FETI-DP for Stokes-Mortar-Darcy Systems

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1 Introduction and problem setting

We consider the coupling across an interface of a fluid flow and a porous media flow. The differential equations involve Stokes equations in the fluid region, Darcy equations in the porous region, plus a coupling through an interface with Beaver-Joseph-Saffman transmission conditions, see [2, 8, 6, 1]. The discretization consists of $P2$-$P0$ finite elements in the fluid region, the lowest order triangular Raviart-Thomas finite elements in the porous region, and the mortar piecewise constant Lagrange multipliers on the interface. Due to the small values of the permeability parameter κ of the porous medium, the resulting discrete symmetric saddle point system is very ill conditioned. Preconditioning is needed in order to efficiently solve the resulting discrete system. The purpose of this work is to present some preliminary results on the extension of the modular FETI type preconditioner proposed in [5, 7] to the multidomain FETI-DP case.

Let $\Omega^f,\ \Omega^p \subset \mathbb{R}^n$ be polyhedral subdomains, define $\Omega = \text{int}(\overline{\Omega}^f \cup \overline{\Omega}^p)$ and $\Gamma = \partial\Omega^f \cap \partial\Omega^p$, with outward unit normal vectors $\boldsymbol{\eta}^i$ on $\partial\Omega^i$, $i = f, p$. The tangent vectors on Γ are denoted by $\boldsymbol{\tau}_1$ $(n = 2)$, or $\boldsymbol{\tau}_l$, $l = 1, 2$ $(n = 3)$. The exterior boundaries are $\Sigma^i := \partial\Omega^i \setminus \Gamma$, $i = f, p$. Fluid velocities are denoted by $\boldsymbol{u}^i : \Omega^i \to \mathbb{R}^n$, $i = f, p$, and pressures by $p^i : \Omega^i \to \mathbb{R}$, $i = f, p$.

We consider Stokes equations in the fluid region Ω^f and Darcy equations for the filtration velocity in the porous medium Ω^p.

$$\begin{array}{cc} \text{Stokes equations} & \text{Darcy equations} \\ \left\{\begin{aligned} -\nabla\cdot T(\boldsymbol{u}^f, p^f) &= \boldsymbol{f}^f \text{ in } \Omega^f \\ \nabla\cdot\boldsymbol{u}^f &= g^f \text{ in } \Omega^f \\ \boldsymbol{u}^f &= \boldsymbol{h}^f \text{ on } \Sigma^f \end{aligned}\right. & \left\{\begin{aligned} \boldsymbol{u}^p &= -\tfrac{\kappa}{\nu}\nabla p^p && \text{in } \Omega^p \\ \nabla\cdot\boldsymbol{u}^p &= g^p && \text{in } \Omega^p \\ \boldsymbol{u}^p\cdot\boldsymbol{\eta}^p &= h^p && \text{on } \Sigma^p. \end{aligned}\right. \end{array} \tag{1}$$

Here $T(\boldsymbol{v},p) := -pI + 2\nu \boldsymbol{D}\boldsymbol{v}$, where ν is the fluid viscosity, $\boldsymbol{D}\boldsymbol{v} := \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$ is the linearized strain tensor and κ denotes the rock permeability. We assume that κ is a real positive constant. We impose the following interface matching conditions across Γ (see [2, 8, 6, 1] and references therein):

1. Conservation of mass across Γ: $\boldsymbol{u}^f \cdot \boldsymbol{\eta}^f + \boldsymbol{u}^p \cdot \boldsymbol{\eta}^p = 0$ on Γ.
2. Balance of normal forces across Γ: $p^f - 2\nu \boldsymbol{\eta}^{fT} \boldsymbol{D}(\boldsymbol{u}^f)\boldsymbol{\eta}^f = p^p$ on Γ.
3. Beavers-Joseph-Saffman condition: $\boldsymbol{u}^f \cdot \boldsymbol{\tau}_l = -\frac{\sqrt{\kappa}}{\alpha^f} 2\boldsymbol{\eta}^{fT} \boldsymbol{D}(\boldsymbol{u}^f)\boldsymbol{\tau}_l$, $l = 1, \cdots, n-1$ on Γ.

We require that $\langle g^f, 1\rangle_{\Omega^f} + \langle g^p, 1\rangle_{\Omega^p} - \langle \boldsymbol{h}^f \cdot \boldsymbol{\eta}^f, 1\rangle_{\Sigma^f} - \langle h^p, 1\rangle_{\Sigma^p} = 0$ which is the compatibility condition (see [6]).

2 Weak Formulation

In this section we present the weak version of the coupled system of partial differential equations introduced above. Without loss of generality, we consider $\boldsymbol{h}^f = \boldsymbol{0}$, $g^f = 0$, $h^p = 0$ and $g^p = 0$ in (1); see [6]. The weak problem is formulated as: *Find* $(\boldsymbol{u}, p, \lambda) \in \boldsymbol{X} \times M_0 \times \Lambda$ *such that for all* $(\boldsymbol{v}, q, \mu) \in \boldsymbol{X} \times M_0 \times \Lambda$ *we have*

$$\begin{cases} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) + b_\Gamma(\boldsymbol{v},\lambda) = f(\boldsymbol{v}) \\ b(\boldsymbol{u},q) \qquad\qquad\qquad\qquad\quad = 0 \\ b_\Gamma(\boldsymbol{u},\mu) \qquad\qquad\qquad\qquad = 0, \end{cases} \tag{2}$$

where $\boldsymbol{X} = \boldsymbol{X}^f \times \boldsymbol{X}^p := H_0^1(\Omega^f, \Sigma^f)^n \times \boldsymbol{H}_0(\mathrm{div}, \Omega^p, \Sigma^p)$ and M_0 is the subset of $M := M^f \times M^p := L^2(\Omega^f) \times L^2(\Omega^p) \equiv L^2(\Omega)$ of pressures with a zero average value in Ω. Here $H_0^1(\Omega^f, \Sigma^f)$ denotes the subspace of $H^1(\Omega^f)$ of functions that vanish on Σ^f. The space $\boldsymbol{H}_0(\mathrm{div}, \Omega^p, \Sigma^p)$ consists of vector functions in $\boldsymbol{H}(\mathrm{div}, \Omega^p)$ with zero normal trace on Σ^p, where $\boldsymbol{H}(\mathrm{div}, \Omega^p) := \left\{ \boldsymbol{v} \in L^2(\Omega^p)^n : \mathrm{div}\, \boldsymbol{v} \in L^2(\Omega^p) \right\}$. For the Lagrange multiplier space we consider $\Lambda := H^{1/2}(\Gamma)$. See [8, 6] for well posedness results. The global bilinear forms are given by

$$a(\boldsymbol{u},\boldsymbol{v}) := a^f_{\alpha^f}(\boldsymbol{u}^f, \boldsymbol{v}^f) + a^p(\boldsymbol{u}^p, \boldsymbol{v}^p) \quad \text{and} \quad b(\boldsymbol{v}, p) := b^f(\boldsymbol{v}^f, p^f) + b^p(\boldsymbol{v}^p, p^p),$$

with local forms $a^f_{\alpha^f}, b^f$ and b^p defined for $\boldsymbol{u}^f, \boldsymbol{v}^i \in \boldsymbol{X}^i$, $p^i, q^i \in M^i$ by

$$a^f_{\alpha^f}(\boldsymbol{u}^f, \boldsymbol{v}^f) := 2\nu(\boldsymbol{D}\boldsymbol{u}^f, \boldsymbol{D}\boldsymbol{v}^f)_{\Omega^f} + \sum_{\ell=1}^{n-1} \frac{\nu\alpha^f}{\sqrt{\kappa}} \langle \boldsymbol{u}^f \cdot \boldsymbol{\tau}_\ell, \boldsymbol{v}^f \cdot \boldsymbol{\tau}_\ell \rangle_\Gamma, \tag{3}$$

$$a^p(\boldsymbol{u}^p, \boldsymbol{v}^p) := (\frac{\nu}{\kappa}\boldsymbol{u}^p, \boldsymbol{v}^p)_{\Omega^p}, \tag{4}$$

$$b^f(\boldsymbol{v}^f, q^f) := -(q^f, \nabla \cdot \boldsymbol{v}^f)_{\Omega^f}, \quad \text{and} \quad b^p(\boldsymbol{v}^p, p^p) := -(p^p, \nabla \cdot \boldsymbol{v}^p)_{\Omega^p}. \tag{5}$$

The weak conservation of mass bilinear form is defined by

$$b_\Gamma(\boldsymbol{v},\mu) := \langle \boldsymbol{v}^f \cdot \boldsymbol{\eta}^f, \mu\rangle_\Gamma + \langle \boldsymbol{v}^p \cdot \boldsymbol{\eta}^p, \mu\rangle_\Gamma, \quad \boldsymbol{v} = (\boldsymbol{v}^f, \boldsymbol{v}^p) \in \boldsymbol{X}, \mu \in \Lambda. \tag{6}$$

The second duality pairing of (6) is interpreted as $\langle \boldsymbol{v}^p \cdot \boldsymbol{\eta}^p, E_{\boldsymbol{\eta}^p}(\mu)\rangle_{\partial\Omega^p}$. Here $E_{\boldsymbol{\eta}^p}$ is any continuous liftin operator from $H^{1/2}(\Gamma)$ to $H^{1/2}(\partial\Omega^p)$; recall that $\Gamma \subset \partial\Omega^p$ and $\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \Omega^p, \Sigma^p)$, see [6]. The functional f in the right-hand side of (2) is defined by $f(\boldsymbol{v}) := f^f(\boldsymbol{v}^f) + f^p(\boldsymbol{v}^p)$, for all $\boldsymbol{v} = (\boldsymbol{v}^f, \boldsymbol{v}^p) \in \boldsymbol{X}$, where $f^i(\boldsymbol{v}^i) := (\boldsymbol{f}^i, \boldsymbol{v}^i)_{L^2(\Omega^i)}$ for $i = f, p$.

The bilinear forms $a^f_{\alpha^f}, b^f$ are associated to the Stokes equations, and the bilinear forms a^p, b^p to the Darcy law. The bilinear form $a^f_{\alpha^f}$ includes interface matching conditions $1.b$ and $1.c$ above. The bilinear form b_Γ is used to impose the weak version of the interface matching condition $1.a$ above.

3 Discretization and Decomposition

From now on we consider only the two-dimensional case. The ideas developed below can be extended to the case of three-dimensional subdomains. We assume that Ω^i, $i = f, p$, are polygonal subdomains. For the fluid region, let $\boldsymbol{X}^{h,f}$ and $M^{h,f}$ be $P2/P0$ triangular finite elements. For the porous region, let $\boldsymbol{X}^{h,p}$ and $M^{h,p}$ be the lowest order Raviart-Thomas finite elements based on triangles. Define $\boldsymbol{X}_h := \boldsymbol{X}^{h,f} \times \boldsymbol{X}^{h,p} \subset \boldsymbol{X}$ and $M_h := M^{h,f} \times M^{h,p} \subset M_0$. We assume that the boundary conditions are included in the definition of the finite element spaces, i.e., for $\boldsymbol{v}^f \in \boldsymbol{X}^{h,f}$ we have $\boldsymbol{v}^f = \boldsymbol{0}$ on the exterior fluid boundary Σ^f and for $\boldsymbol{v}^p \in \boldsymbol{X}^{h,p}$ we have that $\boldsymbol{v}^p_h \cdot \boldsymbol{\eta}^p = 0$ on the porous exterior boundary Σ^p.

With the discretization chosen before we obtain the following symmetric saddle point linear system

$$\left[\begin{array}{c|c|c} K^f & 0 & M^{fT} \\ \hline 0 & K^p & M^{pT} \\ \hline M^f & M^p & 0 \end{array}\right] \left[\begin{array}{c} \boldsymbol{u}^f \\ p^f \\ \hline \boldsymbol{u}^p \\ p^p \\ \hline \lambda \end{array}\right] = \left[\begin{array}{cc|cc|c} A^f & B^{fT} & 0 & 0 & C^{fT} \\ B^f & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & A^p & B^{pT} & -C^{pT} \\ 0 & 0 & B^p & 0 & 0 \\ \hline C^f & 0 & -C^p & 0 & 0 \end{array}\right] \left[\begin{array}{c} \boldsymbol{u}^f \\ p^f \\ \hline \boldsymbol{u}^p \\ p^p \\ \hline \lambda \end{array}\right] = \left[\begin{array}{c} \boldsymbol{f}^f \\ g^f \\ \hline \boldsymbol{f}^p \\ g^p \\ \hline 0 \end{array}\right] \tag{7}$$

with matrices A^i, B^i, C^i defined by

$$a^i(\boldsymbol{u}^i, \boldsymbol{v}^i) = \boldsymbol{v}^{iT} A^i \boldsymbol{u}^i, \qquad b^i(\boldsymbol{u}^i, q^i) = q^{iT} B^i \boldsymbol{u}^i, \quad (\boldsymbol{u}^i \cdot \boldsymbol{\eta}^f, \mu)_\Gamma = \mu^T C^i \boldsymbol{u}^i,$$

and vectors $\boldsymbol{f}^i, g^i$ given by $f^i(\boldsymbol{v}^i) = \boldsymbol{v}^{iT} \boldsymbol{f}^i$, $g^i(q^i) = q^{iT} g^i$, $i = f, p$. Matrix A^f corresponds to ν times the discrete version of the linearized stress tensor on Ω^f. Note that in the case $\alpha^f > 0$, the bilinear form $a^f_{\alpha^f}$ in (3) includes a boundary term. The matrix A^p corresponds to ν/κ times a discrete L^2-norm on Ω^p. Matrix $-B^i$ is the discrete divergence in Ω^i, $i = f, p$, and matrices C^f and C^p correspond to the matrix form of the discrete conservation of mass on Γ. Note that ν can be viewed as a scaling factor since it appears in both matrices A^f and A^p, therefore, ν plays no role for the preconditioning.

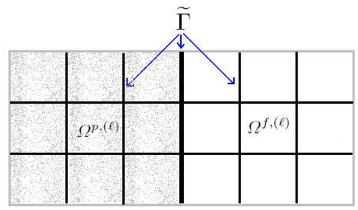

Fig. 1. Global interface $\widetilde{\Gamma}$ that includes all local interfaces and the Stokes/Darcy interface Γ.

Let $\{\Omega^{i,(\ell)}\}_{\ell=1}^{N^i}$ be geometrically conforming substructures of Ω^i, $i = f, p$. We also assume that $\{\Omega^{f,(\ell)}\}_{\ell=1}^{N^f} \cup \{\Omega^{p,(\ell)}\}_{\ell=1}^{N^p}$ forms a geometrically conforming decomposition of Ω, hence, the two decompositions are aligned on the Stokes/Darcy interface Γ, see Figure 1. We define the local inner interfaces as $\Gamma^{i,(\ell)} = \partial\Omega^{i,(\ell)} \setminus \partial\Omega^i$, $\ell = 1, \ldots, N^i$, $i = f, p$. We also define the global interface

$$\widetilde{\Gamma} = \left(\bigcup_{\ell=1}^{N^f} \Gamma^{f,(\ell)} \right) \cup \left(\bigcup_{\ell=1}^{N^p} \Gamma^{p,(\ell)} \right) \cup \Gamma \equiv (\Gamma^f) \cup (\Gamma^p) \cup \Gamma.$$

On the Stokes region $\Omega^{f,(\ell)}$ we consider the following partition of the degrees of freedom,

$$\begin{bmatrix} \boldsymbol{u}_I^{f,(\ell)} \\ p_I^{f,(\ell)} \\ u_{\widetilde{\Gamma}}^{f,(\ell)} \\ \bar{p}^{f,(\ell)} \end{bmatrix} \quad \begin{array}{l} \text{Interior velocities in } \Omega^{f,(\ell)} + \text{tangential velocities on } \partial\Omega^{f,(\ell)} \backslash \Gamma, \\ \text{Interior pressures with zero average in } \Omega^{f,(\ell)}, \\ \text{Interface velocities on } \Gamma^{f,(\ell)} + \text{normal velocities on } \partial\Omega^{f,(\ell)} \cap \Gamma, \\ \text{Constant pressure in } \Omega^{f,(\ell)}. \end{array}$$

Analogously, on the Darcy region $\Omega^{p,(\ell)}$ we use,

$$\begin{bmatrix} \boldsymbol{u}_I^{p,(\ell)} \\ p_I^{p,(\ell)} \\ u_{\widetilde{\Gamma}}^{p,(\ell)} \\ \bar{p}^{p,(\ell)} \end{bmatrix} \quad \begin{array}{l} \text{Interior velocities in } \Omega^{p,(\ell)}, \\ \text{Interior pressures with zero average in } \Omega^{p,(\ell)}, \\ \text{Normal velocities on } \Gamma^{p,(\ell)} + \text{normal velocities on } \partial\Omega^{p,(\ell)} \cap \Gamma, \\ \text{Constant pressure in } \Omega^{p,(\ell)}. \end{array}$$

Then, for $i = f, p$, we have the block structure:

$$A^i = \begin{bmatrix} A_{II}^i & A_{\Gamma I}^{iT} \\ A_{\Gamma I}^i & A_{\Gamma\Gamma}^i \end{bmatrix}, \quad B^i = \begin{bmatrix} B_{II}^i & B_{\Gamma I}^{iT} \\ 0 & \bar{B}^{iT} \end{bmatrix} \text{ and } C^i = \begin{bmatrix} 0 \; 0 \; \widetilde{C}^i \; 0 \end{bmatrix}.$$

The $(2,1)$ entry of B^i corresponds to integrating an interior velocity against a constant pressure, therefore, it vanishes due to the divergence theorem.

Following [9] we choose the following matrix representation in each subdomain $\Omega^{i,(\ell)}$, $i = f, p$,

$$K^{i,(\ell)} = \left[\begin{array}{cc|cc} A_{II}^{i,(\ell)} & B_{II}^{i,(\ell)T} & A_{\Gamma I}^{i,(\ell)T} & 0 \\ B_{II}^{i} & 0 & B_{I\Gamma}^{i,(\ell)} & 0 \\ \hline A_{\Gamma I}^{i,(\ell)} & B_{I\Gamma}^{i,(\ell)T} & A_{\Gamma\Gamma}^{i,(\ell)} & \bar{B}^{i,(\ell)T} \\ 0 & 0 & \bar{B}^{i,(\ell)} & 0 \end{array}\right] = \left[\begin{array}{c|c} K_{II}^{i,(\ell)} & K_{\Gamma I}^{i,(\ell)T} \\ \hline K_{\Gamma I}^{i,(\ell)} & K_{\Gamma\Gamma}^{i,(\ell)} \end{array}\right]. \qquad (8)$$

4 Dual Formulation

In order to simplify the notation and since there is no danger of confusion, we will denote the finite element functions and the corresponding vector representation by the same symbols. Let $\boldsymbol{X}^{i,(\ell)}$, $M^{i,(\ell)}$ be the finite element spaces $\boldsymbol{X}_h$ and M_h restricted to subdomain $\Omega^{i,(\ell)}$, $i = f, p$, $\ell = 1, \ldots, N^i$. Define the product spaces,

$$\boldsymbol{W} = \boldsymbol{W}^f \otimes \boldsymbol{W}^p = \bigotimes_i \bigotimes_\ell \boldsymbol{X}^{i,(\ell)}$$

and $Q = M^f \otimes M^p = \bigotimes_i \bigotimes_\ell M^{i,(\ell)}$. Functions in $\boldsymbol{W}$ do not satisfy any continuity requirement on the subdomains corners or edges. In particular they do not satisfy continuity on Stokes/Stokes edges, or continuity of normal component on Darcy/Darcy edges, neither discrete continuity of normal fluxes on Stokes/Darcy edges. The linear operator $K = \mathrm{diag}(K^f, K^p)$ in (7) defined on the pair of spaces $(\boldsymbol{X}_h, M_h)$, can be extended to the pair $(\boldsymbol{W}, Q)$ defined above. The resulting matrix will be a block diagonal.

Primal degrees of freedom and definition of $\widetilde{\boldsymbol{W}}$: now we introduce our primal degrees of freedom, as is usual in the constructions of FETI-DP [4] and BDDC methods [3]. The primal degrees of freedom are selected accordingly for Stokes and Darcy substructures. On the fluid side, the primal degrees of freedom are given by the fluid velocity field at the substructure corners and by the mean value of both components over each Stokes/Stokes edge on Γ^f; see [10, 9]. For the porous side, the primal degrees of freedom consist of the mean value of the normal flux on each Darcy/Darcy edge on Γ^p; see [11]. For the Stokes/Darcy interface Γ, the primal degrees of freedom consist of the mean value of the normal (either Stokes or Darcy velocity) flux on each Stokes/Darcy edge on Γ; see [7]. The $\widetilde{\boldsymbol{W}}$ is the subspace of $\boldsymbol{W}$ made of functions that are continuous on the primal degrees of freedom described above.

Once the linear operator $K = \mathrm{diag}(K^f, K^p)$ in (7) is extended to $(\boldsymbol{W}, Q)$, it can be restricted to an operator $\widetilde{K}$ acting on $(\widetilde{\boldsymbol{W}}, Q)$. The matrix form of $\widetilde{K}$ is no longer block diagonal but it will have a block structure with small interaction between blocks associated to different subdomains; see [9] In the FETI-DP method we will need the inverse action of $\widetilde{K}$. This inverse action can be obtained by solving a small coarse problem and a (either Darcy or Stokes) local problems for each subdomains.

Functions in $\widetilde{\boldsymbol{W}}$ do not satisfy the dual continuity requirements on $\widetilde{\Gamma}$. The dual continuity requirements can be enforced using additional FETI-Lagrange multipliers μ on $\widetilde{\Gamma} \setminus \Gamma$ and the Stokes-Mortar-Darcy-Lagrange multipliers on Γ just as before. We obtain the linear system

$$\begin{bmatrix} \widetilde{K} & \widetilde{B}^T \\ \widetilde{B} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \widetilde{\lambda} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \tag{9}$$

where the vector $\widetilde{\lambda}$ includes all Lagrange multiplier degrees of freedom. Matrix $\widetilde{B}$ has entries $+1, -1, 0$ for the degrees of freedom associated Γ^f and Γ^p. On the Stokes/Darcy interface Γ, we ensure that the flux continuity across Stokes/Darcy edges on Γ coincides with the last equation of (7). For that, we use the same Lagrange multipliers, up to the constant functions, as for the Stokes-Mortar-Darcy system (7). We now eliminate all degrees of freedom but the ones associated to the Lagrange multipliers to obtain a dual formulation,

$$\widetilde{B}\widetilde{K}^{-1}\widetilde{B}^T\widetilde{\lambda} = \widetilde{F}\widetilde{\lambda} = b = \widetilde{B}\widetilde{K}^{-1}b \tag{10}$$

where $\widetilde{\lambda} \in \mathrm{Rank}(\widetilde{B})$. Note that applying $\widetilde{K}^{-1}$ requires the solution of a Stokes/Darcy problem with a block structure and very little coupling between blocks; see [9].

4.1 Dirichlet Preconditioner

Let us define

$$S_{\widetilde{\Gamma}}^{D} := \mathrm{diag}(S_{\widetilde{\Gamma}}^{f}, S_{\widetilde{\Gamma}}^{p}) \quad \text{where} \quad S_{\widetilde{\Gamma}}^{i} =: \sum_{\ell=1}^{N^i} R^{i,(\ell)T} D_1^{i,(\ell)} S_{\widetilde{\Gamma}}^{i,(\ell)} D_1^{i,(\ell)} R^{i,(\ell)} \tag{11}$$

and $S_{\widetilde{\Gamma}}^{i,(\ell)}$ is defined from (8) via

$$S^{i,(\ell)} = \begin{bmatrix} S_{\widetilde{\Gamma}}^{i,(\ell)} & \bar{B}^{i,(\ell)T} \\ \bar{B}^{i,(\ell)} & 0 \end{bmatrix} := K_{\Gamma\Gamma}^{i,(\ell)} - K_{\Gamma I}^{i,(\ell)} \left(K_{II}^{i,(\ell)}\right)^{-1} K_{\Gamma I}^{i,(\ell)T},$$

$$I_{\widetilde{\Gamma}}^{D} := \mathrm{diag}(I_{\widetilde{\Gamma}}^{f}, I_{\widetilde{\Gamma}}^{p}) \quad \text{where} \quad I_{\widetilde{\Gamma}}^{i} =: \sum_{\ell=1}^{N^i} R^{i,(\ell)T} D_2^{i,(\ell)} I_{\widetilde{\Gamma}}^{i,(\ell)} D_2^{i,(\ell)} R^{i,(\ell)} \tag{12}$$

and $I^{i,(\ell)}_{\widetilde{\Gamma}}$ is an identity matrix. We propose the following preconditioners:

$$\widetilde{B}(S^{D}_{\widetilde{\Gamma}} + I^{D}_{\widetilde{\Gamma}})\widetilde{B}^{T}. \tag{13}$$

In (11) we choose the diagonal matrix $D_1^{i,(\ell)}$ with entries 1/2 on both sides of Stokes/Stokes and Darcy/Darcy edges, the value zero at the Stokes corners, and the values γ_1^f (Stokes side) and γ_1^p (Darcy side) on the Stokes/Darcy edges. In (12) we choose the diagonal matrix $D_2^{i,(\ell)}$ entries equal to γ_2^f (Stokes side) and γ_2^p (Darcy side) on the Stokes/Darcy edges, and entries equal zero elsewhere.

5 Numerical Results

In this section we present representative numerical results concerning the performance of the FETI-DP methods introduced before. We consider $\Omega^f = (1,2)\times(0,1)$ and $\Omega^p = (0,1)\times(0,1)$. We set $\mu = 1$. See [6] for examples of exact solutions and compatible divergence and boundary data. We use Conjugate Gradient (CG) and Preconditioned Conjugate Gradient (PCG) with the Dirichlet preconditioner (13) to solve the linear system (10). In our test problems we run (CG) PCG until the initial residual is reduced by a factor of 10^{-6}.

In our first experiment we fix $H/h = 4$ or $H/h = 8$ and run CG and PCG for different values of $H = H^f = H^p$ and different values of κ. See Table 1 for the FETI-DP method with and without a preconditioner. We observe the preconditioned FETI-DP method with $\gamma_1^f = 0$, $\gamma_1^p = 1$, $\gamma_2^p = 0$ and $\gamma_2^f = 0$ is robust with respect to the number of subdomains and size of the subdomains when the κ is not very small. We repeat the experiment above with $\gamma_1^f = 0$, $\gamma_1^p = 0$, $\gamma_2^f = 0$ and $\gamma_2^p = 1 + H/h$ and present the number of iterations and estimate condition numbers in Table 2. With this choice of paramenters we obtain a robust preconditioner for κ small. Analysis of the FETI-DP methods presented here as well as the design of more sophisticated FETI-DP solvers are currently being studied by the authors.

$\kappa \downarrow N \rightarrow$	2×2	4×4	8×8
1	5(27)	7(57)	8(66)
10^{-2}	7(13)	8(22)	8(36)
10^{-4}	11(47)	19(52)	15(33)
10^{-6}	18(74)	34(131)	43(157)

$\kappa \downarrow N \rightarrow$	2×2	4×4	8×8
1	6(62)	9(98)	10(104)
10^{-2}	8(23)	10(40)	10(64)
10^{-4}	20(70)	20(61)	16(36)
10^{-6}	29(150)	60(259)	79(275)

Table 1. Right: PCG iteration number for different number of subdomains. CG iteration number in parenthesis. Here $\frac{H}{h} = 4$, $H^f = H^p = H = \frac{1}{N}$, $\gamma_1^f = 0$, $\gamma_1^p = 1$, $\gamma_2^f = 0$, $\gamma_2^p = 0$. Left: $\frac{H}{h} = 8$.

$\kappa \downarrow N \rightarrow$	2×2	4×4	8×8
1	9(4.4e+2)	15 (1.8e+3)	22 (7.0e+3)
10^{-2}	7(5.5e+0)	12 (1.9e+1)	16 (7.1e+1)
10^{-4}	7(3.2e+0)	8 (4.6e+0)	8 (4.6e+0)
10^{-6}	7(3.4e+0)	9 (5.7e+0)	10 (6.7e+0)

$\kappa \downarrow N \rightarrow$	2×2	4×4	8×8
1	18(3.2e+3)	32(1.3e+4)	40(5.2e+4)
10^{-2}	14(3.3e+1)	24(1.3e+2)	30(5.2e+2)
10^{-4}	10(8.3e+0)	12(1.3e+1)	14(1.7e+1)
10^{-6}	11(8.3e+0)	13(1.2e+1)	15(1.5e+1)

Table 2. Top: PCG iteration and condition number for different number of subdomains. $\frac{H}{h} = 4$, $H^f = H^p = H = \frac{1}{N}$, $\gamma_1^f = 0$, $\gamma_1^p = 0$, $\gamma_2^f = 1$, $\gamma_2^p = 1 + H/h$. Bottom: $\frac{H}{h} = 8$

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