FETI-DP METHOD FOR DG DISCRETIZATION OF ELLIPTIC PROBLEMS WITH DISCONTINUOUS COEFFICIENTS

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Abstract. In this paper a discontinuous Galerkin (DG) discretization of an elliptic two-dimensional problem with discontinuous coefficients is considered. The problem is posed on a polygonal region Ω which is a union of disjoint polygonals Ω_i of diameter $O(H_i)$ and forms a geometrically conforming partition of Ω. The discontinuities of the coefficients are assumed to occur only across $\partial\Omega_i$. Inside of each substructure Ω_i, a conforming finite element space on a quasiuniform triangulation with triangular elements and mesh size $O(h_i)$ is introduced. To handle the nonmatching meshes across $\partial\Omega_i$, a discontinuous Galerkin discretization is considered. For solving the resulting discrete problem, a FETI-DP method is designed and analyzed. It is established that the condition number of the method is estimated by $C(1 + \max_i \log H_i/h_i)^2$ with a constant C independent of h_i, H_i and the jumps of the coefficients. The method is well suited for parallel computations and it can be straightforwardly extended to three-dimensional problems.

Key words. Interior penalty discretization, discontinuous Galerkin, elliptic problems with discontinuous coefficients. finite element method, FETI-DP algorithms, preconditioners

AMS subject classifications. 65F10, 65N20, 65N30

1. Introduction. In this paper a discontinuous Galerkin (DG) approximation of an elliptic problem with discontinuous coefficients is considered. The problem is posed on a polygonal region Ω which is a union of disjoint polygonal subregions Ω_i of diameter $O(H_i)$ and forms a geometrically partition of Ω, i.e., for $i \neq j$, $\partial\Omega_i \cap \partial\Omega_j$ is empty or is a common corner or edge of $\partial\Omega_i$ and $\partial\Omega_j$, where an edge means a curve of continuous intervals. The discontinuities of the coefficients are assumed to occur only across $\partial\Omega_i$. The problem is approximated by a conforming finite element method (FEM) on a matching triangulation inside each Ω_i, with h_i as mesh parameter, and nonmatching meshes are allowed to occur across $\partial\Omega_i$. This kind of composite discretization is motivated by the local regularity of the solution of the problem being discussed. A discrete problem is formulated using symmetric DG methods with a interior penalty term on $\partial\Omega_i$, see [4, 8, 26]. The main goal of this paper is to design and analyze a FETI-DP method for the resulting discrete problem. To the best of our knowledge, the FETI-DP method has never been considered before in the literature for DG discretizations.

The first FETI-DP method for standard continuous Galerkin discretization was introduced in [11] and is a nonoverlapping domain decomposition method that enforces continuity of the solution at subdomain interfaces by Lagrange multipliers except at subdomain corners where the continuity is enforced directly by assigning a unique value for the functions at each corner. The first mathematical analysis of the method was provided in [25]. The method was further improved by enforcing the continuity directly on averages across the edges or faces on subdomain interfaces [12, 19], see also [28], resulting in better parallel scalability for three-dimensional problems. FETI-DP

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methods, and also similar methods such as FETI, BDDC and BDD, have been tested very successfully and analyzed theoretically for a variety of problems and their discretizations, see [28] and references therein.

The main goal of this paper is to develop a FETI-DP methodology for DG methods and apply it to a nontrivial problem. More specifically, we consider a scalar second order elliptic problem with discontinuous coefficients with nonmatching meshes across the subdomains. The discontinuities of the coefficients do not necessarily satisfy the quasi-monotonicity condition on the jumps of the coefficients, see [10]. We expect that the methodology developed here can be extended to a large class of problems and DG discretizations, see [4, 16, 26].

In this paper, the DG discrete problem is reduced to the Schur complement problem with respect to unknowns on the closure of $\partial\Omega_i \backslash \partial\Omega$, for $i = 1, \cdots, N$. For that, discrete harmonic functions defined in a special way, i.e., in the DG sense, are used. The FETI-DP method for DG discretization is designed and analyzed using the approach from [28, 25], which is used there for standard conforming discretizations. Let $\Gamma^{(i)}$ be the union of all edges $\bar{F}_{ij}$ and $\bar{F}_{ji}$ which are common to Ω_i and Ω_j, where $\bar{F}_{ij}$ and $\bar{F}_{ji}$ refer to the Ω_i and Ω_j sides, respectively, and let $\Gamma := \cup_{i=1}^{N} \Gamma^{(i)}$. The method consists of decomposing Γ into overlapping interfaces $\Gamma^{(i)}, i = 1, \cdots, N$. We note that each $\Gamma^{(i)}$ has unknowns (degrees of freedom) corresponding to nodal points on the closure $\partial\Omega_i \backslash \partial\Omega$ and on the $\bar{F}_{ji} \subset \partial\Omega_j$. Next we impose continuity of the unknowns which correspond to corners of Ω_i and common endpoints of $\bar{F}_{ji}$. These unknowns are called primal. The remaining unknowns on $\Gamma^{(i)}$ and $\Gamma^{(j)}$ are called dual and have jumps, hence, Lagrange multipliers are introduced to eliminate these jumps. For the dual system with Lagrange multipliers, a special block diagonal preconditioner is designed. It leads to independent local problems on $\Gamma^{(i)}$ for $i = 1, \cdots, N$. It is proved that the proposed method is almost optimal with a condition number estimate bounded by $C \max_i (1 + \log H_i/h_i)^2$, where C does not depend on h_i, h_j, h_i/h_j, the number of subdomains Ω_i and the jumps in the coefficients. The method can be straightforwardly extended to DG discretizations of three-dimensional problems. The FETI-DP method developed here complements the BDDC methodology for DG discretizations developed recently in [9]. We note that the discretization used there (based on the harmonic average of the coefficients) is not the same as the one we consider here, additionally, the constraints used there are based on edges while here are based on corners. We point out that the introduction of corner constraints eliminates the interface condition assumption required in [9]. We note that other types of preconditioners have been considered for solving DG discretizations. In connection with block diagonal or overlapping Schwarz methods see for example [13, 14, 22, 6, 1, 2, 7, 24] while for multilevel preconditioners [15, 17, 23, 21, 20, 5]. We note that these preconditioners do not use discrete harmonic extensions and do not belong to the family of iterative substructuring type of methods such as FETI, FETI-DP, BDD, BDDC and Neumann-Neumann.

The paper is organized as follows. In Section 2 the differential problem and a DG discretization are formulated. In Section 3, the Schur complement problem is derived using discrete harmonic functions defined in a special way (in the DG sense). In Section 4, the so-called FETI-DP method is introduced, i.e., the Schur complement problem is reformulated by imposing continuity for the primal variables and by using

Lagrange multipliers at the dual variables. Finally, a special block diagonal preconditioner is defined. The main results of the paper are Theorem 4.2 and Lemma 4.4. Section 6 is devoted to the implementation of the FETI-DP method.

2. Differential and discrete problems. In this section we discuss the continuous and discrete problems we take into consideration for preconditioning.

2.1. Differential problem. Consider the following problem: *Find* $u^*_{ex} \in H^1_0(\Omega)$ *such that*

$$a(u^*_{ex}, v) = f(v) \quad \forall v \in H^1_0(\Omega) \tag{2.1}$$

where

$$a(u,v) := \sum_{i=1}^{N} \int_{\Omega_i} \rho_i(x) \nabla u \cdot \nabla v \, dx \quad \text{and} \quad f(v) := \int_{\Omega} f v \, dx.$$

We assume that $\overline{\Omega} = \cup_{i=1}^{N} \overline{\Omega}_i$ and the substructures Ω_i are disjoint shaped regular polygonal subregions of diameter $O(H_i)$. We assume the partition $\{\Omega_i\}_{i=1}^{N}$ is geometrically conforming, i.e., $\forall i \neq j$ the intersection $\partial\Omega_i \cap \partial\Omega_j$ is empty or is a common corner or edge of Ω_i and Ω_j, where here and below an edge means a curve of continuous intervals while its endpoints are called corners and the the collection of these corners on $\partial\Omega_i$ are refered to corners of Ω_i. We assume $f \in L^2(\Omega)$, and for simplicity of presentation let $\rho_i(x)$ be a positive constant ρ_i.

2.2. Discrete problem. Let us introduce a shape regular and quasiuniform triangulation in each Ω_i with triangular elements and h_i as mesh parameter. The resulting triangulation on Ω is in general nonmatching across $\partial\Omega_i$. Let $X_i(\Omega_i)$ be the regular finite element (FE) space of piecewise linear and continuous functions in Ω_i. Note that we do not assume that functions in $X_i(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$. Define

$$X(\Omega) := X_1(\Omega_1) \times X_2(\Omega_2) \cdots \times X_N(\Omega_N).$$

Let us denote by $\mathcal{E}_i^0$ the collection of all edges of Ω_i which are shared by other subdomains, and denote by $\mathcal{E}_i^{\partial}$ the collection of edges of Ω_i which belong to $\partial\Omega$. The set of all edges of Ω_i is denoted by $\mathcal{E}_i$. Note that $\mathcal{E}_i = \mathcal{E}_i^0$ for all Ω_i which do not intersect $\partial\Omega$ by an edge. A discrete problem obtained by DG method, see [26, 4, 8], is of the form: *Find* $u^* = \{u_i^*\}_{i=1}^{N} \in X(\Omega)$ *such that*

$$a_h(u^*, v) = f(v) \;\; \forall v = \{v_i\}_{i=1}^{N} \in X(\Omega) \tag{2.2}$$

where

$$a_h(u,v) := \sum_{i=1}^{N} a^{(i)}(u,v) \;\text{ and }\; f(v) := \sum_{i=1}^{N} \int_{\Omega_i} f v_i \, dx, \tag{2.3}$$

$$a^{(i)}(u,v) := a_i(u,v) + s_i(u,v) + p_i(u,v), \tag{2.4}$$

and

$$a_i(u,v) := \int_{\Omega_i} \rho_i \nabla u_i \cdot \nabla v_i \, dx, \tag{2.5}$$

$$s_i(u,v) := \sum_{F_{ij}\subset \mathcal{E}_i} \int_{F_{ij}} \frac{1}{l_{ij}} \left(\rho_i \frac{\partial u_i}{\partial n}(v_j - v_i) + \rho_i \frac{\partial v_i}{\partial n}(u_j - u_i) \right) ds, \tag{2.6}$$

$$p_i(u,v) := \sum_{F_{ij}\subset \mathcal{E}_i} \int_{F_{ij}} \frac{\delta}{l_i} \frac{\rho_i}{h_{ij}} (u_j - u_i)(v_j - v_i)\, ds \tag{2.7}$$

where we set $l_{ij} = 2$ when $\bar{F}_{ij} := \partial\Omega_i \cap \partial\Omega_j$ is a common edge of Ω_i and Ω_j, and let $h_{ij} := 2h_ih_j/(h_i+h_j)$, i.e., the harmonic average of h_i and h_j. The notation $F_{ij} \subset \mathcal{E}_i$ includes also boundary edges on $\partial\Omega_i \cap \partial\Omega$ where we set $l_{i\partial} = 1$, and let $u_\partial = 0, v_\partial = 0$ and $h_{i\partial} := h_i$. The partial derivative $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega_i$ and δ is the penalty positive parameter.

We introduce the bilinear forms

$$d_i(u,v) := a_i(u,v) + p_i(u,v) \tag{2.8}$$

and

$$d_h(u,v) := \sum_{i=1}^{N} d_i(u,v), \tag{2.9}$$

and note that the norm defined by $d_h(\cdot,\cdot)$ is a broken norm in $X(\Omega)$ with weights given by ρ_i and $\frac{\delta}{l_{ij}}\frac{\rho_i}{h_{ij}}$. For $u = \{u_i\}_{i=1}^N \in X(\Omega)$ this discrete norm is defined by

$$\| u \|_h^2 := d_h(u,u) = \sum_{i=1}^{N} \left\{ \rho_i \| \nabla u_i \|_{L^2(\Omega_i)}^2 + \sum_{F_{ij}\subset \mathcal{E}_i} \frac{\delta}{l_{ij}} \frac{\rho_i}{h_{ij}} \int_{F_{ij}} (u_i - u_j)^2 ds \right\}.$$

It is known that there exists a $\delta_0 = O(1) > 0$ and a positive constant $c < 1$ such that for every $\delta \geq \delta_0$, we obtain $|s_i(u,u)| \leq c d_i(u,u)$ and $\sum_i |s_i(u,u)| \leq c d_h(u,u)$, and therefore, the following lemma is valid:

LEMMA 2.1. *There exists $\delta_0 > 0$ such that for $\delta \geq \delta_0$ and for all $u \in X(\Omega)$ we have*

$$\gamma_0 d_i(u,u) \leq a^{(i)}(u,u) \leq \gamma_1 d_i(u,u), \quad i = 1, \cdots, N \tag{2.10}$$

and

$$\gamma_0 d_h(u,u) \leq a_h(u,u) \leq \gamma_1 d_h(u,u). \tag{2.11}$$

Here, γ_0 and γ_1 are positive constants independent of the ρ_i, h_i, H_i and u. For the proof we refer to [8].

Lemma 2.1 implies that the discrete problem (2.2) is elliptic and continuous, therefore, the solution exists and it is unique and stable. An optimal a priori error estimate of this method was established in [3, 4] for the continuous coefficient case. When the coefficients are discontinuous across substructures and/or when h_i and h_j are not necessary of the same order, the following result is established in [8].

LEMMA 2.2. *Let u^*_{ex} and u^* be the solution of (2.1) and (2.2). If $u^*_{ex}|_{\Omega_i} \in H^q(\Omega_i)$, $i = 1, \cdots N$, and $3/2 \leq q \leq 2$, then*

$$\| u^*_{ex} - u^* \|_h^2 \leq C \sum_{i=1}^{N} \rho_i \Big(h_i^{2(q-1)} + \sum_{F_{ij} \subset \mathcal{E}_i^0} \frac{h_j}{h_i} h_j^{2(q-1)}\Big) \, \|u^*_{ex}\|^2_{H^q(\Omega_i)}$$

where C is independent of h_i, H_i, ρ_i and u^*_{ex}.

3. Schur complements systems and discrete harmonic extensions. The first step of many iterative substructuring solvers, such as the FETI-DP method that we consider in this paper, requires the elimination of the unknowns associated with the interior of the subdomains. In this section, we describe this step for DG discretizations.

We introduce some notation and then formulate (2.2) as a variational problem with constraints. Let us introduce

$$\Omega^{(i)} := \overline{\Omega}_i \cup \{\cup_{F_{ij} \subset \mathcal{E}_i^0} \bar{F}_{ji}\}$$

i.e., the union of $\overline{\Omega}_i$ and the $\bar{F}_{ji} \subset \partial\Omega_j$ for $F_{ji} = F_{ij}$ with $F_{ij} \subset \partial\Omega_i \backslash \partial\Omega$, and let

$$\Gamma^{(i)} := \cup_{F_{ij} \subset \mathcal{E}_i^0} (\bar{F}_{ij} \cup \bar{F}_{ji}).$$

Note that F_{ij} and F_{ji} are treated separately in spite of geometrically they are the same. Sometimes we use the notation F_{ijh} and F_{jih} to refer the sets of nodal points of the triangulation on F_{ij} and F_{ji} with parameters h_i and h_j, respectively, and $\bar{F}_{ijh}$ and $\bar{F}_{jih}$ when the endpoints are included.

Let $W_i(\Omega^{(i)})$ be the FE space of functions defined by nodal values of $\Omega^{(i)}$, i.e., an element $u^{(i)} \in W_i(\Omega^{(i)})$ is defined by $u_i^{(i)}$, the values at nodal points of $\overline{\Omega}_i$, and by $u_j^{(i)}$, the values at nodal points of $\bar{F}_{jih}$ for $F_{ji} = F_{ij} \subset \mathcal{E}_i^0$. Here and below we use the same notation to denote both FE functions and their vectors representations. Note that $a^{(i)}(\cdot, \cdot)$, see (2.4), is defined on $W_i(\Omega^{(i)}) \times W_i(\Omega^{(i)})$ with corresponding stiffness matrix $A^{(i)}$ given by

$$a^{(i)}(u^{(i)}, v^{(i)}) = \langle A^{(i)} u^{(i)}, v^{(i)} \rangle \quad u^{(i)}, v^{(i)} \in W_i(\Omega^{(i)}) \tag{3.1}$$

where $\langle u^{(i)}, v^{(i)} \rangle$ denotes the ℓ_2 inner product associated to nodes of $\Omega^{(i)}$. Let us represent $u^{(i)}$ as $u^{(i)} = (u_I^{(i)}, u_\Gamma^{(i)})$ where $u_\Gamma^{(i)}$ represents values of $u^{(i)}$ at nodal points of $\Gamma^{(i)}$ and $u_I^{(i)}$ represents the interior nodal values on $I_i := \Omega^{(i)} \backslash \Gamma^{(i)}$, hence, let us represent $W_i(\Omega^{(i)}) = W_i(I_i) \oplus W_i(\Gamma^{(i)})$. Using the representation $u^{(i)} = (u_I^{(i)}, u_\Gamma^{(i)})$, the matrix $A^{(i)}$ can be represented as

$$A^{(i)} = \begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)}, \end{pmatrix} \tag{3.2}$$

where the first block of rows and columns corresponds to the nodal points of I_i while the second block of rows and columns corresponds to the nodal points of $\Gamma^{(i)}$.

The Schur complement of $A^{(i)}$ with respect to $u_\Gamma^{(i)}$ is of the form:

$$S^{(i)} \equiv S_{\Gamma\Gamma}^{(i)} := A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)}(A_{II}^{(i)})^{-1}A_{I\Gamma}^{(i)} \tag{3.3}$$

and let us denote

$$S := \text{diag}\{S^{(1)}, S^{(2)}, \cdots, S^{(N)}\}. \tag{3.4}$$

Note that $S^{(i)}$ satisfies

$$\langle S^{(i)}u_\Gamma^{(i)}, u_\Gamma^{(i)}\rangle \ = \ \min \ a^{(i)}(w^{(i)}, w^{(i)}) \tag{3.5}$$

subject to $w^{(i)} = (w_I^{(i)}, w_\Gamma^{(i)}) \in W_i(\Omega^{(i)})$ with $w_\Gamma^{(i)} = u_\Gamma^{(i)}$ on $\Gamma^{(i)}$. The bilinear form $a^{(i)}(\cdot,\cdot)$ is symmetric and nonnegative, see Lemma 2.1. The minimizing function satisfying (3.5) is called discrete harmonic in the sense of $a^{(i)}(\cdot,\cdot)$ or in the sense of $\mathcal{H}^{(i)}$. An equivalent definition of the minimizing function $\mathcal{H}^{(i)}u_\Gamma^{(i)}$ is given by the solution of

$$a^{(i)}(\mathcal{H}^{(i)}u_\Gamma^{(i)}, v^{(i)}) = 0, \qquad v^{(i)} \in \overset{o}{W}_i(\Omega^{(i)}) \tag{3.6}$$

$$\mathcal{H}^{(i)}u_\Gamma^{(i)} = u_\Gamma^{(i)} \quad \text{on} \ \ \Gamma^{(i)} \tag{3.7}$$

where $\overset{o}{W}_i(\Omega^{(i)})$ is the subspace of $W_i(\Omega^{(i)})$ of functions which vanish on $\Gamma^{(i)}$. We note that for substructures Ω_i which intersect $\partial\Omega$ by edges, the nodal values on $\partial\Omega_i\backslash\Gamma^{(i)}$ are treated as unknowns.

Let $\mathcal{H}_i u_\Gamma^{(i)} \in W_i(\Omega^{(i)})$ be the standard discrete harmonic function of $u_\Gamma^{(i)} \in W_i(\Gamma^{(i)})$ in the sense of $a_i(\cdot,\cdot)$, see (2.5). We note that the extensions $\mathcal{H}_i$ and $\mathcal{H}^{(i)}$ differ from each other since $\mathcal{H}_i u_\Gamma^{(i)}$ depends only on the nodal values of $u_\Gamma^{(i)}$ on $\partial\Omega_i \cap \Gamma^{(i)}$ while $\mathcal{H}^{(i)}u_\Gamma^{(i)}$ depends on the nodal values on all $\Gamma^{(i)}$. The following lemma shows the equivalence (in the energy form defined by $d_i(\cdot,\cdot)$) between discrete harmonic functions in the sense of $\mathcal{H}_i$ and in the sense of $\mathcal{H}^{(i)}$; for the proof see Lemma 4.1 of [9]. This equivalence allows us to take advantages of all the discrete Sobolev results known for $\mathcal{H}_i$ discrete harmonic extensions.

Lemma 3.1. *For $u^{(i)} \in W_i(\Gamma^{(i)})$, it holds that*

$$d_i(\mathcal{H}_i u_\Gamma^{(i)}, \mathcal{H}_i u_\Gamma^{(i)}) \le d_i(\mathcal{H}^{(i)}u_\Gamma^{(i)}, \mathcal{H}^{(i)}u_\Gamma^{(i)}) \le C d_i(\mathcal{H}_i u_\Gamma^{(i)}, \mathcal{H}_i u_\Gamma^{(i)}) \tag{3.8}$$

where C is a positive constant independent of h_i, H_i, ρ_i and $u^{(i)}$.

The next corollary follows directly from Lemma 3.1 and Lemma 2.1.

Corollary 3.2. *For $u^{(i)} \in W_i(\Gamma^{(i)})$, it holds that*

$$C_0 d_i(\mathcal{H}_i u_\Gamma^{(i)}, \mathcal{H}_i u_\Gamma^{(i)}) \le a^{(i)}(\mathcal{H}^{(i)}u_\Gamma^{(i)}, \mathcal{H}^{(i)}u_\Gamma^{(i)}) \le C_1 d_i(\mathcal{H}_i u_\Gamma^{(i)}, \mathcal{H}_i u_\Gamma^{(i)}) \tag{3.9}$$

where C_0 and C_1 are positive constants independent of $h_i, \mathcal{H}_i$, ρ_i and $u^{(i)}$.

Let us introduce the product space

$$W(\Omega) := \prod_{i=1}^{N} W_i(\Omega^{(i)}), \tag{3.10}$$

i.e., $u \in W(\Omega)$ implies that $u = \{u^{(i)}\}_{i=1}^N$ with $u^{(i)} \in W_i(\Omega^{(i)})$.

We now consider the subspace $\hat{W}(\Omega) \subset W(\Omega)$ as the space of functions which are continuous on

$$\Gamma := \cup_{i=1}^N (\cup_{F_{ij} \subset \mathcal{E}_i^0} \bar{F}_{ij}) = \cup_{i=1}^N \Gamma^{(i)}. \tag{3.11}$$

Definition 3.3. *(Space $\hat{W}(\Omega)$, continuity on Γ). We say that $u = \{u^{(i)}\}_{i=1}^N \in W(\Omega)$ is continuous on $\Gamma^{(i)}$ if $u_j^{(i)} = u_j^{(j)}$ on $\bar{F}_{jih}$ and $u_i^{(j)} = u_i^{(i)}$ on $\bar{F}_{ijh}$ for all $F_{ij} \subset \mathcal{E}_i^0$. We say that u is continuous on Γ if it is continuous on all $\Gamma^{(i)}$ $i = 1, \cdots, N$. The space of continuous functions on Γ is denoted by $\hat{W}(\Omega)$.*

Note that there is a one-to-one correspondence between vectors in the spaces $X(\Omega)$ and $\hat{W}(\Omega)$. We introduce the restriction matrices $R_{\Omega^{(i)}} : X(\Omega) \to W_i(\Omega^{(i)})$ which assign the vector values of $u = \{u_i\}_{i=1}^N \in X(\Omega)$ into the vector values of $u^{(i)}$ at the nodes of $\Omega^{(i)}$ only. Note that $u = \{u^{(i)}\}_{i=1}^N \in \hat{W}(\Omega)$. Hence, we can represent uniquely $u \in X(\Omega)$ as $u = \{u^{(i)}\}_{i=1}^N \in \hat{W}(\Omega)$. Note that the discrete problem (2.2) can be written as a system of algebraic equations

$$\hat{A}u^* = f \tag{3.12}$$

with $u^* \in X(\Omega)$ and $f = \{f_i\}_{i=1}^N \in X(\Omega)$, where f_i is the load vector associated with individual subdomains Ω_i. The stiffness matrix $\hat{A}$ can be obtained by restricting the block diagonal matrix, see (3.2)

$$A := \text{diag}\{A^{(1)}, A^{(2)}, \cdots, A^{(N)}\} \tag{3.13}$$

from $W(\Omega)$ to $\hat{W}(\Omega)$, that is,

$$\hat{A} = \sum_{i=1}^{N} R_{\Omega^{(i)}}^T A^{(i)} R_{\Omega^{(i)}}.$$

Note that $\hat{A}$ is no longer block diagonal since there are couplings between substructures due to the continuity on Γ.

Note also that $X(\Omega)$ can be componentwise represented by $X(\Omega) = X(I) \oplus X(\Gamma)$ where $I = \cup_i I_i$ and Γ defined in (3.11), and also $W(\Omega)$ can be represented by $W(\Omega) = \hat{W}(I) \oplus W(\Gamma)$ where

$$\hat{W}(I) = \prod_{i=1}^{N} \hat{W}_i(I_i) \text{ and } W(\Gamma) = \prod_{i=1}^{N} W_i(\Gamma^{(i)}).$$

We introduce the restriction matrices $R_{I_i} : X(I) \to \hat{W}_i(I_i)$ and $R_{\Gamma^{(i)}} : X(\Gamma) \to W_i(\Gamma^{(i)})$ by assigning values of u_I to $u_i^{(i)}$ at nodes of I_i, and values of u_Γ to $u_{\Gamma^{(i)}}$ on nodes of $\Gamma^{(i)}$, respectively. By eliminating the variable u_I^* from (3.12), see (3.2) and (3.3), it is easy to see that

$$\hat{S} u_\Gamma^* = \hat{g}_\Gamma \tag{3.14}$$

where

$$\hat{S} = \sum_{i=1}^{N} R_{\Gamma^{(i)}}^T S^{(i)} R_{\Gamma^{(i)}} \quad \text{and} \quad \hat{g}_\Gamma = f_\Gamma - \sum_{i=1}^{N} R_{\Gamma^{(i)}}^T A_{\Gamma I}^{(i)} (A_{II}^{(i)})^{-1} f_i. \tag{3.15}$$

It is also easy to see from (3.6) and (3.7) that

$$\begin{pmatrix} v_I^{(i)} \\ v_\Gamma^{(i)} \end{pmatrix}^T \begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)}. \end{pmatrix} \begin{pmatrix} \hat{\mathcal{H}}^{(i)} u_\Gamma^{(i)} \\ u_\Gamma^{(i)} \end{pmatrix} = \langle S^{(i)} u_\Gamma^{(i)}, v_\Gamma^{(i)} \rangle. \tag{3.16}$$

Note that $\hat{W}(\Gamma)$ is the natural space for defining $\langle \hat{S}\ \cdot, \cdot \rangle$ due to (3.15), (3.16) and the continuity of $\hat{W}(\Gamma)$ on Γ.

4. FETI-DP with corner constraints. We now design a FETI-DP method for solving (3.14). We follow the abstract approach described in pages 160-167 in [28].

We introduce the nodal points associated to the corners unknowns by

$$\mathcal{V} := \cup_{i=1}^{N} \mathcal{V}_i \qquad \text{where} \qquad \mathcal{V}_i := \{\cup_{F_{ij} \subset \mathcal{E}_i^0} \partial F_{ij}\}. \tag{4.1}$$

Let $\tilde{W}(\Gamma)$ be the subspace of $W(\Gamma)$ of functions which are continuous on $\mathcal{V}$ in the sense that the finite element functions $u = \{u^{(i)}\}_{i=1}^N \in W(\Gamma)$ satisfy

$$u_i^{(i)}(x) = u_i^{(j)}(x) \quad \text{for all } \ x \in \partial F_{ij} \ \text{ for all } \ F_{ij} \subset \mathcal{E}_i^0 \tag{4.2}$$

and

$$u_j^{(j)}(x) = u_j^{(i)}(x) \quad \text{for all } \ x \in \partial F_{ji} \ \text{ for all } \ F_{ji} = F_{ij} \subset \mathcal{E}_i^0. \tag{4.3}$$

Here and below $F_{ji} = F_{ij} \subset \mathcal{E}_i^0$ means that F_{ji} is such that $F_{ji} = F_{ij}$ with $F_{ij} \subset \mathcal{E}_i^0$.

Note that

$$\hat{W}(\Gamma) \subset \tilde{W}(\Gamma) \subset W(\Gamma). \tag{4.4}$$

Note that a function $u \in \tilde{W}(\Gamma)$ does not imply that $u_i^{(i)} = u_j^{(i)}$ at points of $\partial F_{ij} = \partial F_{ji}$.

Let $\tilde{A}$ be the stiffness matrix which is obtained by restricting the matrix A defined in (3.13) from $W(\Omega)$ to $\tilde{W}(\Omega)$. Note that $\tilde{A}$ is no longer block diagonal since there are couplings between variables at points of $\mathcal{V}$. We represent $u \in \tilde{W}(\Omega)$ as $u = (u_I, u_\Pi, u_\triangle)$ where the subscript I refers to the interior degrees of freedom at nodal points of all $\Omega_i \cup \mathcal{E}_i^\partial$, the Π to the corners of all $\mathcal{V}_i$, and the $\triangle$ to the remaining nodal points, i.e. of the $\Gamma^{(i)} \backslash \mathcal{V}$. The vector $u = (u_I, u_\Pi, u_\triangle) \in \tilde{W}(\Omega)$ is obtained from the vector $u = (u_I, u_\Gamma) \in W(\Omega)$ using the equations (4.2), (4.3), i.e., the continuity of u on $\mathcal{V}$. Using the decomposition $u = (u_I, u_\Pi, u_\triangle) \in \tilde{W}(\Omega)$ we can partition $\tilde{A}$ as

$$\tilde{A} = \begin{pmatrix} A_{II} & A_{I\Pi} & A_{I\triangle} \\ A_{\Pi I} & A_{\Pi\Pi} & A_{\Pi\triangle} \\ A_{\triangle I} & A_{\triangle\Pi} & A_{\triangle\triangle} \end{pmatrix}. \tag{4.5}$$

We note that the only couplings across subdomains are through the variables Π where the matrix $A_{\Pi\Pi}$ is subassembled.

A Schur complement of $\tilde{A}$ with respect to the $\triangle$-unknowns (eliminating the I- and the Π-unknowns) is of the form

$$\tilde{S} := A_{\triangle\triangle} - (A_{\triangle I}\ A_{\triangle\Pi}) \begin{pmatrix} A_{II} & A_{I\Pi} \\ A_{\Pi I} & A_{\Pi\Pi} \end{pmatrix}^{-1} \begin{pmatrix} A_{I\triangle} \\ A_{\Pi\triangle} \end{pmatrix}. \tag{4.6}$$

A vector $u \in \tilde{W}(\Gamma)$ can uniquely be represented by $u = (u_\Pi, u_\triangle)$, therefore, we can represent $\tilde{W}(\Gamma) = \hat{W}_\Pi(\Gamma) \oplus \tilde{W}_\triangle(\Gamma)$, where $\hat{W}_\Pi(\Gamma)$ refers to the Π-degrees of freedom of $\tilde{W}(\Gamma)$ while $\tilde{W}_\triangle(\Gamma)$ to the $\triangle$-degrees of freedom of $\tilde{W}(\Gamma)$. The vector space $\tilde{W}_\triangle(\Gamma)$ can be decomposed as

$$\tilde{W}_\triangle(\Gamma) = \prod_{i=1}^{N} W_{i,\triangle}(\Gamma^{(i)}) \tag{4.7}$$

where the local space $W_{i,\triangle}(\Gamma^{(i)})$ refers to the degrees of freedom associated to the nodes of $\Gamma^{(i)} \backslash \mathcal{V}$. Hence, a vector $u \in \tilde{W}(\Gamma)$ can be represented as $u = (u_\Pi, u_\triangle)$ with $u_\Pi \in \hat{W}_\Pi(\Gamma)$ and $u_\triangle = \{u_\triangle^{(i)}\}_{i=1}^N \in \tilde{W}_\triangle(\Gamma)$ where $u_\triangle^{(i)} \in W_{i,\triangle}(\Gamma^{(i)})$. Note that $\tilde{S}$, see (4.6), is defined on the space $\tilde{W}_\triangle(\Gamma)$, and the following lemma follows (cf. Lemma 6.22 in [28] and Lemma 4.2 in [25]):

Lemma 4.1. *Let $u_\triangle \in \tilde{W}_\triangle(\Gamma)$ and let $\tilde{S}$ and $\tilde{A}$, defined in (4.6) and (4.5), respectively. Then,*

$$\langle \tilde{S} u_\triangle, u_\triangle \rangle \ = \min \langle \tilde{A} w, w \rangle \tag{4.8}$$

where the minimum is taken over $w = (w_I, w_\Pi, u_\triangle) \in \tilde{W}(\Omega)$.

Let us take $u \in \tilde{W}(\Gamma)$ as $u = (u_\Pi, u_\triangle)$ with $u_\Pi \in \hat{W}_\Pi(\Gamma)$ and $u_\triangle \in \tilde{W}_\triangle(\Gamma)$. We have $u_\triangle = \{u_\triangle^{(i)}\}_{i=1}^N$ with $u_\triangle^{(i)} \in W_{i,\triangle}(\Gamma^{(i)})$. The vector $u_\triangle^{(i)} \in W_{i,\triangle}(\Gamma^{(i)})$ can be partitioned as

$$u_\triangle^{(i)} = \{(u_\triangle^{(i)})_i, \{(u_\triangle^{(i)})_j\}_{F_{ji}=F_{ij}\subset\mathcal{E}_i^0}\}$$

where

$$(u_\triangle^{(i)})_i = u^{(i)}_{\triangle|_{\partial\Omega_i\backslash\mathcal{E}_i^0}} \ \text{ and } \ (u_\triangle^{(i)})_j = u^{(i)}_{\triangle|_{F_{ji}}}.$$

In order to define the jumping matrices $B_\triangle$, consider $u_\triangle \in \tilde{W}_\triangle(\Gamma)$ such that

$$u^{(i)}_{\triangle|_{F_{ij}}} - u^{(j)}_{\triangle|_{F_{ij}}} = 0, \ \text{ i.e., } \ (u_\triangle^{(i)})_i - (u_\triangle^{(j)})_i = 0 \ \text{ on } \ F_{ijh} \ \text{ for } \ F_{ij} \subset \mathcal{E}_i^0, \tag{4.9}$$

$$u^{(j)}_{\triangle|_{F_{ji}}} - u^{(i)}_{\triangle|_{F_{ji}}} = 0, \text{ i.e., } (u^{(j)}_{\triangle})_j - (u^{(i)}_{\triangle})_j = 0 \text{ on } F_{jih} \text{ for } F_{ji} = F_{ij} \subset \mathcal{E}^0_i,$$

and we denote these constraints, taking them for all $i = 1, \cdots, N$, by

$$B_\triangle u_\triangle = 0, \qquad B_\triangle = (B^{(1)}_\triangle, B^{(2)}_\triangle, \cdots, B^{(N)}_\triangle). \tag{4.10}$$

The rectangular matrix $B^{(i)}_\triangle$ consists of columns of $B_\triangle$ attributed to the (i)-th component of the product space $\tilde{W}_\triangle(\Gamma)$. The entries of the rectangular matrix consist of values of $\{0, 1, -1\}$, and $B_\triangle u_\triangle$, for $u_\triangle \in \tilde{W}_\triangle(\Gamma)$, measures the jump of $u_\triangle$ across the $\triangle$-nodes. It is easy to see that $B_\triangle$ is full rank.

We can reformulate the problem (3.14) as the variational problem with constraints in $\tilde{W}_\triangle(\Gamma)$ space: *Find* $u^*_\triangle \in \tilde{W}_\triangle(\Gamma)$ *such that*

$$J(u^*_\triangle) = \min J(v_\triangle) \tag{4.11}$$

subject to $v_\triangle \in \tilde{W}_\triangle(\Gamma)$ with constraints $B_\triangle v_\triangle = 0$, where

$$J(v_\triangle) := 1/2\langle \tilde{S} v_\triangle, v_\triangle\rangle - \langle \tilde{g}_\triangle, v_\triangle\rangle \tag{4.12}$$

where $\tilde{S}$ is defined in (4.6) and

$$\tilde{g}_\triangle := \tilde{f}_\triangle - (A_{\triangle I}\ A_{\triangle \Pi}) \begin{pmatrix} A_{II} & A_{I\Pi} \\ A_{\Pi I} & A_{\Pi\Pi} \end{pmatrix}^{-1} \begin{pmatrix} f_I \\ \hat{f}_\Pi \end{pmatrix}. \tag{4.13}$$

Here we note that $f = \{f_i\}_{i=1}^N \in X(\Omega)$ can be represented $f = (f_I, \hat{f}_\Pi, \tilde{f}_\triangle)$, where $\tilde{f}_\triangle = \{\tilde{f}_{i,\Delta}\}_{i=1}^N$ and $\tilde{f}_{i,\Delta}$ are the load vectors associated with the individual subdomains Ω_i, i.e., the entries $\tilde{f}_{i,\Delta}$ are defined as $\int_{\Omega_i} f v^i_\Delta dx$ when v^i_Δ are the canonical basis functions of $\tilde{W}_{i,\Delta}(\Gamma^{(i)})$. Note that $\tilde{S}$ is symmetric and positive definite since $\tilde{A}$ has these properties; see also Lemma 4.1. Introducing Lagrange multipliers $\lambda \in V$, where $V := \text{range}(B_\triangle)$, the problem (4.11) reduces to the saddle point problem of the form: *Find* $u^*_\triangle \in \tilde{W}_\triangle(\Gamma)$ *and* $\lambda^* \in V$ *such that*

$$\begin{cases} \tilde{S} u^*_\triangle & + & B^T_\triangle \lambda^* & = & \tilde{g}_\triangle \\ B_\triangle u^*_\triangle & & & = & 0. \end{cases} \tag{4.14}$$

Hence, (4.14) reduces to

$$F\lambda^* = g \tag{4.15}$$

where

$$F := B_\triangle \tilde{S}^{-1} B^T_\triangle, \qquad g := B_\triangle \tilde{S}^{-1} \tilde{g}_\triangle. \tag{4.16}$$

When λ^* is computed, $u^*_\triangle$ can be found by solving the problem

$$\tilde{S} u^*_\triangle = g - B^T_\triangle \lambda^*. \tag{4.17}$$

4.1. Dirichlet Preconditioner. We now define the FETI-DP preconditioner. Let $S_{\triangle}^{(i)}$ be the Schur complement of $S^{(i)}$, see (3.3), restricted to $W_{i,\triangle}(\Gamma^{(i)}) \subset W_i(\Gamma^{(i)})$, i.e., taken $S^{(i)}$ on functions of $W_i(\Gamma^{(i)})$ which vanish on $\mathcal{V} \cap \Gamma^{(i)}$. Let

$$S_{\triangle} := \text{diag}\{S_{\triangle}^{(i)}\}_{i=1}^{N}. \tag{4.18}$$

In other words, $S_{\triangle}^{(i)}$ is obtained from $S^{(i)}$ by deleting rows and columns corresponding to nodal values at nodal points of $\mathcal{V} \cap \Gamma^{(i)}$.

Let us introduce diagonal scaling matrices $D_{\triangle}^{(i)}$ for $i = 1, \cdots, N$. The $D_{\triangle}^{(i)}$ maps $W_{i,\triangle}(\Gamma^{(i)})$ into itself and the diagonal entries are defined by

$$D_{\triangle}^{(i)} = \frac{\rho_j^{\gamma}}{\rho_i^{\gamma} + \rho_j^{\gamma}} \text{ on } F_{ijh} \cup F_{jih}, \text{ for all } F_{ij} \subset \mathcal{E}_i^0 \text{ with } F_{ji} = F_{ij} \subset \mathcal{E}_i^0 \tag{4.19}$$

for $\gamma \in [1/2, \infty)$, see [27], and define

$$B_{D,\triangle} := (B_{\triangle}^{(1)} D_{\triangle}^{(1)}, \cdots, B_{\triangle}^{(N)} D_{\triangle}^{(N)}). \tag{4.20}$$

Let

$$P_{\triangle} := B_{D,\triangle}^T B_{\triangle} \tag{4.21}$$

which maps $\tilde{W}_{\triangle}(\Gamma)$ into itself. It is easy to check that for $w_{\triangle} = \{w_{\triangle}^{(i)}\}_{i=1}^{N} \in \tilde{W}_{\triangle}(\Gamma)$ with $w_{\triangle}^{(i)} \in W_{i,\triangle}(\Gamma^{(i)})$ and $w_{\triangle}^{(j)} \in W_{j,\triangle}(\Gamma^{(j)})$ it holds:

$$(P_{\triangle} w_{\triangle}(x))_i^{(i)} = \frac{\rho_j^{\gamma}}{\rho_i^{\gamma} + \rho_j^{\gamma}} [(w_{\triangle}^{(i)}(x))_i - (w_{\triangle}^{(j)}(x))_i], \qquad x \in F_{ijh} \tag{4.22}$$

$$(P_{\triangle} w_{\triangle}(x))_j^{(i)} = \frac{\rho_j^{\gamma}}{\rho_i^{\gamma} + \rho_j^{\gamma}} [(w_{\triangle}^{(i)}(x))_j - (w_{\triangle}^{(j)}(x))_j], \qquad x \in F_{jih}. \tag{4.23}$$

Note that

$$(P_{\triangle} w_{\triangle}(x))_j^{(j)} = \frac{\rho_i^{\gamma}}{\rho_i^{\gamma} + \rho_j^{\gamma}} [(w_{\triangle}^{(j)}(x))_j - (w_{\triangle}^{(i)}(x))_j], \qquad x \in F_{jih} \tag{4.24}$$

$$(P_{\triangle} w_{\triangle}(x))_i^{(j)} = \frac{\rho_i^{\gamma}}{\rho_i^{\gamma} + \rho_j^{\gamma}} [(w_{\triangle}^{(j)}(x))_i - (w_{\triangle}^{(i)}(x))_i], \qquad x \in F_{ijh}. \tag{4.25}$$

It is easy to see that $P_{\triangle}$ preserves jumps in the sense that

$$B_{\triangle} P_{\triangle} = B_{\triangle}. \tag{4.26}$$

From this follows that $P_{\triangle}$ is a projection ($P_{\triangle}^2 = P_{\triangle}$).

In the FETI-DP method the preconditioner is of the form

$$M^{-1} := B_{D,\triangle} S_{\triangle} B_{D,\triangle}^T = \sum_{i=1}^{N} B_{\triangle}^{(i)} D_{\triangle}^{(i)} S_{\triangle}^{(i)} D_{\triangle}^{(i)} (B_{\triangle}^{(i)})^T. \tag{4.27}$$

Note that M^{-1} is a block-diagonal matrix and each block is invertible since $S_\triangle^{(i)}$ and $D_\triangle^{(i)}$ are invertible and $B_\triangle^{(i)}$ is a full rank matrix. The following theorem holds:

THEOREM 4.2. *For any $\lambda \in V$ it holds that*

$$\langle M\lambda, \lambda\rangle \;\le\; \langle F\lambda, \lambda\rangle \;\le\; C(1+\log\frac{H}{h})^2\langle M\lambda,\lambda\rangle \tag{4.28}$$

where C is a positive constant independent of h_i, H_i, λ and the jumps of ρ_i. Here and below, $\log(\frac{H}{h}) := \max_{i=1}^N \log(\frac{H_i}{h_i})$.

Proof. By the general abstract theory for FETI-DP developed by [25], see also Theorem 6.35 of [28], the proof of the theorem follows by checking Lemma 4.3 and Lemma 4.4 below. The proofs of these lemmas are presented separately below. □

LEMMA 4.3. *For $u_\triangle \in \tilde{W}_\triangle(\Gamma)$ it follows that*

$$\langle \tilde{S}u_\triangle, u_\triangle\rangle \;\le\; \langle S_\triangle u_\triangle, u_\triangle\rangle. \tag{4.29}$$

Proof. The proof follows from Lemma 4.1 and from

$$\langle \tilde{S}u_\triangle, u_\triangle\rangle = \min\langle \tilde{A}w, w\rangle \;\le\; \min\langle \tilde{A}v, v\rangle \;=\; \langle S_\triangle u_\triangle, u_\triangle\rangle \tag{4.30}$$

where the minima are taken over $w = (w_I, w_\Pi, u_\triangle) \in \tilde{W}(\Omega)$ and $v = (v_I, 0, u_\triangle) \in \tilde{W}(\Omega)$. □

LEMMA 4.4. *For any $u_\triangle \in \tilde{W}_\triangle(\Gamma)$, it holds that*

$$\| P_\triangle u_\triangle \|_{S_\triangle}^2 \le C(1+\log\frac{H}{h})^2 \| u_\triangle \|_{\tilde{S}}^2 \tag{4.31}$$

where C is a positive constant independent of h_i, H_i, $u_\triangle$ and the jumps of ρ_i.

Proof. We first consider the case when the the edges is a single interval only. Let $u_\triangle \in \tilde{W}_\triangle(\Gamma)$ and let $u = (u_\Pi, u_\triangle) \in \tilde{W}(\Gamma)$ be the solution of

$$\langle \tilde{S}u_\triangle, u_\triangle\rangle \;=\; \min\langle Sw, w\rangle =: \langle Su, u\rangle, \tag{4.32}$$

where the minimum is taken over $w = (w_\Pi, u_\triangle) \in \tilde{W}(\Gamma)$ with $w_\Pi \in \hat{W}_\Pi(\Gamma)$ and S is defined in (3.4).

Let us represent the u as $\{u^{(i)}\}_{i=1}^N \in W(\Gamma)$ where $u^{(i)} \in W_i(\Gamma^{(i)})$. Let $I_{F_{ij}}u^{(i)}$ be the linear function on $\bar{F}_{ijh}$ and $\bar{F}_{jih}$ defined by values of $u^{(i)}$ at ∂F_{ij} and ∂F_{ji}, respectively. Let $\hat{u} = \{\hat{u}^{(i)}\}_{i=1}^N$ where $\hat{u}^{(i)} \in W_i(\Gamma^{(i)})$ be defined as

$$\hat{u}_i^{(i)} = I_{F_{ij}} u_i^{(i)} \;\text{ on }\; \bar{F}_{ijh} \;\text{ for all }\; F_{ij} \subset \mathcal{E}_i^0$$

and

$$\hat{u}_j^{(i)} = I_{F_{ji}} u_j^{(i)} \;\text{ on }\; \bar{F}_{jih} \;\text{ for all }\; F_{ji} = F_{ij} \subset \mathcal{E}_i^0$$

Note that $\hat{u} \in \hat{W}(\Gamma)$. Therefore, representing $\hat{u} = (\hat{u}_\Pi, \hat{u}_\triangle)$ we have $B_\triangle \hat{u}_\triangle = 0$. Using this we have, see (4.21),

$$P_\triangle u_\triangle \equiv B_{D,\triangle}^T B_\triangle u_\triangle = B_{D,\triangle}^T B_\triangle(u_\triangle - \hat{u}_\triangle) = P_\triangle(u_\triangle - \hat{u}_\triangle).$$

Define $v \in W(\Gamma)$ to be equal to $P_\triangle(u_\triangle - \hat{u}_\triangle)$ at the $\triangle$-nodes and $u_\triangle - \hat{u}_\triangle$, which is equal to zero, at the Π-nodes. Let us represent $v = \{v^{(i)}\}_{i=1}^N$ with $v^{(i)} \in W_i(\Gamma^{(i)})$. We have

$$\| P_\triangle u_\triangle \|^2_{S_\triangle} = \| v \|^2_S = \sum_{i=1}^N \| v^{(i)} \|^2_{S^{(i)}} \tag{4.33}$$

in view of the definition of $S_\triangle^{(i)}$ and S, see (4.18) and (3.4), hence, to prove the lemma it remains to show that

$$\sum_{i=1}^N \| v^{(i)} \|^2_{S^{(i)}} \le C(1 + \log H/h)^2 \|u\|^2_S \tag{4.34}$$

since by (4.32) we obtain (4.31). By Corollary 3.2 we need to show

$$\sum_{i=1}^N \tilde{d}_i(v^{(i)}, v^{(i)}) \le C(1 + \log H/h)^2 \sum_{i=1}^N \tilde{d}_i(u^{(i)}, u^{(i)}) \tag{4.35}$$

where, see (2.8),

$$\tilde{d}_i(v^{(i)}, v^{(i)}) := d_i(\mathcal{H}_i v^{(i)}, \mathcal{H}_i v^{(i)})$$

and so

$$\tilde{d}_i(v^{(i)}, v^{(i)}) = \rho_i \| \nabla v_i^{(i)} \|^2_{L^2(\Omega_i)} + \sum_{F_{ij} \subset \mathcal{E}_i} \frac{\rho_i \delta}{l_{ij} h_{ij}} \| v_i^{(i)} - v_j^{(i)} \|^2_{L^2(F_{ij})}, \tag{4.36}$$

where $v_i^{(i)} = \mathcal{H}_i v^{(i)}$ and $u_i^{(i)} = \mathcal{H}_i u^{(i)}$ inside the subdomains Ω_i.

We first estimate the first term of (4.36). We have

$$\| \nabla v_i^{(i)} \|^2_{L^2(\Omega_i)} \le C \sum_{F_{ij} \subset \mathcal{E}_i^0} \| v_i^{(i)} \|^2_{H_{00}^{1/2}(F_{ij})} \tag{4.37}$$

by the well-known estimate, see [28], and the fact that $v_i^{(i)} = 0$ at corners of $\partial\Omega_i$. Note that for subdomains Ω_i which intersect $\partial\Omega$ by edges we use the obvious inequality

$$\| \nabla v_i^{(i)} \|^2_{L^2(\Omega_i)} \le \| \nabla \tilde{v}_i^{(i)} \|^2_{L^2(\Omega_i)}$$

where $\tilde{v}_i^{(i)}$ is the stardard discrete harmonic extension on Ω_i with $\tilde{v}_i^{(i)} = v_i^{(i)}$ on $\mathcal{E}_i^0$ and $\tilde{v}_i^{(i)} = 0$ on $\mathcal{E}_i^\partial$. For the case $F_{ij} \subset \mathcal{E}_i^0$, we use (4.22) to get

$$\begin{aligned} \rho_i \| v_i^{(i)} \|^2_{H_{00}^{1/2}(F_{ij})} &= \frac{\rho_i \rho_j^{2\gamma}}{(\rho_i^\gamma + \rho_j^\gamma)^2} \| (u-\hat{u})_i^{(i)} - (u-\hat{u})_i^{(j)} \|^2_{H_{00}^{1/2}(F_{ij})} \le \\ &\le 3 \, \{ \rho_i \| (u-\hat{u})_i^{(i)} \|^2_{H_{00}^{1/2}(F_{ij})} + \rho_j \| (u-\hat{u})_j^{(j)} \|^2_{H_{00}^{1/2}(F_{ji})} + \\ &+ \frac{\rho_i \rho_j^{2\gamma}}{(\rho_i^\gamma + \rho_j^\gamma)^2} \| (u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)} \|^2_{H_{00}^{1/2}(F_{ij})} \} \end{aligned} \tag{4.38}$$

where we have used that $\frac{\rho_i\rho_j^{2\gamma}}{(\rho_i^\gamma+\rho_j^\gamma)^2} \le \min\{\rho_i,\rho_j\}$ if $\gamma \in [1/2,\infty)$, see [27].

For estimating the first term of the right-hand side of (4.38), it is well-known that, see [28],

$$\rho_i \parallel (u-\hat{u})_i^{(i)} \parallel^2_{H_{00}^{1/2}(F_{ij})} \le C(1+\log\frac{H_i}{h_i})^2\rho_i \parallel \nabla u_i^{(i)} \parallel^2_{L^2(\Omega_i)} \le \le C(1+\log\frac{H_i}{h_i})^2 d_i(u^{(i)},u^{(i)}) \tag{4.39}$$

since $\hat{u}_i^{(i)} = I_{F_{ij}}u_i^{(i)}$ is the linear interpolant of $u_i^{(i)}$ on F_{ij} with values $u_i^{(i)}$ on ∂F_{ij}. The estimate for the second term of the right-hand side of (4.38) is similar.

It remains to estimate the third term of the right-hand side of (4.38). We have

$$\parallel (u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)} \parallel^2_{H_{00}^{1/2}(F_{ij})} = = |(u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)}|^2_{H^{1/2}(F_{ij})} + \int_{F_{ij}} \frac{((u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)})^2}{dist(s,\partial F_{ij})}ds. \tag{4.40}$$

The first term of (4.40) is estimated as follows. Let Q_i be the L_2 projection on $X_i(F_{ij})$, the restriction of $X_i(\partial\Omega_i)$ to $\bar{F}_{ij}$ with h_i -triangulation on F_{ij}. Using the inverse inequality, the $H^{1/2}$ and L_2 stabilities of the L_2 projection we have

$$\begin{aligned}|(u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)}|^2_{H^{1/2}(F_{ij})} &\le C\ \{|Q_i(u_i^{(j)}-u_j^{(j)})|^2_{H^{1/2}(F_{ij})} + \\ &+|Q_i(u_j^{(j)}-\hat{u}_j^{(j)})|^2_{H^{1/2}(F_{ij})} + |(u-\hat{u})_j^{(j)}|^2_{H^{1/2}(F_{ij})} + |\hat{u}_i^{(j)}-\hat{u}_j^{(j)}|^2_{H^{1/2}(F_{ij})}\} \\ &\le C\{\frac{1}{h_i} \parallel u_i^{(j)}-u_j^{(j)} \parallel^2_{L^2(F_{ij})} + |\hat{u}_i^{(j)}-\hat{u}_j^{(j)}|^2_{H^{1/2}(F_{ij})} + |(u-\hat{u})_j^{(j)}|^2_{H^{1/2}(F_{ij})}\} \le \\ &\le C\{\frac{1}{h_i} \parallel u_i^{(j)}-u_j^{(j)} \parallel^2_{L^2(F_{ij})} + \max_{\partial F_{ij}}(u_i^{(j)}-u_j^{(j)})^2 + (1+\log\frac{H_j}{h_j})^2 \parallel \nabla u_j^{(j)} \parallel^2_{L^2(\Omega_j)}\}\end{aligned} \tag{4.41}$$

since $\hat{u}_i^{(j)}$ and $\hat{u}_j^{(j)}$ are linear on F_{ij} and F_{ji} and $F_{ij} = F_{ji}$. The second term of RHS of last inequality of (4.41) is estimated as follows. Let $\bar{u}_j^{(j)}$ be the average of $u_j^{(j)}$ on F_{ji}. We obtain

$$\begin{aligned}\max_{\partial F_{ij}}(u_i^{(j)}-u_j^{(j)})^2 &\le 3\ \{\ \max_{\partial F_{ij}}(Q_i(u_i^{(j)}-u_j^{(j)}))^2 + \\ &+\max_{\partial F_{ij}}(Q_i(u_j^{(j)}-\bar{u}_j^{(j)}))^2 + \max_{\partial F_{ij}}(u_j^{(j)}-\bar{u}_j^{(j)})^2\} \le \\ &\le C\{\frac{1}{h_i} \parallel u_i^{(j)}-u_j^{(j)} \parallel^2_{L^2(F_{ij})} + \max_{\partial F_{ij}}(Q_i(u_j^{(j)}-\bar{u}_j^{(j)}))^2 + \max_{F_{ji}}(u_j^{(j)}-\bar{u}_j^{(j)})^2\ \}.\end{aligned} \tag{4.42}$$

Hence, using a discrete Sobolev inequality, see [28], and the $H^{1/2}$ stability of the L^2 projection we obtain

$$\max_{\partial F_{ij}}(Q_i(u_j^{(j)}-\bar{u}_j^{(j)}))^2 \le C(1+\log\frac{H_i}{h_i})|u_j^{(j)}|^2_{H^{1/2}(F_{ji})} \le C(1+\log\frac{H_i}{h_i}) \parallel \nabla u_j^{(j)} \parallel^2_{L^2(\Omega_j)}$$

and so

$$\max_{\partial F_{ij}}(u_i^{(j)}-u_j^{(j)})^2 \le C\{\frac{1}{h_i} \parallel u_i^{(j)}-u_j^{(j)} \parallel^2_{L^2(F_{ij})} + + (1+\log\frac{H}{h}) \parallel \nabla u_j^{(j)} \parallel^2_{L^2(\Omega_j)}\}. \tag{4.43}$$

Substituting this into (4.41), we get

$$
\begin{aligned}
(4.44)\qquad & |(u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)}|_{H^{1/2}F(_{ij})} \leq \\
& \leq C(1+\log\frac{H}{h})^2\{\|\nabla u_j^{(j)}\|^2_{L^2(\Omega_j)} + \frac{1}{h_i}\|u_i^{(j)} - u_j^{(j)}\|^2_{L^2(F_{ij})}\}.
\end{aligned}
$$

We now estimate the second term of (4.40) as follows. In order to simplify notation we take F_{ij} as the interval [0, H]. Note that

$$
\begin{aligned}
(4.45)\qquad & \int_{F_{ij}} \frac{((u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)})^2}{dist(s,\partial F_{ij})} ds \leq \\
& \leq C\{\int_0^{H/2} \frac{((u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)})^2}{s} ds + \int_{H/2}^{H} \frac{((u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)})^2}{(H-s)} ds\}.
\end{aligned}
$$

Let us estimate the first term of RHS of (4.45). Let $h_i \leq h_j$ (the proof for $h_i > h_j$ is similar).

$$
\begin{aligned}
& \int_0^{H/2} \frac{((u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)})^2}{s} ds = \int_0^{h_i} \frac{((u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)})^2}{s} ds + \\
& + \int_{h_i}^{H/2} \frac{((u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)})^2}{s} ds \leq C\{[(u-\hat{u})_i^{(j)}(h_i) - (u-\hat{u})_j^{(j)}(h_i)]^2 + \\
& +(1+\log\frac{H_i}{h_i}) \max_{F_{ij}}((u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)})^2\} \leq \\
& \leq C(1+\log\frac{H_i}{h_i}) \max_{F_{ij}}((u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)})^2 \leq \\
& \leq C(1+\log\frac{H_i}{h_i})\{\max_{F_{ij}}(u_i^{(j)} - u_j^{(j)})^2 + \max_{F_{ij}}(\hat{u}_i^{(j)} - \hat{u}_j^{(j)})^2\} \leq \\
& \leq C(1+\log\frac{H_i}{h_i}) \max_{F_{ij}}(u_i^{(j)} - u_j^{(j)})^2 \leq \\
& \leq C(1+\log\frac{H}{h})^2\{\frac{1}{h_i}\|u_i^{(j)} - u_j^{(j)}\|^2_{L^2(F_{ij})} + \|\nabla u_j^{(j)}\|^2_{L^2(\Omega_j)}\}.
\end{aligned}
$$

We have used (4.42) and (4.43) to obtain the last inequality. Using this estimate in (4.45) we get

$$
\begin{aligned}
(4.46)\qquad & \int_{F_{ij}} \frac{((u-\hat{u})_i^{(j)} - (u-\hat{u})_i^{(j)})^2}{dist(s,\partial F_{ij})} ds \leq \\
& C(1+\log\frac{H}{h})^2\{\|\nabla u_j^{(j)}\|^2_{L^2(\Omega_j)} + \frac{1}{h_i}\|u_i^{(j)} - u_j^{(j)}\|^2_{L^2(F_{ij})}\}.
\end{aligned}
$$

Substituting (4.44) and (4.46) for i and j into (4.40) we get

$$
\begin{aligned}
(4.47)\qquad & \|(u-\hat{u})_i^{(j)} - (u-\hat{u})_j^{(j)}\|^2_{H^{1/2}_{00}(F_{ij})} \leq \\
& C(1+\log\frac{H}{h})^2\{\|\nabla u_j^{(j)}\|^2_{L^2(F_{ij})} + \frac{1}{h_i}\|u_i^{(j)} - u_j^{(j)}\|^2_{L^2(F_{ij})}\}.
\end{aligned}
$$

Substituting this and (4.39) into (4.38) we get

$$\rho_i \parallel v_i^{(i)} \parallel^2_{H^{1/2}_{00}(F_{ij})} \leq C(1+\log\frac{H}{h})^2\{\tilde{d}_i(u^{(i)},u^{(i)})+\tilde{d}_j(u^{(j)},u^{(j)})\} \tag{4.48}$$

It remains to estimate the second term of the right-hand side of (4.36). The case $F_{ij} \in \mathcal{E}_i^\partial$ is trivial. For the case $F_{ij} \subset \mathcal{E}_i^0$ we have, see (4.22) - (4.23),

$$\begin{aligned}
&\rho_i \parallel v_i^{(i)} - v_j^{(i)} \parallel^2_{L^2(F_{ij})} = \frac{\rho_i \rho_j^{2\gamma}}{(\rho_i^\gamma + \rho_j^\gamma)^2} \times \\
&\times \parallel [(u-\hat{u})_i^{(i)} - (u-\hat{u})_i^{(j)}] + [(u-\hat{u})_j^{(j)} - (u-\hat{u})_j^{(i)}] \parallel^2_{L^2(F_{ij})} \leq \\
&\leq 2\rho_i \parallel u_i^{(i)} - u_j^{(i)} \parallel^2_{L^2(F_{ij})} + 2\rho_j \parallel u_i^{(j)} - u_j^{(j)} \parallel^2_{L^2(F_{ij})}
\end{aligned} \tag{4.49}$$

since $\hat{u}_i^{(i)} = \hat{u}_i^{(j)}$ and $\hat{u}_j^{(j)} = \hat{u}_j^{(i)}$ on F_{ij} and F_{ji}, respectively. Hence

$$\begin{aligned}
&\sum_{F_{ij} \subset \mathcal{E}_i^0} \frac{\rho_i \delta}{l_{ij} h_{ij}} \parallel v_i^{(i)} - v_j^{(i)} \parallel^2_{L^2(F_{ij})} \leq \\
&\leq C \sum_{F_{ij} \subset \mathcal{E}_i^0} \frac{\delta}{l_{ij} h_{ij}} \{\rho_i \parallel u_i^{(i)} - u_j^{(i)} \parallel^2_{L^2(F_{ij})} + \rho_j \parallel u_i^{(j)} - u_j^{(j)} \parallel^2_{L^2(F_{ij})}\} \\
&\leq C \sum_{F_{ij} \subset \mathcal{E}_i^0} \{\tilde{d}_i(u^{(i)},u^{(i)}) + \tilde{d}_j(u^{(j)},u^{(j)})\}.
\end{aligned} \tag{4.50}$$

Substituting (4.48) and (4.50) into (4.36) we get

$$\tilde{d}_i(v^{(i)},v^{(i)}) \leq C(1+\log\frac{H}{h})^2\{\tilde{d}_i(u^{(i)},u^{(i)}) + \sum_{F_{ij} \subset \mathcal{E}_i} \tilde{d}_j(u^{(j)},u^{(j)})\}. \tag{4.51}$$

Summing the inequalities (4.51) for i from 1 to N and noting that the number of edges of each subdomain can be bounded independently of N, we obtain (4.35) and (4.34).

The proof also works with minor modifications for the case when F_{ij} is a continuous curve of intervals. For that, we should consider discrete Sobolev tools for non straight edges, see for instance [18], and interpret $I_{F_{ij}} u^{(i)}$ and $I_{F_{ji}} u^{(j)}$ as the linear function with respect to parametrized path on the edge defined by the nodal value of $u^{(i)}$ or $u^{(j)}$ at ∂F_{ij} and ∂F_{ji}. □

5. Implementation. The problem (4.15) can be solved efficiently by the preconditioned conjugate gradient method with the preconditioner M defined in (4.27). To simplify the presentation we only discuss the Richardson's method. For the system of algebraic equations, see (4.15),

$$F\lambda^* = g \tag{5.1}$$

the Richardson iterative method is of the form: with given λ_0 and for $k = 0, 1, \cdots$

$$\lambda_{k+1} = \lambda_k - \tau_{opt} M^{-1}(F\lambda_k - g) \tag{5.2}$$

where $\tau_{opt} = 2/(C(1+\log H/h)^2+1))$, see (4.28). We need to compute first

$$\tilde{r}_k := F\lambda_k - g = B_\triangle \tilde{S}^{-1}(B_\triangle^T \lambda_k - \tilde{g}_\triangle) = B_\triangle \tilde{S}^{-1}\tilde{g}_k \tag{5.3}$$

and then, see (4.27),

$$r_k := M^{-1}\tilde{r}_k = \sum_{i=1}^{N} B_\triangle^{(i)} D_\triangle^{(i)} S_\triangle^{(i)} D_\triangle^{(i)} (B_\triangle^{(i)})^T \tilde{r}_k \tag{5.4}$$

where $\tilde{r}_k = \{\tilde{r}_k^{(i)}\}_{i=1}^N \in \tilde{W}_\triangle(\Gamma)$ with $\tilde{r}_k^{(i)} \in W_{i,\triangle}(\Gamma^{(i)})$.

To compute $\tilde{r}_k$ we need to solve a system

$$\tilde{S} v_k = \tilde{g}_k. \tag{5.5}$$

Note that $\tilde{S}$ is a global matrix but with very weak couplings only through the corners of substructures Ω_i for $i = 1, \cdots, N$. Hence, the system with $\tilde{S}$ is solved by a special algorithm based on the Cholesky factorization. For the conforming case this algorithm is described in the book [28], see pp. 166-167. We modify this algorithm to solve (5.5). The main modification corresponds to the fact that in our case in a common corner of substructures Ω_i we have multiple unknowns while in the conforming case we have only one value. A computation of $B_\triangle^T v$, see (4.9) and (4.10), for a given $v = \{v^{(i)}\}_{i=1}^N$, reduces to multiply the rectangular matrices $(B_\triangle^{(i)})^T$ with entries $\{0, 1, -1\}$ by the subvector of $v^{(i)}$ belonging to $W_{i,\triangle}^{(i)}(\Gamma^{(i)})$.

To compute $M^{-1}\tilde{r}_k$, see (5.4), we need to compute for $i = 1, \cdots, N$

$$S_\triangle^{(i)} D_\triangle^{(i)} (B_\triangle^{(i)})^T \tilde{r}_k =: S_\triangle^{(i)} \tilde{v}_k^{(i)}. \tag{5.6}$$

A computation of $\tilde{v}_k^{(i)} := D_\triangle^{(i)} (B_\triangle^{(i)})^T \tilde{r}_k$ reduces to a multiplication of $\tilde{r}_k$ by $(B_\triangle^{(i)})^T$ and then by the diagonal scaling matrix $D_\triangle^{(i)}$, see (4.19). In turn, a computation of $S_\triangle^{(i)} \tilde{v}_k^{(i)}$ is reduced to solving a local problem defined on $\Gamma^{(i)}$, see (4.18), with zero values of $\tilde{v}^{(i)}$ at the corners $\mathcal{V}$ in $\Omega^{(i)}$. These problems involve the solution the problem $S^{(i)}$ on each $\Omega^{(i)}$ with Dirichlet data on $\Gamma^{(i)}$. We point out that the local problems are independent so they can be solved in parallel.

Finally compute

$$\lambda_{k+1} = \lambda_k - \tau_{opt} r_k$$

with the computed above r_k.

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