, \theta(f(z,\cdot))\rangle, \qquad \theta \in C(\mathbb{R}). \tag{6.10}$$

In this way, the equation (6.9) can also be rewritten as

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \psi(x,t)\langle \mu_{z,x,t}, \theta\rangle \underline{\Delta\varphi}(z)\, d\mathfrak{m}(z)\, dx\, dt = 0, \tag{6.11}$$

where $\theta(\lambda) = |\lambda - \alpha|$.

On the other hand, inserting in the integral equation defining weak solution of (6.1) with a test function as above, we easily get, letting $\varepsilon \to 0$, that (6.11) holds when θ is any affine function. Therefore, we deduce that (6.11) holds for finite linear combinations of affine functions and functions of the form $|\cdot - \alpha|$, $\alpha \in \mathbb{R}$. Since these combinations generate the piecewise affine functions, we finally conclude that (6.11) holds for all $\theta \in C(\mathbb{R})$.

Set $F(z) := \int_{\mathbb{R}^{n+1}_+} \psi(x,t)\langle \mu_{z,x,t}, \theta\rangle\, dx\, dt$ and observe that $\int_{\mathcal{K}} F(z)\underline{\Delta\varphi}(z)\, d\mathfrak{m}(z) = 0$, for all φ such that φ, $\Delta\varphi \in \mathcal{A}(\mathbb{R}^n)$. Then, we can apply Lemma 3.2 to obtain that F is equivalent to a constant for all $\theta \in C(\mathbb{R})$. Using this fact and defining

$$\mu_{x,t} := \int_{\mathcal{K}} \mu_{z,x,t}\, d\mathfrak{m}(z) \in \mathcal{M}(K'),$$

we have, in particular,

$$\int_{\mathbb{R}^{n+1}_+} \psi(x,t)\langle \mu_{z,x,t}, \theta\rangle\, dx\, dt = \int_{\mathcal{K}}\int_{\mathbb{R}^{n+1}_+} \psi(x,t)\langle \mu_{z,x,t}, \theta\rangle\, dx\, dt\, d\mathfrak{m}(z) = \int_{\mathbb{R}^{n+1}_+} \psi(x,t)\langle \mu_{x,t}, \theta\rangle\, dx\, dt,$$

for a.e. $z \in \mathcal{K}$, for all $\theta \in C(\mathbb{R})$.

Hence,

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+} \langle \mu_{x,t}, \int_{\mathcal{K}} W(z,\cdot)\, d\mathfrak{m}(z)\rangle \psi(x,t)\, dx\, dt &= \sum_i \mathfrak{m}(\mathcal{K}_i) \int_{\mathbb{R}^{n+1}_+} \langle \mu_{x,t}, \theta_i\rangle \psi(x,t)\, dx\, dt \\
&= \sum_i \mathfrak{m}(\mathcal{K}_i) \int_{\mathbb{R}^{n+1}_+} \langle \mu_{z,x,t}, \theta_i\rangle \psi(x,t)\, dx\, dt = \sum_i \int_{\mathcal{K}}\int_{\mathbb{R}^{n+1}_+} \langle \mu_{z,x,t}, \theta_i\rangle \chi_{\mathcal{K}_i}(z)\psi(x,t)\, dx\, dt\, d\mathfrak{m}(z) \\
&= \int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \langle \mu_{z,x,t}, W(z,\cdot)\rangle \psi(x,t)\, d\mathfrak{m}(z)\, dx\, dt
\end{aligned} \tag{6.12}$$

for any function $W(\lambda, z) = \sum_i \theta_i(\lambda)\chi_{\mathcal{K}_i}(z)$, where $\theta_i \in C(K')$, $\mathcal{K}_i$ is any Borelian subset of $\mathcal{K}$, and $\chi_{\mathcal{K}_i}$ is the characteristic function of $\mathcal{K}_i$. By approximation (6.12) holds for any $W \in C(\mathcal{K} \times K')$.

3. From (6.8), taking the limit as $\varepsilon \to 0$, passing to a subnet if necessary, we get

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \langle \nu_{z,x,t}, |\cdot - \psi_\alpha(z)|\rangle \varphi_t + \langle \nu_{z,x,t}, |f(z,\cdot) - f(z,\psi_\alpha(z))|\rangle \Delta\varphi(z)\, d\mathfrak{m}(z)\, dx\, dt + \int_{\mathbb{R}^n}\int_{\mathcal{K}} |u_0(z,x) - \psi_\alpha(z)|\varphi(x,0)\, d\mathfrak{m}(z)\, dx \ge 0 \tag{6.13}$$

for all $\alpha \in \mathbb{R}$ and for all $0 \le \varphi \in C_c^\infty(\mathbb{R}^{n+1})$.

We define $I(\rho,\alpha)$ and $G(\rho,\alpha)$ by

$$I(\rho,\alpha) := \int_{\mathcal{K}} |g(z,\rho) - g(z,\alpha)|\, d\mathfrak{m}(z), \tag{6.14}$$

$$G(\rho,\alpha) := |\rho - \alpha|. \tag{6.15}$$

Now, setting $\theta(\rho) = |g(z,\rho) - g(z,\alpha)|$, we have

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \langle \nu_{z,x,t}, |\cdot - \psi_\alpha(z)|\rangle \varphi_t\, d\mathfrak{m}(z)\, dx\, dt = \int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \langle \nu_{z,x,t}, \theta(f(z,\cdot))\rangle \varphi_t\, d\mathfrak{m}(z)\, dx\, dt$$
$$= \int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \langle \mu_{z,x,t}, |g(z,\cdot) - g(z,\alpha)|\rangle \varphi_t\, d\mathfrak{m}(z)\, dx\, dt.$$

Using (6.12), we obtain

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \langle \nu_{z,x,t}, |\cdot - \psi_\alpha(z)|\rangle \varphi_t\, d\mathfrak{m}(z)\, dx\, dt &= \int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \langle \mu_{z,x,t}, |g(z,\cdot) - g(z,\alpha)|\rangle \varphi_t\, d\mathfrak{m}(z)\, dx\, dt \\ &= \int_{\mathbb{R}^{n+1}_+} \Big\langle \mu_{x,t}, \int_{\mathcal{K}} |g(z,\cdot) - g(z,\alpha)|\, d\mathfrak{m}(z)\Big\rangle \varphi_t\, dx\, dt \\ &= \int_{\mathbb{R}^{n+1}_+} \langle \mu_{x,t}, I(\cdot,\alpha)\rangle \varphi_t\, dx\, dt.\end{aligned} \tag{6.16}$$

Analogously,

$$\int_{\mathbb{R}^{n+1}_+}\int_{\mathcal{K}} \langle \nu_{z,x,t}, |f(z,\cdot) - f(z,\psi_\alpha(z))|\rangle \Delta\varphi(x,t)\, d\mathfrak{m}(z)\, dx\, dt = \int_{\mathbb{R}^{n+1}_+} \langle \mu_{x,t}, G(\cdot,\alpha)\rangle \Delta\varphi(x,t)\, dx\, dt. \tag{6.17}$$

Using (6.16) and (6.17) in (6.13), we have

$$\int_{\mathbb{R}^{n+1}_+} \langle \mu_{x,t}, I(\cdot,\alpha)\rangle \varphi_t + \langle \mu_{x,t}, G(\cdot,\alpha)\rangle \Delta\varphi\, dx\, dt + \int_{\mathbb{R}^n}\int_{\mathcal{K}} |u_0(z,x) - \psi_\alpha(z)|\varphi(x,0)\, d\mathfrak{m}(z)\, dx \ge 0, \tag{6.18}$$

for all $0 \le \varphi \in C_c^\infty(\mathbb{R}^{n+1})$ and all $\alpha \in \mathbb{R}$.

Now, choosing $\varphi(x,t) = \delta_h(t)\phi(x)$, with $0 \le \phi \in C_c^\infty(\mathbb{R}^n)$ and $\delta_h(t) = \max\left\{\frac{h-|t|}{h}, 0\right\}$ for $h > 0$ in (6.18), we obtain

$$\overline{\lim_{h\to 0}}\frac{1}{h}\int_0^h \int_{\mathbb{R}^n} \langle \mu_{x,t}, I(\cdot,\alpha)\rangle \phi\, dx\, dt \le \int_{\mathbb{R}^n}\int_{\mathcal{K}} |u_0(z,x) - \psi_\alpha(z)|\phi\, d\mathfrak{m}(z)\, dx. \tag{6.19}$$

Using the flexibility provided by ϕ in (6.19), we deduce that the same inequality holds if $\alpha \in L^\infty(\mathbb{R}^n)$ and $\phi = \chi_{B_R}, R > 0$.

We have that $\varphi_0(x) = f(z, u_0(z,x))$ is independent of z. Taking $\alpha(x) = \varphi_0(x)$ and recalling that $u_0(z,x) = g(z, \alpha(x))$, we have $\alpha(x) = \bar{f}(\bar{u}(x,0))$. Using this and $\psi_\alpha(z) = g(z,\alpha)$ in (6.19), we obtain that

$$\lim_{h\to 0} \frac{1}{h} \int_0^h \int_{B_R} \langle \mu_{x,t}, I(\cdot, \bar{f}(\bar{u}(x,0)))\rangle \, dx\, dt = 0, \qquad \forall R > 0. \tag{6.20}$$

4. By using the Theorem 5.1 with $u_1 = u_\varepsilon$ and $u_2(x) = \psi_\alpha(\frac{x}{\varepsilon})$, for all $0 \le \varphi \in C_c^\infty(\mathbb{R}_+^{n+1})$ we get

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} B_{\vartheta_\delta}^{\psi_\alpha(\frac{x}{\varepsilon})}(\frac{x}{\varepsilon}, u_\varepsilon(x,t))\varphi_t \, dx\, dt \\
&\quad - \int_{\mathbb{R}_+^{n+1}} H_\delta(f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_\alpha(\frac{x}{\varepsilon})))\nabla[f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_\alpha(\frac{x}{\varepsilon}))] \cdot \nabla\varphi \, dx\, dt \\
&\quad = \int_{\mathbb{R}_+^{n+1}} |\nabla[f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_\alpha(\frac{x}{\varepsilon}))]|^2 H_\delta'(f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_\alpha(\frac{x}{\varepsilon})))\varphi \, dx\, dt.
\end{aligned} \tag{6.21}$$

Now, we let $\alpha = \xi(y,s) := \bar{f}(\bar{u}(y,s))$, take $0 \le \phi \in C_c^\infty((\mathbb{R}_+^{n+1})^2)$, integrate in y, s, and send $\delta \to 0$, to get

$$\begin{aligned}
&\int_{(\mathbb{R}_+^{n+1})^2} |u_\varepsilon(x,t) - \psi_{\xi(y,s)}(\frac{x}{\varepsilon})|\phi_t - \nabla_x |f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon}))| \cdot \nabla_x \phi \, dx\, dt\, dy\, ds \\
&\quad = \lim_{\delta\to 0} \int_{(\mathbb{R}_+^{n+1})^2} \Big\{ |\nabla_x[f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]|^2 \\
&\qquad\qquad \times H_\delta'(f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi \Big\} \, dx\, dt\, dy\, ds.
\end{aligned}$$

Then we use Theorem 4.2 on multiscale Young measures to obtain, as $\varepsilon \to 0$,

$$\begin{aligned}
&\int_{(\mathbb{R}_+^{n+1})^2} \langle \mu_{x,t}, I(\cdot, \xi(y,s))\rangle \phi_t + \langle \mu_{x,t}, G(\cdot, \xi(y,s))\rangle \Delta_x \phi \, dx\, dt\, dy\, ds \\
&\quad = \lim_{\varepsilon\to 0} \lim_{\delta\to\ 0} \int_{(\mathbb{R}_+^{n+1})^2} \Big\{ |\nabla_x[f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]|^2 \\
&\qquad\qquad \times H_\delta'(f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi \Big\} \, dx\, dt\, dy\, ds.
\end{aligned} \tag{6.22}$$

5. Observe that $\nabla_y[f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon}))] = \nabla_y \xi(y,s)$. Hence

$$0 = \int_{\mathbb{R}_+^{n+1}} \nabla_y[f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon}))] \cdot \nabla_x[H_\delta(f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon}))\phi] \, dx\, dt,$$

which implies that

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} \Big\{ \nabla_y[f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon}))] \cdot \nabla_x[f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon}))] \\
&\qquad \times H_\delta'(f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi \Big\} \, dx\, dt \\
&= -\int_{\mathbb{R}_+^{n+1}} \Big\{ \nabla_y[f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon}))] \cdot \nabla_x \phi \\
&\qquad \times H_\delta(f(\frac{x}{\varepsilon}, u_\varepsilon(x,t)) - f(\frac{x}{\varepsilon}, \psi_{\xi(y,s)}(\frac{x}{\varepsilon}))) \Big\} \, dx\, dt.
\end{aligned}$$

Integrating in y,s and letting $\delta\to0$, we have

$$\int_{(\mathbb{R}^{n+1}_+)^2}|f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))|\operatorname{div}_y\nabla_x\phi\,dx\,dt\,dy\,ds$$
$$=\lim_{\delta\to0}\int_{(\mathbb{R}^{n+1}_+)^2}\Big\{\nabla_y[f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]\cdot\nabla_x[f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]$$
$$\times H_\delta'(f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi\Big\}\,dx\,dt\,dy\,ds.$$

By Theorem 4.2, as $\varepsilon\to0$, we get

$$\int_{(\mathbb{R}^{n+1}_+)^2}\langle\mu_{x,t},G(\cdot,\xi(y,s))\rangle\operatorname{div}_y\nabla_x\phi\,dx\,dt\,dy\,ds \tag{6.23}$$
$$=\lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{(\mathbb{R}^{n+1}_+)^2}\Big\{\nabla_y[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]\cdot\nabla_x[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]$$
$$\times H_\delta'(f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon})))\phi\Big\}\,dx\,dt\,dy\,ds.$$

Similarly, we have also that $f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))=f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-\xi(y,s)$ and thus

$$\nabla_x[f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]=\nabla_xf(\frac{x}{\varepsilon},u_\varepsilon(x,t)),$$

is independent of y. Hence, by integrating first in (y,s) and then (x,t), proceeding as above in obtaining (6.23), yields the equality

$$\int_{(\mathbb{R}^{n+1}_+)^2}\langle\mu_{x,t},G(\cdot,\xi(y,s))\rangle\operatorname{div}_x\nabla_y\phi\,dx\,dt\,dy\,ds \tag{6.24}$$
$$=\lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{(\mathbb{R}^{n+1}_+)^2}\Big\{\nabla_x[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]\cdot\nabla_y[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]$$
$$\times H_\delta'(f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon})))\phi\Big\}\,dx\,dt\,dy\,ds$$

where u_ε and ξ are functions of x,t and y,s, respectively.

6. Let $\bar u$ be the weak solution of (6.6). From (5.10) in Lemma 5.2, we have

$$\int_{\mathbb{R}^{n+1}_+}|l-\bar u(y,s)|\phi_s+\operatorname{sgn}(\bar f(l)-\bar f(\bar u(y,s)))\nabla_y\bar f(\bar u)\cdot\nabla_y\phi\,dy\,ds \tag{6.25}$$
$$=\lim_{\delta\to0}\int_{\mathbb{R}^{n+1}_+}|\nabla_y\bar f(\bar u)|^2H_\delta'(\bar f(l)-\bar f(\bar u(y,s)))\phi\,dy\,ds,\qquad\text{for all } l\in\mathbb{R}.$$

Now, let $k:=\bar f(l)$ and notice that $l=\int_{\mathcal K}g(z,\bar f(l))\,d\mathfrak m(z)$ and that $\bar u(y,s)=\int_{\mathcal K}g(z,\xi(y,s))\,d\mathfrak m(z)$. Thus,

$$\int_{\mathbb{R}^{n+1}_+}|l-\bar u(y,s)|\phi_s\,dy\,ds=\int_{\mathbb{R}^{n+1}_+}\Big|\int_{\mathcal K}(g(z,k)-g(z,\xi(y,s))\,d\mathfrak m(z)\Big|\phi_s\,dy\,ds$$
$$=\int_{\mathbb{R}^{n+1}_+}\Big(\int_{\mathcal K}|g(z,k)-g(z,\xi(y,s))|\,d\mathfrak m(z)\Big)\phi_s\,dy\,ds=\int_{\mathbb{R}^{n+1}_+}I(k,\xi(y,s))\phi_s\,dy\,ds.$$

Also,

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} \operatorname{sgn}(\bar f(l)-\bar f(\bar u(y,s)))\nabla_y \bar f(\bar u)\cdot\nabla_y\phi\,dy\,ds\\
&=-\int_{\mathbb{R}_+^{n+1}} \nabla_y|\bar f(l)-\bar f(\bar u(y,s))|\cdot\nabla_y\phi\,dy\,ds=\int_{\mathbb{R}_+^{n+1}}|k-\xi(y,s)|\Delta_y\phi\,dy\,ds\\
&=\int_{\mathbb{R}_+^{n+1}} G(k,\xi(y,s))\Delta_y\phi\,dy\,ds.
\end{aligned}$$

Besides, since $\nabla_y\xi(y,s)=\nabla_y[f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} |\nabla_y\bar f(\bar u)|^2 H_\delta'(\bar f(l)-\bar f(\bar u(y,s)))\phi\,dy\,ds=\int_{\mathbb{R}_+^{n+1}}|\nabla_y\xi(y,s)|^2H_\delta'(k-\xi(y,s))\phi\,dy\,ds\\
&=\int_{\mathbb{R}_+^{n+1}}|\nabla_y f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))|^2H_\delta'(k-\xi(y,s))\phi\,dy\,ds.
\end{aligned}$$

Using the two previous equalities in (6.25) we obtain

$$\int_{\mathbb{R}_+^{n+1}} I(k,\xi(y,s))\phi_s+G(k,\xi(y,s))\Delta_y\phi\,dy\,ds=\lim_{\delta\to0}\int_{\mathbb{R}_+^{n+1}}|\nabla_y f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))|^2H_\delta'(k-\xi(y,s))\phi\,dy\,ds.$$

for all $k\in\mathbb{R}$ and all $0\le\phi\in C_c^\infty((\mathbb{R}_+^{n+1})^2)$.

We take $k=f(\frac{x}{\varepsilon},u_\varepsilon(x,t))$ in the above equality and integrate in x,t to get

$$\begin{aligned}
&\int_{(\mathbb{R}_+^{n+1})^2} I(f(\frac{x}{\varepsilon},u_\varepsilon(x,t)),\xi(y,s))\phi_s+G(f(\frac{x}{\varepsilon},u_\varepsilon(x,t)),\xi(y,s))\Delta_y\phi\,dx\,dt\,dy\,ds\\
&=\lim_{\delta\to0}\int_{(\mathbb{R}_+^{n+1})^2}\Big\{|\nabla_y[f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]|^2\\
&\qquad\times H_\delta'(f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi\Big\}\,dx\,dt\,dy\,ds.
\end{aligned}\tag{6.26}$$

Applying Theorem 4.2, letting $\varepsilon\to0$, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_{(\mathbb{R}_+^{n+1})^2} I(f(\frac{x}{\varepsilon},u_\varepsilon(x,t)),\xi(y,s))\phi_s\,dx\,dt\,dy\,ds\\
&=\int_{(\mathbb{R}_+^{n+1})^2}\int_{\mathcal{K}}\langle\nu_{z,x,t},I(f(z,\cdot),\xi(y,s))\rangle\phi_s\,d\mathfrak{m}(z)\,dx\,dt\,dy\,ds\\
&=\int_{(\mathbb{R}_+^{n+1})^2}\int_{\mathcal{K}}\langle\mu_{z,x,t},I(\cdot,\xi(y,s))\rangle\phi_s\,d\mathfrak{m}(z)\,dx\,dt\,dy\,ds=\int_{(\mathbb{R}_+^{n+1})^2}\langle\mu_{x,t},I(\cdot,\xi(y,s))\rangle\phi_s\,dx\,dt\,dy\,ds
\end{aligned}$$

Similarly

$$\lim_{\varepsilon\to0}\int_{(\mathbb{R}_+^{n+1})^2} G(f(\frac{x}{\varepsilon},u_\varepsilon(x,t)),\xi(y,s))\Delta_y\phi\,dx\,dt\,dy\,ds=\int_{(\mathbb{R}_+^{n+1})^2}\langle\mu_{x,t},G(\cdot,\xi(y,s))\rangle\Delta_y\phi\,dx\,dt\,dy\,ds.$$

Using the last two equalities in (6.26), we get

$$\int_{(\mathbb{R}_+^{n+1})^2}\langle\mu_{x,t},I(\cdot,\xi(y,s))\rangle\phi_s+\langle\mu_{x,t},G(\cdot,\xi(y,s))\rangle\Delta_y\phi\,dx\,dt\,dy\,ds \tag{6.27}$$
$$=\lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{(\mathbb{R}_+^{n+1})^2}\Big\{|\nabla_y[f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon}))]|^2$$
$$\times H_\delta'(f(\frac{x}{\varepsilon},u_\varepsilon(x,t))-f(\frac{x}{\varepsilon},\psi_{\xi(y,s)}(\frac{x}{\varepsilon})))\phi\Big\}\,dx\,dt\,dy\,ds.$$

7. We now prove that

$$\int_{\mathbb{R}_+^{n+1}}\langle\mu_{x,t},I(\cdot,\xi(x,t))\rangle\varphi_t+\langle\mu_{x,t},G(\cdot,\xi(x,t))\rangle\Delta\varphi\,dx\,dt\ge0, \tag{6.28}$$

for all $0\le\varphi\in C_c^\infty(\mathbb{R}_+^{n+1})$.

By adding (6.22) and (6.23), we deduce that

$$\int_{(\mathbb{R}_+^{n+1})^2}\langle\mu_{x,t},I(\cdot,\xi)\rangle\phi_t+\langle\mu_{x,t},G(\cdot,\xi)\rangle\big(\Delta_x\phi+\operatorname{div}_y\nabla_x\phi\big)\,dx\,dt\,dy\,ds \tag{6.29}$$
$$=\lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{(\mathbb{R}_+^{n+1})^2}\Big\{|\nabla_x[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]|^2$$
$$+\nabla_y[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]\cdot\nabla_x[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]\Big\}$$
$$\times H_\delta'(f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon})))\phi\,dx\,dt\,dy\,ds,$$

where $u_\varepsilon=u_\varepsilon(x,t)$, $\xi=\xi(y,s)$.

The sum of (6.27) and (6.24)) gives

$$\int_{(\mathbb{R}_+^{n+1})^2}\langle\mu_{x,t},I(\cdot,\xi)\rangle\phi_s+\langle\mu_{x,t},G(\cdot,\xi)\rangle\big(\Delta_y\phi+\operatorname{div}_x\nabla_y\phi\big)\,dx\,dt\,dy\,ds \tag{6.30}$$
$$=\lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{(\mathbb{R}_+^{n+1})^2}\Big\{|\nabla_y[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]|^2$$
$$+\nabla_y[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]\cdot\nabla_x[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]\Big\}$$
$$\times H_\delta'(f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))\phi\,dx\,dt\,dy\,ds.$$

Finally, taking the sum between (6.29) and (6.30) we obtain

$$\int_{(\mathbb{R}_+^{n+1})^2}\langle\mu_{x,t},I(\cdot,\xi)\rangle(\phi_t+\phi_s)+\langle\mu_{x,t},G(\cdot,\xi)\rangle\big(\Delta_x+\operatorname{div}_y\nabla_x+\operatorname{div}_x\nabla_y+\Delta_y\big)\phi\,dx\,dt\,dy\,ds \tag{6.31}$$
$$=\lim_{\varepsilon\to0}\lim_{\delta\to0}\int_{(\mathbb{R}_+^{n+1})^2}\Big\{\big|\nabla_x[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]$$
$$+\nabla_y[f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon}))]\big|^2H_\delta'(f(\frac{x}{\varepsilon},u_\varepsilon)-f(\frac{x}{\varepsilon},\psi_\xi(\frac{x}{\varepsilon})))\phi\Big\}\,dx\,dt\,dy\,ds\le0.$$

Now, we take $\phi(x,t,y,s):=\varphi(\frac{x+y}{2},\frac{t+s}{2})\rho_j(\frac{x-y}{2})\theta_j(\frac{t-s}{2})$, where $0\le\varphi\in C_c^\infty(\mathbb{R}_+^{n+1})$, and ρ_j, θ_j are classical approximations of the identity in $\mathbb{R}^n$ and $\mathbb{R}$, respectively, as in the doubling of variables method, and observe that

$$\big(\Delta_x + \operatorname{div}_y \nabla_x + \operatorname{div}_x \nabla_y + \Delta_y\big)\phi = \rho_j(\frac{x-y}{2})\theta_j(\frac{t-s}{2})\Delta_x\varphi(\frac{x+y}{2}, \frac{t+s}{2}).$$

Substituting such test function in the inequality in (6.31) and letting $j \to \infty$, we obtain (6.28), proving the assertion.

8. To conclude the proof, we set $\varphi(x,t) = \delta_h(t)\Lambda(x)$ in (6.28), with $0 \le \delta_h \in C_c^\infty(\mathbb{R}_+)$ and Λ given by (5.25). Hence,

$$\begin{aligned}
(6.32) \qquad -\int_{\mathbb{R}_+^{n+1}} \langle \mu_{x,t}, I(\cdot,\xi(x,t))\rangle \delta_h'(t)\Lambda(x)\,dx\,dt &\le \int_{\mathbb{R}_+^{n+1}} \langle \mu_{x,t}, G(\cdot,\xi(x,t))\rangle \delta_h(t)\Delta\Lambda(x)\,dx\,dt \\
&\le (n+1)\int_{\mathbb{R}_+^{n+1}} \langle \mu_{x,t}, G(\cdot,\xi(x,t))\rangle \delta_h(t)\Lambda(x)\,dx\,dt \\
&\le C\int_{\mathbb{R}_+^{n+1}} \langle \mu_{x,t}, I(\cdot,\xi(x,t))\rangle \delta_h(t)\Lambda(x)\,dx\,dt,
\end{aligned}$$

where we use that $G(\cdot,\cdot) \le CI(\cdot,\cdot)$. Defining

$$\gamma(t) := \int_{\mathbb{R}^n} \langle \mu_{x,t}, I(\cdot,\xi(x,t))\rangle \Lambda(x)\,dx,$$

and using (6.32), we get

$$-\int_0^\infty \gamma(s)\delta_h'(s)\,ds \le C\int_0^\infty \gamma(s)\delta_h(s)\,ds.$$

Let $t > 0$ to be a Lebesgue point of the function γ and taking

$$\delta_h(s) = \frac{s-h}{h}\chi_{[h,2h]}(s) - \frac{s-t-h}{h}\chi_{(t,t+h]}(s) + \chi_{(2h,t]}(s),$$

we note that

$$\delta_h' = \frac{1}{h}\chi_{[h,2h]} - \frac{1}{h}\chi_{(t,t+h]}.$$

Hence,

$$(6.33) \qquad \frac{1}{h}\int_t^{t+h} \gamma(s)\,ds - \frac{1}{h}\int_h^{2h}\gamma(s)\,ds \le C\int_0^\infty \gamma(s)\delta_h(s)\,ds.$$

Furthermore, due to the monotonicity of $g(z,\cdot)$ for all z, we have

$$\begin{aligned}
\gamma(s) &= \int_{\mathbb{R}^n} \langle \mu_{x,s}, I(\cdot,\xi(x,s))\rangle \Lambda(x)\,dx \\
&= \int_{\mathbb{R}^n} \Big\{ \langle \mu_{x,s}, I(\cdot,\xi(x,0))\rangle + \langle \mu_{x,s}, I(\cdot,\xi(x,s)) - I(\cdot,\xi(x,0))\rangle \Big\}\Lambda(x)\,dx \\
&\le \int_{\mathbb{R}^n} \Big\{ \langle \mu_{x,s}, I(\cdot,\xi(x,0))\rangle + |\bar u(x,s) - \bar u(x,0)| \Big\}\Lambda(x)\,dx.
\end{aligned}$$

Letting $h \to 0$, taking into account (6.20) and that $\bar u$ is a weak solution, we see that

$$\lim_{h\to 0}\frac{1}{h}\int_0^h \gamma(s)\,ds = 0.$$

Thus, making $h \to 0$ in (6.33), we arrive at

$$\gamma(t) \le C\int_0^t \gamma(s)ds \qquad \text{for a.e. } t \ge 0.$$

Hence, Gronwall's lemma implies $\gamma(t) = 0$ for a.e. $t \geq 0$ which, by the definition of γ, means that $\langle \mu_{x,t}, I(\cdot, \xi(x,t))\rangle = 0$ for a.e. $(x,t) \in \mathbb{R}_+^{n+1}$, and so $\langle \mu_{x,t}, G(\cdot, \xi(x,t))\rangle = 0$ for a.e. $(x,t) \in \mathbb{R}_+^{n+1}$. Therefore, $\mu_{x,t}$ is the Dirac mass concentrated at $\xi(x,t)$ for a.e. $(x,t) \in \mathbb{R}_+^{n+1}$. Recalling the definition of $\mu_{x,t}$ we have also that $\mu_{z,x,t}$ is the Dirac mass concentrated at $\xi(x,t)$ for a.e. (z,x,t), and thus, $\nu_{z,x,t}$ is the Dirac mass concentrated at $g(z, \bar{f}(\bar{u}(x,t)))$ for a.e. (z,x,t). Hence, we can apply Theorem 4.2 to conclude (6.7).

Finally, the fact that the whole sequence u_ε converges in the weak star topology of $L^\infty(\mathbb{R}_+^{n+1})$ to $\bar{u}$ follows from (6.7) observing that, for any $\varphi \in C_c(\mathbb{R}_+^{n+1})$, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+^{n+1}} g(\frac{x}{\varepsilon}, \bar{f}(\bar{u}(x,t)))\varphi(x,t)dxdt &= \int_{\mathbb{R}_+^{n+1}}\int_{\mathcal{K}} g(z, \bar{f}(\bar{u}(x,t)))\varphi(x,t)d\mathfrak{m}(z)\,dx\,dt \\
&= \int_{\mathbb{R}_+^{n+1}}\left(\int_{\mathcal{K}} g(z, \bar{f}(\bar{u}(x,t))\,)d\mathfrak{m}(z)\right)\varphi(x,t)\,dx\,dt \\
&= \int_{\mathbb{R}_+^{n+1}} \bar{u}(x,t)\varphi(x,t)\,dx\,dt,
\end{aligned}$$

by the definition of $\bar{f}$.

□

7. Homogenization of Porous medium type equations on bounded domains

In this section we consider a homogenization problem for a porous medium type equation, similar to the one analyzed in the previous section, but now in a bounded domain. Because of boundary constraints, we consider a flux function of the form $f(x, \frac{x}{\varepsilon}, u)$, depending also on the "slow variable" x, instead of simply $f(\frac{x}{\varepsilon}, u)$. Here, we will use a completely different approach to address the homogenization problem, which will allow us to consider more general initial data, namely, initial data that are not necessarily of the form (6.5). On the other hand, our approach here will require that we restrict ourselves to the case where $\mathcal{A}(\mathbb{R}^n)$ is a regular algebra w.m.v. instead of a general ergodic algebra. As it was shown in Section 3, $\mathrm{FS}(\mathbb{R}^n)$ provides a very encompassing example of regular algebra w.m.v., and it is not even known so far whether there are ergodic algebras that are not regular algebras w.m.v., neither whether there are regular algebras w.m.v. that are not subalgebras of $\mathrm{FS}(\mathbb{R}^n)$. Also, here, for our result on the existence of oscillatory profiles correcting the weak convergence into a strong convergence, we need to ask the flux function to be convex, which was not necessary for the corresponding result in Theorem 6.1. The discussion in this section largely extends the corresponding one in [15] concerning the homogenization of a particular type of the general equation considered here, in the nondegenerate case.

So, let Ω be a bounded open subset of $\mathbb{R}^n$ with smooth boundary. We consider the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta f(x, \frac{x}{\varepsilon}, u), & (x,t) \in Q, \\ u(x,0) = u_0(\frac{x}{\varepsilon}, x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,\infty), \end{cases} \tag{7.1}$$

where $f(x,z,u)$ is such that, if we set $f_\varepsilon(x,u) := f(x, \frac{x}{\varepsilon}, u)$, for each fixed $\varepsilon > 0$, then $f_\varepsilon(x,u)$ satisfies **(f1)**, **(f2)**, **(f3)** of Section 5. Also, $f(x,z,0) = 0$ for all $(x,z) \in \partial\Omega \times \mathbb{R}^n$ (cf. **(f4)** in Section 5). Further, for each $(x,u) \in \Omega \times \mathbb{R}$, $f(x,\cdot,u) \in \mathcal{A}(\mathbb{R}^n)$, where now $\mathcal{A}(\mathbb{R}^n)$ is a given regular algebra w.m.v., and there exists a continuous function $g : \Omega \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $g(x,z,f(x,z,u)) = u$, $f(x,z,g(x,z,v)) = v$, and, for each $(x,u) \in \Omega \times \mathbb{R}$, $g(x,\cdot,u) \in \mathcal{A}(\mathbb{R}^n)$.

Define $\bar f : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$p = \fint g(x, z, \bar f(x, p))\, dz.$$

It is easy to see that $\bar f$ satisfies **(f1)** and **(f4)**. Nevertheless, in general, it may not satisfy **(f3)**, and it is not clear whether it inherits from f the verification of all the conditions in **(f2)**. Therefore, we cannot use the Theorem 5.4 to assert the existence of a weak solution for the initial-boundary value problem

$$\begin{cases} \partial_t u = \Delta \bar f(x, u), & (x,t) \in Q, \\ u(x, 0) = \fint u_0(z, x)\, dz, & x \in \Omega, \\ u(x, t) = 0, & (x,t) \in \partial\Omega \times [0, \infty). \end{cases} \tag{7.2}$$

That is the reason why, in the next theorem, we characterize the homogenized limit $\bar u$ in a different way. However, since the existence of a weak solution to (7.2) follows from the general compactness result in [24] and **(f4)**, and for uniqueness we only need **(f1)** and **(f4)**, we actually could characterize $\bar u$ as the unique weak solution of (7.2).

We next state and prove the main result of this section. We will use the concept and some basic facts about viscosity solutions of fully nonlinear parabolic equations. We refer to [10] for a general exposition of the theory of viscosity solutions of fully-nonlinear elliptic and parabolic equations.

Before stating the theorem, let us introduce the following notation. Given a function $h \in L^\infty(\Omega)$, we denote by $\Delta^{-1} h$ the solution of the boundary value problem

$$\begin{cases} \Delta v(x) = h(x), & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega. \end{cases} \tag{7.3}$$

Theorem 7.1. *Let $u_\varepsilon(x,t)$ be the weak solution of (7.1). Then, as $\varepsilon \to 0$, u_ε weak star converges in $L^\infty(\Omega \times [0,\infty))$ to $\bar u(x,t)$, which is uniquely defined as follows. Let $\bar U$ be the viscosity solution of*

$$\begin{cases} \partial_t U = \bar f(x, \Delta U), & (x,t) \in Q, \\ U(x, 0) = \bar U_0(x) := \Delta^{-1} \left\{ \fint u_0(z, x) \right\}, & x \in \Omega, \\ U(x, t) = 0, & (x,t) \in \partial\Omega \times [0, \infty). \end{cases} \tag{7.4}$$

Then $\bar U \in L^\infty((0,\infty); W^{2,p}(\Omega))$, for any $1 < p < \infty$, and

$$\bar u(x,t) := \Delta \bar U, \qquad a.e.\ (x,t) \in \Omega \times (0, \infty).$$

Moreover, assuming the existence of a weak solution to (7.2), $\bar u$ is the weak solution of (7.2), and if $f(x, z, \cdot)$ is strictly convex, for all $(x, z) \in \Omega \times \mathbb{R}^n$, then

$$u_\varepsilon(x,t) - g\big(x, \frac{x}{\varepsilon}, \bar f(x, \bar u(x,t))\big) \to 0 \qquad as\ \varepsilon \to 0\ in\ L^1_{loc}(\Omega \times [0,\infty)). \tag{7.5}$$

Proof. 1. The fact that the solutions of (7.1) form a uniformly bounded sequence in $L^\infty(Q)$ follows from the last part of Remark 5.2. To see this, take $\alpha_1 < 0 < \alpha_2$ such that $\psi_{\alpha_1}(x, \frac{x}{\varepsilon}) \le u_0(x, \frac{x}{\varepsilon}) \le \psi_{\alpha_2}(x, \frac{x}{\varepsilon})$. Hence, using (5.30), once for $(u_\varepsilon - \psi_{\alpha_1})_+$ and again for $(u_\varepsilon - \psi_{\alpha_2})_-$, we obtain

$$\psi_{\alpha_1}(x, \frac{x}{\varepsilon}) \le u_\varepsilon(x,t) \le \psi_{\alpha_2}(x, \frac{x}{\varepsilon}),$$

which proves the uniform boundedness of the family $\{u_\varepsilon\}$.

2. Now, let us make a general observation concerning problem (5.3)–(5.4), under assumptions **(f1)**–**(f4)** on f. So, let u be the weak solution of (5.3)–(5.4) and, for each $t \in [0,\infty)$, let $U(\cdot, t) := \Delta^{-1} u(\cdot, t)$. We claim that U is the viscosity solution of

$$\begin{cases} \partial_t U - f(x, \Delta U) = 0, & (x,t) \in Q, \\ U(x, 0) = U_0(x), & x \in \Omega, \\ U(x, t) = 0, & (x,t) \in \partial\Omega \times (0, \infty), \end{cases} \tag{7.6}$$

where $U_0 = \Delta^{-1} u_0$. Indeed, let u_σ be the smooth solution of the corresponding regularized problem (5.34)–(5.35), where we replace ε by σ since here we use ε as the homogenization parameter. For each $t \in [0, \infty)$, let $U_\sigma(\cdot, t) := \Delta^{-1} u_\sigma(\cdot, t)$. Since u_σ and U_σ are smooth, it is clear that the latter is the (viscosity) solution of

$$\begin{cases} \partial_t U_\sigma - f^\sigma(x, \Delta U_\sigma) = 0, & (x,t) \in Q, \\ U_\sigma(x,0) = U_0(x), & x \in \Omega, \\ U_\sigma(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases} \tag{7.7}$$

Since $\{u_\sigma(x,t)\}_{0<\sigma<1}$ is uniformly bounded in $L^\infty(\Omega \times [0,\infty))$, we easily see that the $U_\sigma(x,t)$ form a uniformly bounded sequence in $L^\infty([0,\infty); W^{2,p}(\Omega))$ for all $p \in (1,\infty)$. On the other hand, from (7.7) we easily deduce that $|U_\sigma(x,t) - U_\sigma(x,s)| \le C|t-s|$ for all $x \in \Omega$ for some constant $C > 0$, independent of σ. Hence, we see that U_σ is uniformly bounded in $W^{1,\infty}(\bar{Q})$. In particular, there is a subsequence U_{σ_i} of U_σ converging locally uniformly in $\bar{Q}$ to a function $U \in W^{1,\infty}(\bar{Q})$ which satisfies $U = \Delta^{-1} u$.

It follows in a standard way that U is the viscosity solution of (7.6). Indeed, given any $(x_0, t_0) \in Q$, we consider $\varphi \in C^2(Q)$ such that $U - \varphi$ has a strict local maximum at (x_0, t_0). Since $U_{\varepsilon_i} - \varphi$ converges locally uniformly in $\bar{Q}$ to the function $U - \varphi$, we may obtain a sequence $(x_i, t_i) \in Q$ such that (x_i, t_i) is a point of local maximum of $U_{\varepsilon_i} - \varphi$ and $(x_i, t_i) \to (x_0, t_0)$ as $i \to \infty$. Thus, we have

$$\partial_t \varphi(x_i, t_i) - f^{\varepsilon_i}(x_i, \Delta\varphi(x_i, t_i)) \le 0,$$

from which follows, as $i \to \infty$,

$$\partial_t \varphi(x_0, t_0) - f(x_0, \Delta\varphi(x_0, t_0)) \le 0. \tag{7.8}$$

To relax the assumption of a strict local maximum to just a local maximum we proceed as usual replacing φ by, say, $\tilde{\varphi}(x,t) := \varphi(x,t) + \delta(|x - x_0|^2 + (t - t_0)^2)$ obtaining (7.8) with $\tilde{\varphi}$ instead of φ and that we obtain again (7.8) for φ passing to the limit when $\delta \to 0$. In an entirely similar way we prove the reverse inequality when $U - \varphi$ has a local minimum at (x_0, t_0), so proving that U is a viscosity solution of (7.6).

3. Next we shall study the homogenization of (7.9) using a method motivated by [16]. We define $U_\varepsilon(x,t)$ in $\Omega \times [0,\infty)$ by $U_\varepsilon := \Delta^{-1} u_\varepsilon$ where u_ε is the weak solution of (7.1). By step 2, we have that U_ε is the viscosity solution of

$$\begin{cases} \partial_t U_\varepsilon - f(x, \frac{x}{\varepsilon}, \Delta U_\varepsilon) = 0, & (x,t) \in Q, \\ U_\varepsilon(x,0) = U_{0,\varepsilon}(x), & x \in \Omega, \\ U(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases} \tag{7.9}$$

where $U_{0,\varepsilon} = \Delta^{-1} u_{0,\varepsilon}$, with $u_{0,\varepsilon}(x) = u_0(\dfrac{x}{\varepsilon}, x)$. The same argument used in previous step shows that

$$U_\varepsilon \in L^\infty((0,\infty); W^{2,p}(\Omega)) \bigcap \mathrm{Lip}((0,\infty); L^\infty(\Omega)),$$

and so there is a subsequence U_{ε_i} of U_ε converging locally uniformly in $\bar{Q}$ to a function

$$\bar{U} \in L^\infty((0,\infty); W^{2,p}(\Omega)) \bigcap \mathrm{Lip}((0,\infty); L^\infty(\Omega)),$$

in particular, $\bar{U} \in W^{1,\infty}(\bar{Q})$.

4. We claim that $\bar{U}(x,t)$ is the viscosity solution of the initial-boundary value problem

$$\begin{cases} \partial_t U - \bar{f}(x, \Delta U) = 0, & (x,t) \in Q, \\ U(x,0) = \bar{U}_0(x), & x \in \Omega, \\ U(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty). \end{cases} \tag{7.10}$$

where

$$\bar{U}_0 := \Delta^{-1} \fint u_0(z,x)\, dz.$$

5. Indeed, let $(\hat{x},\hat{t}) \in Q$ and let $\varphi \in C^2(Q)$ be such $\bar{U}-\varphi$ has a local maximum at $(\hat{x},\hat{t})$. Also, let $v_\delta \in \mathcal{A}(\mathbb{R}^n)$ be a smooth function satisfying

$$g\left(\hat{x}, z, \bar{f}(\hat{x},p)\right) - p - \delta \le \Delta_z v_\delta \le g\left(\hat{x}, z, \bar{f}(\hat{x},p)\right) - p + \delta, \tag{7.11}$$

with $p = \Delta\varphi(\hat{x},\hat{t})$, whose existence is asserted by Lemma 3.3. In particular, given any $\delta' > 0$ we can find $\delta > 0$ sufficiently small such that

$$\bar{f}(\hat{x}, \Delta\varphi(\hat{x},\hat{t})) - \delta' \le f(\hat{x}, z, \Delta\varphi(\hat{x},\hat{t}) + \Delta v_\delta(z)) \le \bar{f}(\hat{x}, \Delta\varphi(\hat{x},\hat{t})) + \delta'.$$

Take $\rho > 0$ and let $x_j \in \Omega$ be a point of maximum of

$$U_j(x,\hat{t}) - \varphi(x,\hat{t}) - \varepsilon_j^2 v_\delta(\frac{x}{\varepsilon_j}) - \rho|x-\hat{x}|^2 + \rho,$$

where we denote $U_j = U_{\varepsilon_j}$, such that $x_j \to \hat{x}$ as $j \to \infty$. Such sequence (x_j) exists since U_j converges locally uniformly to $\bar{U}$ and v_δ is bounded. We have

$$\varphi_t(x_j,\hat{t}) - f\left(x_j, \frac{x_j}{\varepsilon_j}, \Delta\varphi(x_j,\hat{t}) + \Delta v_\delta(\frac{x_j}{\varepsilon_j}) + \rho\right) \le 0,$$

and

$$f\left(\hat{x}, \frac{x_j}{\varepsilon_j}, \Delta\varphi(\hat{x},\hat{t}) + \Delta v_\delta(\frac{x_j}{\varepsilon_j})\right) \le \bar{f}(\hat{x}, \Delta\varphi(\hat{x},\hat{t})) + \delta',$$

which, after addition, gives

$$\varphi_t(x_j,\hat{t}) - \bar{f}(\hat{x}, \Delta\varphi(\hat{x},\hat{t})) \le O(|x_j-\hat{x}|) + O(\rho) + \delta'.$$

Hence, letting $j \to \infty$ first, and then letting $\rho, \delta' \to 0$, we obtain

$$\varphi_t(\hat{x},\hat{t}) - \bar{f}(\hat{x}, \Delta\varphi(\hat{x},\hat{t})) \le 0.$$

The reverse inequality, when $\bar{U}-\varphi$ has a local minimum at $(\hat{x},\hat{t})$, follows in an entirely similar way, which concludes the proof of the claim.

6. By the uniqueness of the viscosity solution of (7.10)(see for instance [10], Theorem 8.2), we conclude that the whole sequence $U_\varepsilon(x,t)$ converges uniformly to $\bar{U}(x,t)$. Let $\bar{u} := \Delta\bar{U}$. Given any $\varphi \in C_c^\infty(Q)$, we have

$$\int_Q u_\varepsilon(x,t)\varphi(x,t)\,dx\,dt = \int_Q \Delta U_\varepsilon \varphi\,dx\,dt = \int_Q U_\varepsilon \Delta\varphi\,dx\,dt \overset{\varepsilon\to 0}{\longrightarrow}$$
$$\int_Q \bar{U}\Delta\varphi\,dx\,dt = \int_Q \bar{u}\varphi\,dx\,dt.$$

Consequently, $u_\varepsilon(x,t)$ converges in the weak-$*$ topology of $L^\infty(Q)$ to $\bar{u} = \Delta\bar{U}(x,t)$, which concludes the proof of the first part of the theorem.

7. Now, let us assume the existence of a weak solution $\tilde{u}$ of (7.2), which actually follows from the compactness result in [24]. Let $\tilde{U} := \Delta^{-1}\tilde{u}$. As it was done above, we easily prove that $\tilde{U}$ is the viscosity solution of (7.10). Therefore, $\tilde{U} \equiv \bar{U}$, and so $\tilde{u} = \bar{u}$.

8. We are going to prove (7.5) under the additional assumption that $f(x,z,\cdot)$ is strictly convex for all $(x,z) \in \Omega \times \mathbb{R}^n$. We first observe that the identity

$$\partial_t U_\varepsilon - f(x, \frac{x}{\varepsilon}, \Delta U_\varepsilon) = 0,$$

holds in the sense of distributions in Q. Indeed, for any $\varphi \in C_0^\infty((0,\infty); H_0^1(\Omega))$, we have

$$\int_Q u_\varepsilon\varphi_t - \nabla f(x, \frac{x}{\varepsilon}, u_\varepsilon) \cdot \nabla\varphi\,dx\,dt = 0. \tag{7.12}$$

Given $\phi \in C_0^\infty(Q)$, we take $\varphi = \Delta^{-1}\phi$ in (7.12), use $u_\varepsilon = \Delta U_\varepsilon$ and integration by parts, to obtain that

$$\int_Q U_\varepsilon \phi_t + f(x, \frac{x}{\varepsilon}, \Delta U_\varepsilon)\phi \, dx \, dt = 0, \tag{7.13}$$

holds for any $\phi \in C_0^\infty(Q)$. Similarly, since $\bar{u}$ is the weak solution of (7.2), we have

$$\int_Q \bar{U}\phi_t + \bar{f}(x, \Delta\bar{U})\phi \, dx \, dt = 0, \tag{7.14}$$

for any $\phi \in C_0^\infty(Q)$.

Using $\phi(x,t)\varphi(\frac{x}{\varepsilon})$ with $\phi \in C_0^\infty(Q)$ and $\varphi \in \mathcal{A}(\mathbb{R}^n)$, as the test function ϕ in (7.13), which is clearly possible, and taking the limit along a suitable subnet $\varepsilon(d)$, $d \in D$, we obtain by Theorem 4.2

$$\int_0^\infty \int_\Omega \int_{\mathcal{K}} \{\langle \nu_{x,t,z}, f(x,z,\cdot)\rangle - \bar{f}(x, \Delta\bar{U})\}\phi(x,t)\varphi(z) \, d\mathfrak{m}(z) \, dx \, dt = 0,$$

where $\mathcal{K}$ is the compactification of $\mathbb{R}^n$ associated with $\mathcal{A}(\mathbb{R}^n)$, and we have used (7.14). Since ϕ and φ are arbitrary, we have

$$\langle \nu_{x,t,z}, f(x,z,\cdot)\rangle = \bar{f}(x, \Delta\bar{U}) = f\big(x, z, g(x, z, \bar{f}(x, \Delta\bar{U}))\big), \quad \text{for a.e. } (x,t,z) \in Q \times \mathcal{K}.$$

Since $f(x,z,\cdot)$ is strictly convex for all $(x,z) \in \Omega \times \mathbb{R}^n$, we conclude that

$$\nu_{x,t,z} = \delta_{g(x,z,\bar{f}(x,\Delta\bar{U}))}, \quad \text{for a.e. } (x,t,z) \in Q \times \mathcal{K}.$$

and this implies through Theorem 4.3 that (7.5) holds. □

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