GERMS OF INTEGRABLE FORMS AND VARIETIES OF MINIMAL DEGREE

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ABSTRACT. We study the subvariety of integrable 1-forms in a finite dimensional vector space $W \subset \Omega^1(\mathbb{C}^n, 0)$. We prove that the irreducible components with dimension comparable with the rank of W are of minimal degree.

1. INTRODUCTION

Let $(\mathbb{C}^n, 0)$ be the germ of $\mathbb{C}^n$ at the origin. For $q \in \{0, ..., n\}$, $\Omega^q(\mathbb{C}^n, 0)$ will stand for the space of germs of holomorphic q-differential forms at $0 \in \mathbb{C}^n$.

In this work we are interested in describing the intersection of the set of integrable 1-forms in $\Omega^1(\mathbb{C}^n, 0)$ with a finite dimensional vector space $W \subset \Omega^1(\mathbb{C}^n, 0)$. In more concrete terms, our main objects of study are the projective varieties

$$I_W = \left\{ [\omega] \in \mathbb{P}(W) \,|\, \omega \wedge d\omega = 0 \right\}$$

where W is as above and $\mathbb{P}(W)$ is the space of complex lines in W.

Our motivation steams from the study of the irreducible components of the space of foliations on $\mathbb{P}^n$, see [5] and references therein. In the existing literature the usual approach to study the space of foliations on $\mathbb{P}^n$ passes through the recognition of distinguishing features of some classes of foliations, and the proof of the stability of these features under small deformations. In this note, instead of looking at the foliations we focus directly on the defining equations of I_W. For that sake we make use of a simple idea presented in [2] reminiscent of Steiner's construction of rational normal curves, see Section 2.

In order to state our main results we need first to introduce the **rank** of a finite vector space $W \subset \Omega^1(\mathbb{C}^n, 0)$. By definition, rank$(W)$ is the greatest integer r for which the natural map

$$\bigwedge^r W \longrightarrow \Omega^r(\mathbb{C}^n, 0)$$

is not the zero map. Notice that $\mathrm{rank}(W) \leq \min(\dim W, n)$.

Theorem 1. *Let $W \subset \Omega^1(\mathbb{C}^n, 0)$ be a finite dimensional vector space and let Σ be an irreducible component of I_W. If the codimension of Σ in $\mathbb{P}(W)$ is at most* $\mathrm{rank}(W) - 2$ *then Σ is a variety of minimal degree.*

Recall that a variety is of minimal degree if its degree exceeds by one its codimension in its linear span, that is

$$X \text{ is of minimal degree} \iff \deg X = \dim \mathrm{Span}(X) - \dim X + 1 \,.$$

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They are well understood, and in particular are completely classified (see [7] and references therein). Any variety of minimal degree is either a linear subspace, a quadric hypersurface (eventually singular), a rational normal scroll, the Veronese surface in $\mathbb{P}^5$ or a cone over such a surface. When the rank and the dimension of W coincide we explore this classification to obtain the more precise result below.

Theorem 2. *If* $\operatorname{rank}(W) = \dim W = n$ *then every irreducible component of* I_W *is either a linear subspace or a rational normal curve in its linear span.*

Theorem 2 turns out to be sharp as the concrete examples in Section 4 testify. In Section 5 we characterize when a given rational normal curve of integrable 1-forms is an irreducible component of I_W using a beautiful geometric construction due to Gelfand and Zakharevich, see Corollary 5.2.

It has to be pointed out that the hypothesis on the rank is rather restrictive, and one should not expect similar results about the space of foliations on projective varieties. For example, it is well known that for a fixed integer $d \geq 1$, foliations induced by generic pencils of degree d hypersurfaces in $\mathbb{P}^n$, $n \geq 3$, spread the irreducible component $R_n(d,d)$ of the space of foliations of degree $2d-2$. Its codimension in the projective space $\mathbb{P}H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2d))$ is

$$\underbrace{(n+1)N_{2d-1} - N_{2d} + n - 1}_{\dim \mathbb{P}H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(2d))} \quad - \quad \underbrace{2N_d - 2}_{\dim R_n(d,d)} , \text{ where } N_k = \binom{n+k}{k} - 1$$

while its degree, according to [6, Section 5.1], is

$$\frac{1}{N_d - 1}\binom{2N_d - 2}{N_d} .$$

In particular, for n or d sufficiently large, it is clear that the degree is considerably greater than the codimension. It does not seem to be easy to infer properties of the degree and/or geometry of the irreducible components of the space of foliations on projective varieties from Theorem 2. Nevertheless, at the other extreme of the spectrum of compact complex manifolds, there are the manifolds of algebraic dimension zero. Recall that the algebraic dimension of compact complex manifold X, commonly denoted by $a(X)$, is the transcendence degree over $\mathbb{C}$ of its field of meromorphic functions. For this class of manifolds Theorem 2 has the following consequence.

Corollary 3. *Let X be a compact complex manifold and $\mathcal{L}$ be a line-bundle over it. If $a(X) = 0$ then the irreducible components of the space of codimension one foliations with conormal bundle $\mathcal{L}$ are either linear subspaces or rational normal curves.*

Proof. We are interested in the irreducible components of

$$\left\{ [\omega] \in \mathbb{P}H^0(X, \Omega^1_X \otimes \mathcal{L}) \,\middle|\, \omega \wedge d\omega = 0 \right\} .$$

Localizing at a generic point $x \in X$, the sections of $\Omega^1_X \otimes \mathcal{L}$ determine germs of holomorphic 1-forms that span a finite dimensional vector space W of $\Omega^1(X,x) \simeq \Omega^1(\mathbb{C}^n, 0)$ of dimension m. If $\bigwedge^m W \to \Omega^m(X,x)$ is the zero map then there exists meromorphic functions $a_1, \ldots, a_m \in \mathbb{C}(X)$ and a basis $\omega_1, \ldots, \omega_m$ of $H^0(X, \Omega^1_X \otimes \mathcal{L})$ such that $a_1\omega_1 + \ldots + a_m\omega_m = 0$. But the hypothesis $\mathbb{C}(X) = \mathbb{C}$ leads to a contradiction that implies $\dim W = \operatorname{rank}(W)$. The corollary follows from Theorem 2. $\square$

2. Rational normal curves and the proof of Theorem 1

2.1. Steiner's construction of rational normal curves.

A rational normal curve in $\mathbb{P}^n$ is nothing more than a smooth non-degenerate rational curve of degree n. Up to projective automorphisms there is only one rational normal curve in $\mathbb{P}^n$, and it can be seen as the image of natural morphism

$$\mathbb{P}^1 \longrightarrow \operatorname{Sym}^n \mathbb{P}^1 \simeq \mathbb{P}^n$$
$$p \longmapsto \underbrace{p + \cdots + p}_{n \text{ times}} \, .$$

Notice that this map is induced by the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(n)|$.

Given $n+3$ points in general position in $\mathbb{P}^n$, that is no $n+1$ points among them are contained in a hyperplane, there is a unique rational normal curve containing them. This curve can be synthetically constructed through the following procedure which can be traced back to Steiner. Let $p_1, \ldots, p_{n+3}$ be the $n+3$ points under consideration, and for i ranging from 1 to n let Π_i be the $\mathbb{P}^{n-2}$ spanned by the points $p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n$. For a fixed i there is a pencil of hyperplanes containing Π_i. The elements of this pencil can be written as $H_i(s : t) = \{sF_i + tG_i = 0\}$ where $(s : t) \in \mathbb{P}^1$ and F_i, G_i are linear forms on $\mathbb{C}^{n+1}$. These linear forms can be chosen in order that $p_{n+1} \in H_i(0 : 1)$, $p_{n+2} \in H_i(1 : 0)$ and $p_{n+3} \in H_i(1 : 1)$. It turns out that the map

$$(s : t) \longmapsto \bigcap_{i=1}^{n} H_i(s : t)$$

parameterizes the unique rational normal curve through $p_1, \ldots, p_{n+3}$. Indeed $(0 : 1)$, $(1 : 0)$ and $(1 : 1)$ are mapped to p_{n+1}, p_{n+2} and p_{n+3} respectively. Furthermore, for $i = 1, \ldots, n$, there exists one and only one hyperplane in the pencil $H_i(s : t)$ containing p_i, and p_i belongs to $H_j(s : t)$ for every $j \neq i$ and every $(s : t) \in \mathbb{P}^1$.

2.2. Rational normal curves of integrable 1-forms.

The following proposition is a rephrasing of the codimension one case of [2, Thm. 4.1]. The result, in codimension one as well as in arbitrary codimension, is originally due to Panasyuk [12] and settles a conjecture of Zakharevich [14]. In all these works rational normal curves of integrable 1-forms appear under the label of Veronese webs, a terminology introduced in [8].

Proposition 2.1. *Let $W \subset \Omega^1(\mathbb{C}^{n+1}, 0)$ be a finite dimensional vector space with $\dim W = \operatorname{rank}(W)$. If there are $\dim W + 2$ classes of integrable 1-forms in general position in $\mathbb{P}(W)$ then the unique rational normal curve through them parametrizes classes of integrable 1-forms.*

Proof. For $\dim W \leq 2$, the proposition is evident. So we will assume that $\dim W \geq 3$. Moreover, after taking generic hyperplane sections, we can also assume that $\dim W = \operatorname{rank}(W) = n + 1$. Let $p_1 = [\omega_1], \ldots, p_{n+3} = [\omega_{n+3}]$ be $n + 3$ points in general position in $\mathbb{P}(W)$. Since $\operatorname{rank}(W) = n+1$, there exist germs of meromorphic vector fields $v_1, \ldots, v_{n+1}$ satisfying

$$\omega_i(v_j) = \delta_{ij}, \qquad i, j \in \{1, \ldots, n+1\},$$

where δ_{ij} is the Kroenecker symbol. The hyperplanes in $\mathbb{P}(W)$ are in one to one correspondence with the lines in the space V generated by $v_1, \ldots, v_{n+1}$. To wit, V is a concrete realization of the dual of W.

The hyperplanes $H_i(s : t)$ containing $p_1, p_{i-1}, p_{i+1}, p_n$ are thus defined by the linear family of vector fields

$$\zeta_i(s, t) = s(av_i + bv_{n+1}) + t(cv_i + dv_{n+1})$$

where a, b, c, d are complex numbers satisfying $ad - bc \neq 0$. Hence the unique rational normal curve through $[\omega_1], \ldots, [\omega_{n+3}]$ is parametrized by

$$\langle \zeta_1(s, t) \wedge \cdots \wedge \zeta_n(s, t) , \, \omega_1 \wedge \ldots \wedge \omega_{n+1} \rangle .$$

where $\langle \cdot, \cdot \rangle$ stands for the natural inner product.

Suppose now that the 1-forms $\omega_1, \ldots, \omega_{n+3}$ are integrable. If this is the case then for every $i, j \in \{1, \ldots, n\}$

$$[\zeta_i(s, t), \zeta_j(s, t)] \wedge \zeta_1(s, t) \wedge \cdots \wedge \zeta_n(s, t)$$

vanishes at $n + 3$ distinct points $(s : t) \in \mathbb{P}^1$. But its coefficients have degree $n + 2$ in the variables (s, t). Thus the above expression vanishes identically, which proves the proposition. $\qquad \square$

2.3. **Proof of Theorem 1.** Replace W by a generic vector subspace W' of dimension equal to the codimension of Σ plus two. Thus, since W is generic, $\dim W' = \operatorname{rank}(W')$ and $\mathbb{P}(W')$ intersects Σ at a curve C. Moreover, we can assume that C is an irreducible component of $I_{W'}$.

If Σ is not of minimal degree then C is also not of minimal degree. Replacing W' by the linear span of C and applying Proposition 2.1 to sufficiently many points in C away from the other irreducible components of $I_{W'}$ one arrives at a contradiction which proves the theorem. $\qquad \square$

3. Varieties of minimal degree and the Proof of Theorem 2

Suppose $W \subset \Omega^1(\mathbb{C}^n, 0)$ is vector subspace satisfying $\dim W = \operatorname{rank}(W)$, and let Σ be an irreducible component of I_W of dimension at least two. Theorem 1 implies that Σ is a variety of minimal degree. If it is not a linear subspace of $\mathbb{P}(W)$ then, after replacing W by a generic vector subspace of appropriate dimension, we can assume that Σ has dimension exactly two and it is a not a plane linearly embedded in $\mathbb{P}(W)$. Moreover, it is harmless to assume that $\mathbb{P}(W)$ is the linear span of Σ.

To prove Theorem 2 we aim at a contradiction. To obtain it we will analyze each of the classes of surfaces of minimal degree. But first we recall in detail their classification.

3.1. **Surfaces of minimal degree.** If $X \subset \mathbb{P}^n$ is a surface of minimal degree then X is $\mathbb{P}^2$, or the embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ through the complete linear system $|\mathcal{O}_{\mathbb{P}^2}(2)| \simeq \mathbb{P}^5$, or a rational normal scroll $S(a, b)$ with $(a, b) \in \mathbb{N}^2 - \{(0, 0)\}$, and $a + b + 1 = n$.

The rational normal scrolls $S(a, b) \subset \mathbb{P}^{a+b+1}$ can be described as follows. First consider two disjoint linear subspaces $\mathbb{P}^a$ and $\mathbb{P}^b$ in $\mathbb{P}^n$. Consider now two rational normal curves $C_a \subset \mathbb{P}^a$ and $C_b \subset \mathbb{P}^b$, and let $\varphi_a : \mathbb{P}^1 \to C_a$ and $\varphi_b : \mathbb{P}^1 \to C_b$ be their parametrizations. In case $i = 0$, $\varphi_i : \mathbb{P}^1 \to C_0 \subset \mathbb{P}^0$ is nothing more then the constant map. In all other cases φ_i is an isomorphic embedding. The rational normal scroll $S(a, b)$ is the union of the lines $\overline{\varphi_a(t)\, \varphi_b(t)}$ for t varying in $\mathbb{P}^1$. Note that when $a = 0$ we have a cone over a rational normal curve in $\mathbb{P}^{n-1}$.

3.2. **Veronese surface.** We start the case by case analysis, excluding Veronese surfaces.

Lemma 3.1. *The surface Σ is not a Veronese surface.*

Proof. Assume Σ is a Veronese surface. Consider eight points in general position contained in Σ but not contained in any other irreducible component of I_W. Let C be the unique rational normal curve C passing through them.

On the one hand C is not contained in Σ, since $\deg C = 5$ is odd and every curve in Σ has even degree. Indeed, intersecting a curve in Σ with an hyperplane is the same as intersecting its pre-image under the Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5$ with a conic.

On the other hand, Proposition 2.1 ensures that $C \subset I_W$. The choice of the eight points implies C must also be contained in Σ. This contradiction proves the lemma. $\square$

3.3. **Pencils of integrable 1-forms.** Now we turn our attention to the possibility of Σ be a rational normal scroll. We will first exclude the degenerate cases $\Sigma = S(0, n-1)$, $n \geq 3$. Notice that these cases are characterized by their non-smoothness.

Lemma 3.2. *The surface Σ is smooth.*

Proof. If Σ is not smooth then it must be the cone $S(0, n-1)$ over a rational normal curve in $\mathbb{P}^{n-1}$ with $n \geq 3$. The idea is to look at the line of integrable 1-forms through the vertex of $S(0, n-1)$. For that sake, let ω_0 be a representative of the vertex and $\omega_1, \ldots, \omega_n$ be representatives of points in Σ away from the vertex such that these $(n+1)$ differential forms constitute a basis of W.

It will convenient to assume that all the non-zero 1-forms in W are non-zero at the origin. Notice that this can be achieved after taking representatives and localizing outside the singular locus of $\omega_0 \wedge \ldots \wedge \omega_n$, which is non-zero thanks to the assumption $\dim W = \mathrm{rank}(W)$. Therefore there exists a choice of coordinates $x_0, \ldots, x_n$ in $\mathbb{C}^{n+1}$ for which $\omega_i = g_i dx_i$ where $g_0, \ldots, g_n$ are suitable germs of invertible functions. Furthermore, after dividing all the 1-forms by g_0, we can also assume that $\omega_0 = dx_0$.

For a fixed $i \in \{1, \ldots, n\}$, consider the linear family $s\omega_0 + t\omega_i$ of integrable 1-forms parametrized by $(s, t) \in \mathbb{C}^2$. It is well known, see for instance [3], that there exists a unique meromorphic 1-form η_i such that

$$d\left(s\omega_0 + t\omega_i\right) = \eta_i \wedge \left(s\omega_0 + t\omega_i\right)$$

for every $(s, t) \in \mathbb{C}^2$. When $(s, t) = (1, 0)$, the above equation reads as $\eta_i \wedge dx_0 = 0$. The differentiation of this identity leads to $d\eta_i \wedge dx_0 = 0$. Combining these two identities with the one obtained when $(s, t) = (0, 1)$, one promptly infers that $\eta_i = h_i(x_0, x_i)dx_0$ for a suitable two variables function h_i.

Let now $\omega = \sum_{i=1}^{n} \lambda_i \omega_i$ be another integrable 1-form distinct from the previous ones. Of course, there exists such 1-form since we are assuming that Σ has dimension two. As before we consider the linear family $s\omega_0 + t\omega$ and the corresponding 1-form $\eta = hdx_0$ satisfying

$$d\omega = \eta \wedge \omega = \sum_{i=1}^{n} \lambda_i h f_i dx_0 \wedge dx_i \,.$$

Comparing this last identity with

$$d\omega = \sum_{i=1}^{n} \lambda_i d\omega_i = \sum_{i=1}^{n} \lambda_i \eta_i \wedge \omega_i = \sum_{i=1}^{n} \lambda_i h_i f_i dx_0 \wedge dx_i$$

one deduces that $h_i = h = h(x_0)$. Thus all the elements in W are integrable contradicting the hypothesis that $S(0, n-1)$, $n \geq 3$, is an irreducible component of I_W. $\square$

We have shown slightly more. The proof above also shows the following

Lemma 3.3. *If a rational normal scroll of the form $S(0,k)$, $k \geq 1$, is contained in I_W then its linear span is also contained in I_W.*

Notice that in the the extremal case $k = 1$, $S(0,1)$ is nothing more than $\mathbb{P}^2$.

3.4. **Projections versus restrictions and the proof of Theorem 2.** To conclude the proof of Theorem 2 it remains to consider the rational normal scrolls $S(a,b)$ with $a, b \geq 1$. This is done in the next proposition.

Proposition 3.4. *If a rational normal scroll of the form $S(a,b)$ is contained in I_W then its linear span is also contained in I_W.*

Proof. Assume $\mathbb{P}(W)$ coincides with the linear span of $S(a,b)$. We will proceed by induction, with the basis being given by Lemma 3.3. To prove the result for $S(a,b)$, with $a, b \geq 1$, assume it holds for $S(a-1, b)$ and $S(a, b-1)$.

We can suppose, see the proof of Lemma 3.2, that every non-zero 1-form in W is non-zero at the origin. Thus, if $\omega \in W$ is an integrable 1-form then it defines a smooth foliation $\mathcal{F}_\omega$ on $(\mathbb{C}^{n+1}, 0)$. Let $L \simeq (\mathbb{C}^n, 0)$ be an arbitrary leaf of $\mathcal{F}_\omega$. Notice that we are abusing the notation here. The leaf L does not necessarily passes through the origin of $\mathbb{C}^{n+1}$. We are thinking in terms of a representative of ω defined on a connected neighborhood of the origin where the foliation $\mathcal{F}_\omega$ is defined by a submersion with connected fibers, and L is an arbitrary fiber of such submersion.

If $\iota : L \to \mathbb{C}^{n+1}$ denotes its inclusion into $\mathbb{C}^{n+1}$, then $W_L := \iota^* W$ is a vector space of $\Omega^1(\mathbb{C}^n, 0)$ satisfying $\dim W_L = \dim W - 1$ and $\operatorname{rank}(W_L) = \operatorname{rank}(W) - 1$. The induced rational map

$$\iota^* : \mathbb{P}(W) \dashrightarrow \mathbb{P}(W_L)$$

is nothing more than the linear projection centered at $[\omega_0]$. Notice that I_{W_L} is contained in the image of I_W.

Suppose $S(a,b)$ is an irreducible component of I_W and that $[\omega]$ belongs either to C_a or C_b in $S(a,b)$. The projection of $S(a,b)$ centered at a point in C_a, resp. C_b, is clearly $S(a-1, b)$, resp. $S(a, b-1)$. By induction hypothesis I_{W_L} must coincide with $\mathbb{P}(W_L)$. Since L is arbitrary, this implies that for every $\alpha \in W$ the 4-form $\omega \wedge \alpha \wedge d\alpha$ is identically zero.

Let $\omega_1, \ldots, \omega_4 \in W$ be four linearly independent 1-forms with classes in $C_a \cup C_b$. The argument above shows that for every $\alpha \in W$ and every $i \in \{1, 2, 3, 4\}$, the 4-form $\alpha \wedge d\alpha \wedge \omega_i = 0$. Thus $\alpha \wedge d\alpha = 0$ for any $\alpha \in W$. The proposition follows. $\square$

4. Examples

4.1. Left-invariant 1-forms on Lie groups.

Let G be a complex Lie group and $\mathfrak{g}$ be its Lie algebra. The vector space of left-invariant 1-forms on G is naturally identified with $W = \mathfrak{g}^*$. The classes of integrable 1-forms in $\mathbb{P}W$ are in one to one correspondence with codimension one Lie subalgebras of $\mathfrak{g}$.

For example, if $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ then the irreducible components of $I_{\mathfrak{g}^*} \subseteq \mathbb{P}(\mathfrak{g}^*)$ are easily described: if α, β, γ is one basis of $\mathfrak{g}^*$ satisfying $d\alpha = \alpha \wedge \beta$, $d\beta = \alpha \wedge \gamma$, $d\gamma = \beta \wedge \gamma$, then $\omega = x\alpha + y\beta + z\gamma \in \mathfrak{g}^*$ is integrable if and only if

$$(2xz - y^2)\alpha \wedge \beta \wedge \gamma = 0.$$

Thus $I_{\mathfrak{g}^*} \subseteq \mathbb{P}(\mathfrak{g}^*)$ has only one irreducible component which is a conic.

More generally, if $\mathfrak{g}$ is any Lie algebra then main result of [11] implies that the irreducible components of $I_{\mathfrak{g}^*} \subseteq \mathbb{P}(\mathfrak{g}^*)$ are either linear subspaces or conics of the type described above.

4.2. Godbillon-Vey sequences.

Another natural source of rational curves of integrable 1-forms is the *development* of foliations with finite Godbillon-Vey sequence as studied in [4]. Given a meromorphic integrable 1-form on a projective manifold X (or more generally pseudo-parallelizable compact manifold) there exists a sequence of 1-forms $(\omega_0, \omega_1, \ldots, \omega_k, \ldots)$ such that the *formal* 1-form (defined on X times a formal neighborhood of the origin of $\mathbb{C}$)

$$\Omega = dz + \sum_{i=0}^{\infty} \frac{z^i}{i!} \omega_i \, ,$$

is integrable and $\omega_0 = \omega$. A sequence with such properties is called a Godbillon-Vey sequence of ω, and Ω is a development of ω. When this sequence is finite, *i.e.* $\omega_i = 0$ for $i > i_0$, the restriction of Ω to $\{z = \text{const.}\}$ produces a rational normal curve of integrable 1-forms in the vector space W generated by $\omega_0, \ldots, \omega_{i_0}$.

When $i_0 = 2$ we are in a situation not essentially different from the example associate to sl$(2, \mathbb{C})$. In this case the foliation induced by ω is transversely projective, and at neighborhood of a generic point of X there is a a map to SL$(2, \mathbb{C})$ such that the sequence $(\omega_0, \omega_1, \omega_2)$ is the pull-back of a sequence of left-invariant 1-forms on SL$(2, \mathbb{C})$.

When $i_0 > 2$, although one can obtain rational normal curves of degree equal to $\dim \mathbb{P}(W)$, no example of this kind fall under our hypothesis. Indeed, according to [4, Lemma 2.3], $\omega_i \wedge \omega_j = 0$ for every $i, j \geq 2$. In particular $\text{rank}(W) \leq 3$.

4.3. Rational normal curves of arbitrary degree.

Fix an integer $n \geq 2$. Set $\omega_0 = dx_0$ and, for j ranging from 1 to n, set

$$\omega_j = f_j dx_j \quad \text{where} \quad f_j = (j+1) + j(x_0 + \cdots + x_n) \, .$$

Consider the vector space $W \subset \Omega^1(\mathbb{C}^{n+1}, 0)$ generated by $\omega_0, \ldots, \omega_n$. Clearly $\dim W = \text{rank}(W) = n + 1$.

Notice that $\omega_1, \ldots, \omega_n$ are all integrable 1-forms. A computation shows that $\omega_{n+1} = \sum \omega_i$ as well as $\omega_{n+2} = \sum_{i=0}^{n} \frac{1}{(i+2)} \omega_i$ are also integrable. Thus, according to Proposition 2.1, the unique rational normal curve C through $[\omega_0], \ldots, [\omega_{n+2}]$ is contained in I_W. But, as another computation shows, the 1-form $\sum_{i=0}^{n} (i+1)\omega_i$ is not integrable. Hence Theorem 2 implies that C is an irreducible component of I_W.

5. Gelfand-Zakharevich correspondence

Although concrete, the previous example says nothing about the underlying geometry of rational normal curves of integrable 1-form. Here we are going to review a beautiful geometric construction from [9, pages 79–80], that puts in correspondence analytic equivalence classes of germs of holomorphic surfaces along smooth rational curves endowed with a morphisms to $\mathbb{P}^1$, and rational normal curves of integrable 1-forms. Using this correspondence we will characterize when rational normal curves are irreducible components of I_W in terms of properties of the associated surface.

5.1. **From rational normal curves to surfaces.** Set X equal to $(\mathbb{C}^{n+1}, 0)$. As above, X should be thought as a sufficiently small connected neighborhood of the origin. Let $W \subset \Omega^1(X)$ be a vector subspace of dimension and rank equal to $n+1$. Suppose that all the non-zero 1-forms in W are non-singular at every point of X.

Let $\gamma : \mathbb{P}^1 \to \mathbb{P}(W)$ be a parametrization of a rational normal curve of integrable 1-forms in $\mathbb{P}(W)$. For any $\lambda \in \mathbb{P}^1$, let $\mathcal{F}_\lambda$ be the foliation associated to $\gamma(\lambda)$. Since the 1-forms in W have no singular points, $\mathcal{F}_\lambda$ is a smooth foliation on X and as such has leaf space naturally isomorphic to $(\mathbb{C}, 0)$. Considering the union of the leaf spaces of all the foliations $\mathcal{F}_\lambda$ with λ varying in $\mathbb{P}^1$, one obtains a germ of complex surface $X^{(2)}$. To each point $x \in X$, there is a smooth rational curve C_x corresponding to the union of the leaves of the foliations $\mathcal{F}_\lambda$ through x. Let $C = C_0$ the curve corresponding to the origin $0 \in X$. It is not hard to see that $C^2 = C \cdot C_x = n$: just take the point x in the intersection of leaves of $\mathcal{F}_{\lambda_1}, \ldots, \mathcal{F}_{\lambda_n}$ through the origin. Notice also that $X^{(2)}$ comes endowed with a holomorphic map $\pi : X^{(2)} \to \mathbb{P}^1$ that associates to a leaf of $\mathcal{F}_\lambda$ the point $\lambda \in \mathbb{P}^1$. Of course the restriction $\pi_{|C} : C \to \mathbb{P}^1$ is an isomorphism.

If $\Gamma \subset X \times X^{(2)}$ is the point-leaf correspondence, that is

$$\Gamma = \left\{ (x, L) \in X \times X^{(2)} \,\middle|\, x \in L \right\},$$

and $\rho_1 : \Gamma \to X$, $\rho_2 : \Gamma \to X^{(2)}$ are the natural projections then: for any $\lambda \in \mathbb{P}^1$ and any leaf $L \subset X$ of $\mathcal{F}_\lambda$, $\rho_2 \rho_1^{-1}(L)$ is a point of $X^{(2)}$; and for any section $\sigma : \mathbb{P}^1 \to X^{(2)}$, the intersection

$$\bigcap_{\lambda \in \mathbb{P}^1} \rho_1\big(\rho_2^{-1}\big(\sigma(\lambda)\big)\big)$$

is a point of X, see [9, Theorem 2.2].

The triple $(X^{(2)}, C, \pi)$ will be called the Gelfand-Zakharevich triple associate to the rational normal curve of integrable 1-forms $\gamma(\mathbb{P}^1)$. On the one hand the pair $(X^{(2)}, C)$, seen as a germ of surface along a rational curve modulo isomorphisms, does not depend on the parametrization of the rational normal curve. On the other hand, the morphism π does depend on the parametrization but its equivalence class modulo composition on the left with automorphism of $\mathbb{P}^1$ does not. In other words, the linear system that defines π does not depend on the parametrization. Thus, it is fair to say that the Gelfand-Zakharevich triple is canonically associated to the rational normal curve $\gamma(\mathbb{P}^1)$.

5.2. **From surfaces to rational normal curves.** Start now with a triple (S, C, π), where S is a germ of smooth surface S along a smooth rational curve C of self-intersection n and endowed with a morphism $\pi : S \to \mathbb{P}^1$, and assume $\pi_{|C} : C \to \mathbb{P}^1$ is a isomorphism.

Deformation theory tell us that the space of deformations $\mathcal{X}$ of C is smooth and $\mathcal{X} \simeq (H^0(C, N_C), 0) \simeq (\mathbb{C}^{n+1}, 0)$. To each $\lambda \in \mathbb{P}^1$, let $\mathcal{F}_\lambda$ be the foliation of $\mathcal{X}$ which has as leaves deformations of C intersecting $\pi^{-1}(\lambda)$ at a fixed point. It is possible to show that there exists a vector space $W \subset \Omega^1(\mathcal{X}, 0)$ of dimension and rank equal to $n+1$, and a family of foliations $\mathcal{F}_\lambda$ parametrized by a rational normal curve of integrable 1-forms contained in $\mathbb{P}(W)$, see [9, Theorem 2.3]. Hence, the two constructions just presented are inverse two each other modulo the respective natural equivalence relations.

5.3. **Rational normal curves as irreducible components of** I_W**.** Let now $W \subset \Omega^1(\mathbb{C}^{n+1}, 0)$ be a vector space of dimension and rank equal to $n+1$, and $C \subset \mathrm{I}_W$ be a rational normal curve. If (S, C, π) is the Gelfand-Zakharevich triple associated to C then S has algebraic dimension[1] $a(S)$ equal to one or two. Indeed, $a(S)$ it is at least one because the morphism $\pi : S \to \mathbb{P}^1$ induces a inclusion of $\mathbb{C}(\mathbb{P}^1)$ into $\mathbb{C}(S)$; and $a(S)$ is at most two because $C^2 > 0$ what allow us to apply [1, Théorème 6] or [10, Theorem 6.7].

Theorem 5.1. *Assume $n \geq 2$. The algebraic dimension of S is two if and only if I_W coincides with $\mathbb{P}(W)$.*

Proof. If I_W coincides with $\mathbb{P}(W)$ then the same arguments used to prove Lemma 3.2 imply that W is in a suitable system of coordinates the vector space generated by $hdx_0, \ldots, hdx_n$ for a fixed meromorphic function h. In these coordinates, the foliations induced by elements of W globalize to smooth foliations on $\mathbb{C}^{n+1}$. The leaf space of each one of these foliations is isomorphic to $\mathbb{C}$, and the Gelfank-Zakharevich triple is isomorphic to $(E(\mathcal{O}_{\mathbb{P}^1}(n)), C_0, \pi)$ where $E(\mathcal{O}_{\mathbb{P}^1}(n))$ is the total space of $\mathcal{O}_{\mathbb{P}^1}(n)$, C_0 is the zero section, and $\pi : E(\mathcal{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1$ is the natural projection. Since $E(\mathcal{O}_{\mathbb{P}^1}(n))$ is an algebraic surface its algebraic dimension is at least two. Thus $\mathrm{I}_W = \mathbb{P}(W)$ implies $a(S) = 2$.

Suppose now that $a(S) = 2$. Therefore, there exists a projective surface Z containing S as an open subset. Moreover, if $i : S \to Z$ is the inclusion then the Theorems of Andreotti and Hartshorne refereed to above imply that the induced morphism $i^* : \mathbb{C}(Z) \to \mathbb{C}(S)$ is surjective. Thus the morphism $\pi : S \to \mathbb{P}^1$ extends to a rational map, still denoted by π, $\pi : Z \dashrightarrow \mathbb{P}^1$. Since its indeterminacies, if any, are away from C, it is harmless to assume that π is indeed a regular map defined on all of Z.

We claim that the surface Z is a rational surface and that the fibers of π are rational curves. The arguments are essentially the same as the ones laid down in [13, Section 5.4.3] which we refer for further details. First notice that the abundance of rational curves on Z implies that there are no holomorphic 1-forms on it. Hence linear and algebraic equivalence coincide thanks to Hodge theory. After blowing-up Z at n distinct points of C, one obtains a fibered surface $\overline{\pi} : \overline{Z} \to \mathbb{P}^1$ containing a section $\overline{C}$ of self-intersection zero which moves in a linear system of projective dimension one. This suffices to show that fibers of $\overline{\pi}$, and hence also the fibers of π, are rational curves. Successive contractions of the (-1)-curves on the fibers of π that do not intersect the curve C lead us to a relative minimal model Z_0 of Z which has to be the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$. The complement of the section of self-intersection $-n$ is isomorphic to $E(\mathcal{O}_{\mathbb{P}^1}(n))$ with the curve C identified with

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[1]As in the case of compact surfaces we are considering the algebraic dimension of S as the transcendence degree over $\mathbb{C}$ of its field of germs of meromorphic functions $\mathbb{C}(S)$.

C_0. Thus we conclude that the Gelfand-Zakharevich triple (S, C, π) extends to the triple $(E(\mathcal{O}_{\mathbb{P}^1}(n)), C_0, \pi)$ associate to $W = \oplus_{i=0}^n \mathbb{C} dx_i$. The naturalness of Gelfand-Zakharevich correspondence implies the result. $\qquad \square$

Corollary 5.2. *Assume $n \geq 2$. The curve C is an irreducible component of I_W if and only if the algebraic dimension of S is one.*

When $n = 1$ all the elements of $\mathbb{P}(W) = \mathbb{P}^1$ correspond to integrable 1-forms. Neverthless, there is a natural analogue of Theorem 5.1 in this case. It reads as: $a(S) = 2$ if and only if the there exists a closed meromorphic 1-form η such that $d\omega = \eta \wedge \omega$ for every $\omega \in W$. The reader can easily infer such result from the proof of Theorem 5.1. Notice that only the first paragraph has to be adapted, the remaining of the proof works as it is.

§

It would be interesting to investigate if, and if yes how, the Gelfand-Zakharevich correspondence globalizes when studying rational normal curves of foliations on compact complex manifolds. For instance a structure theorem for foliations in these curves along the lines of [3] would be a welcome addition to the literature.

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