Axiom A versus Newhouse phenomena for Benedicks-Carleson toy models

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1 Introduction

Uniform hyperbolicity has been a long standing paradigm of complete dynamical description: any dynamical system such that the tangent bundle over its Limit set (the accumulation points of any orbit) splits into two complementary subbundles which are uniformly forward (respectively backward) contracted by the tangent map can be completely described from a geometrical and topological point of view.

Nevertheless, uniform hyperbolicity is a property less universal than it was initially thought: *there are open sets in the space of dynamics containing only non-hyperbolic systems.* Actually, Newhouse showed that for smooth surface diffeomorphisms, the unfolding of a *homoclinic tangency* (a non transversal intersection of stable and unstable manifolds of a periodic point) generates open sets of diffeomorphisms such that their Limit set is non-hyperbolic (see [N1], [N2], [N3]).

To explain his construction, firstly we recall that the stable and unstable sets

$$W^s(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to \infty\},$$

$$W^u(p) = \{y \in M : \ dist(f^n(y), f^n(p)) \to 0 \text{ as } n \to -\infty\}$$

are C^r-injectively immersed submanifolds when p is a hyperbolic periodic point of f.

Definition 1 *Let $f : M \to M$ be a diffeomorphism. We say that f exhibits a homoclinic tangency if there is a hyperbolic periodic point p of f such that the stable and unstable manifolds of p have a non-transverse intersection.*

It is important to say that a homoclinic tangency is (locally) easily destroyed by small perturbation of the invariant manifolds. To get open sets of diffeomorphisms with persistent homoclinic tangencies, Newhouse considers certain systems where the homoclinic tangency is associated to an invariant hyperbolic set with large fractal dimension. In particular, he studied the intersection of the local stable and unstable manifold of a hyperbolic set (for instance, a classical horseshoe), which, roughly speaking, can be visualized as a product of two Cantor sets whose thickness are large. Newhouse's construction depends on how this fractal invariant varies with perturbations of the dynamics, and actually this is the main reason that his construction works in the C^2−topology. In fact, Newhouse argument is based on the continuous dependence of the thickness with respect to C^2 perturbations. A similar construction in the C^1−topology leading to same phenomena is unknown (indeed, some results in the *opposite* direction can be found in [U] and [M]). In this setting, it was conjectured by Smale that

Axiom A surface diffeomorphisms are open and dense in $Diff^1(M)$.

In the present paper, we consider a special set of maps acting on a two dimensional rectangle. For this special type of systems, we show that, if one deals in C^2−topology, there are open set of diffeomorphisms which are not hyperbolic, while in the C^1−topology, the Axiom A property is open and dense.

A typical family where the Newhouse's phenomena hold is the so called Hénon maps. In fact, it was proved in [U2] that, for certain parameter of this family, the unfolding of a tangency leads to an open set of non-hyperbolic diffeomorphisms.

Numerical simulations indicate that the attractor of the Hénon map (i.e., the closure of the unstable manifold of its fixed saddle point) has the structure of the product of a line segment and a Cantor set with small dimension (when a certain parameter b is close to zero). Although it is a great oversimplification (and many of the later difficulties on the analysis of Hénon attractors arise because of the roughness of such approximation), this idea gives a very good understanding of the geometry of the Hénon map. As a guide to what follows, it is worth to point out that Benedicks and Carleson [BC, section 3, p. 89] have constructed a model where the point moves on a pure product space $(-1, 1) \times K$ where K is the Cantor set obtained by repeated iteration of the division proportions $(b, 1-2b, b)$ and the dynamics on $(-1, 1)$ is given by a family of quadratic maps: in fact, the dynamical system over $(-1, 1)$ act as a movement on a fan of lines, where each line has its own x-evolution, while it is contracted in the y-direction (see figure 1).

More precisely, consider a one parameter family $\{f_y\}_{y\in[0,1]}$ such that

$$f_y : [-1, 1] \to [-1, 1]$$

is a C^r−unimodal map verifying that 0 is the critical point and $f_y(0)$ is the maximum value of f_y for all $y \in [0, 1]$. We denote by $\mathcal{D}^r$ the set of families of C^r−unimodal maps satisfying the conditions stated above.

Let $k : [0, a]\cup[b, 1] \to [0, 1]$ be a C^r function such that $k(0) = 0 = k(1)$, $k(a) = 1 = k(b)$ and $|k'| > \gamma > 1$. Put

$$K(x, y) = \begin{cases} K_+(y) & \text{if } x > 0, \\ K_-(y) & \text{if } x < 0, \end{cases}$$

where $K_+ = (k_{/[0,a]})^{-1}, K_- = (k_{/[b,1]})^{-1}$.

The bulk of this article is the study of the dynamics of $F : ([-1, 1] \setminus \{0\} \times [0, 1]) \to [-1, 1] \times [0, 1]$ given by

$$F(x, y) = (f(x, y), K(x, y)) = (f_y(x), K_{sgn(x)}(y)). \qquad (1)$$

We denote by $\mathcal{D}^r$ the set of such maps F with the "usual" C^r-topology. Observe that the line $x = 0$ is a discontinuity line of any $F \in \mathcal{D}^r$, so that we are dealing with a one to one maps F which are C^r−diffeomorphisms only on $[-1, 1] \setminus \{0\} \times [0, 1] \to [-1, 1] \times [0, 1]$. In particular, we should tell some few words about the precise definition of the C^r-topology in this context:

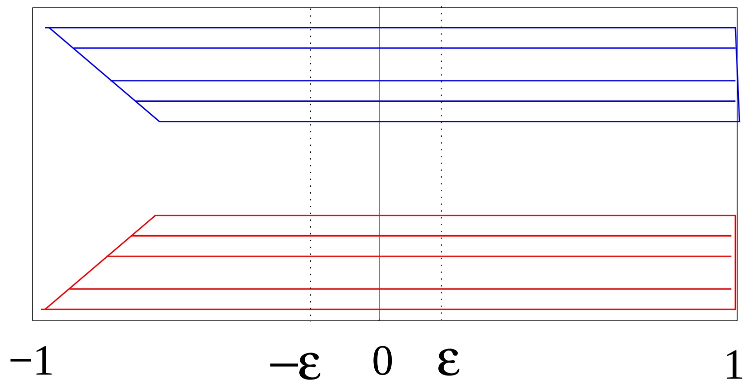

Figure 1: Dynamics of F.

Definition 2 *Given $F, \widetilde{F} \in \mathcal{D}^r$, consider $\{f_y\}_{y\in[0,1]}$ and $k : [0,a] \cup [b,1] \to [0,1]$ (resp., $\{\widetilde{f_y}\}_{y\in[0,1]}$ and $\widetilde{k} : [0,\widetilde{a}] \cup [\widetilde{b},1] \to [0,1]$) the functions associated to F (resp., $\widetilde{F}$). We say that F and $\widetilde{F}$ are C^r-close if $\{f_y\}_{y\in[0,1]}$ is C^r-close to $\{\widetilde{f_y}\}_{y\in[0,1]}$ in the usual manner, a is close to $\widetilde{a}$, b is close to $\widetilde{b}$ and k is C^r-close to $\widetilde{k}$.*

Now, let us recall that a set Λ is called *hyperbolic* for a dynamical system f if it is compact, f-invariant and the tangent bundle $T_\Lambda M$ admits a decomposition $T_\Lambda M = E^s \oplus E^u$ invariant under Df and there exist $C > 0$, $0 < \lambda < 1$ such that

$$|Df^n_{/E^s(x)}| \le C\lambda^n \text{ and } |Df^{-n}_{/E^u(x)}| \le C\lambda^n \;\; \forall x \in \Lambda, \;\; n \in \mathbb{N}.$$

Moreover, a diffeomorphism is called *Axiom A* if the non-wandering set is hyperbolic and it is the closure of the periodic points. In the sequel, $\Omega(F)$ denotes the non-wandering set and $L(F)$ the limit set.

At this point, we are ready to state our main results:

Theorem A. *For $r \ge 2$, there exists an open set $\mathcal{U} \subset \mathcal{D}^r$ such that, for any $F \in \mathcal{U}$, the limit set $L(F)$ of F is not a hyperbolic set. Moreover, there exists a residual set $\mathcal{R} \subset \mathcal{U}$ such that any $F \in \mathcal{R}$ has infinitely many periodic sinks.*

On the other hand, in the C^1-topology, the opposite statement holds:

Theorem B. *There exists an open and dense set $\mathcal{V} \subset \mathcal{D}^1$ such that $\Omega(F)$ is a hyperbolic set (and F is Axiom A) for any $F \in \mathcal{V}$.*

Concerning the proof of these results, a fundamental role will be played by certain points in the line $\{x = 0\}$:

Definition 3 *Given $F \in \mathcal{D}^r$, consider $k : [0,a] \cup [b,1] \to [0,1]$ the Cantor map related to F and denote by K_0 the Cantor set induced by k. For any $y \in K_0$, we call*

$$c_y^\pm = (0^\pm, y)$$

a critical point *of F.*

The relevance of this concept becomes clear from the following simple remark:

Remark 1.1 *It follows from the definition that, if $c_y^{\pm} \in L(F)$ and $c_y^{\pm}$ is not a periodic sink, then $L(F)$ is not hyperbolic.*

Closing this introduction, we give the organization of the paper:

- In section 2, we follow the same ideas of Newhouse to construct a C^2-open set $\mathcal{U}$ where the critical points can not be removed from the limit set, so that the proof of theorem A can be derived from the combination of this fact and the remark 1.1.

- In section 3, the proof of theorem B is presented. Morally speaking, our basic idea is inspired by a proof of Jakobson's theorem [J] (of C^1-density of hyperbolicity among unimodal maps of the interval) along the lines sketched in the book of de Melo and van Strien [dMvS]: namely, in the one-dimensional setting, one combines Mañé's theorem [M1] (giving the hyperbolicity of compact invariant sets far away from critical points of a C^2 Kupka-Smale interval map) with an appropriate C^1-perturbation to force the critical point to fall into the basin of a periodic sink. In our two-dimensional setting, we start by showing that the points of the limit set staying away from the critical line $\{x = 0\}$ belong to a hyperbolic set; this is done by proving that any compact set disjoint from the critical line exhibits a dominated splitting and then it is used theorem B in [PS1] (which is the two-dimensional generalization of Mañé's theorem [M1]) to conclude hyperbolicity. Next, we exploit a recent theorem of Moreira [M] about the non-existence of C^1-stable intersections of Cantor sets plus the geometry of the maps $F \in \mathcal{D}^1$ to prove a dichotomy for the critical points of a generic F: either critical points fall into the basins of a finite number of periodic sinks or they return to some small neighborhood of the critical line. Finally, we prove the critical points returning close enough to the critical line can be absorbed by the basins of a finite number of periodic sinks after a C^1-perturbation; thus, we conclude that the limit set of a generic $F \in \mathcal{D}^1$ is the union of an hyperbolic set with a finite number of periodic sinks, i.e., a generic $F \in \mathcal{D}^1$ is Axiom A.

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2 Proof of theorem A

The strategy is similar to the arguments of [N1] (see also [PT]).

Given $0 < t < 1$ and $m \geq m_0 = m_0(t)$ (where $m_0(t)$ is a large integer to be chosen later), we put $\delta_m := 1/(2^m - 1)$, $\epsilon_m := \sin(\pi\delta_m/2)$ and we select a parameter ρ_m such that $1 - \cos(\pi\delta_m) < t\rho_m/2 < 1 - \cos(\pi(1-\delta_m)/2^{m-1})$ (e.g., $\rho_m := 2(1 - \cos(3\pi\delta_m/2))/t$ works for $m_0(t)$ sufficiently large). Next, we take $\mu_m : [0,1] \to [0,1]$ a C^2-map such that $\mu_m(y) = \mu_m(1-y)$ and $\mu_m(y) = 1 - \sqrt{1 - \rho_m y/2}$ for every $y \in [0, \frac{t}{2}]$ and we define

$$F^t(x,y) = (f_{\epsilon_m}(x,y), K^t(x,y)), \tag{2}$$

with

$$K^t(x,y) := \begin{cases} (k^t_{/[0,\frac{t}{2}]})^{-1}(y) & \text{if } x > 0, \\ (k^t_{/[1-\frac{t}{2},1]})^{-1}(y) & \text{if } x < 0, \end{cases}$$

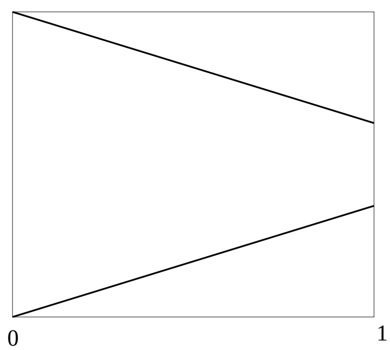
where k^t is the map

$$k^t(y) = \begin{cases} 2y/t & \text{if } 0 \le y \le t/2, \\ 2(1-y)/t & \text{if } 1-\frac{t}{2} \le y \le 1. \end{cases}$$

and $f_{\epsilon_m}(x,y)$ is a C^2 family of unimodal maps such that

$$f_{\epsilon_m}(x,y) = \begin{cases} 1-2x^2 & \text{if } |x| \ge \epsilon_m, \\ 1-\mu_m(y) & \text{at } x=0, \end{cases}$$

Also, let $K_0 = K_0^t$ be the Cantor set induced by $k = k^t$. See figure 2.



$$y \to \tfrac{t}{2}y,\ x > 0;\ \ y \to 1-\tfrac{t}{2}y,\ x < 0$$

Figure 2: Dynamics of K^t.

To simplify the exposition, firstly we consider the proof of theorem A only for maps $F = F^t$ of the form (2). Then, we explain how the general case follows from the previous one (at the end of this section).

We begin by recalling some classical facts about dynamically defined Cantor sets and their thickness. For a more detailed explanation, see [PT].

Definition 4 *We say that a Cantor set $\mathcal{K} \subset \mathbb{R}$ is* dynamically defined *if it is the maximal invariant set of a $C^{1+\alpha}$ expanding map with respect to a given Markov partition.*

Definition 5 *A* gap *(resp.* bounded gap*) of a Cantor set $\mathcal{K}$ is a connected component (resp., bounded connected component) of $\mathbb{R}-\mathcal{K}$. Given U a bounded gap of $\mathcal{K}$ and $u \in \partial U$, we call the* bridge *C of $\mathcal{K}$ at u to the maximal interval such that $u \in \partial C$ and C contains no point of a gap U' with $|U'| \ge |U|$. The thickness of $\mathcal{K}$ at u is $\tau(\mathcal{K},u) = |C|/|U|$ and the* thickness *$\tau(\mathcal{K})$ of $\mathcal{K}$ is the infimum over $\tau(\mathcal{K},u)$ for all boundary points u of bounded gaps.*

Remark 2.1 *For the Cantor sets K_0^t induced by the maps k^t above, it is not hard to see that $0 < \tau(K_0^t) = t/2(1-t) < \infty$.*

Remark 2.2 *The quadratic map $f_2(x) := 1-2x^2$ has arbitrarily thick dynamically defined Cantor sets. In fact, using the fact that $1-2x^2$ is conjugated to the complete* tent *map*

$$T_2(x) := \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2, \\ 2-2x & \text{if } 1/2 \leq x \leq 1, \end{cases}$$

via the explicit *conjugation $h(x) = -\cos(\pi x)$, we can exhibit thick Cantor sets as follows. Denote by $\widetilde{I}_2^{(m)} := [h(2\delta_m), h((1-\delta_m)/2^{m-2})]$ and put $\widetilde{I}_i^{(m)} := f_2(\widetilde{I}_{i-1}^{(m)})$ for $i = 3, \ldots, m$. As it is explained in the section 2 of chapter 6 of Palis-Takens book [PT], the intervals $h^{-1}(\widetilde{I}_2^{(m)}), \ldots, h^{-1}(\widetilde{I}_m^{(m)})$ form a Markov partition of a dynamically defined Cantor set K_m of thickness $\tau(K_m) = 2^{m-1} - 3$ associated to the tent map $T_2(x)$, and,* a fortiori, *$\widetilde{K}_m := h(K_m)$ are dynamically defined Cantor sets associated to f_2 (and Markov partition $\widetilde{I}_2^{(m)}, \ldots, \widetilde{I}_m^{(m)}$) such that $\tau(\widetilde{K}_m) \to \infty$ (as $m \to \infty$).*

Remark 2.3 *Let $\mathcal{K}(\psi)$ be the dynamically defined Cantor set associated to a $C^{1+\alpha}$ expanding map ψ. If ϕ is $C^{1+\alpha}$-close to ψ, then the thickness of $\mathcal{K}(\phi)$ is close to the thickness of $\mathcal{K}(\psi)$. In other words, the thickness of dynamically defined Cantor sets $\mathcal{K}$ depend continuously on $\mathcal{K}$ (with respect to the $C^{1+\alpha}$-topology). See [PT].*

Now we state Newhouse's *gap lemma* ensuring that two *linked* Cantor sets with large thickness should intersect somewhere:

Lemma 2.1 (Gap Lemma [N1]) *Given two Cantor sets $\mathcal{K}_1$ and $\mathcal{K}_2$ of $\mathbb{R}$ such that*

$$\tau(\mathcal{K}_1)\tau(\mathcal{K}_2) > 1,$$

then one of the following possibilities occurs:

- *$\mathcal{K}_1$ is contained in a gap of $\mathcal{K}_2$;*
- *$\mathcal{K}_2$ is contained in a gap of $\mathcal{K}_1$;*
- *$\mathcal{K}_1 \cap \mathcal{K}_2 \neq \emptyset$.*

For later reference, we recall the following definition:

Definition 6 *We say that $\mathcal{K}_1$ and $\mathcal{K}_2$ are* linked *if neither $\mathcal{K}_1$ is contained in a gap of $\mathcal{K}_2$ nor $\mathcal{K}_2$ is contained in a gap of $\mathcal{K}_1$.*

After these preliminaries, we can complete the discussion of this section as follows.

End of the proof of theorem A:

We observe that, since $F = F^t$ is the product map $F^t(x,y) = (1-2x^2, K^t_{sgn(x)}(y))$ at the region $([-1,\epsilon_m] \cup [\epsilon_m, 1]) \times [0,1]$, it follows that $\Lambda_{\epsilon_m} := \widetilde{K}_m \times K_0^t$ is a hyperbolic set of F^t. Moreover, the stable lamination $W^s(\Lambda_{\epsilon_m})$ is composed by vertical lines passing through $\widetilde{K}_m \times \{0\}$ and the unstable lamination $W^u(\Lambda_{\epsilon_m})$ is composed by horizontal lines passing through $\{0\} \times K_0^t$. We divide the construction of $\mathcal{U}$ into three steps.

Step 1: From remarks 2.1 and 2.2, given $0 < t < 1$, we can choose $m_0(t) \in \mathbb{N}$ large such that, for every $m \geq m_0(t)$, it holds

$$\tau(\widetilde{K}_m)\tau(K_0^t) > 1.$$

Step 2: Consider the following line segment:

$$L^+ := F^2(\{0^+\} \times [0, t/2]) = \{(1 - 2(1 - \mu_m(y))^2, \frac{t^2}{4}y)\}_{y \in [0,t/2]} = \{(-1 + \rho_m y, \frac{t^2}{4}y)\}_{y \in [0,t/2]}.$$

In the sequel, L^+ plays the role of a *line of tangencies*: more precisely, we introduce

$$\tilde{K}^s = (f_2^{-1}(\widetilde{I}_2^{(m)} \cap \widetilde{K}_m) \times [0, 1]) \cap L^+, \qquad \tilde{K}^u = F^2(\{0^+\} \times K_0^t) = W^u_{loc}(\Lambda_{\epsilon_m}) \cap L^+.$$

We claim that $\tilde{K}^s \cap \tilde{K}^u \neq \emptyset$. In fact, since L^+ is a straight line segment transversal to both horizontal and vertical foliations, we obtain that $\tau(\tilde{K}^s) = \tau(\widetilde{K}_m)$ and $\tau(\tilde{K}^u) = \tau(K_0^t)$, so that $\tau(\tilde{K}^s)\tau(\tilde{K}^u) > 1$ (by step 1). Hence, by Newhouse gap lemma 2.1, it suffices to show that $\tilde{K}^s$ and $\tilde{K}^u$ are linked. However, it is not hard to see that this follows from our choice of ρ_m. Indeed, from the definitions of $\tilde{K}^s$ and $\tilde{K}^u$, we get that $\tilde{K}^s$ and $\tilde{K}^u$ are linked if and only if the vertical projection $\overline{K}^s := f_2^{-1}(\widetilde{I}_2^{(m)} \cap \widetilde{K}_m)$ of $\tilde{K}^s$ is linked to the vertical projection $\overline{K}^u$ of $\tilde{K}^u$. On the other hand, $\overline{K}^s$ and $\overline{K}^u$ are linked because their convex hulls are linked: more precisely, the convex hull I^s of $\overline{K}^s$ is $f_2^{-1}(\widetilde{I}_2^{(m)}) = [-\cos(\pi\delta_m), -\cos(\pi(1 - \delta_m)/2^{m-1})]$ and the convex hull I^u of $\overline{K}^u$ is $[0, -1 + t\rho_m/2]$, so that our choice of ρ_m verifying

$$1 - \cos(\pi\delta_m) < t\rho_m/2 < 1 - \cos(\pi(1 - \delta_m)/2^{m-1})$$

implies that I^s and I^u are linked.

Next, we notice that $\tilde{K}^s \cap \tilde{K}^u \neq \emptyset$ means that $F^2(c_y^+) \in W^s_{loc}(\Lambda_{\epsilon_m})$ for some critical point c_y^+, $y \in K_0^t$. It follows that c_y^+ is a non-periodic critical point belonging to the Limit set $L(F)$. Therefore, from remark 1.1, the Limit set is not hyperbolic.

Step 3: Finally, we claim that any sufficiently small C^2 neighborhood $\mathcal{U} \subset \mathcal{D}^2$ of the map $F = F^t$ constructed above fits the conclusion of the first part of theorem A. Indeed, this is a consequence of the following known facts:

1. The hyperbolic basic set Λ_{ϵ_m} has a continuation to an invariant hyperbolic basic set $\Lambda_{\epsilon_m}(G)$ of G;

2. The Cantor sets $\tilde{K}^s$ and $\tilde{K}^u$ have unique continuations to Cantor sets $\tilde{K}^s(G)$ and $\tilde{K}^u(G)$. Moreover, these Cantor sets are $C^{1+\alpha}-$close to $\tilde{K}^s$ and $\tilde{K}^u$ respectively;

3. Thus, the Cantor sets $\tilde{K}^s(G)$ and $\tilde{K}^u(G)$ have thickness close to the thickness of $\tilde{K}^s$ and $\tilde{K}^u$ respectively; by continuity of the thickness (see remark 2.3), it follows that $\tau(\tilde{K}^u(G))\tau(\tilde{K}^u(G)) > 1$;

4. From Newhouse gap lemma 2.1, it follows that $\tilde{K}^s(G) \cap \tilde{K}^u(G) \neq \emptyset$;

5. Hence, there are (non-periodic) critical points contained in the Limit set of G, and so, by remark 1.1, it is not hyperbolic.

At this point, it remains only to prove the second part of theorem A, namely, the existence of a residual set $\mathcal{R} \subset \mathcal{U}$ such that any $F \in \mathcal{R}$ has infinitely many sinks.

Let $\mathcal{U}_n \subset \mathcal{U}$ be the (open) subset of maps $F \in \mathcal{U}$ with n attracting periodic orbits (at least) and $\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{U}_n$. In this notation, our task is reduced to show the next proposition.

Proposition 2.1 *$\mathcal{U}_n$ is dense for every $n \in \mathbb{N}$.*

We start our argument with the following notion.

Definition 7 *We say that $F \in \mathcal{D}^r$ exhibits a "heteroclinic tangency" if there are two periodic points p and q of F such that*

1. *there exists $c^{\pm} = (0^{\pm}, y_p) \in W^u(p)$;*
2. *there exists $k > 0$ such that $F^k(c^{\pm}) \in W^s_{loc}(q)$ and $F^j(c^{\pm})$ does not intersect the critical line for $j < k$;*
3. *the unstable manifold of q intersects transversally the stable manifold of p.*

The relevance of the heteroclinic tangencies becomes apparent in the next lemma.

Lemma 2.2 *Let $F \in \mathcal{D}^r$ with a heteroclinic tangency. Then, there exists $G \in \mathcal{D}^r$ arbitrarily C^r close to F having an attracting periodic point near the heteroclinic tangency.*

Proof: From the facts that $c^{\pm} \in W^u(p)$ and the unstable manifold of q intersects transversally the stable manifold of p, it follows that there exists $c_n^{\pm} = (0^{\pm}, y_n)$ such that $F^{-k_n}(c_n^{\pm}) \to F^k(c^{\pm})$ (for some appropriate sequence k_n) and $c_n^{\pm} \to c^{\pm}$. Hence, we can take a C^r small perturbation G of F such that $G^k(c_n^{\pm}) = F^{-k_n}(c_n^{\pm})$ and $G = F$ along the orbit $F^{-j}(c_n^{\pm})$ for $j = 1, \dots, k_n$ (provided that n is large enough). In particular, we have that $c_n^{\pm} = G^{k+k_n}(c_n^{\pm})$ is a super-attracting periodic point of G of period $k + n$. This ends the proof.

■

On the other hand, heteroclinic tangencies are frequent inside $\mathcal{U}$.

Lemma 2.3 *Let $F \in \mathcal{U}$. Then, there exists G C^r close to F exhibiting a heteroclinic tangency.*

Proof: This is an immediate consequence of the construction of $\mathcal{U}$: given $F \in \mathcal{U}$, we can find $x_1, x_2 \in \Lambda_{\epsilon_m}$ and a critical point $c^{\pm} \in W^u_{loc}(x_1)$ such that $F^k(c^{\pm}) \in W^s_{loc}(x_2)$; let p and q be periodic points in Λ_{ϵ_m} close to x_1 and x_2 (resp.) so that their local unstable and stable manifolds are close to the corresponding invariant manifolds of x_1 and x_2 (resp.), and they are homoclinically related; in this situation, after a proper small perturbation, we can find G C^r-close to F such that G has a heteroclinic tangency involving p and q.

■

Finally, the proof of the desired proposition follows from a direct combination of the two previous lemmas.

Proof of proposition 2.1: It is proved by induction. Given $F \in \mathcal{U}_n$, we can use the lemma 2.3 to find G C^r close to F keeping the same number n of attracting periodic points of G_n such that G_{n+1} has a heteroclinic tangency. By lemma 2.2, we can unfold this tangency to create a new sink, i.e., we can find $H \in \mathcal{U}_{n+1}$ C^r close to G. The result follows.

$\blacksquare$

This completes the proof of theorem A.

3 Proof of theorem B

Before giving the proof of theorem B, we briefly outline the strategy. Given $\epsilon > 0$, let us take $U_\epsilon = ([-1, -\epsilon] \cup [\epsilon, 1]) \times [0, 1]$ and

$$\Lambda_\epsilon = \Omega(F) \cap \bigcap_{n \in \mathbb{Z}} F^n(U_\epsilon).$$

Strategy of the proof.

1. For any $\epsilon > 0$, we show that, C^1-generically, the set Λ_ϵ is composed by a locally maximal hyperbolic set and a finite number of periodic attracting points. This is performed in subsection 3.1 (see theorem 3.1).

2. We show that, C^1-generically, any critical point either it is contained in the basin of attraction of the sinks (of step 1 above) or it returns to $[-\epsilon, \epsilon] \times [0, 1]$. This is performed in subsection 3.2.

3. Later, we produce a series of C^1-perturbations (of size proportional to ϵ) in the way to create a finite number of periodic sinks such that their basins contain the critical points. This is performed in subsection 3.3.

4. From items 1, 2 and 3, it follows that $\Omega(F) \subset \Lambda_\epsilon \cup \{p_1, ..., p_k\}$, where each p_i is a periodic attracting point ($i = 1, \dots, k$), and therefore it is concluded that $\Omega(F)$ is hyperbolic (and F is Axiom A).

3.1 Hyperbolicity of Λ_ϵ

Theorem 3.1 *Let $\epsilon > 0$ be a positive constant. Then, for a C^1-generic $F \in \mathcal{D}^2$, Λ_ϵ contains a finite number of periodic attracting points and the complement of the basin of attraction of them $\hat{\Lambda}_\epsilon$ exhibits a hyperbolic splitting $T\hat{\Lambda}_\epsilon = E^s \oplus E^u$ such that E^s is contractive, E^u is expansive (and, in fact, $E^u = \mathbb{R} \cdot (1, 0)$).*

The proof of this result uses the notion of dominated splitting and theorem B in [PS1]. Firstly, we revisit the definition of dominated splittings:

Definition 8 *An f-invariant set Λ has a* dominated splitting *if we can decompose its tangent bundle into two invariant subbundles $T_\Lambda M = E \oplus F$ such that:*

$$\|Df^n_{/E(x)}\| \cdot \|Df^{-n}_{/F(f^n(x))}\| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0. \tag{3}$$

with $C > 0$ and $0 < \lambda < 1$.

Secondly, we recall that Pujals and Sambarino [PS1] proved that any compact invariant set exhibiting dominated splitting of a generic C^2 surface diffeomorphism is hyperbolic:

Theorem 3.2 ([PS1]) *Let $f \in Diff^2(M^2)$ be a C^2-diffeomorphism of a compact surface and $\Lambda \subset \Omega(f)$ a compact invariant set exhibiting a dominated splitting. Assume that all periodic points in Λ are hyperbolic of saddle type. Then, Λ can be decomposed into a hyperbolic set and a finite number of normally hyperbolic periodic closed curves whose dynamical behaviors are C^2-conjugated to irrational rotations.*

We claim that it suffices to prove the next proposition in order to conclude the proof of theorem 3.1:

Proposition 3.1 *Let $\epsilon > 0$ be a positive constant. Then, for a C^1-generic $F \in \mathcal{D}^2$, the set Λ_ϵ contains a finite number of periodic attracting points, the complement of the basins of attraction of them $\hat{\Lambda}_\epsilon$ exhibits a dominated splitting $T\hat{\Lambda}_\epsilon = E^s \oplus F$ such that E^s is contractive (after $n_0 = n_0(\epsilon)$ iterations), F is spanned by $(1,0)$, and all periodic points in $\hat{\Lambda}_\epsilon$ are hyperbolic of saddle type.*

In fact, since it is immediate that there are no periodic closed curves inside Λ_ϵ whose dynamical behavior are conjugated to irrational rotations[1], we can put proposition 3.1 and theorem 3.2 together so that the hyperbolicity of Λ_ϵ follows.

Thus, we devote most of the rest of this subsection to the proof of this proposition. Let us begin with some useful notation. Given (x_0, y_0), we denote by $(x_i, y_i) := F^i(x_0, y_0)$; also, we write the derivative of a map $F(x,y) = (f(x,y), K(x,y))$ of the form (1) as

$$DF = \begin{pmatrix} f_x & f_y \\ 0 & K_y \end{pmatrix}.$$

In particular, it follows that

$$DF^n(x_0, y_0) = \begin{pmatrix} A_n & B_n \\ 0 & D_n \end{pmatrix},$$

where

$$A_n := \Pi_{i=0}^{n-1} f_x(x_i, y_i) \quad , D_n := \Pi_{i=0}^{n-1} K_y(x_i, y_i),$$

$$B_n = \sum_{j=0}^{n-1} f_y(x_j, y_j)\, \Pi_{i=0}^{j-1} K_y(x_i, y_i)\, \Pi_{i=j+1}^{n-1} f_x(x_i, y_i).$$

[1] Because in this case $F = \mathbb{R} \cdot (1,0)$ should be a tangent line of such closed curve C at some point. Combining this fact with the minimality of the dynamics on C and the continuity of dominated splitting (besides the invariance of $\mathbb{R} \cdot (1,0)$), we obtain that the whole curve C is tangent to the line field F, a contradiction.

In the sequel, we fix two positive constants λ_0, λ_1 such that $\lambda_0 < \lambda_1 < 1$ and

$$|K_y| < \lambda_0.$$

Concerning the proof of proposition 3.1, we observe that $(1,0)$ is an invariant direction by DF and, moreover, it is the natural candidate to be the expansive one. Therefore, the existence of a dominated splitting follows once we build up an invariant cone field around $(1,0)$. To perform this task, first we need the next lemma.

Lemma 3.1 *For any $F \in \mathcal{D}^1$, there exist a finite number of attracting periodic points with trajectory in Λ_ϵ and a positive integer $n_0 = n_0(\varepsilon)$ such that, for any $(x_0, y_0) \in \Lambda_\epsilon$ outside the basins of attraction of those periodic points, it holds*

$$|A_n| = \Pi_{i=0}^{n-1} |f_x(x_i, y_i)| > \lambda_1^n,$$

whenever $n > n_0$.

In order to do not interrupt the flow of ideas, we postpone the proof of the lemma. Assuming momentarily this lemma, we are able to prove the desired proposition.
Proof of proposition 3.1. Let b be a positive constant such that

$$|f_y| < b.$$

Take n_0 the integer provided by lemma 3.1 and let R_0 be a positive constant[2] such that, for any $m < n_0$ and any point $(x_0, y_0) \in \Lambda_\epsilon$, it holds

$$\Pi_{i=0}^{m} |f_x(x_i, y_i)| > R_0^{-1}.$$

Now, for all $(x_0, y_0) \in \Lambda_\epsilon$ outside the basins of the attracting points of lemma 3.1, let us bound B_n for $n > n_0$:

$$\begin{aligned} |B_n| &\leq \sum_{j=0}^{n-1} |f_y(x_j, y_j)| \Pi_{i=0}^{j-1} |K_y(x_i, y_i)| \, \Pi_{i=j+1}^{n-1} |f_x(x_i, y_i)| \qquad (4) \\ &= \sum_{j=0}^{n-1} |f_y(x_j, y_j)| \cdot |D_j| \cdot \frac{|A_n|}{|A_{j+1}|} \\ &< R_0 b n_0 |A_n| \frac{1}{1-\lambda_0} + b|A_n| \sum_{j=n_0}^{n-1} \frac{\lambda_0^j}{\lambda_1^j} \\ &< R_0 b n_0 |A_n| \frac{1}{1-\lambda_0} + b|A_n| \frac{1}{\lambda_1 - \lambda_0}. \end{aligned}$$

Using this estimate, we claim that the cone field $C(\gamma_0) = C(\mathbb{R} \cdot (1,0), \gamma_0)$ is a forward invariant cone field for sufficiently small $\gamma_0 > 0$. In fact, take $\gamma_0 > 0$ small and let us consider

$$v_n = DF^n(1, \gamma) = (A_n + \gamma B_n, \gamma D_n),$$

[2]Such a constant R_0 always exists since $f(x, y) \in \mathcal{U}^r$ is a family of unimodal maps whose (unique) critical point is $x = 0$ and $(x_i, y_i) \in U_\epsilon$ implies $|x_i| \geq \epsilon$.

where $|\gamma| < \gamma_0$. The slope of v_n with respect to $(1,0)$ is

$$|\mathrm{slope}(v_n,(1,0))| = \frac{|\gamma D_n|}{|A_n + \gamma B_n|}.$$

Note that the estimate (4) implies

$$|A_n + \gamma B_n| > |A_n| - \gamma_0 |B_n| > |A_n| \left(1 - \gamma_0 b (R_0 n_0 + 1)\cdot(\lambda_1 - \lambda_0)^{-1}\right).$$

Hence, if γ_0 is small so that

$$1 - \gamma_0 b(R_0 n_0 + 1)(\lambda_1 - \lambda_0)^{-1} > \frac{1}{2},$$

using lemma 3.1, we conclude that

$$\mathrm{slope}(v_n,(1,0)) < \gamma_0 \frac{2|D_n|}{|A_n|} < 2\left(\frac{\lambda_0}{\lambda_1}\right)^n \cdot \gamma_0.$$

Thus, assuming n_0 large so that $(\lambda_0/\lambda_1)^{n_0} < 1/4$ and taking $\gamma_1 = 2(\lambda_0/\lambda_1)^{n_0}\gamma_0$, we see that, for any $n > n_0$,

$$DF^n(C(\gamma_0)) \subset C(\gamma_1) \subset C(\gamma_0/2).$$

In other words, $C(\gamma_0)$ is a forward invariant cone field and the existence of a dominated splitting $E^s \oplus \mathbb{R}\cdot(1,0)$ is guaranteed (over the set $\hat{\Lambda}_\epsilon$ of points outside the basins of the attracting points of lemma 3.1).

Next, we show that E^s is uniformly contracted: for every $(x_0,y_0) \in \Lambda_\epsilon$, we fix $e_0^{(s)} = (u_0^s, v_0^s) \in E^s_{(x_0,y_0)}$ with $\|e_0^{(s)}\| = 1$ and we put $DF^n(x_0,y_0)\cdot e_0^{(s)} := \pm\lambda_n^s \cdot e_n^{(s)} \in E^s_{(x_n,y_n)}$. Next, we compute the determinant of DF^n:

$$|A_n \cdot D_n| = |\det DF^n| = \frac{|DF^n\cdot(1,0)\wedge DF^n\cdot e_0^{(s)}|}{|(1,0)\wedge e_0^{(s)}|} = \frac{|A_n|\cdot|\lambda_n^{(s)}|\cdot|v_n^{(s)}|}{|v_0^{(s)}|},$$

where $|u \wedge v|$ denotes the area of the rectangle determined by the vectors u and v. Because the direction E^s does not belong to the cone field $C(\gamma_0)$ and $|v_0^{(s)}| \le \|e_0^{(s)}\| = 1$, we get

$$|\lambda_n^{(s)}| = |D_n| \frac{|v_0^{(s)}|}{|v_n^{(s)}|} \le \frac{1}{\gamma_0}|D_n|.$$

Since $|D_n| \le \lambda_0^n$ for all $n \in \mathbb{N}$, this proves that for all $F \in \mathcal{D}^1$ of the form (1) such that $|K_y| < \lambda_0$ and for any $\lambda_0 < \lambda_1 < 1$, the set Λ_ϵ is the union of a finite number of sinks and a set $\hat{\Lambda}_\epsilon$ exhibiting a dominated decomposition $E^s \oplus F$ where E^s is contractive (after n_0 iterates) and $F = (1,0)\cdot\mathbb{R}$ satisfies $DF^n(1,0) = (A_n, 0)$ where $|A_n| > \lambda_1^n$ (for $n > n_0$).

Finally, it remains only to see that, for a C^1-generic $F \in \mathcal{D}^2$, it is possible to take $\hat{\Lambda}_\epsilon$ such that all periodic points in $\hat{\Lambda}_\epsilon$ are hyperbolic of saddle type. However, this is a consequence of a simple argument (compare with [PS1, p. 966]): recall that, by the transversality theorem, all periodic points hyperbolic of a C^1-generic $F \in \mathcal{D}^2$ are hyperbolic; it follows that for such a $F \in \mathcal{D}^2$, the compact invariant subset $\Lambda_\epsilon^{(0)} := \Lambda_\epsilon - \{p \in$

$\Lambda_\epsilon : p$ is a periodic sink$\} \subset \Omega(F)$ only contains hyperbolic periodic points of saddle type. Furthermore, $\Lambda_\epsilon^{(0)}$ admits a dominated splitting (since $\Lambda_\epsilon^{(0)} \subset \hat{\Lambda}_\epsilon$). Thus, we obtain from theorem 3.2 that $\Lambda_\epsilon^{(0)}$ is a hyperbolic set. We claim that $P_\epsilon(F) := \Lambda_\epsilon - \Lambda_\epsilon^{(0)}$ is finite (so that $\Lambda_\epsilon^{(0)} = \hat{\Lambda}_\epsilon$ and, *a fortiori*, all periodic points of $\hat{\Lambda}_\epsilon$ are hyperbolic of saddle-type). Indeed, if $\# P_\epsilon(F) = \infty$, we have $\emptyset \neq \overline{P_\epsilon(F)} - P_\epsilon(F) \subset \Lambda_\epsilon^{(0)}$. However, since $\Lambda_\epsilon^{(0)}$ is hyperbolic, we can select a compact neighborhood U of Λ_ϵ such that the maximal invariant of U is hyperbolic. Thus, we get that, up to removing a finite number of periodic sinks, $P_\epsilon(F) \subset U$, a contradiction with the hyperbolicity of the maximal invariant subset of U. This completes the proof of the proposition 3.1.

$\blacksquare$

Closing the proof of the hyperbolicity of Λ_ϵ, we prove the statement of lemma 3.1.

Proof of lemma 3.1. It is enough to apply the following lemma due to Pliss (see [Pl], [M2]).

Lemma 3.2 (Pliss) *Given* $0 < \gamma_0 < \gamma_1 < 1$ *and* $a > 0$*, there exist* $n_0 = n_0(\gamma_0, \gamma_1, a)$ *and* $l = l(\gamma_0, \gamma_1, a) > 0$ *such that, for any sequences of numbers* $\{a_i\}_{0 \leq i \leq n}$ *with* $n_0 < n$, $a^{-1} < a_i < a$ *and* $\Pi_{i=0}^n a_i < \gamma_0^n$*, there are* $1 \leq n_1 < n_2 < \ldots < n_r \leq n$ *with* $r > ln$ *and such that*

$$\Pi_{i=n_j}^k a_i < \gamma_1^{k-n_j} \quad n_j \leq k \leq n.$$

In fact, let us consider the set of points $(z, w) \in \Lambda_\epsilon$ such that

$$\liminf_{n \to \infty} \frac{1}{n} \log |A_n(z, w)| < \log \sqrt{\lambda_1}. \tag{5}$$

Since, for any $(z_i, w_i) = F^i(z, w) \in \Lambda_\epsilon$, it holds $|z_i| \geq \epsilon$, we can use the lemma 3.2 twice to obtain that there exists a subsequence of forward iterates of (z, w) accumulating on some point (x_0, y_0) which has a subsequence of forward iterates

$$\{(x_{n_j}, y_{n_j})\}_{j>0} = \{F^{n_j}(x_0, y_0)\}$$

such that any (x_{n_j}, y_{n_j}) satisfies

$$|A_n(x_{n_j}, y_{n_j})| < \sqrt{\lambda_1^n}, \ \forall\, n > 0.$$

Using the same type of calculation of estimative (4), we get, for any $j > 0$,

$$\Pi_{i=0}^n ||DF(x_{i+n_j}, y_{i+n_j})|| < (1 + b(\sqrt{\lambda_1} - \lambda_0)^{-1})\sqrt{\lambda_1^n}, \ \forall\, n > 0.$$

By standard arguments it follows that, for any $\sqrt{\lambda_1} < \lambda_2 < 1$, there exists $\gamma = \gamma(\lambda_1, \lambda_2)$ such that

$$F^n(B_\gamma(x_{n_j}, y_{n_j})) \subset B_{\lambda_2^n \gamma}(F^n(x_{n_j}, y_{n_j}))$$

for all $j, n > 0$. Taking q_0 an accumulation point of $\{(x_{n_j}, y_{n_j})\}$, it is not hard to see that

$$F^j(B_{\frac{\gamma}{2}}(q_0)) \subset B_{\lambda_2^j \frac{\gamma}{2}}(F^j(q_0))$$

for any $j > 0$ and there exists a positive integer $m = m(q_0)$ such that

$$F^m(B_{\frac{\gamma}{2}}(q_0)) \subset B_{\lambda_2^m \frac{\gamma}{2}}(q_0).$$

Therefore, it follows that:

1. there is an unique attracting periodic point[3] p_0 inside $B_{\frac{\gamma}{2}}(q_0)$,

2. the basin of attraction of p_0 contains $B_{\frac{\gamma}{2}}(q_0)$,

3. the point (x_0, y_0) and the initial point (z, w) verifying (5) belong the basin of attraction p_0;

Since the number of attracting periodic point with local basin of attraction with radius larger than $\frac{\gamma}{2}$ is finite, we conclude that there are a finite number of periodic attracting points whose basins contain the points of Λ_ϵ verifying (5). This concludes the proof of the lemma.

$\blacksquare$

Remark 3.1 *For later use, we observe that the hyperbolic sets Λ_ϵ can be assumed to be locally maximal. More precisely, we claim that there exists a locally maximal hyperbolic set $\Lambda_\epsilon \subset \widetilde{\Lambda}_\epsilon \subset U_{\epsilon/2}$. Indeed, fix $\gamma = \gamma(\epsilon) > 0$ a positive small constant such that the local stable manifold $W^s_\gamma(p)$ of any point $p \in \Lambda_{\epsilon/2}$ is the graph of a real function of the y-coordinate defined over an interval of length $\delta = \delta(\epsilon) > 0$. Next, we take $k = k(\epsilon) > 0$ a large integer so that the lengths of the 2^k intervals $I_1^{(k)}, \dots, I_{2^k}^{(k)}$ of the kth stage of the construction of the Cantor set K_0 are $< \delta/2$. Note that we can suppose that $W^s_\gamma(p) \subset U_{\epsilon/2}$ for any $p \in \Lambda_{\epsilon/2} \cap U_{3\epsilon/4}$. Now, for each $j = 1, \dots, 2^k$, we consider the stable lamination $\mathcal{F}^s_{j,\pm} = \{W^s_\gamma(p) \cap [-1,1] \times I_j^{(k)}\}_{p \in \hat{\Lambda}_{\epsilon/2} \cap U_{3\epsilon/4}}$. Given $\ell \in \mathcal{F}^s_{j,\pm}$, we denote by $R^{(k)}_{j,\pm}(\ell)$ the rectangle delimited by the four lines $\{\pm 1\} \times [0,1]$, $[-1,1] \times \partial I_j^{(k)}$ and ℓ. Given $\ell, \widetilde{\ell} \in \mathcal{F}^s_{j,\pm}$, we say that $\ell \prec \widetilde{\ell}$ if and only if $R^{(k)}_{j,\pm}(\ell) \subset R^{(k)}_{j,\pm}(\widetilde{\ell})$. Observe that $\prec$ is a* total order[4] *of $\mathcal{F}^s_{j,\pm}$. Thus, for each $j = 1, \dots, 2^k$, we can define $\ell_{j,\pm} \in \mathcal{F}^s_{j,\pm}$ the* outermost *stable leaf of $\hat{\Lambda}_{\epsilon/2} \cap U_{3\epsilon/4} \cap [-1,1] \times I_j^{(k)}$ as the unique leaf of $\mathcal{F}^s_{j,\pm}$ such that $\ell \prec \ell_{j,\pm}$ for all $\ell \in \mathcal{F}^s_{j,\pm}$. Consider the family of rectangles $R^{(k)}_{j,\pm}(\epsilon) := R^{(k)}_{j,\pm}(\ell_{j,\pm})$. Finally, let $\widetilde{\Lambda}_\epsilon$ be the maximal invariant set associated to this family of rectangles. It follows that $\widetilde{\Lambda}_\epsilon$ has local product structure (because $W^s_{loc}(p) \cap W^u_{loc}(q) \in R^{(k)}_{j,\pm}(\epsilon)$ when $p, q \in \widetilde{\Lambda}_\epsilon \cap R^{(k)}_{j,\pm}(\epsilon)$) and $\Lambda_\epsilon \subset \widetilde{\Lambda}_\epsilon \subset U_{\epsilon/2}$ (because $\Lambda_\epsilon \cap [-1,1] \times I_j^{(k)} \subset R^{(k)}_{j,\pm} \subset U_{\epsilon/2}$). This proves our claim.*

Remark 3.2 *We claim that $\widetilde{\Lambda}_\epsilon$ is the maximal invariant set of $U_\epsilon \cup \widetilde{R_\epsilon}$, where $\widetilde{R_\epsilon} = \{R^{(k)}_{j,\pm}(\ell_{j,\pm})\}$ is the family of rectangles introduced in the previous remark. Indeed, given z a point whose orbit $\mathcal{O}(z)$ stays in $U_\epsilon \cup \widetilde{R_\epsilon}$, we note that $z \in \Lambda_{\epsilon/2}$ (since $U_\epsilon \cup \widetilde{R_\epsilon} \subset U_{\epsilon/2}$). On the other hand, we have two possibilities:*

- *$\mathcal{O}(z) \subset \widetilde{R_\epsilon}$: this means that $z \in \widetilde{\Lambda}_\epsilon$;*
- *there exists $y \in \mathcal{O}(z) - \widetilde{R_\epsilon}$: this means that $y \in (U_\epsilon \cap \Lambda_{\epsilon/2}) - \widetilde{R_\epsilon}$, a contradiction (since, by definition, $U_\epsilon \cap \Lambda_{\epsilon/2} \subset U_{3\epsilon/4} \cap \Lambda_{\epsilon/2} \subset \widetilde{R_\epsilon}$).*

[3] Actually, using that $(x_{n_j}, y_{n_j}) = F^{n_j}(x_0, y_0) \to q_0$, it can be concluded that q_0 is the periodic point.

[4] Because any two distinct stable leaves are disjoint and $\partial\ell \subset [-1,1] \times \partial I_j^{(k)}$ for any $\ell \in \mathcal{F}^s_{j,\pm}$.

In particular, it follows that the positive orbit $\mathcal{O}^+(p)$ of every point $p \notin W^s(\widetilde{\Lambda_\epsilon})$ escapes any sufficiently small neighborhood *of $U_\epsilon \cup \widetilde{R_\epsilon}$. In fact, if the positive orbit of a given point p stays forever inside a small neighborhood W of $U_\epsilon \cup \widetilde{R_\epsilon}$, its accumulation points always belong to the maximal invariant set $\Lambda(W)$ of W. However, since the maximal invariant $\widetilde{\Lambda_\epsilon}$ of $U_\epsilon \cup \widetilde{R_\epsilon}$ is locally maximal (by remark 3.1), $\Lambda(W) = \widetilde{\Lambda_\epsilon}$ for any small neighborhood W of $U_\epsilon \cup \widetilde{R_\epsilon}$. Hence, $p \in W^s(\widetilde{\Lambda_\epsilon})$, an absurd.*

Before proceeding further, we use a fundamental result of C. G. Moreira to improve the geometry of the isolating neighborhood of $\widetilde{\Lambda_\epsilon}$.

Theorem 3.3 ([M]) *Generically in the C^1-topology, a pair of C^1-dynamically defined Cantor sets are disjoint. In particular, the arithmetic difference of a C^1 generic pair of C^1-dynamically defined Cantor sets has empty interior (so that it is also a Cantor set).*

More precisely, combining our theorem 3.1 with this theorem, we have the following consequence:

Corollary 3.1 *Fix $\epsilon > 0$. Then, for a C^1-generic $F \in \mathcal{D}^2$, the maximal invariant set Λ_ϵ of U_ϵ is a locally maximal hyperbolic set such that $int(U_\epsilon)$ is an isolating neighborhood of Λ_ϵ.*

Proof: Let $F \in \mathcal{D}^2$ a C^1-generic map verifying theorem 3.1. We consider a finite Markov partition $\{P_i\}_{i=1}^M$ of $\widetilde{\Lambda_{\epsilon/2}}$ with small diameter. We take $p_i \in P_i \cap \widetilde{\Lambda_{\epsilon/2}}$ and we define $E_i := E^s(p_i)$. Since the stable foliation of F restricted to P_i is $C^{1+\alpha}$-close to the foliation of P_i by straight lines with direction E_i when the diameter of the Markov partition is small, we can assume, up to performing a $C^{1+\alpha}$-perturbation of the unimodal family $f(x,y)$, that the stable foliation of F restricted to P_i is the foliation by straight lines parallel to E_i. Recall that the angle between the stable directions E_i and the unstable (horizontal) directions is uniformly bounded away from zero. In particular, we also have a system of coordinates on each P_i (given by the horizontal foliation and the foliation by lines parallel to E_i) where we can write $F|_{P_i}(x,y) = (f_i(x), K_{sgn(x)}(y))$ and $\widetilde{\Lambda_{\epsilon/2}} \cap P_i$ is a product of two dynamically defined Cantor sets, i.e., $\widetilde{\Lambda_{\epsilon/2}} \cap P_i = K_i^s \cdot (1,0) + K_i^u \cdot (\mu_i, 1)$ with K_i^s, K_i^u dynamical Cantor sets of the real line and $(\mu_i, 1) \in E_i$.

In this context, the fact that the verticals $\{\pm\epsilon\} \times [0,1]$ don't intersect $\widetilde{\Lambda_{\epsilon/2}}$ is equivalent to $\pm\epsilon \notin K_i^s + \mu_i \cdot K_i^u$ for every $i = 1, \dots, M$. However, this property can be achieved by a C^1-typical perturbation $\widehat{F}$ of $F \in \mathcal{D}^2$: by Moreira's theorem 3.3, we can choose, for each i, a $(\hat{K}_i^s, \hat{f}_i)$ C^1-dynamically defined Cantor set C^1-close to (K_i^s, f_i) so that $\pm\epsilon \notin \hat{K}_i^s + \mu_i \cdot K_i^u$, and, consequently, $\widehat{F}|_{P_i}(x,y) := (\hat{f}_i(x), K_{sgn(x)}(y)) \in \mathcal{D}^2$ has the desired property.

∎

3.2 (Quasi) Critical points eventually return

Definition 9 *Given $\epsilon > 0$, we call any point $(\pm\epsilon, y)$ with $y \in K_0$ a ϵ-quasi-critical point (or simply quasi-critical point).*

Now we'll use again C. G. Moreira's fundamental result (theorem 3.3) to show that, for a C^1 generic $F \in \mathcal{D}^2$, any quasi critical point returns to the "critical region". In other words, roughly speaking, the next result states that we can avoid in the C^1 topology the thickness obstruction (responsible for C^2 Newhouse phenomena).

Lemma 3.3 *Let $\epsilon > 0$ be a positive constant. Then, for a C^1 generic $F \in \mathcal{D}^2$, there exists $m_0 \in \mathbb{N}$ such that any quasi-critical point $(\pm\epsilon, y) \in \{\pm\epsilon\} \times K_0$ satisfies*

$$F^{m_y}(\pm\epsilon, y) \in R_\epsilon := (-\epsilon, \epsilon) \times [0,1], \quad F^m(\pm\epsilon, y) \notin (-\epsilon, \epsilon) \times [0,1], \; 0 < m < m_y$$

for some positive integer $m_y \leq m_0$ or it is contained in the basin of attraction of some of the (finitely many) attracting periodic points of theorem 3.1.

Proof: Take $F \in \mathcal{D}^2$ with the properties described during the proof of corollary 3.1. Since the maximal invariant set Λ_ϵ of U_ϵ is the union of a finite number of periodic sinks and a hyperbolic set $\hat{\Lambda}_\epsilon$ of saddle type, we see that our task is equivalent to show that

$$\left(\bigcup_{k\geq 0} F^k(\{\pm\epsilon\} \times K_0)\right) \bigcap W^s_{loc}(\hat{\Lambda}_\epsilon) = \emptyset$$

for a C^1-typical $F \in \mathcal{D}^2$. Keeping this goal in mind, given $N \in \mathbb{N}$, we define

$$\mathcal{G}_N := \{F \in \mathcal{D}^2 : \left(\bigcup_{k=0}^{N} F^k(\{\pm\epsilon\} \times K_0)\right) \bigcap W^s_{loc}(\hat{\Lambda}_\epsilon) = \emptyset\}.$$

It follows that the proof of the lemma is complete once we show that $\mathcal{G}_N$ is C^1-dense (because it is clearly C^1-open). Observe that $\mathcal{G}_0$ is C^1 dense because Λ_ϵ is locally maximal with isolating neighborhood U_ϵ for a C^1-typical F (in view of the corollary 3.1). Assuming that $\mathcal{G}_{N-1}$ is C^1-dense for some $N \geq 1$, we claim that $\mathcal{G}_N$ is also C^1-dense. In fact, given $F \in \mathcal{G}_{N-1}$, we can refine the Markov partition $\{P_i\}_{i=1}^M$ (appearing in the proof of corollary 3.1) so that $F^j(\{\pm\epsilon\} \times K_0) \cap P_i = \emptyset$ for every $0 \leq j \leq N-1$. Next, for every $p \in \{\pm\epsilon\} \times [0,1]$, we denote by $E(p)$ the tangent line of the C^2 curve $F^N(\{\pm\epsilon\} \times [0,1])$ at the point p. Note that $E(p)$ is a C^1 function of $p \in \{\pm\epsilon\} \times [0,1]$. Therefore, since K_0 is a C^2 dynamical Cantor set of Hausdorff dimension $HD(K_0) < 1$, we see that, without loss of generality, one can assume the directions E_i of the (straight lines) stable foliations of $P_i \cap W^s_{loc}(\hat{\Lambda}_\epsilon)$. Furthermore, by compactness, we can also fix a Markov partition $I_1, \dots, I_k$ of K_0 of sufficiently small diameter so that the directions E_i are still transversal to the finite collection of C^2 curves $F^N(\{\pm\epsilon\} \times I_l)$ for every $i = 1, \dots, M$ and $l = 1, \dots, k$. At this stage, we write

$$P_i \cap F^N(\{\pm\epsilon\} \times K_0) = P_i \cap \bigcup_{b=1}^{a(i)} F^N(\{\pm\epsilon\} \times I_{l(b,i)})$$

and we observe that, by transversality, the projection of each $F^N(\{\pm\epsilon\} \times (I_{l(b,i)} \cap K_0))$ along the direction E_i gives a $C^{1+\alpha}$ dynamical Cantor set $L_{b,i}$. Moreover, we note that $P_i \cap F^N(\{\pm\epsilon\} \times K_0) \cap W^s_{loc}(\hat{\Lambda}_\epsilon) \neq \emptyset$ if and only if $K^s_i \cap (\cup_{b=1}^{a(i)} L_{b,i}) \neq \emptyset$ (where K^s_i is the stable

Cantor set introduced during the proof of corollary 3.1). Using Moreira's theorem 3.3, we obtain $(\widetilde{K_i^s}, \widetilde{f_i})$ dynamical Cantor sets C^1-close to (K_i^s, f_i) such that $\widetilde{K_i^s} \cap (\cup_{b=1}^{a(i)} L_{b,i}) = \emptyset$ for every i. It follows that $\widetilde{F}|P_i(x,y) := (\widetilde{f_i}(x), K_{sgn(x)}y)$ (in the linearizing coordinates inside each P_i) is C^1-close to $F \in \mathcal{G}_{N-1}$ and $\widetilde{\Lambda}_\epsilon \cap P_i = \widetilde{K_i^s} \times K_0$ (in the same linearizing coordinates). In particular, by construction, we have $\widetilde{F} \in \mathcal{G}_N$. This ends the argument.

∎

Remark 3.3 *In the previous statement, we deal with the returns to the critical strip of "quasi-critical" points $(\pm\epsilon, y)$, $y \in K_0$, instead of critical points $c_y^\pm$. The technical reason behind this procedure will be clear in the next section (when we perform the "flatness" perturbation to force critical points to fall into the basins of sinks).*

3.3 Creating sinks whose large basins contain all critical points

Lemma 3.4 *For a C^1-generic $F \in \mathcal{D}^2$, the critical points $c_y^\pm \in \{0^\pm\} \times K_0$ belong to the union of the basins of a finite number of periodic sinks of F.*

Proof: Fix $F \in \mathcal{D}^2$ be a C^1-generic map satisfying the properties of the lemma 3.3. Given $\delta > 0$, we'll find a C^1-perturbation of F with size δ whose critical points belong to the basins of finitely many critical points. In this direction, we take $\epsilon > 0$ sufficiently small such that $|f_y'(x)| < \delta/2$ for every $|x| \leq \epsilon$ and $y \in [0,1]$. Now, we perturb F to make it "flat" in the critical strip $\overline{R_\epsilon} := [-\epsilon, \epsilon] \times [0,1]$, i.e., we define

$$g(x,y) = \begin{cases} f(x,y) & \text{if } |x| \geq \epsilon \\ f(\pm\epsilon, y) & \text{if } |x| \leq \epsilon \end{cases}$$

and $G(x,y) := (g(x,y), K_{sgn(x)}(y))$. Observe that, although $G \notin \mathcal{D}^1$ because $g(x,y)$ is not unimodal, G is $\delta/2$-close to F in the Lipschitz norm and $G = F$ outside the critical strip R_ϵ. In particular, the pieces of orbits of F and G are equal while they stay outside R_ϵ. Hence, since F satisfies the lemma 3.3, we have that G satisfies the same properties, namely, its quasi-critical points $\{\pm\epsilon\} \times K_0$ return to the critical region R_ϵ (after a bounded number of iterates) or they fall into the basins of finitely many periodic sinks (inside Λ_ϵ). We claim that the quasi-critical points returning to R_ϵ belong to the basins of finitely many *super-attracting* periodic sinks of G. Indeed, by compactness and continuity, we can take a Markov partition $I_1, \dots, I_k$ of K_0 of small diameter and some integers $r_1, \dots, r_k$ so that every quasi-critical point $p \in \{\pm\epsilon\} \times I_l$ return to R_ϵ or fall into the basin of a sink after exactly r_l iterates. Since the pieces of orbits of F and G outside R_ϵ are the same, and the piece of the G-orbit outside R_ϵ of a point $(x,y) \in R_\epsilon$ equals to the piece of F-orbit outside R_ϵ of the point $(\pm\epsilon, y)$, we obtain that G send the boxes $[-\epsilon, \epsilon] \times I_l$ strictly inside another (*a priori* different box) $[-\epsilon, \epsilon] \times I_j$ or inside the basin of a periodic sink after r_l iterates (exactly), so that our claim follows. Finally, we complete the proof by noticing that, although $G \notin \mathcal{D}^1$, one can *slightly* "undo" the "flat" perturbation in order to get a $H \in \mathcal{D}^2$ such that its critical points belong to the basin of finitely many periodic sinks and H is $\delta/2$-close to G in the Lipschitz norm (and, *a fortiori*, $H \in \mathcal{D}^2$ is δ-close to $F \in \mathcal{D}^2$ in the C^1-topology).

∎

3.4 End of the proof of theorem B

By combining the corollary 3.1 and the lemma 3.4, we get that the non-wandering set $\Omega(F)$ of a C^1-typical $F \in \mathcal{D}^2$ can be written as $\Omega(F) = \Lambda_\epsilon \cup \{p_1, \ldots, p_k\}$ where $p_1, \ldots, p_k$ are periodic sinks of F whose (large) basins contain a ϵ-neighborhood of the critical set, i.e., $\Omega(F)$ is a hyperbolic set.

Thus, the proof of theorem B will be complete once we can show the following claim: a Kupka-Smale $F \in \mathcal{D}^1$ such that $\Omega(F)$ is hyperbolic is Axiom A. However, this is a consequence of the following argument of Pujals and Sambarino [PS1, p. 966]: $\Omega(F)$ hyperbolic implies $L(F)$ hyperbolic, so that the results of Newhouse [N4] say that periodic points are dense in $L(F)$ and we can do spectral decomposition of $L(F)$. Hence, we can show that $\Omega(F) = L(F)$ whenever we can verify the no-cycles condition. Since our phase space is two-dimensional, a cycle can only occur among basic sets of saddle-type. However, since F is Kupka-Smale, the intersections of invariant manifolds involved in this cycle are transversal, an absurd.

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