Regularity results for semimonotone operators

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May 17, 2010

Abstract

We introduce the concept of ρ-semimonotone point-to-set operators in Hilbert spaces. This notion is symmetrical with respect to the graph of T, as is the case for monotonicity, but not for other related notions, like e.g. hypomonotonicity, of which our new class is a relaxation. We give a necessary condition for ρ-semimonotonicity of T in terms of Lispchitz continuity of $[T+\rho^{-1}I]^{-1}$ and a sufficient condition related to expansivity of T. We also establish surjectivity results for maximal ρ-semimonotone operators.

keywords: hypomonotonicity, surjectivity, prox-regularity, semimonotonicity.

1 Introduction

Before introducing the class of ρ-semimonotone operators we recall the concept of monotonicity and a few of its relaxations.

Definition 1. *Let H be a Hilbert space, $T : H \to \mathcal{P}(H)$ a point-to-set operator and $G(T)$ its graph.*

i) T is said to be monotone iff

$$\langle x-y, u-v\rangle \geq 0, \qquad \forall (x,u),(y,v) \in G(T).$$

ii) T is said to be maximal monotone if it is monotone and additionally $G(T) = G(T')$ for all monotone operator $T' : H \to \mathcal{P}(H)$ such that $G(T) \subset G(T')$.

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iii) For $\rho \in \mathbb{R}_{++}$, T is said to be ρ-hypomonotone iff

$$\langle x-y, u-v\rangle \geq -\rho\|x-y\|^2, \qquad \forall (x,u),(y,v) \in G(T).$$

iv) For $\rho \in \mathbb{R}_{++}$, T is said to be maximal ρ-hypomonotone if it is ρ-hypomonotone and additionally $G(T) = G(T')$ for all ρ-hypomonotone operator $T' : H \to \mathcal{P}(H)$ such that $G(T) \subset G(T')$.

v) T is said to be premonotone iff

$$\langle x-y, u-v\rangle \geq -\sigma(y)\|x-y\|, \qquad \forall (x,u),(y,v) \in G(T),$$

where $\sigma : H \to \mathbb{R}$ is a positive valued function defined over the whole space H.

Next we introduce the class of operators which are the main subject of this paper.

Definition 2. *Let $T : H \to \mathcal{P}(H)$ be a point-to-set operator, $G(T)$ its graph and $\rho \in (0,1)$ a real number.*

i) T is said to be ρ-semimonotone iff

$$\langle x-y, u-v\rangle \geq -\frac{\rho}{2}\left(\|x-y\|^2 + \|u-v\|^2\right) \tag{1}$$

for all $(x,u),(y,v) \in G(T)$.

ii) T is said to be maximal ρ-semimonotone if it is ρ-semimonotone and additionally $G(T) = G(T')$ for all ρ-semimonotone operator $T' : H \to \mathcal{P}(H)$ such that $G(T) \subset G(T')$.

The concepts of hypomonotonicity and premonotonicity were introduced in [5] and [2] respectively. We mention that a notion of maximal premonotonicity has also been introduced in [2], but the definition is rather technical and thus we prefer to omit it.

We mention that we restrict the range of the parameter ρ to the interval $(0,1)$ because all operators turn out to be ρ-semimonotone for $\rho \geq 1$, as can be easily verified.

It is clear that monotone operators are both premonotone and ρ-hypomonotone for all $\rho > 0$, and that ρ-hypomonotone operators with $\rho \in (0, 1/2)$ are 2ρ-semimonotone. It is also elementary that T is ρ-hypomonotone iff $T + \rho I$ is monotone (I being the identity operator in H).

In order to have a clearer view of the relation among these notions, it is worthwhile to look at the special case of self-adjoint linear operators in the finite dimensional case. If $\Lambda(A)$ is the spectrum (i.e., set of eigenvalues) of the self-adjoint linear operator $A : H \to H$, it is well known that A is monotone iff $\Lambda(A) \subset [0,\infty)$ and it follows easily from the comment above that A is ρ-hypomonotone iff $\Lambda(A) \subset [-\rho,\infty)$. On the other hand, linear premonotone operators are just monotone. It is also elementary that A is ρ-semimonotone iff

$$\Lambda(A) \subset (-\infty, -\eta(\rho)] \cup [-\beta(\rho) + \infty),$$

with $0 < \beta(\rho) < \eta(\rho)$ given by (5) and (7), i.e., the eigenvualues of self-adjoint ρ-semimonotone operators can lie anywhere on the real line, excepting for an open interval around $-1/\rho$ contained in the negative halfline.

One of the main properties of maximal monotone operators is related to the regularization of the inclusion problem consisting of finding $x \in H$ such that $b \in T(x)$, with T monotone and $b \in H$. Such problem may have no solution, or an infinite set of solutions, but the problem $b \in (T + \lambda I)(x)$ is well posed in Hadamard's sense for all $\lambda > 0$, meaning that there exists a unique solution, and it depends continuously on b. This is a consequence of Minty's Theorem (see [4]), which states that for a maximal monotone operator T, the operator $T + \lambda I$ is onto, and its inverse is Lipschitz continuous with constant $L = \lambda^{-1}$, (and henceforth point-to-point), for all $\lambda > 0$.

When the notion of monotonicity is relaxed, one expects to preserve at least some version of Minty's result. In the case of hypomonotonicity, the fact that $T + \rho I$ is monotone when T is ρ-hypomonotone easily implies that Minty's result holds for maximal ρ-hypomonotone operators whenever λ belongs to (ρ, ∞), with the Lipschitz constant of $(T + \lambda I)^{-1}$ taking the value $(\lambda - \rho)^{-1}$.

The situation is more complicated when T is premonotone. Examples of premonotone operators T defined on the real line such that $T + \lambda I$ fails to be monotone for all $\lambda > 0$ have been presented in [2]. Nevertheless, the following surjectivity result has been proved in [2]: when T is maximal premonotone and H is finite dimensional then $T + \lambda I$ is onto for all $\lambda > 0$. Minty's Theorem cannot be invoked in this case, and the proof uses an existence result for equilibrium problems originally established in [3] and extended later on in [1].

Before discussing the ρ-semimonotone case, it might be illuminating to look at the surjetivity issue in the one-dimensional case. It is easy to check that $T + \lambda I$ is strictly increasing when T is monotone and $\lambda > 0$, or T is ρ-hypomonotone and $\lambda > \rho$, and furthermore the values of the regularized operator $T + \lambda I$ go from $-\infty$ to $+\infty$. The surjectivity is then an easy consequence of the maximality of the graph $G(T)$. When T is pre-monotone, $T + \lambda I$ may fail to be increasing for all $\lambda > 0$ (see Example 3 in [2]), but still it holds that the operator values go from $-\infty$ to $+\infty$, and the surjectivity is also guaranteed. This is not the case for ρ-semimonotone operators. Not only a ρ-semimonotone operator T defined on $\mathbb{R}$ may be such that $T + \lambda I$ fails to be monotone for all $\lambda > 0$, but T, and even $T + \lambda I$, may happen to be strictly decreasing! (see Example 1 below). We will nevertheless manage to establish regularity of $T + \lambda I$ when T is ρ-semimonotone and λ belongs to a certain interval $(\beta(\rho), \eta(\rho)) \subset (0, +\infty)$, with $\beta(\rho)$, $\eta(\rho)$ as in (5), (7) below (in the case of T like in Example 1, the surjectivity will be a consequence of the fact that T is strictly decreasing). We cannot invoke Minty's result in an obvious way, since $T + \lambda I$ will not in general be monotone; rather, the proof will proceed through the analysis of the regularity properties of the operator $[T + \beta(\rho) I]^{-1} + \gamma(\rho) I$, with $\gamma(\rho)$ as in (6) below.

2 Semimonotone operators

In this section we will establish several properties of semimonotone operators. We start our analysis with some elementary ones.

Proposition 1. *An operator $T : H \to \mathcal{P}(H)$ is ρ-semimonotone if and only if the operator T^{-1} is ρ-semimonotone; furthermore, T is maximal ρ-semimonotone if and only if T^{-1} is maximal ρ-semimonotone.*

Proof. The result follows immediately from Definition 2, taking into account that $(x, u) \in G(T)$ iff $(u, x) \in G(T^{-1})$. □

We mention that monotonicity of T is also equivalent to monotonicity of T^{-1}, but the similar statement fails to hold for ρ-hypomonotone operators. In fact, one of the motivations behind the introduction of the class of ρ-semimonotone operators is the preservation of this symmetry property enjoyed by monotone operators.

Proposition 2. *If $T : H \to \mathcal{P}(H)$ is ρ-semimonotone and α belongs to $(\rho, 1/\rho)$ then αT is $\bar{\rho}$-semimonotone with $\bar{\rho} = \rho \max\{\alpha, 1/\alpha\}$.*

Proof. Note first that $\bar{\rho}$ belongs to $(0, 1)$. Let $\bar{T} = \alpha T$ and take $(x, \bar{u}), (y, \bar{v}) \in G(\bar{T})$. By definition of $\bar{T}$, there exist $u \in T(x), v \in T(y)$ such that $\bar{u} = \alpha u, \bar{v} = \alpha v$. By ρ-semimonotonicity of T

$$\langle x - y, \bar{u} - \bar{v} \rangle = \alpha \langle x - y, u - v \rangle \geq -\frac{\rho}{2} \left(\alpha \|x - y\|^2 + \alpha \|u - v\|^2 \right)$$

$$= -\frac{\rho}{2} \left(\alpha \|x - y\|^2 + \frac{1}{\alpha} \|\bar{u} - \bar{v}\|^2 \right) \geq -\frac{\bar{\rho}}{2} \left(\|x - y\|^2 + \|\bar{u} - \bar{v}\|^2 \right),$$

establishing $\bar{\rho}$-semimonotonicity of $\bar{T} = \alpha T$. □

Proposition 3. *If $T : H \to \mathcal{P}(H)$ is δ-semimonotone for some $\delta \in (0, 1)$, then T is ρ-semimonotone for all $\rho \in (\delta, 1)$.*

Proof. Elementary. □

Proposition 4. *If $T : H \to \mathcal{P}(H)$ (or $T^{-1} : H \to \mathcal{P}(H)$) is δ-hypomonotone with $\delta \in (0, 1/2)$, then T is 2δ-semimonotone. Moreover, if both T and T^{-1} are δ-hypomonotone with $\delta \in (0, 1)$ then T is δ-semimonotone.*

Proof. Elementary. □

Remark 1. *We mention that a δ-hypomonotone operator T with $\delta \geq 1/2$, may fail to be ρ-semimonotone for all ρ, but the operator $A = \dfrac{\rho}{2\delta} T$ is ρ-semimonotone for all $\rho \in (0, 1)$.*

Proposition 5. *An operator $T : H \to \mathcal{P}(H)$ is ρ-semimonotone if and only if*

$$\|x - y + u - v\|^2 \geq (1-\rho)\left(\|x-y\|^2 + \|u-v\|^2\right), \qquad \forall (x,u),\ (y,v) \in G(T).$$

Proof. Elementary. $\square$

Proposition 6. *If $T : H \to \mathcal{P}(H)$ is maximal ρ-semimonotone then its graph is closed (in the strong topology).*

Proof. Elementary. $\square$

2.1 The one dimensional case

We study in this section ρ-semimonotone real valued functions, providing a simple characterization that helps in the construction of a key example and also suggests the line to follow in order to study the general case.

Lemma 1. *Given $\rho \in (0,1)$ define $\theta(\rho)$ as*

$$\theta(\rho) = \rho^{-1}\sqrt{1-\rho^2}. \tag{2}$$

A function $f : X \subset \mathbb{R} \to \mathbb{R}$ is ρ-semimonotone if and only if $g : X \to \mathbb{R}$ defined by $g(x) = f(x) + \rho^{-1}x$ satisfies

$$\|g(x) - g(y)\| \geq \theta(\rho)\|x - y\| \tag{3}$$

for all $x, y \in X$, or equivalently, $g^{-1} = (f + \rho^{-1}I)^{-1}$ is Lipschitz continuous with constant $\theta(\rho)^{-1}$.

Proof. Assume that $f : X \to \mathbb{R}$ is ρ-semimonotone and define $g(x) = f(x) + \rho^{-1}x$. By Definition 2, for all $x, y \in X$

$$(x-y)[f(x) - f(y)] \geq -\frac{\rho}{2}\left((x-y)^2 + [f(x)-f(y)]^2\right)$$

or, equivalently,

$$\frac{f(x)-f(y)}{x-y} = \frac{(x-y)[f(x)-f(y)]}{(x-y)^2} \geq -\frac{\rho}{2}\left(1 + \left[\frac{f(x)-f(y)}{x-y}\right]^2\right). \tag{4}$$

Take any $x \neq y \in X$ and define $t = \dfrac{f(x)-f(y)}{x-y}$. Then, (4) is equivalent to $t \geq -\frac{\rho}{2}(1+t^2)$, i.e.,

$$\begin{aligned}\frac{\rho}{2}t^2 + t + \frac{\rho}{2} \geq 0 \iff &\ t \leq t_1 = \frac{-1-\sqrt{1-\rho^2}}{\rho} \quad \text{or} \quad t \geq t_2 = \frac{-1+\sqrt{1-\rho^2}}{\rho} \\ \iff &\ \left|t + \frac{1}{\rho}\right| \geq \frac{\sqrt{1-\rho^2}}{\rho} = \theta(\rho).\end{aligned}$$

Since for any $x \neq y$,

$$\frac{f(x)-f(y)}{x-y}+\frac{1}{\rho}=\frac{f(x)-f(y)}{x-y}+\frac{1}{\rho}\cdot\frac{x-y}{x-y}=\frac{g(x)-g(y)}{x-y},$$

the proof is complete. □

Example 1. Fix $\rho \in (0,1)$, $\delta \geq \rho^{-1}\sqrt{1-\rho^2}$, and define $g:\mathbb{R}\to\mathbb{R}$ as $g(x)=-\delta x-\frac{1}{3}x^3$. Then

$$g'(x)=-\delta-x^2 \implies |g'(x)|=\delta+x^2\geq\delta$$

for all $x\in\mathbb{R}$. Thus, g verifies (3). Hence,

$$f(x)=g(x)-\frac{1}{\rho}x=-\left(\delta+\frac{1}{\rho}\right)x-\frac{1}{3}x^3$$

is a ρ-semimonotone function, in view of Lemma 1. On the other hand, for all $\lambda\in\mathbb{R}$ the function $h(x)=f(x)+\lambda x$ fails to be non-decreasing, and hence $f+\lambda I$ is not monotone.

3 Prox-regularity properties

The surjectivity properties of $T+\lambda I$ for a ρ-semimonotone operator T are related to its connection with the operator $[T+\beta I]^{-1}+\gamma I$, presented in the next theorem.

Theorem 2. *Let I be the identity operator in H. Take $\rho\in(0,1)$ and define $\beta,\gamma,\eta\in\mathbb{R}_{++}$ as*

$$\beta=\beta(\rho)=\frac{1-\sqrt{1-\rho^2}}{\rho}, \tag{5}$$

$$\gamma=\gamma(\rho)=\frac{\rho}{2\sqrt{1-\rho^2}}, \tag{6}$$

$$\eta=\eta(\rho)=\frac{1}{\gamma}+\beta=\frac{1+\sqrt{1-\rho^2}}{\rho}. \tag{7}$$

i) An operator $T:H\to\mathcal{P}(H)$ is ρ-semimonotone if and only if the operator $(T+\beta I)^{-1}+\gamma I$ is monotone.

ii) An operator $T:H\to\mathcal{P}(H)$ is maximal ρ-semimonotone if and only if the operator $(T+\beta I)^{-1}+\gamma I$ is maximal monotone.

Proof. Consider $A : H \times H \to H \times H$ defined as $A(x,u) = (u - \gamma x, (1+\beta\gamma)x - \beta u)$. It is elementary that A is invertible, with $A^{-1}(x,u) = (u + \beta x, (1+\beta\gamma)x + \gamma u)$. Let $(\bar{x}, \bar{u}) = A(x,u)$ and $\bar{T} = (T + \beta I)^{-1} + \gamma I$. We claim that $(x,u) \in G(\bar{T})$ if and only if $(\bar{x}, \bar{u}) \in G(T)$. We proceed to prove the claim: $(x,u) \in G(\bar{T})$ iff $u \in (T+\beta I)^{-1}(x) + \gamma x$ iff $\bar{x} = u - \gamma x \in (T+\beta I)^{-1}(x)$ iff $x \in (T+\beta I)(\bar{x}) = T(\bar{x}) + \beta \bar{x}$ iff $\bar{u} = x - \beta\bar{x} \in T(\bar{x})$ iff $(\bar{x}, \bar{u}) \in G(T)$.

The claim is established and we proceed with the proof of (i). Consider pairs $(x,u), (y,v) \in G(\bar{T})$ and let $(\bar{x}, \bar{u}) = A(x,u)$ as before, and also $(\bar{y}, \bar{v}) = A(y,v)$. Observe that $\bar{T}$ is monotone if and only if, for all $(x,u), (y,v) \in G(\bar{T})$, it holds that

$$0 \leq \langle x - y, u - v\rangle = \langle (\bar{u} + \beta\bar{x}) - (\bar{v} + \beta\bar{y}), [(1+\gamma\beta)\bar{x} + \gamma\bar{u}] - [(1+\gamma\beta)\bar{y} + \gamma\bar{v}]\rangle =$$

$$(1 + 2\gamma\beta)\langle \bar{x} - \bar{y}, \bar{u} - \bar{v}\rangle + (1+\gamma\beta)\beta\|\bar{x} - \bar{y}\|^2 + \gamma\|\bar{u} - \bar{v}\|^2, \tag{8}$$

using the definition of $(\bar{x}, \bar{u}), (\bar{y}, \bar{v})$ and the formula of A^{-1} in the first equality. Note that the inequality in (8) is equivalent to

$$\langle \bar{x} - \bar{y}, \bar{u} - \bar{v}\rangle \geq -\frac{(1+\gamma\beta)\beta}{1 + 2\gamma\beta}\|\bar{x} - \bar{y}\|^2 - \frac{\gamma}{1 + 2\gamma\beta}\|\bar{u} - \bar{v}\|^2 =$$

$$-\frac{\rho}{2}\left(\|\bar{x} - \bar{y}\|^2 + \|\bar{u} - \bar{v}\|^2\right), \tag{9}$$

using (5), (6) in the equality. In view of the claim above and the invertibility of A, $(\bar{x}, \bar{u}), (\bar{y}, \bar{v})$ cover $G(T)$ when $(x,u), (y,v)$ run over $G(\bar{T})$. Thus, we conclude from (1) that the inequality in (9) is equivalent to the ρ-semimonotonicity of T.

We proceed now with the proof of (ii): In view of (i), if we can add a pair (x,u) to $G(\bar{T})$ while preserving the monotonicity of $\bar{T}$, then we can add the pair $(\bar{x}, \bar{u}) = A(x,u)$ to $G(T)$ and preserve the ρ-semimonotonicity of T, and viceversa. It follows that the maximal monotonicity of $\bar{T}$ is equivalent to the maximal ρ-semimonotonicity of T. □

Corollary 1. *If $T : H \to \mathcal{P}(H)$ is maximal ρ-semimonotone then the operator $(T + \beta I)^{-1} + \mu I$ is onto for all $\mu > \gamma(\rho)$, where $\gamma(\rho)$ is given by* (6).

Proof. By Theorem 2(ii), $\bar{T} = (T + \beta I)^{-1} + \gamma I$, with $\beta(\rho)$ as in (5), is maximal monotone. Since

$$(T + \beta I)^{-1} + \mu I = [(T + \beta I)^{-1} + \gamma I] + (\mu - \gamma)I = \bar{T} + (\mu - \gamma)I$$

and $\mu - \gamma > 0$, the result follows from Minty's Theorem. □

Corollary 2. *If $T : H \to \mathcal{P}(H)$ is maximal ρ-semimonotone then the operator $T + \lambda I$ is onto for all $\lambda \in (\beta(\rho), \eta(\rho))$, where $\beta(\rho)$ and $\eta(\rho)$ are given by* (5) *and* (7) *respectively.*

Proof. Fix $\beta(\rho), \gamma(\rho)$ and $\eta(\rho)$ as in (5)-(7). Given $\lambda \in (\beta, \eta)$, define $\mu = (\lambda - \beta)^{-1} > 0$. In view of (7), $\lambda < \eta$ implies that $\mu > \gamma$. By Corollary 1, $(T + \beta I)^{-1} + \mu I$ is onto. Fix $y \in H$. We must exhibit some $z \in H$ such that $y \in (T+\lambda I)(z)$. Since $(T+\beta I)^{-1}+\mu I$ is onto, there exists $x \in H$ such that $\mu y \in \left[(T+\beta I)^{-1}+\mu I\right](x)$, or equivalently, $\mu(y-x) \in (T+\beta I)^{-1}(x)$, that is to say,

$$x \in (T+\beta I)[\mu(y-x)] \tag{10}$$

Define $z=\mu(y-x)$. In view of (10), $y-\frac{1}{\mu}z = x \in (T+\beta I)(z)$, which is equivalent to

$$y \in \left[T+\left(\beta+\frac{1}{\mu}\right)I\right](z) = (T+\lambda I)(z), \tag{11}$$

in view of the definition of μ. It follows from (11) that the chosen z is an appropriate one, thus establishing the surjectivity of $T+\lambda I$. □

We prove next that if T is ρ-semimonotone then $[T+\lambda I]^{-1}$ is point-to-point and continuous for an apropriate λ.

Theorem 3. *Let $\beta(\rho)$ and $\eta(\rho)$ be given by* (5) *and* (7) *respectively. If $T : H \to \mathcal{P}(H)$ is ρ-semimonotone then the operator $(T+\lambda I)^{-1}$ is Lipschitz continuous for all $\lambda \in (\beta(\rho), \eta(\rho))$, with Lipschitz constant $L(\lambda)$ given by*

$$L(\lambda) = \frac{|1-\rho\lambda| + \sqrt{1-\rho^2}}{2\lambda - \rho(1+\lambda^2)}, \tag{12}$$

and henceforth point-to-point.

Proof. Take $u, v \in H$, $x \in (T+\lambda I)^{-1}(u)$ and $y \in (T+\lambda I)^{-1}(v)$. We must prove that

$$\|x-y\| \le L(\lambda)\|u-v\|. \tag{13}$$

Note that $u-\lambda x \in T(x)$, $v-\lambda y \in T(y)$, so that, applying Definition 2,

$$-\frac{\rho}{2}\left[\|x-y\|^2 + \|u-v-\lambda(x-y)\|^2\right] \le \langle (u-\lambda x)-(v-\lambda y), x-y\rangle =$$

$$\langle u-v, x-y\rangle - \lambda\|x-y\|^2. \tag{14}$$

Expanding the last term in the leftmost expression of (14) and rearranging, we get

$$\left[\lambda - \frac{\rho}{2}\left(1+\lambda^2\right)\right]\|x-y\|^2 - \frac{\rho}{2}\|u-v\|^2 \le (1-\lambda\rho)\langle u-v, x-y\rangle \le$$

$$|1-\lambda\rho|\|u-v\|\|x-y\|. \tag{15}$$

From the fact that $\lambda \in (\beta,\eta)$, it follows easily that $\lambda - \frac{\rho}{2}(1+\lambda^2) > 0$, so that, taking $u=v$ in (15), we obtain that $x=y$, and henceforth (13) holds when $u=v$. Otherwise, define

$$\omega = \frac{\|x-y\|}{\|u-v\|}$$

and observe that the inequality in (15) is equivalent to

$$\left[2\lambda - \rho\left(1+\lambda^2\right)\right]\omega^2 - 2|1-\lambda\rho|\omega - \rho \le 0. \tag{16}$$

Again, the fact that $\lambda \in (\beta, \eta)$ guarantees that the coefficient of ω^2 in the left hand side of (16) is positive, so that (16) holds iff ω belongs to the interval whose extrems are the two roots of the quadratic in the left hand side of (16), namely

$$\omega_1 = \frac{|1-\rho\lambda| - \sqrt{1-\rho^2}}{2\lambda - \rho(1+\lambda^2)}, \qquad \omega_2 = \frac{|1-\rho\lambda| + \sqrt{1-\rho^2}}{2\lambda - \rho(1+\lambda^2)}.$$

It is not hard to check that $\omega_1 < 0 < \omega_2$; the right inequality is immediate, and the left one follows easily from the fact that λ belongs to $(\beta(\rho), \eta(\rho))$. Since $\omega = \|x-y\|/\|u-v\|$ is positive, we conclude that (16) is equivalent to $\omega \le \omega_2$, which is itself equivalent to (13), in view of the definition of $L(\lambda)$, given in (12). The fact that $(T+\lambda I)^{-1}$ is point-to-point is an immediate consequence of (13). □

Corollary 3. *If $T : H \to \mathcal{P}(H)$ is ρ-semimonotone then the operator $(T^{-1} + \lambda I)^{-1}$ is Lipschitz continuous for all $\lambda \in (\beta(\rho), \eta(\rho))$, with Lipschitz constant $L(\lambda)$ given by* (12). *If in addition T is maximal, then $T^{-1} + \lambda I$ is onto for all $\lambda \in (\beta(\rho), \eta(\rho))$.*

Proof. The result follows from Proposition 1, Corolary 2 and Theorem 3. □

Remark 2. *Note that* $\lim_{\rho\to 1^-} \beta(\rho) = \lim_{\rho\to 1^-} \eta(\rho) = 1$, *and that* $\lim_{\rho\to 0^+} \beta(\rho) = 0$, $\lim_{\rho\to 0^+} \eta(\rho) = +\infty$, *so that the "regularity window" of a ρ-semimonotone operator T (i.e., the interval of values of λ for which $T+\lambda I$ is onto and its inverse is Lipschitz continuous), approaches the whole positive halfline when ρ approaches* 0, *i.e., when T approaches plain monotonicity, and reduces to a thin interval around* 1 *when ρ approaches* 1 *(remember that when $\rho = 1$ the inequality in* (1) *holds for any operator T, meaning that no "regularity window" can occur for $\rho = 1$).*

Remark 3. *Observe that*

$$0 < \beta(\rho) < \rho < 1 < \frac{1}{\rho} < \eta(\rho)$$

for all $\rho \in (0,1)$, so that 1, ρ *and* ρ^{-1} *always belong to the "regularity window" of a ρ-semimonotone operator T. We present next the values of the Lipschitz constant $L(\lambda)$ of $(T+\lambda I)^{-1}$ for the case in which λ takes these three special values:*

$$L(1) = \frac{1}{2}\left(1 + \sqrt{\frac{1+\rho}{1-\rho}}\right), \qquad L(\rho) = \frac{1}{\rho}\left(1 + \frac{1}{\sqrt{1-\rho^2}}\right), \qquad L\left(\frac{1}{\rho}\right) = \frac{\rho}{\sqrt{1-\rho^2}}.$$

We state next that the characterization of semimonotonicity presented in Lemma 1 for the one dimensional case is a necessary condition for the general case.

Corollary 4. *If $T : H \to \mathcal{P}(H)$ is ρ-semimonotone then the operator $(T + \rho^{-1}I)^{-1}$ is Lipschitz continuous with Lipschitz constant equal to $\theta(\rho)^{-1}$, where $\theta(\rho)$ is given by* (2).

Proof. The result follows from Theorem 3 and Remark 3 with $\lambda = \rho^{-1}$. □

A sufficient condition can be stated in terms of expansivity of T. We prove next that if T is expansive, with expansivity constant larger than or equal to $\eta(\rho)$ as given by (7) (an assumption stronger than Lipschitz continuity of $(T + \rho^{-1}I)^{-1}$ with Lipschitz constant equal to $\theta(\rho)^{-1}$), then T is ρ-semimonotone.

Proposition 7. *Take $\rho \in (0, 1)$. If $T : H \to \mathcal{P}(H)$ is ν-expansive with $\nu \geq \eta(\rho)$, then T is ρ-semimonotone.*

Proof. Fix $u \in T(x)$ and $v \in T(y)$, with $x \neq y$. Define $t = \dfrac{\|u - v\|}{\|x - y\|}$. Then $t \geq \nu$ because T is ν-expansive. Therefore $t \geq t_2 = \dfrac{1 + \sqrt{1 - \rho^2}}{\rho}$, where t_2 is the largest root of the quadratic $\dfrac{\rho}{2}t^2 - t + \dfrac{\rho}{2}$, as in the proof of Lemma 1. Thus,

$$\frac{\rho}{2}\left(t^2 + 1\right) \geq t \Longrightarrow \frac{\rho}{2}\left(\|x - y\|^2 + \|u - v\|^2\right) \geq \|u - v\|\|x - y\| \geq -\langle x - y, u - v\rangle$$

for all $x \neq y$. Since the inequality in (1) is trivially valid when $x = y$, the result holds. □

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