

Resonance Webs of Hyperplane Arrangements

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Abstract.

The irreducible components of the first resonance variety of a hyperplane arrangement naturally determine a resonance web. In this paper we initiate the study of these resonance webs with emphasis on their space of abelian relations.

§1. Introduction

Let $\mathbf{A} = \{H_1, \dots, H_r\}$ be an arrangement of $r \geq 1$ hyperplanes in \mathbb{P}^n . The complement of \mathbf{A} is an affine variety that will be denoted by $M = M(\mathbf{A})$. It is a result of Arnold [1] (for the braid arrangement) and Brieskorn [5] (for an arbitrary hyperplane arrangement) that the cohomology ring of M , $H^\bullet(M, \mathbb{Z})$, is torsion free and generated, as a graded algebra, by the degree one elements determined by the classes of the logarithmic differential forms

$$(2\pi i)^{-1} \left(d \log \frac{h_i}{h_r} \right) \quad \text{for } i \in \{1, \dots, r-1\}$$

where h_1, \dots, h_r are the linear polynomials in $\mathbb{C}[x_0, \dots, x_n]$ defining the hyperplanes in \mathbf{A} .

Given $a \in H^1(M) = H^1(M, \mathbb{C})$, consider the complex $(H^\bullet(M), a)$ with arrows given by multiplication by a :

$$0 \longrightarrow H^0(M) \longrightarrow H^1(M) \longrightarrow H^2(M) \longrightarrow \dots \longrightarrow H^n(M) \longrightarrow 0.$$

The resonance varieties of M , or \mathbf{A} , are defined as

$$\mathbf{R}^i(M) = \mathbf{R}^i(\mathbf{A}) = \{a \in H^1(M), | h^i(H^\bullet(M), a) \neq 0\}.$$

The paper [14] provides the following nice description of the first resonance variety $\mathbf{R}^1(M)$. The irreducible components of $\mathbf{R}^1(M)$ are precisely the maximal isotropic subspaces of $H^1(M)$ for the quadratic form

$$\wedge : H^1(M, \mathbb{C}) \otimes H^1(M, \mathbb{C}) \longrightarrow H^2(M, \mathbb{C})$$

having dimension at least two. Moreover, the irreducible components of $R^1(M)$ of dimension k are in correspondence with pencils of hypersurfaces on \mathbb{P}^n having exactly $k + 1$ elements with support contained in arrangement. In particular, for each irreducible component Σ of the resonance variety there is a unique (singular holomorphic) foliation \mathcal{F}_Σ on \mathbb{P}^n defined by the corresponding pencil of hypersurfaces.

The interest of the study of resonance varieties is amplified by its relation with the cohomology jumping loci of rank one local systems on M . The characteristic variety $\text{Char}^i(M)$ of M is the subvariety of $\text{Hom}(\pi_1(M), \mathbb{C}^*)$ defined as

$$\text{Char}^i(M) = \{ \rho \in \text{Hom}(\pi_1(M), \mathbb{C}^*) \mid h^i(M, \mathbb{C}_\rho) \neq 0 \},$$

where \mathbb{C}_ρ is the rank one local system determined by ρ . The relation with the resonance varieties is given by the following Theorem [2, 11]: the exponential map

$$\begin{aligned} \exp : H^1(M) &\longrightarrow \text{Hom}(\pi_1(M), \mathbb{C}^*) \\ a &\longmapsto \left(\gamma \mapsto \exp \left(2\pi i \int_\gamma a \right) \right) \end{aligned}$$

defines isomorphisms between the germs $(R^i(M), 0)$ and the germs $(\text{Char}^i(M), \mathbf{1})$, where $\mathbf{1} \in \text{Hom}(\pi_1(M), \mathbb{C}^*)$ is the trivial representation.

The study of the foliation \mathcal{F}_Σ for irreducible components $\Sigma \subset R^1(M)$ leded the author and S. Yuzvinsky (see [21]) to obtain bounds for the dimension of Σ . Although there is now (specially after [27]) a reasonably clear picture about each of the irreducible components of $R^1(M)$, it is not very clear how the totality of them sit inside $H^1(M, \mathbb{C})$. In this paper we propose an approach to produce invariants for arrangements that may turn out to be useful to the study of this question. The underlying idea is fairly simple: instead of looking at the foliations associated to the resonance varieties one at a time, we should look at all of them at the same time. More precisely, we will associate to an arrangement A what we call its resonance web $\mathcal{W}(A)$ – the superposition of all the foliations \mathcal{F}_Σ associate to irreducible components of $R^1(A)$ – and will study its space of abelian relations.

Conversely, many relevant examples for web geometry, specially in what concerns the dimension of the space of abelian relations, appear as resonance webs of certain arrangements. The list starts with Bol exceptional 5-web[4], contains Spence-Kummer exceptional 9-web[25, 23],

and ends with some other exceptional webs presented in [25, 23]. This provides further motivation to pursue the study of resonance webs.

Our main result is Theorem 4.1 which determines the rank of the resonance webs for the braid arrangements which, as we will see in Section 4, correspond to the $\binom{n+3}{4}$ -web on $\mathfrak{M}_{0,n+3}$ (the moduli space of $n+3$ distinct ordered points on \mathbb{P}^1) defined by the $\binom{n+3}{4}$ natural maps

$$\mathfrak{M}_{0,n+3} \longrightarrow \mathfrak{M}_{0,4} \simeq \mathbb{C} \setminus \{0, 1\}.$$

We will also draw some general considerations about the abelian relations of resonance webs and use them to study some of the examples of exceptional webs mentioned above. Although we have no major results on the structure of the space of abelian relations of resonance webs for general arrangements, the blurry picture delineated by the study of these examples is considerably intricate and, we believe, invites further investigation.

Plan of the paper. In Section 2 we define webs, their spaces of abelian relations, and show how to bound the rank of arbitrary webs. We also briefly review the algebraization results for webs of maximal rank, define exceptional webs, and present Bol's example of exceptional planar 5-web. In Section 3 we define resonance webs and initiate the study of their space of abelian relations, more specifically the subspace of polylogarithmic abelian relations generated by tuples of iterated integrals of logarithmic 1-forms with poles on the arrangement. The reader will also find in Section 3 a brief presentation of a couple of basic results from Chen's theory of iterated integrals relevant to our study. Section 4 is devoted to the statement and proof of our main result: the determination of the rank of the resonance webs of the braid arrangements. Section 5 studies the resonance webs of some line arrangements on \mathbb{P}^2 which correspond to the examples by Pirio and Robert. In Section 6 we draw some final considerations about our study and state some questions.

§2. Web geometry

For us a **germ of codimension one k -web** $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ on $(\mathbb{C}^n, 0)$ is a collection of k germs of smooth codimension one holomorphic foliations subjected to the condition that any two distinct foliations $\mathcal{F}_i, \mathcal{F}_j$ have distinct tangent spaces at zero.

Usually in the literature a stronger condition is imposed on the tangent spaces at zero. In the terminology of [7] the tangent spaces are usually assumed to be in strong general position, meaning that for

any $1 \leq m \leq n$ the intersection of tangent spaces at zero of m distinct foliations \mathcal{F}_i have codimension m .

Perhaps the most studied invariant of a germ of codimension one web \mathcal{W} is its space of abelian relations $\mathcal{A}(\mathcal{W})$. If we chose integrable 1-forms ω_i inducing the foliations \mathcal{F}_i then $\mathcal{A}(\mathcal{W})$ is equal to

$$\left\{ (\eta_i)_{i=1}^k \in (\Omega^1(\mathbb{C}^n, 0))^k \mid \forall i \, d\eta_i = 0, \eta_i \wedge \omega_i = 0 \text{ and } \sum_{i=1}^k \eta_i = 0 \right\}.$$

If $u_i : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ are local submersions defining the foliations \mathcal{F}_i then, after integration, the abelian relations can be read as functional equations of the form $\sum_{i=1}^k g_i(u_i) = 0$ for some germs of holomorphic functions $g_i : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$. Thus we can interpret the abelian relations of \mathcal{W} as functional equations (of a rather special kind) among the first integrals of the foliations defining it.

2.1. Rank of webs

Clearly $\mathcal{A}(\mathcal{W})$ is a vector space and its dimension is commonly called the **rank** of \mathcal{W} and is denoted by $\text{rank}(\mathcal{W})$. We will now explain how one can bound the rank of arbitrary codimension one webs. This is a classical subject in web geometry and has been treated by Bol ($n = 2$) and Chern (for webs in strong general position) $n \geq 3$ in the decade of 1930, and more recently by Cavalier-Lehmann for ordinary webs, see the definition below. Here we will deal with arbitrary codimension one webs. This section is a summary of [20, Section 2.2].

For every $i \in \{1, \dots, k\}$ let ω_i be a germ of 1-form defining \mathcal{F}_i and satisfying $\omega_i(0) \neq 0$. For any positive integer j define $\mathcal{L}^j(\mathcal{W})$ as the subspace of the \mathbb{C} -vector space $\text{Sym}^j(\Omega_0^1(\mathbb{C}^n))$ generated by the j -th symmetric powers of the exterior forms $\omega_i(0)$ with $i \in \{1, \dots, k\}$. Set $\ell^j(\mathcal{W}) = \dim \mathcal{L}^j(\mathcal{W})$.

Alternatively, if $u_i : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ are local submersions defining the foliations \mathcal{F}_i , and h_i are their linear terms then

$$(1) \quad \ell^j(\mathcal{W}) = \dim \left(\mathbb{C}h_1^j + \dots + \mathbb{C}h_k^j \right).$$

Notice that the integer $\ell^j(\mathcal{W})$ is bounded by k and by the dimension of the vector space of homogeneous polynomials of degree j in n variables, i.e.

$$(2) \quad \ell^j(\mathcal{W}) \leq \max \left\{ k, \binom{n+j-1}{n-1} \right\}.$$

In the terminology of [7], a germ of k -web \mathcal{W} on $(\mathbb{C}^n, 0)$ is ordinary if and only if $\ell^j(\mathcal{W}) = \max\left\{k, \binom{n+j-1}{n-1}\right\}$ for every positive integer j .

A good lower bound is harder to obtain. For webs in strong general position there is a lemma by Castelnuovo [20, Proposition 2.2.2] which says that

$$(3) \quad \ell^j(\mathcal{W}) \geq \min(k, j(n-1) + 1).$$

For arbitrary webs it is not possible improve this lower bound beyond its specialization to $n = 2$. That is for an arbitrary k -web, $\ell^j(\mathcal{W}) \geq \min(k, j + 1)$.

The argument used to prove the proposition below is borrowed from Trépreau's proof [26] of Chern's bound for the rank of webs in strong general position.

Proposition 2.1. *If \mathcal{W} is an arbitrary k -web on $(\mathbb{C}^n, 0)$ then*

$$\text{rk}(\mathcal{W}) \leq \sum_{j=1}^{\infty} \max(0, k - \ell^j(\mathcal{W}))$$

and the sum involves only finitely many non-zero terms.

Proof. The space of abelian relations of \mathcal{W} admits a natural filtration $\mathcal{A}(\mathcal{W}) = \mathcal{A}^0(\mathcal{W}) \supseteq \mathcal{A}^1(\mathcal{W}) \supseteq \dots \supseteq \mathcal{A}^j(\mathcal{W}) \supseteq \dots$, where

$$\mathcal{A}^j(\mathcal{W}) = \ker \left\{ \mathcal{A}(\mathcal{W}) \longrightarrow \left(\frac{\Omega^1(\mathbb{C}^n, 0)}{\mathfrak{m}^j \cdot \Omega^1(\mathbb{C}^n, 0)} \right)^k \right\},$$

and \mathfrak{m} is the maximal ideal of $\mathbb{C}\{x_1, \dots, x_n\}$.

One can easily verify [20, Lemma 2.2.6] that

$$(4) \quad \dim \frac{\mathcal{A}^j(\mathcal{W})}{\mathcal{A}^{j+1}(\mathcal{W})} \leq k - \dim \left(\mathbb{C} \cdot h_1^{j+1} + \dots + \mathbb{C} \cdot h_k^{j+1} \right)$$

where h_i is as in (1). The bound follows. Moreover, as $\ell^j(\mathcal{W}) \geq \min(k, j + 1)$ there are only finitely many non-zero terms at the summation above. Q.E.D.

The proposition above combine with the lower bounds previously discussed allow us to recover the bounds for the rank of germs of codimension one webs available elsewhere.

Corollary 2.1. Let \mathcal{W} be a germ of k -web on $(\mathbb{C}^n, 0)$. The assertions below hold true.

- (1) **(Bol's bound)** If $n = 2$ then

$$\text{rank}(\mathcal{W}) \leq \frac{k(k-1)}{2}.$$

- (2) **(Chern's bound)** If $n \geq 3$ and \mathcal{W} is in strong general position then

$$\text{rank}(\mathcal{W}) \leq \sum_{j=1}^{\infty} \max(0, k - j(n-1) - 1).$$

- (3) **(Cavalier-Lehmann's bound)** If $n \geq 3$ and \mathcal{W} is an ordinary web then

$$\text{rank}(\mathcal{W}) \leq \sum_{j=1}^{\infty} \max\left(0, k - \binom{n+j-1}{n-1}\right).$$

The number at right-hand side of Chern's bound is Castelnuovo's bound $\pi(n, k)$ for the arithmetic genus of irreducible, non-degenerated curves of degree k on \mathbb{P}^n .

2.2. Algebraic and algebraizable webs

An important class of examples of webs is the class of webs dual to projective curves. If C is reduced degree k projective curve on \mathbb{P}^n then for every general hyperplane H_0 a germ of codimension one k -web \mathcal{W}_C is naturally defined on (\mathbb{P}^n, H_0) through projective duality. More precisely \mathcal{W}_C is defined by the germs of submersions $p_i : (\mathbb{P}^n, H_0) \rightarrow C$ characterized by

$$H \cdot C = p_1(H) + p_2(H) + \cdots + p_k(H)$$

for every H sufficiently close to H_0 .

Abel's addition Theorem says that for every $p_0 \in C$ and every holomorphic 1-form $\omega \in H^0(C, \omega_C)$ the sum

$$\int_{p_0}^{p_1(H)} \omega + \int_{p_0}^{p_2(H)} \omega + \cdots + \int_{p_0}^{p_k(H)} \omega$$

does not depend on H . One can reformulate this statement as

$$\sum_{i=1}^k p_i^* \omega = 0.$$

It follows that $(p_1^* \omega, \dots, p_k^* \omega)$ can be interpreted as an abelian relation of \mathcal{W}_C . Consequently there is an injection of $H^0(C, \Omega_C^1)$ into $\mathcal{A}(\mathcal{W}_C)$.

There is a converse to Abel's addition Theorem (due to Lie, Poincaré, Darboux, Wirtinger see [20, Chapter 4]) which implies that this injection is indeed an isomorphism.

2.3. Exceptional webs

For k -webs in strong general position on $(\mathbb{C}^n, 0)$, $n \geq 3$, the maximality of rank implies that the web is algebraizable (biholomorphic to a web obtained from a projective curve through duality as explained above) when $k \leq n+1$ or $k \geq 2n$. This was proved by Bol for $n = 3$ and for $n > 3$ it is a recent result of Trépreau, see [26]. The planar case ($n = 2$) is rather special in what concerns the classification of webs of maximal rank. For $k \leq 4$ it is well-known that planar webs of maximal rank are algebraizable, the proof for $k = 4$ can be traced back to Lie's work on double translation surfaces, and for $k = 3$ is due to Blaschke-Howe. In sharp contrast, [18] exhibit examples of non-algebraizable planar k -webs of maximal rank are presented for every $k \geq 5$. Further infinite families of examples appear in [19]. Despite the recent advances, see for instance [15, 16, 18, 19], the classification of exceptional (non-algebraizable and of maximal rank) planar k -webs is wide open.

So far we have focused on germs of webs, but we can consider webs globally defined on a complex variety X . For our purposes it will be sufficient to consider completely decomposable webs, that is webs \mathcal{W} which can be globally presented as the superposition of k pairwise distinct global foliations $\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$. For the general definition of global webs see [20, Chapter 1].

Given a global web \mathcal{W} , there exists a subvariety $\Lambda = \Lambda(\mathcal{W}) \subset X$ such that for every $x \in X \setminus \Lambda$ the germ of web \mathcal{W}_x obtained by localizing \mathcal{W} at x is a germ of codimension one web and the rank of \mathcal{W}_x is independent of $x \in X \setminus \Lambda$, see [24, Theorem 1.2.2]. More precisely, over $X \setminus \Lambda$ the space of abelian relations of \mathcal{W} is a local system of k -uple of germs of closed 1-forms. Therefore it still makes sense to talk about the rank of \mathcal{W} for global webs \mathcal{W} .

The first example of exceptional web dates back to the 1936 and was found by Bol, see [4]. It is the global 5-web \mathcal{B}_5 defined on \mathbb{P}^2 formed by the superposition of 4 pencils of lines with base points in general position and one pencil of conics through these four base points. We will explain below that this web is naturally associated to an arrangement of lines on \mathbb{P}^2 .

§3. Resonance webs

Let $\mathbf{A} = \{H_1, \dots, H_r\}$ be an arrangement of $r \geq 1$ hyperplanes in \mathbb{P}^n . Recall from the introduction that $\mathbf{R}^1(\mathbf{A})$, the first resonance variety of \mathbf{A} , is the union of the maximal isotropic subspaces of $(H^1(M), \wedge)$ of dimension at least two.

For $i \in \{1, \dots, r\}$, let $h_i \in \mathbb{C}[x_0, \dots, x_n]$ be a linear polynomial defining H_i . From now on we will identify $H^\bullet(M)$ with the algebra generated by the logarithmic 1-forms $(d \log h_i - d \log h_r)$ with $i \in \{1, \dots, r-1\}$.

Before defining our main object of study, let us give a brief idea on how one can associate a pencil of hypersurfaces with $d+2$ completely decomposable fibers to an irreducible component Σ of $\mathbf{R}^1(\mathbf{A})$ of dimension d . Let $\omega_1, \dots, \omega_d$ be a basis of Σ . As Σ is isotropic there exists non-constant rational functions $h_{ij} \in \mathbb{C}(x_0, \dots, x_n)$ ($i \neq j$) such that $\omega_i = h_{ij} \omega_j$. Differentiating this last expression one obtains $0 = dh_{ij} \wedge \omega_j$. Thus the level sets of the functions h_{ij} are tangent to the distribution determined by ω_i for every $i \in \{1, \dots, d\}$. Stein factorization Theorem ensures the existence of a rational map $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$ such that $df \wedge dh_{ij} = df \wedge \omega_i = 0$ for every $i, j \in \{1, \dots, d\}$. Moreover, there exists d linearly independent logarithmic 1-forms on \mathbb{P}^1 , say η_1, \dots, η_d , such that $f^*(\eta_i)_\infty \subset \mathbf{A}$, and $f^*(\eta_i) = \omega_i$ for every $i \in \{1, \dots, d\}$. The maximality of Σ implies that the cardinality of

$$P_\Sigma = \bigcup_{i \in \underline{d}} (\eta_i)_\infty$$

is equal to $d+2$. Thus the rational map $f = f_\Sigma$ determines a pencil of hypersurfaces with $d+2$ fibers contained in the support of the arrangement. Moreover, the restriction of f to $M = \mathbb{P}^n \setminus \mathbf{A}$ is a regular morphism

$$f|_M : M \rightarrow C_\Sigma,$$

where $C_\Sigma = \mathbb{P}^1 \setminus P_\Sigma$.

Let \mathcal{F}_Σ be the foliation on M (or on \mathbb{P}^n) determined by the level sets of f_Σ . We define $\mathcal{W}(\mathbf{A})$, the **resonance web** of \mathbf{A} as the global web on M (or on \mathbb{P}^n) obtained by the superposition of the foliations \mathcal{F}_Σ with Σ ranging over the irreducible components of $\mathbf{R}^1(\mathbf{A})$.

Example 3.1. Consider the arrangement on \mathbb{P}^2 defined by the polynomial $\{xyz(x-z)(y-z)(x-y) = 0\}$. The points of its complement can be interpreted as isomorphism classes of 5 ordered points on \mathbb{P}^1 . To wit, the point $(x : y : 1) \in \mathbb{P}^2$ satisfying $x - y \neq 0, x \neq 0, 1$, and

$y \neq 0, 1$ naturally correspond to the 5-tuple $(0, 1, \infty, x, y) \in (\mathbb{P}^1)^5$. Because of that, we will denote this arrangement by $\mathcal{A}_{0,5}$, and its complement in \mathbb{P}^2 by $\mathfrak{M}_{0,5}$. The resonance variety of $\mathfrak{M}_{0,5}$ have five irreducible components and the associated morphism are the five forgetting maps $\mathfrak{M}_{0,5} \rightarrow \mathfrak{M}_{0,4}$, sending isomorphism classes of five ordered points on \mathbb{P}^1 to isomorphism classes of four ordered points on \mathbb{P}^1 . It is a simple exercise to verify that fibers of four of these maps form pencils of lines with base points in general position, and that the fibers of one of them is a pencil of conics through these base points. Thus Bol's exceptional 5-web \mathcal{B}_5 is nothing more than $\mathcal{W}(\mathcal{A}_{0,5})$.

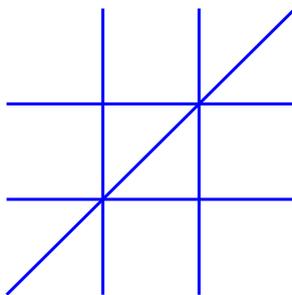


Fig. 1. Affine trace of the arrangement $\mathcal{A}_{0,5}$.

Due to the important role of Bol's 5-web in the development of web geometry it is natural to enquire about the rank of the resonance webs for arbitrary arrangements. To determine the rank of an arbitrary web is a daunting task. The only general method dates back to Abel (see [24, 23] for a modern account) and involves lengthy algebraic manipulations which lead to linear differential equations of high order that have to be solved. Although implementable, in practice such method is not computationally efficient and cannot deal with k -webs when k is large, say $k > 10$.

An alternative approach to compute the rank of certain webs has been devised by Giles Robert. Loosely speaking it restricts the search of abelian relations to a certain class of differential forms defined from iterated integrals of logarithmic differentials. Here we follow we will explore this approach, adding some topology/combinatorics of arrangements to the picture.

3.1. Logarithmic abelian relations

Consider the morphism

$$\begin{aligned} \Psi_1 : \bigoplus_I H^1(C_\Sigma) &\longrightarrow H^1(M) \\ (\eta_\Sigma) &\mapsto \sum f_\Sigma^* \eta_\Sigma. \end{aligned}$$

where Σ ranges over all the irreducible components of $R^1(A)$. Let $\text{Log}^1 \mathcal{W}(A)$ be its kernel, i.e., $\text{Log}^1 \mathcal{W}(A) = \ker \Psi_1$.

Proposition 3.1. *The vector space $\text{Log}^1 \mathcal{W}(A)$ embeds into the space of abelian relations of $\mathcal{W}(A)$.*

Proof. Let $(\eta_\Sigma) \in \ker \Psi_1$ be a non-zero element. Each η_Σ corresponds to a logarithmic 1-form on \mathbb{P}^1 . The pull-backs under the corresponding rational map $f_\Sigma^* \eta_\Sigma$ are closed logarithmic 1-forms on \mathbb{P}^n , and $f_\Sigma^* \eta_\Sigma$ defines a distribution tangent to the foliation \mathcal{F}_Σ . Thus $(f_\Sigma^* \eta_\Sigma) \in \mathcal{A}(\mathcal{W}(A))$ as wanted. Q.E.D.

3.2. Chen's theory of iterated integrals

A natural place to look for further abelian relations is to consider differential forms with logarithmic coefficients, that is, differential forms like

$$\log(z) \frac{dz}{z-1}.$$

The right formalism to deal with such objects is Chen's theory of iterated integrals.

In this paragraph M will be an arbitrary connected complex manifold. Given a path $\gamma : [0, 1] \rightarrow M$ and a collection of 1-forms $\omega_1, \dots, \omega_k$ the iterated integral of $\omega_1 \otimes \dots \otimes \omega_k$ along γ is defined as

$$\int_\gamma \omega_1 \otimes \dots \otimes \omega_k = \int_{\Delta_k} p_1^* \omega_1 \wedge \dots \wedge p_k^* \omega_k$$

where $p_i : M^k \rightarrow M$ is the projection on the i -th factor and Δ_k is the image of standard simplex in \mathbb{R}^k on M^k under the map $\underbrace{\gamma \times \dots \times \gamma}_{k \text{ times}}$.

Even if the 1-forms ω_i are closed this integral **does** depend on the path and not just on its homotopy class. It is a result of Chen [8, Theorem 4.1.1] that the elements of $\Omega^1(M)^{\otimes k}$ whose iterated integral do not depend on the representative in a given homotopy class are in

the intersection of the kernels of the linear maps, $i \in \underline{k-1}$,

$$\begin{aligned} \Omega^1(M)^{\otimes k} &\longrightarrow \Omega^1(M)^{\otimes i-1} \otimes \Omega^2(M) \otimes \Omega^1(M)^{\otimes (k-i-1)} \\ \omega_1 \otimes \cdots \otimes \omega_k &\mapsto \omega_1 \otimes \cdots \otimes \omega_{i+1} \wedge \omega_{i+2} \otimes \omega_{i+3} \otimes \cdots \otimes \omega_k, \end{aligned}$$

with the kernels of the linear maps, $i \in \underline{k}$,

$$\begin{aligned} \Omega^1(M)^{\otimes k} &\longrightarrow \Omega^1(M)^{\otimes i-1} \otimes \Omega^2(M) \otimes \Omega^1(M)^{\otimes (k-i)} \\ \omega_1 \otimes \cdots \otimes \omega_k &\mapsto \omega_1 \otimes \cdots \otimes d\omega_{i+1} \otimes \omega_{i+2} \otimes \cdots \otimes \omega_k. \end{aligned}$$

If $B^k(M)$ denotes this intersection then every element of $B^k(M)$ gives rise to a function on the universal covering of M through (iterated) integration. Thus we can interpret the elements of $B^k(M)$ as closed 1-forms on the universal covering of M by considering the differential of this function.

Moreover, Chen also proved that if we consider a vector subspace V of $\Omega^1(M)$ formed by closed 1-forms with no non-zero exact forms then the iterated integrals define an injection of $\bigoplus_{k \geq 1} V^{\otimes k} \cap B^k(M)$ into the space of holomorphic functions on the universal covering of M . In particular, when M is the complement of a hyperplane arrangement, this is the case for $H^1(M)$ seen as a vector subspace of $\Omega^1(M)$.

It is also interesting to observe that $B(M) = \bigoplus_{k=1}^{\infty} B^k(M)$ admits a natural structure of $\pi_1(M)$ -module, with action defined by analytic continuation of the iterated integrals. Notice that the summands $B^k(M)$ are not $\pi_1(M)$ -invariant when $k \geq 2$. But the terms of the filtration

$$F^\bullet : F^k(M) = \bigoplus_{i=1}^k B^i(M)$$

are $\pi_1(M)$ -invariant.

3.3. Polylogarithmic abelian relations

It is natural to extend the construction of Section 3.1 to arbitrary iterated integrals of logarithmic 1-forms. For each $i \geq 1$, consider the morphism

$$\begin{aligned} \Psi_i : \bigoplus_I H^1(C_\Sigma)^{\otimes i} &\longrightarrow H^1(M)^{\otimes i} \\ (\eta_\Sigma) &\mapsto \sum f_\Sigma^* \eta_\Sigma. \end{aligned}$$

where, as before, Σ ranges over all the irreducible components of $\mathbb{R}^1(\mathbf{A})$. Define $\text{Log}^i \mathcal{W}(\mathbf{A})$ as its kernel, i.e., $\text{Log}^i \mathcal{W}(\mathbf{A}) = \ker \Psi_i$. Define also

$\text{Log}^\infty \mathcal{W}(A)$ as the direct sum

$$\text{Log}^\infty \mathcal{W}(A) = \bigoplus_{i=1}^{\infty} \text{Log}^i \mathcal{W}(A).$$

Proposition 3.2. *If we consider the localization of the web $\mathcal{W}(A)$ at a generic point of M then the vector space $\text{Log}^\infty \mathcal{W}(A)$ embeds into the space of abelian relations of germ $\mathcal{W}(A)$. Moreover, the analytic continuation of this embedding gives rise to a local system of abelian relations globally defined on M .*

Proof. It suffices to combine the proof of Proposition 3.1 with the properties of iterated integrals recalled on Section 3.2 Q.E.D.

Let $k_d(A)$ be the number of irreducible components of $R^1(A)$ of dimension d and $k(A)$ be the total number of irreducible components of $R^1(A)$. Notice that the resonance web of A is a $k(A)$ -web.

Corollary 3.1. The following inequalities hold true:

$$\begin{aligned} \text{rank}(\mathcal{W}(A)) &\geq \dim \text{Log}^\infty(A), \\ \dim \text{Log}^\infty(A) &\leq \frac{(k(A) - 1)(k(A) - 2)}{2}, \\ \dim \text{Log}^i(A) &\geq \sum_d d^i k_d(A) - \dim B^i(M) \cap H^1(M)^{\otimes i}. \end{aligned}$$

In particular $\text{Log}^\infty(A)$ is a finite dimensional vector space.

Proof. The first inequality follows Proposition 3.2. The second follows from the first combined with Bol's bound (Corollary 2.1) for the rank of planar webs. To proof the third inequality it suffices to notice that Chen's *integrability conditions* are trivially satisfied by collections of 1-forms on a curve. Thus the morphism Ψ_k factors as in the diagram below

$$\begin{array}{ccc} & & N^k(M) \\ & \nearrow & \downarrow \\ \bigoplus_{\Sigma} H^1(C_{\Sigma})^{\otimes k} & \xrightarrow{\Psi_k} & H^1(M)^{\otimes k} \\ & & \downarrow \\ & & H^2 \otimes H^{1 \otimes k-2} \oplus \dots \oplus H^{1 \otimes k-2} \otimes H^2 \end{array}$$

where $H^i = H^i(M)$ and $N^k(M) = B^k(M) \cap H^1(M)^{\otimes k}$. The corollary follows. Q.E.D.

Since M is the complement of a hyperplane arrangement, $H^\bullet(M)$ is generated in degree one. Consequently

$$\dim N^2(M) = h^1(M)^2 - h^2(M).$$

In general we do not know how to control the dimensions of the vector spaces $N^i(M)$ when $i \geq 3$. Nevertheless for fiber type arrangements there is the following Künneth type formula which is a corollary of [6, Theorem 3.38].

Proposition 3.3. *If \mathcal{A} is a fiber type arrangement on \mathbb{P}^n with exponents $\{e_1, \dots, e_n\}$ then*

$$\dim N^i(M) = \sum e_1^{j_1} \cdots e_n^{j_n}$$

where the sum is over all ordered n -uples $0 \leq j_1 \leq \cdots \leq j_n$ with $j_1 + \cdots + j_n = i$.

As will be made clear by the examples in Section 5 the bound for the rank given by the first and third inequalities of Corollary 3.1 is rather crude and does not capture many otherwise easily predicible abelian relations. Nevertheless, we will need not more than these crude bounds to determine the rank of the resonance webs of the braid arrangements.

§4. Resonance webs of the braid arrangements

For $n \geq 2$, let $\mathbf{A}_{0,n+3}$ be the arrangement of hyperplanes on \mathbb{P}^n defined by the vanishing of the polynomial

$$\left(\prod_{i=0}^n x_i \right) \left(\prod_{i=1}^n (x_i - x_0) \right) \left(\prod_{1 \leq i < j \leq n} (x_i - x_j) \right).$$

It is the quotient of the braid arrangement B_{n+2} on \mathbb{P}^{n+2}

$$\prod_{0 \leq i < j \leq n+1} (y_i - y_j)$$

by its center $\{y_0 = y_1 = \dots = y_{n+1}\}$. The resonance variety of $\mathbf{A}_{0,n+3}$ is isomorphic to the resonance variety of B_{n+2} , and the resonance web of B_{n+2} is a linear pull-back of the resonance web of $\mathbf{A}_{0,n+3}$. Consequently, both webs have isomorphic space (local system) of abelian relations.

The complement of $\mathbf{A}_{0,n+3}$ will be denoted by $\mathfrak{M}_{0,n+3}$ and can be identified with the moduli space of $(n+3)$ -uples of pairwise distinct

ordered points of \mathbb{P}^1 . The resonance variety of $\mathfrak{M}_{0,n+3}$ has $\binom{n+3}{4}$ irreducible components which are in correspondence with the forgetful maps

$$\mathfrak{M}_{0,n+3} \longrightarrow \mathfrak{M}_{0,4}.$$

Thus the resonance web of $\mathcal{W}(A_{0,n+3})$ is a $\binom{n+3}{4}$ -web on $\mathfrak{M}_{0,n+3} \subset \mathbb{P}^n$. The main result of this paper is the determination of the rank of $\mathcal{W}(A_{0,n+3})$ given below.

Theorem 4.1. *For every $n \geq 2$ the equality*

$$\text{rank}(\mathcal{W}(A_{0,n+3})) = 3 \binom{n+3}{4} - \binom{n+2}{3} - \binom{n+1}{2} - n.$$

holds true.

The remaining of this Section is devoted to the proof of this Theorem. It will be convenient to work in the affine chart $x_0 = 1$.

4.1. Upper bound for the rank

For each ordered 4-uple of ordered integers $1 \leq \alpha < \beta < \gamma < \delta \leq n+3$ consider the map

$$\begin{aligned} \rho_{\alpha\beta\gamma\delta} : \mathfrak{M}_{0,n+3} &\longrightarrow \mathfrak{M}_{0,4} \\ (x_1, \dots, x_{n+3}) &\longmapsto (x_\alpha, x_\beta, x_\gamma, x_\delta) \end{aligned}$$

where the points in the source and the target represent isomorphism classes.

Since $\mathfrak{M}_{0,4} = \mathbb{C} - \{0, 1\}$ each of these $\binom{n+3}{4}$ maps define isotropic subspaces of $H^1(\mathfrak{M}_{0,n+3})$, namely $\rho_{\alpha\beta\gamma\delta}^* H^1(\mathfrak{M}_{0,4}) \subset H^1(\mathfrak{M}_{0,n+3})$. It can be verified that these isotropic subspaces are maximal, and that there are no other resonance varieties for $A_{0,n+3}$, see [11].

Consider now $\mathcal{W} = \mathcal{W}(A_{0,n+3})$, the germification of the resonance web of $A_{0,n+3}$ at a generic point of $\mathfrak{M}_{0,n+3} \subset \mathbb{C}^n$. It will be useful to consider the following subwebs:

- (1) For each $\alpha \in \underline{n+3}$ let \mathcal{W}_α be the $\binom{n+2}{3}$ -subweb defined by the maps ρ_I where I ranges over all the ordered 4-uples containing α . Similarly for distinct α, β and α, β, γ in the above range let $\mathcal{W}_{\alpha,\beta}$ be the $\binom{n+1}{2}$ -subweb and $\mathcal{W}_{\alpha,\beta,\gamma}$ be the n -subweb where I ranges over the ordered 4-uples that contains α, β and α, β, γ respectively.
- (2) For each $\alpha \in \underline{n+3}$ let \mathcal{W}^α be the pull back of $\mathcal{W}(A_{0,n+2})$ under the morphism

$$\rho^\alpha : \mathfrak{M}_{0,n+3} \longrightarrow \mathfrak{M}_{0,n+2}$$

that forgets the α -th point.

We combine the two constructions above and define

$$\mathcal{W}_{\beta,\gamma}^\alpha = \mathcal{W}^\alpha \cap \mathcal{W}_{\beta,\gamma}$$

where the intersection of the webs is the web formed by the common foliations of both webs.

Proposition 4.1. *For each $a \in \{1, 2, 3, 4\}$ and every ordered subset I of $\underline{n+3}$ of cardinality $4 - a$, $\mathcal{L}^a(\mathcal{W})$ is isomorphic to $\mathcal{L}^a(\mathcal{W}_I)$. Moreover, the dimension of $\mathcal{L}^a(\mathcal{W})$ is given by the formula*

$$\ell^a(\mathcal{W}) = \ell^a(\mathcal{W}_I) = \binom{n+a-1}{n-1}.$$

Proof. The proof will be by a double induction on n and a .

When $n = 2$, $\mathcal{W}_I = \mathcal{W}(A_{0,5})_I$ is a $(a+1)$ -web on \mathbb{C}^2 . Therefore $\mathcal{L}^a(\mathcal{W})$ contains $\mathcal{L}^a(\mathcal{W}_I)$, and equation (2) implies

$$a+1 \geq \ell^a(\mathcal{W}) \geq \ell^a(\mathcal{W}_I) = a+1.$$

The result follows in this case.

When $a = 1$, we can assume that $I = (n+1, n+2, n+3)$ and therefore by normalizing the points of $M_{0,n+3}$ in such way that the last three are $0, 1, \infty$ we have that the n foliations defining \mathcal{W}_I are defined by the morphisms $(x_1, \dots, x_n, 0, 1, \infty) \mapsto (x_i, 0, 1, \infty)$. Clearly $\mathcal{L}(\mathcal{W}) = \mathcal{L}(\mathcal{W}_I) = \bigoplus_{i=1}^n \mathbb{C}x_i$.

Suppose now that $a \geq 2$ and $n \geq 3$. Assume $I \subset \{n+1, n+2, n+3\}$, $j \in \{1, \dots, n\}$, and $(x_{n+1}, x_{n+2}, x_{n+3}) = (0, 1, \infty)$. Consider the linear map defined by the derivation $\frac{\partial}{\partial x_j}$ from $\mathbb{C}_a[x_1, \dots, x_n] \simeq \text{Sym}^a \Omega^1(\mathbb{C}^n, 0)$ to $\mathbb{C}_{a-1}[x_1, \dots, x_n] \simeq \text{Sym}^{a-1} \Omega^1(\mathbb{C}^n, 0)$. It induces the following diagram with exact rows where the unlabeled arrows are the natural inclusions.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{L}^a(\mathcal{W}_I^j) & \longrightarrow & \mathcal{L}^a(\mathcal{W}_I) & \xrightarrow{\frac{\partial}{\partial x_j}} & \mathcal{L}^{a-1}(\mathcal{W}_{I \cup \{j\}}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L}^a(\mathcal{W}^j) & \longrightarrow & \mathcal{L}^a(\mathcal{W}) & \xrightarrow{\frac{\partial}{\partial x_j}} & \mathcal{L}^{a-1}(\mathcal{W}_j) & \longrightarrow & 0 \end{array}$$

Notice that \mathcal{W}^j is isomorphic to $\mathcal{W}(A_{0,n+2})$ and therefore, by induction hypothesis, the leftmost vertical arrow is an isomorphism. The induction hypothesis also implies that the vector spaces $\mathcal{L}^{a-1}(\mathcal{W}_j)$ and $\mathcal{L}^{a-1}(\mathcal{W}_{I \cup \{j\}})$ are both isomorphic to $\mathcal{L}^{a-1}(\mathcal{W})$. Thus the rightmost

arrow is also an isomorphism. It follows that the middle vertical arrow is also an isomorphism and has dimension given by

$$\ell^a(\mathcal{W}) = \binom{(n-1)+a-1}{(n-1)-1} + \binom{n+(a-1)-1}{n-1} = \binom{n+a-1}{n-1}.$$

The proposition follows. Q.E.D.

As a consequence we obtain that for every $n \geq 2$ the webs $\mathcal{W}(\mathbf{A}_{0,n+3})$ are ordinary webs.

Corollary 4.1. For every $n \geq 2$ the inequality

$$\text{rank}(\mathcal{W}(\mathbf{A}_{0,n+3})) \leq 3 \binom{n+3}{4} - \binom{n+2}{3} - \binom{n+1}{2} - n.$$

holds true.

Proof. It suffices to combine Propositions 2.1 and 4.1. Q.E.D.

4.2. Lower bound for the rank

As there are exactly $\binom{n}{2} + 2n + 1$ hyperplanes in the (projective) arrangement $\mathbf{A}_{0,n+3}$, $h^1(\mathfrak{M}_{0,n+3}) = \binom{n}{2} + 2n$. Notice also that $h^1(\mathfrak{M}_{0,4}) = 2$. Since $k(\mathbf{A}_{0,n+3}) = k_2(\mathbf{A}_{0,n+3})$, Corollary 3.1 implies

$$(5) \quad \begin{aligned} \dim \text{Log}^1 \mathcal{W}(\mathbf{A}_{0,n+3}) &\geq 2 \binom{n+3}{4} - h^1(\mathfrak{M}_{0,n+3}) \\ &= 2 \binom{n+3}{4} - \binom{n}{2} + 2n. \end{aligned}$$

Similarly,

$$\dim \text{Log}^2 \mathcal{W}(\mathbf{A}_{0,n+3}) \geq \binom{n+3}{4} \cdot h^1(\mathfrak{M}_{0,4})^2 - \dim N^2(\mathfrak{M}_{0,n+3}).$$

It is a result of Arnold [1] that the Poincar3 polynomial of $\mathfrak{M}_{0,n+3}$ is $P(t) = (1+2t)(1+3t) \cdots (1+(n+1)t)$. Therefore the dimension of $N^2(\mathfrak{M}_{0,n+3})$ is equal to

$$h^1(\mathfrak{M}_{0,n+3})^2 - h^2(\mathfrak{M}_{0,n+3}) = \left(\binom{n}{2} + 2n \right)^2 - P''(0)/2.$$

Consequently

$$(6) \quad \dim AR_{\text{log}}^2(\mathcal{A}_{0,n+3}) \geq 4 \binom{n+3}{4} - h^1(M_{0,n+3})^2 + h^2(M_{0,n+3}).$$

4.3. Proof of Theorem 4.1

It is not hard to prove by induction that summing the righthand side of the inequalities (5) and (6) one obtains $3\binom{n+3}{4} - \binom{n+2}{3} - \binom{n+1}{2} - n$. Therefore, Proposition 3.2 implies that

$$\begin{aligned} \text{rank}(\mathcal{W}(\mathcal{A}_{0,n+3})) &\geq \dim AR_{\log}^1(\mathcal{A}_{0,n+3}) + \dim AR_{\log}^2(\mathcal{A}_{0,n+3}) \\ &\geq 3\binom{n+3}{4} - \binom{n+2}{3} - \binom{n+1}{2} - n. \end{aligned}$$

Combining this lower bound with the upper bound given in Corollary 4.1 concludes the proof of Theorem 4.1. Q.E.D.

As the webs $\mathcal{W}(\mathcal{A}_{0,n+3})$ attains Cavalier-Lehmann's bound, it is natural to ask if they are algebraizable or, more generally, linearizable. As the leaves of algebraic webs are contained hypersurfaces, every algebraizable web is linearizable but the converse is not always true. Pirio in [22] proved that the webs are not $\mathcal{W}(\mathcal{A}_{0,n+3})$ linearizable. We refer to this work and references therein for an in depth discussion about linearization issues for webs.

§5. Examples

We start with a simple example. Consider \mathbf{A}_0 as the arrangement of 9 lines on \mathbb{P}^2 with affine trace presented in Figure 2. If we suppose that the triple point is at the origin of \mathbb{C}^2 then pencil of lines through it corresponds to an irreducible component of the resonance variety of dimension two. There are other two triple points at the line at infinity and they also correspond to irreducible components of the resonance variety of dimension two. There are no other irreducible components of $R^1(\mathbf{A}_0)$. The resonance web $\mathcal{W}(\mathbf{A}_0)$ is the 3-web determined by the superposition of the foliations given by the level sets of the functions $x, y, x/y$. It clearly has rank one as

$$d \log x - d \log y - d \log \frac{x}{y} = 0,$$

but $\dim \text{Log}^\infty \mathcal{W}(\mathbf{A}_0) = 0$ as one can promptly verify.

5.1. More polylogarithmic abelian relations

Given an arrangement of hyperplanes \mathbf{A} on \mathbb{P}^n we define the **resonance closure** of \mathbf{A} , denoted by $\overline{\mathbf{A}}$, as the arrangement of hypersurfaces on \mathbb{P}^n characterized by the following property: $H \in \overline{\mathbf{A}}$ if and only $H \in \mathbf{A}$, or there exists two distinct irreducible components Σ_1, Σ_2 of $R^1(\mathbf{A})$ such that $\dim f_{\Sigma_1}(H) = \dim f_{\Sigma_2}(H) = 0$. In other words either H belongs to

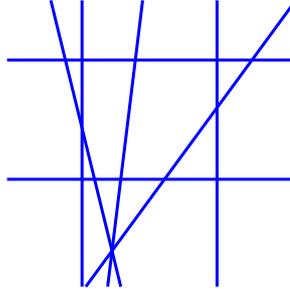


Fig. 2. $1 = \text{rank}(\mathcal{W}(A)) < \dim \text{Log}^\infty \mathcal{W}(A) = 0$.

the original arrangement A or it is invariant by two distinct foliations of the web $\mathcal{W}(A)$. The complement of \overline{A} will be denoted by \underline{M} .

Example 5.1. The resonance closure of the example A_0 is obtained by adding the lines $\{x = 0\}$ and $\{y = 0\}$. A more interesting (family of) examples is obtained by considering the (family of) arrangement(s) K_5 determine by the 10 lines joining 5 points in \mathbb{P}^2 in general position. It can be verified that the resonance variety of K_5 has 10 irreducible components: 5 of dimension 3 corresponding to pencils the lines through the points, and 5 of dimension 2 corresponding to the pencils of conics through 4 of the 5 points. If C denotes the conic through the 5 points then $\overline{K_5} = K_5 \cup \{C\}$.

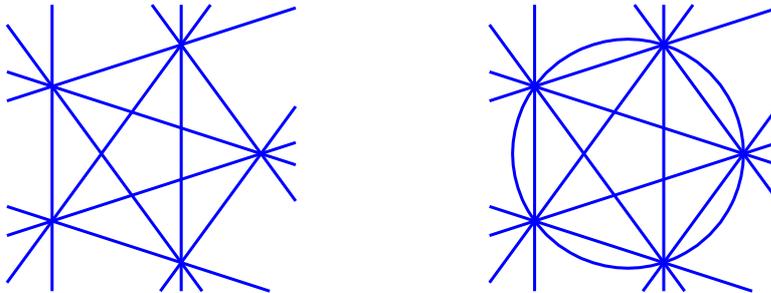


Fig. 3. On the left K_5 , on the right $\overline{K_5}$. The line at infinity is not included in the arrangements above.

These definitions are relevant for us because of the following remark.

Proposition 5.1. *If $\mathcal{W}(A)$ is the resonance web of an arrangement A then every germ of abelian relation of $\mathcal{W}(A)$ extends to the universal covering of \underline{M} .*

Proof. Let $\Lambda = \Lambda(\mathcal{W}) \subset \mathbb{P}^n$ be the subvariety given by [24, Theorem 1.2.2] that satisfies: for every $x \in \mathbb{P}^n \setminus \Lambda$ the germ of web \mathcal{W}_x obtained by localizing $\mathcal{W}(A)$ at x is a germ of codimension one web and the rank of \mathcal{W}_x is independent of $x \in \mathbb{P}^n \setminus \Lambda$.

Suppose there exists an irreducible component V of Λ which is not contained in \overline{A} and let $U \subset \mathbb{P}^n$ be a small neighborhood of a generic point $p \in V$. Let q be a point in U close to p and consider γ a loop based q generating $\pi_1(U \setminus V, q) \simeq \mathbb{Z}$. If $(\eta_1, \dots, \eta_k) \in \mathcal{A}(\mathcal{W}_q)$ is a germ of abelian relation then the action of γ on it is trivial. Indeed, according to our hypothesis V is invariant by at most one of the foliations determining the web, say \mathcal{F}_1 . Thus for every $i = 1 \dots k(A)$, the leaf of \mathcal{F}_i through p is transverse to V , and consequently γ is homotopic to a path contained the leaf \mathcal{F}_i through q . As analytic continuation along paths contained in leaves leaves unchanged the corresponding 1-form, the analytic continuation along γ of (η_1, \dots, η_k) yields $(\tilde{\eta}_1, \eta_2, \dots, \eta_k)$ for some 1-form $\tilde{\eta}_1$. Since these 1-forms sum up to zero, $\eta_1 = \tilde{\eta}_1$. Thus any abelian relation defined on a neighborhood of q extends to $U \setminus V$. A standard argument in foliation theory, shows that holomorphic first integrals defined at the neighborhood of any point of V extends naturally through non invariant subvarieties. Hence the closed 1-forms η_2, \dots, η_k extend to the whole neighborhood U . The same will hold for η_1 because $\eta_1 = -\sum_{i=2}^k \eta_i$. Q.E.D.

If Σ is an irreducible component of $R^1(A)$ we define its closure $\overline{\Sigma}$ as the maximal isotropic subspace of $H^1(\underline{M})$ containing the image of Σ under the natural inclusion $H^1(M) \rightarrow H^1(\underline{M})$. The **closure of the resonance variety** of A is the subvariety of $H^1(\underline{M})$ defined as

$$\overline{R^1(A)} = \bigcup_{\Sigma \subset R^1(A)} \overline{\Sigma}.$$

Notice that $\overline{R^1(A)} \subset R^1(\overline{A})$ but the equality does not hold in general.

To each irreducible component $\overline{\Sigma} \subset \overline{R^1(A)}$ there is a set of points of \mathbb{P}^1 , $P_{\overline{\Sigma}} \supset P_{\Sigma}$ with complement $\underline{C}_{\overline{\Sigma}}$ satisfying $(f_{\Sigma})^* H^1(\underline{C}_{\overline{\Sigma}}) = \overline{\Sigma}$. Analogously to what have been done in Sections 3.1 and 3.3, we can

consider

$$\begin{aligned} \overline{\Psi}_i : \bigoplus_{\underline{\Sigma}} H^1(\underline{C}_{\Sigma})^{\otimes i} &\longrightarrow H^1(\underline{M})^{\otimes i} \\ (\eta_{\Sigma}) &\mapsto \sum f_{\Sigma}^* \eta_{\Sigma}. \end{aligned}$$

and define, for every $i \in \mathbb{N}$, $\overline{\text{Log}^i \mathcal{W}(\mathbf{A})} = \ker \overline{\Psi}_i$. We also define $\overline{\text{Log}^{\infty} \mathcal{W}(\mathbf{A})}$ as the direct sum $\bigoplus_i \overline{\text{Log}^i \mathcal{W}(\mathbf{A})}$.

Of course, we also have a Proposition 3.2:

Proposition 5.2. *If we consider the localization of the web $\mathcal{W}(\mathbf{A})$ at a generic point of M then the vector space $\overline{\text{Log}^{\infty} \mathcal{W}(\mathbf{A})}$ embeds into the space of abelian relations of germ $\mathcal{W}(\mathbf{A})$. Moreover, the analytic continuation of this embedding gives rise to a local system of abelian relations globally defined on \underline{M} .*

Of course, we can write lower bounds for $\overline{\text{Log}^i \mathcal{W}(\mathbf{A})}$ analogous to the ones given by Corollary 3.1.

If $\mathbf{A} = \{C_1, \dots, C_r\}$ is an arrangement of curves on \mathbb{P}^2 it is still a simple matter to determine the dimension of the second cohomology group of the complement. If $t_i(\mathbf{A})$ denotes the number of singular points in the support of \mathbf{A} through which there exactly i branches, and $\chi(\widetilde{C}_i)$ ($i = 1, \dots, r$) are the Euler characteristic of the normalizations of the curves in the arrangement then [10, Proposition 2.4],

$$(7) \quad h^1(M) = r - 1 \quad \text{and} \quad h^2(M) = 1 + \sum_{i=2}^{\infty} (i-1)t_i(\mathbf{A}) - \sum_{i=1}^r (\chi(\widetilde{C}_i) - 1).$$

We can use these observations (together with some computer algebra) to recover the following (unpublished) joint result of Pirio and Robert.

Theorem 5.1. *The resonance web of K_5 is exceptional.*

Proof. Notice that the closures of the irreducible components of $\mathbb{R}^1(K_5)$ all have dimension 3. As $h^1(\underline{M}) = 10$, we obtain the lower bound

$$\overline{\text{Log}^1 \mathcal{W}(K_5)} \geq 10 \cdot 3 - h^1(\overline{M}) = 30 - 10 = 20.$$

To control $\overline{\text{Log}^2 \mathcal{W}(K_5)}$, we need to know the dimensions of $N^2(\underline{M}) = \ker\{H^1(\underline{M})^{\otimes 2} \rightarrow H^2(\underline{M})\}$. For an arbitrary arrangement of curves the cohomology algebra of the complement has no reason to be generated in degree 1, but for $\underline{M} = \mathbb{P}^2 \setminus K_5$, [10, Theorem 2.46] implies it is the case. Thus

$$\overline{\text{Log}^2 \mathcal{W}(K_5)} \geq 10 \cdot 9 - h^1(\overline{M})^2 + h^2(\overline{M}) = 90 - 100 + 25 = 15.$$

The inequalities above turn out to be equalities. To determine the dimension of $\overline{\text{Log}}^3\mathcal{W}(\mathcal{K}_5)$ one has *just* to compute the dimension of the kernel of a 3375×243 matrix. A brute-force calculation¹ shows that

$$\dim \overline{\text{Log}}^3\mathcal{W}(\mathcal{K}_5) = 1.$$

Thus $\mathcal{W}(\mathcal{K}_5)$ attains Bol's bound $9 \cdot 8/2 = 36$. To prove it is non-algebraizable it suffices to apply [19, Proposition 2.1]. Q.E.D.

Notice that \mathcal{K}_5 is indeed a 2-parameter family of 10-webs. Although the combinatorics is fixed, the moduli space of isomorphisms classes of 5 point on \mathbb{P}^2 has dimension 2.

5.2. Rational abelian relations

The inclusion of $\overline{\text{Log}}^\infty\mathcal{W}(\mathcal{A})$ into $\mathcal{A}(\mathcal{W}(\mathcal{A}))$ does not exhaust the space of abelian relations of $\mathcal{W}(\mathcal{A})$ in general, unlike when $\mathcal{A} = \mathcal{A}_{0,5}$ or $\mathcal{A} = \mathcal{K}_5$. The simplest example is when \mathcal{A} is an arrangement of 9 lines with 3 aligned threefold intersection points and all the other intersections are ordinary. In this case the resonance variety has only three local components, each of them having dimension two. Supposing that these three points are $(0 : 1 : 0)$, $(1 : 0 : 0)$, and $(1 : 1 : 0)$ then the corresponding foliations are defined by the 1-forms dx , dy , and $dx + dy$. As they satisfy $(dx) + (dy) - (dx + dy) = 0$, it is clear that $\mathcal{W}(\mathcal{A})$ has rank one but $\dim \overline{\text{Log}}^\infty\mathcal{W}(\mathcal{A}) = 0$.

Define $\overline{\text{Rat}}\mathcal{W}(\mathcal{A})$ as the kernel of the linear map

$$\begin{aligned} \overline{\Upsilon} : \bigoplus_{\Sigma} \frac{\mathbb{C}(\mathbb{P}^1)}{\mathbb{C}} &\longrightarrow \frac{\mathbb{C}(\mathbb{P}^n)}{\mathbb{C}} \\ (g_{\Sigma}) &\mapsto \sum f_{\Sigma}^*(g_{\Sigma}). \end{aligned}$$

where $\mathbb{C}(\mathbb{P}^n)$ stands for the field of rational functions on \mathbb{P}^n , and the summation Σ run over all the irreducible components of $\mathcal{R}^1(\mathcal{A})$. Coordinate-wise differentiation injects $\ker \overline{\Upsilon}$ into $\mathcal{A}(\mathcal{W}(\mathcal{A}))$. Notice that its image intersects $\overline{\text{Log}}^\infty\mathcal{W}(\mathcal{A})$ only at zero. Therefore, we have the following lower bound for the rank of $\mathcal{W}(\mathcal{A})$

$$(8) \quad \text{rank}(\mathcal{W}(\mathcal{A})) \geq \dim \overline{\text{Log}}^\infty\mathcal{W}(\mathcal{A}) + \dim \overline{\text{Rat}}\mathcal{W}(\mathcal{A}).$$

We do not know how to give general lower bounds for $\dim \overline{\text{Rat}}\mathcal{W}(\mathcal{A})$. We have only the following simple result.

¹Maple script available at www.impa.br/~jvp/artigos.html.

Lemma 5.1. *Let A be a planar arrangement, and let $\{\ell_1, \dots, \ell_m\}$ be the lines in its closure. If n_i is the number of local irreducible components $\overline{\Sigma} \subset \overline{R^1(A)}$ containing ℓ_i in its support then*

$$\dim \overline{\text{Rat}}\mathcal{W}(A) \geq \sum_{i=1}^n \frac{(n_i - 1)(n_i - 2)}{2}.$$

Proof. Suppose that ℓ_1 is the line at infinity. The foliations associated to the n_1 components of $\overline{R^1(A)}$ containing ℓ_1 are defined by dh_1, \dots, dh_{n_1} where h_i is a linear form. To prove the lemma it suffices to observe that for $p \geq 1$, the kernel of the maps

$$(a_1, \dots, a_{n_1}) \mapsto \sum_{i=0}^{n_1} a_i (h_i)^p$$

will correspond to rational abelian relations with polar set contained in ℓ_1 . Q.E.D.

Of course one can do better by considering rational functions with poles on fibers of f_Σ for non-local components of $R^1(A)$.

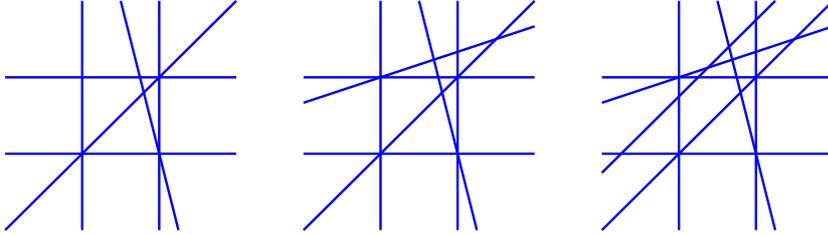


Fig. 4. Arrangements B_6, B_7 , and B_8 .

Example 5.2. Let B_{5+n} , $n = 1, 2, 3$, be the arrangements obtained from $A_{0,5}$ by adding n generic lines, each through a distinct double point, see Figure 4. The resonance webs are the webs B_6, B_7 and B_8 considered by Robert and Pirio, and proved to be exceptional by them.

The rank of B_6 can be computed easily as follows. As it contains $A_{0,5}$ it has rank at least 6. Adding the pencil of lines through a double point of the support of $A_{0,5}$ we have that $\overline{B_6} = B_6$ contain two lines with three triple points, thus Lemma 5.1 implies $\dim \overline{\text{Rat}}\mathcal{W}(B_6) \geq 2$. The proof of Theorem 4.1 tell us that $\Psi_1 : \oplus H^1(C_\Sigma) \rightarrow H^1(M)$ and $\Psi_2 : \oplus H^1(C_\Sigma)^{\otimes 2} \rightarrow H^1(M)^{\otimes 2}$, with $\Sigma \subset R^1(A_{0,5})$, are both surjective.

If h_1, h_2 are the equations of the lines intersecting at the double point under consideration then $d \log h_1 - d \log h_2$ and $(d \log h_1 - d \log h_2)^{\otimes 2}$ belong (respectively) to the image of Ψ_1 and Ψ_2 . Thus

$$\dim \text{Log}^\infty \mathcal{W}(\mathcal{B}_6) \geq \dim \text{Log}^\infty \mathcal{W}(\mathcal{A}_{0,5}) + 2.$$

Putting all together we deduce that $\text{rank } \mathcal{W}(\mathcal{B}_6) \geq 6 + 2 + 2 = 10$. Thus $\mathcal{B}_6 = \mathcal{W}(\mathcal{B}_6)$ is of maximal rank as it attains Bol's bound.

One can deal similarly with \mathcal{B}_7 and \mathcal{B}_8 but Lemma 5.1 does not suffice. One has to consider also rational first integrals for the foliation associated to the non-local component of $R^1(\mathcal{B}_6)$ with poles on one (for \mathcal{B}_7) or three (for \mathcal{B}_8) fibers.

For general arrangements of lines, the inclusion

$$\overline{\text{Log}^\infty} \mathcal{W}(\mathcal{A}) \oplus \overline{\text{Rat}^\infty} \mathcal{W}(\mathcal{A}) \subset \mathcal{A}(\mathcal{W}(\mathcal{A}))$$

is strict. In the next two sections we will consider two examples of line arrangements on \mathbb{P}^2 with resonance webs having abelian relations which support this claim.

5.3. Mixed abelian relations

In the same way that we looked for abelian relations among collections of iterated integrals of logarithmic 1-forms we can also look for them at collections of iterated integrals of arbitrary rational 1-forms. In this section we will consider an one-parameter family of arrangements with resonance webs having abelian relations of this form.

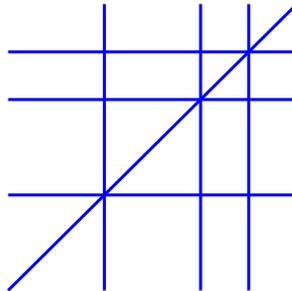


Fig. 5. The arrangement P.

For $\lambda \in \mathbb{C} \setminus \{0, 1\}$, let $P = P_\lambda$ be the arrangement of 8 lines obtained from $\mathcal{A}_{0,5} = \{xyz(x-y)(x-z)(y-z) = 0\}$ by adding the two extra lines

$\{(x-\lambda z)(y-\lambda z) = 0$, see Figure 5. Its resonance variety has 8 irreducible components: two local of dimension three; three local of dimension two; and three non-local of dimension two determined by pencils of conics. It is closed arrangement in the sense that $\mathbf{P} = \overline{\mathbf{P}}$.

From Corollary 3.1 we see that

$$\dim \text{Log}^1 \mathcal{W}(\mathbf{P}) \geq 11 \text{ and } \dim \text{Log}^2 \mathcal{W}(\mathbf{P}) \geq 5.$$

This inequalities are indeed equalities. Moreover, one can verify that $\dim \text{Rat } \mathcal{W}(\mathbf{P}) = 4$ given us at least a 20-dimensional subspace of the space of abelian relations.

There is still one extra abelian relation, relation G_{21}^a in Section 3.3 of [23], involving the pull-backs under f_Σ of

$$d \left(\frac{\log(x)}{1-x} \right) = -d \log(x-1) + d \log(x) + \frac{\log(x)}{(1-x)^2} dx.$$

The last summand is neither the differential of a rational function, nor an iterated integral of logarithmic 1-forms. Indeed it can be written as $\frac{dx}{x} \otimes \frac{dx}{(1-x)^2}$, and therefore is an iterated integral of rational 1-forms.

5.4. Twisted logarithmic abelian relations

Of course the class of abelian relations with components being iterated integrals of rational 1-forms encompass all previous classes of abelian relations considered. Note that if we consider all the abelian relations in this class we obtain an unipotent local system over the complement of the closure of the arrangement. We believe that the maximal unipotent local system in $\mathcal{A}(\mathcal{W}(\mathbf{A}))$ is exactly the one generated by the abelian relations given by iterated integrals of rational 1-forms.

Our last example, shows that the local system $\mathcal{A}(\mathcal{W}(\mathbf{A}))$ is not in general unipotent. Let \mathbf{F} be the non-Fano arrangement presented in Figure 6. It is a closed arrangement ($\mathbf{F} = \overline{\mathbf{F}}$) and its resonance variety has 9 irreducible components: six of them are local of dimension two, and three of them are determined pencil of conics and also have dimension two. The resonance web $\mathcal{W}(\mathbf{F})$ was studied independently by Piro [23] and Robert [25]. They proved that $\mathcal{W}(\mathbf{F})$ has maximal rank.

Corollary 3.1 implies

$$\dim \text{Log}^1 \mathcal{W}(\mathbf{F}) \geq 12 \text{ and } \dim \text{Log}^2 \mathcal{W}(\mathbf{F}) \geq 9,$$

and Lemma 5.1 implies $\dim \text{Rat } \mathcal{W}(\mathbf{F}) \geq 4$. These bounds are indeed equalities. Moreover, brute-force computation yields $\dim \text{Log}^3 \mathcal{W}(\mathbf{F}) = 2$. Thus we have a 27-dimensional subspace of the space of abelian relations.

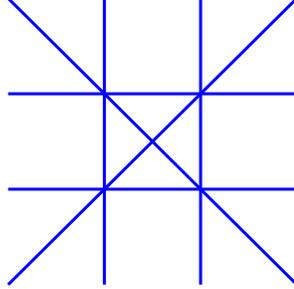


Fig. 6. The non-Fano arrangement.

The missing abelian relation comes from the intersection of the irreducible components of $\text{Char}^1(M)$ determined by the three non-local components. In order to explain this abelian relation we will digress a little.

§

Let A be an arrangement of hypersurfaces on \mathbb{P}^n and M be its complement. Recall from the introduction the morphism

$$\begin{aligned} \exp : H^1(M) &\longrightarrow \text{Hom}(\pi_1(M), \mathbb{C}^*) \\ a &\longmapsto \left(\gamma \mapsto \exp \left(2\pi i \int_{\gamma} a \right) \right). \end{aligned}$$

For a 1-form $\omega \in H^1(M)$, let $\varrho_{\omega} : \pi_1(M) \rightarrow \mathbb{C}^*$ be the representation $\exp(\omega)$ and \mathbb{C}_{ω} the corresponding rank one local system.

The \mathbb{C} -sheaf \mathbb{C}_{ω} admits the following resolution

$$0 \longrightarrow \mathbb{C}_{\omega} \longrightarrow \mathcal{O}_M \xrightarrow{\nabla_{\omega}} \Omega^1(M) \xrightarrow{\nabla_{\omega}} \Omega^2(M) \xrightarrow{\nabla_{\omega}} \cdots \longrightarrow 0$$

where $\Omega^{\bullet}(M)$ are the sheaves of holomorphic differentials on M and $\nabla_{\omega} : \Omega^{\bullet}(M) \rightarrow \Omega^{\bullet+1}(M)$ is given by the formula $\nabla_{\omega}(\alpha) = d\alpha - \omega \wedge \alpha$. Since M is Stein, the sheaves $\Omega^{\bullet}(M)$ are acyclic and consequently

$$H^i(M, \mathbb{C}_{\omega}) = \frac{\ker \nabla_{\omega} : H^0(M, \Omega^i(M)) \rightarrow H^0(M, \Omega^{i+1}(M))}{\nabla_{\omega}(H^0(M, \Omega^{i-1}(M)))}.$$

If $\omega_1 - \omega_2 = d \log F$ for some $F \in H^0(M, \mathcal{O}_M)$ then $\mathbb{C}_{\omega_1} \simeq \mathbb{C}_{\omega_2}$ and the corresponding resolutions relate through the diagram

$$(9) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & \mathbb{C}_{\omega_1} & \longrightarrow & \mathcal{O}_M & \xrightarrow{\nabla_{\omega_1}} & \Omega^1(M) & \xrightarrow{\nabla_{\omega_1}} & \Omega^2(M) & \xrightarrow{\nabla_{\omega_1}} & \cdots & \longrightarrow & 0 \\ & & & & \downarrow F^{-1} & & \downarrow F^{-1} & & \downarrow F^{-1} & & & & \\ 0 & \longrightarrow & \mathbb{C}_{\omega_1} & \longrightarrow & \mathcal{O}_M & \xrightarrow{\nabla_{\omega_2}} & \Omega^1(M) & \xrightarrow{\nabla_{\omega_2}} & \Omega^2(M) & \xrightarrow{\nabla_{\omega_2}} & \cdots & \longrightarrow & 0 \end{array}$$

where the vertical arrows are multiplication by F^{-1} .

For $\alpha \in \Omega^1(M)$ we have $\nabla_{\omega}(\alpha) = 0$ if and only if the (multi-valued) 1-form $\exp(\int \omega) \alpha$ is closed. Moreover, if ω belongs to some irreducible component Σ of $\mathbb{R}^1(A)$ then for every $\alpha \in \Sigma$ we have that the 1-form $\exp(\int \omega) \alpha$ is closed.

Let Σ be an irreducible component of $\mathbb{R}^1(A)$ and let $\alpha \in \Sigma$ be a logarithmic 1-form with residues in $\mathbb{Q} \setminus \mathbb{Z}_{>0}$. Let $P : N \rightarrow \mathbb{P}^n$ be the finite abelian covering determined by \mathbb{C}_{α} . Since the monodromy is finite (residues $\in \mathbb{Q}$), N is a quasi-projective variety. I will write N for a smooth

Lemma 5.2. *If $\beta \in \Sigma$ is not a complex multiple of α then $\exp(-\int P^* \alpha) P^* \beta$ is a closed rational 1-form on N which is not exact.*

Proof. Let $f = f_{\Sigma} : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$ be the associate rational map. Recall that for $\alpha, \beta \in \Sigma$ there exist logarithmic 1-forms $\tilde{\alpha}, \tilde{\beta}$ on \mathbb{P}^1 with poles of \mathbb{P}_{Σ} such that $\beta = f_i^* \tilde{\beta}$.

If $p : C \rightarrow \mathbb{P}^1 \setminus \mathbb{P}_{\Sigma}$ is the finite ramified covering determined by $\exp(-\int \tilde{\alpha})$ then it fits in the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{P} & M \\ \tilde{f} \downarrow & & \downarrow f \\ C & \xrightarrow{p} & \mathbb{P}^1 \setminus \mathbb{P}_{\Sigma} \end{array}$$

Clearly $\exp(-\int P^* \alpha) P^* \beta$ is equal to $\tilde{f}^* \exp(-\int p^* \tilde{\alpha}) \tilde{\beta}$. Thus if the former is exact, the same holds for the latter.

Suppose $\exp(-\int p^* \tilde{\alpha}) \tilde{\beta} = dg$ for some rational function on C . The rational function $\tilde{h} = \exp(\int p^* \tilde{\alpha}) g$ is invariant under the covering transformation, thus is equal to $p^* h$ for some rational function on \mathbb{P}^1 . It is a simple matter to verify that $\nabla_{\alpha}(h) = \beta$, and therefore β represents

the zero class in $H^1(\mathbb{P}^1 \setminus P_\Sigma, \mathbb{C}_\alpha)$. But the main result of [13] implies that the complex $(\Omega^\bullet(\mathbb{P}^1 \setminus P_\Sigma), \nabla_\alpha)$ is quasi-isomorphic to the complex $(H^\bullet(\mathbb{P}^1 \setminus P_\sigma, \mathbb{C}), \wedge \alpha)$. Hence the class of β is not zero. Q.E.D.

Proposition 5.3. *Let A be an arrangement of hyperplanes on \mathbb{P}^n and let $\Sigma_1, \dots, \Sigma_r$ be irreducible components of the resonance variety $R^1(A)$. If $\exp(\Sigma_1), \dots, \exp(\Sigma_r)$ intersect at some $\rho \in \text{Hom}(\pi_1(M), \mathbb{C}^*)$ distinct from the trivial representation. If N , the ramified covering of \mathbb{P}^n determined by ρ , is simply-connected then*

$$\dim \frac{\mathcal{A}(\mathcal{W}(A))}{\text{Log}^\infty \mathcal{W}(A) \oplus \text{Rat} \mathcal{W}(A)} \geq -h^1(M, \mathbb{C}_\omega) + \sum_{i=1}^r (\dim \Sigma_i - 1).$$

Proof. Since ω is in the intersection of $\exp(\Sigma_1), \dots, \exp(\Sigma_r)$ there exist non-zero logarithmic 1-forms $\alpha_i \in \Sigma_i$ ($i = 1, \dots, r$) and rational functions $f_{ij} \in \mathcal{O}(M)$ ($i, j = 1, \dots, r$) satisfying

$$\alpha_i - \alpha_j = d \log f_{ij}.$$

In particular, \mathbb{C}_{α_i} is isomorphic to \mathbb{C}_{α_j} for every i, j . It is harmless to assume that all the 1-forms α_i have non-integer residues and $\omega = \alpha_1$.

Since $\Sigma_i \subset \ker \nabla_{\alpha_i}$ it follows from (9) that $(f_{ij})^{-1} \cdot \Sigma_j \in \ker \nabla_{\alpha_i}$ for every i, j . Therefore we can define the map

$$\Lambda : \bigoplus_{i=1}^r \frac{\Sigma_i}{\mathbb{C} \cdot \alpha_i} \longrightarrow H^1(M, \mathbb{C}_{\alpha_1})$$

$$(\beta_i) \mapsto \beta_1 + \sum_{i=2}^r (f_{1i})^{-1} \beta_i.$$

Notice that as we explained before the (multi-valued) 1-forms

$$\exp\left(-\int \alpha_1\right) (f_{1i})^{-1} \beta_i = \exp\left(-\int \alpha_i\right) \beta_i$$

are closed.

If $(\beta_1, \dots, \beta_r)$ belongs to $\ker \Lambda$ then there exists $g \in H^0(M, \mathcal{O}_M)$ for which

$$\beta_1 + \sum_{i=2}^r (f_{1i})^{-1} \beta_i = dg - g \alpha_1$$

or, equivalently, for suitable choices of branches of $\exp(\int -\alpha_i)$ we have that

$$\sum_{i=1}^r \exp\left(\int -\alpha_i\right) (f_{1i})^{-1} \beta_i = d\left(\exp\left(\int \alpha_1\right) g\right)$$

If we pull-back this equation to N using the ramified covering $P : N \rightarrow \mathbb{P}^n$ then all the 1-forms involved are legitimate (univalued) closed rational 1-forms. Since N is simply-connected, the pull-backs \widetilde{f}_Σ of the maps f_Σ have as target rational curves. Thus the 1-forms $\exp(f - \alpha_1)(f_{1i})^{-1}\beta_i$ can be uniquely written as the pull-back by \widetilde{f}_Σ of the sum of an exact rational differential with an logarithmic 1-form. Discarding the rational component, one obtains an identity as above but with zero righthand side. Clearly it is an abelian relation. The linear independence of the corresponding abelian relations for $(\beta_1, \dots, \beta_r)$ varying on a basis of $\ker \Lambda$ follows from Lemma 5.2. Q.E.D.

We do not know if the hypothesis made on the topology of N is necessary to prove the proposition above.

§

Back to the analysis of the example, we have that the exponential of the three non-local components intersect at a representation ρ for which $h^2(M, \mathbb{C}_\rho) = 2$. Moreover, the representation also satisfies the hypothesis of Proposition 5.3. Thus it ensures the existence of the *missing* abelian relation.

In the table below we present the dimensions of the subspaces of the space of abelian relations of the examples studied in this section.

	k	Log^1	$\overline{\text{Log}}^1$	Log^2	$\overline{\text{Log}}^2$	Log^3	$\overline{\text{Log}}^3$	Rat	Mix	Twist
$A_{0,5}$	5	5	5	1	1	0	0	0	0	0
B_6	6	6	6	2	2	0	0	2	0	0
B_7	7	7	7	3	3	0	0	5	0	0
B_8	8	8	8	4	4	0	0	9	0	0
P	8	11	11	5	5	0	0	4	1	0
F	9	12	12	9	9	2	2	4	0	1
K_5	10	16	20	5	15	0	1	0	0	0

§6. Final remarks and some problems

Although we have studied resonance webs for hyperplane arrangements one can study resonance webs for arbitrary hypersurfaces arrangements on \mathbb{P}^n . Even more generally, if one replaces the cohomology algebra $H^\bullet(M)$ by a finite dimensional algebra of differential forms on a quasi-projective variety then one can still talk about resonance varieties. Its irreducible components are still in correspondence with rational maps

to projective curves [3], and consequently one can still define the resonance webs. We are not aware of any exceptional web arising this way that are not pull-backs by rational maps of resonance webs of hyperplane arrangements.

Problem 1. Let S be a projective surface and consider the algebra of holomorphic differentials over it. Are there examples of projective surfaces S for which the corresponding resonance web is of maximal rank ?

§

So far, all the known examples of exceptional webs are global webs on suitable projective surfaces and their local system of abelian relations have quasi-unipotent monodromy. There is even a conjecture spelled out by Hénaut in [16] that this should always be the case. The case of resonance varieties provide a nice toy model for this problem.

Problem 2. Suppose the exponential of the closures of the irreducible components of resonance varieties of an hyperplane arrangement do not intersect except at the trivial representation. Is it true that the monodromy of the local system of abelian relations is unipotent ? Is the local system of abelian relations generated by iterated integrals of rational 1-forms ? More ambitiously, does the iterated integrals of rational 1-forms and of twisted rational 1-forms suffice to exhaust the space of abelian relations ?

§

As there are very few results [18, 19] about the classification of exceptional planar webs, it seems natural to propose the following.

Problem 3. Classify the line arrangements having exceptional resonance webs.

We have no idea how hard this problem is. We do not even know if there is a bound on the number of irreducible components of the resonance variety when the resonance web is maximal.

§

The last problem below has been proposed to us by Pirio.

Problem 4. It is interesting to compare our Theorem 4.1 with Damiano's determination of the rank of the (dimension one) web given by the $(n + 3)$ maps [12]

$$\mathfrak{M}_{0,n+3} \longrightarrow \mathfrak{M}_{0,n+2}.$$

These turn out to attain the corresponding bound for the rank of one-dimensional webs and are also non-linearizable. One can of course ask for the determination of the rank(s) of the codimension q webs defined by the maps

$$\mathfrak{M}_{0,n+3} \longrightarrow \mathfrak{M}_{0,q+3}.$$

For the definitions of webs of arbitrary codimension and their ranks see [9] and [17].

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