Infinite ergodic theory and Non-extensive entropies.

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We bring into account a series of result in the infinite ergodic theory that we believe that they are relevant to the theory of non-extensive entropies.

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I. INTRODUCTION.

In Ref. [33] it has been proposed that Non-extensive entropies can be useful to describe the dynamics of systems with zero Lyapunov exponent but exhibiting some weak form of sensitiveness to initial conditions. This sensitiveness would be described by the $q-$generalized Lyapunov exponent λ_q for some value of q in such a way that the average distance between point after $n-$iterates would be of the order of $\exp_q(\lambda_q n)$ where $\exp_q(t) = [1 + (1-q)t]^{\frac{1}{1-q}}$. In particular, it was conjectured in Ref. [34] that a version of Pesin's theorem for sub-exponential instability would relate the $q-$entropy with the $q-$generalized Lyapunov exponent. More precisely, they would coincide if $\lambda_q > 0$ and $q < 1$ and in this case the average distance increases polynomially.

With this in mind, we are recalling some results in the infinite ergodic theory, meaning the ergodic theory of systems preserving a non-finite measure. The reason to focus on these type of systems is because they exhibit zero Lyapunov exponent and they may exhibit sub-exponential instability. Therefore, they can be analyzed in the framework of non-extensive entropies.

In this direction, our goal is to show that for some systems preserving an infinite measure (and therefore having zero Lyapunov exponent) the next assertions hold:

1. they do not have a unique quantity that describes sub-exponential instabilities;
2. sub-exponential rates can grow faster than polynomial ones.

As a consequence of that, a twofold observation follows: *there is no a single quantity to characterize sub-exponential growth, and it can not necessary be understood in polynomial terms.*

To explain that, the following is done:

1. A class of examples having infinite invariant measure is introduced;
2. It is explained why systems having infinite invariant measure exhibit zero Lyapunov exponents;
3. It is shown how ergodic theorems can be recovered for infinite invariant measure and how the time averages become intrinsically random making difficult to find a unique "generalized Lyapunov" quantity;
4. It is explained how these type of systems displays sub-exponential instabilities and it is shown different examples where the sub-exponential instability is not polynomial.

For this exposition, we follow Ref. [1] and Ref. [35]. We want to point out that in this note, no new theorems for the infinite ergodic theory are provided; we only present some results already proved somewhere else and we recast them to show that they could be relevant for the theory of non-extensive statistical mechanics.

II. INFINITE MEASURE.

Given a map $T : X \to X$ acting on a phase space X, its action can lead to very complicated (chaotic) dynamics. Ergodic theory can be seen as a quantitative theory of dynamical systems, enabling us to rigorously deal with such situations, where it is impossible to predict when exactly some relevant event is going to take place. For example, Birkhoff ergodic theorem, tells us quite precisely how often an event will occur for typical initial states. In fact, a rich quantitative theory is available for systems possessing an invariant finite measure μ, meaning that $\mu \circ T^{-1} = \mu$. Moreover, in case of smooth systems, Birkhoff ergodic theorem allows to characterize the rate of mixing of a system.

However, there do exist systems of interest (not necessarily too exotic), which happen to have an infinite invariant measure, i.e., measure preserved by T and that $\mu(X) = \infty$. The "Infinite Ergodic Theory" focuses on such systems trying to answer the simplest quantitative question of understanding the long-term behavior of occupation times

$$S_n(A) := \Sigma_{k=0}^{n-1} 1_A(T^k(x)) \tag{1}$$

where 1_A is the characteristic function of the set A ($1_A(x) = 1$ if $x \in A$ otherwise the value is zero). The quantity $S_n(A)$ counts the number of visits an orbit pays to A before time n. Slightly more general, we can also look at ergodic sums

$$S_n(f) := \Sigma_{k=0}^{n-1} f(T^k(x)) \tag{2}$$

of measurable functions f.

Some typical examples of infinite measure preserving transformation are:

1. Boole maps, $T : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $T(x) = x - \frac{1}{x}$, where the invariant measure is the Lebesgue measure in the Real line (see Ref. [8] and [4]);

2. Pomeau-Manneville maps, $T : [0,1] \to [0,1]$, $T(x) = x + cx^p mod(1)$, in which zero is a parabolic fixed point ($T'(0) = 1$), and the invariant measure has support in $[0,1]$ but gives infinite measure to the interval (see Ref. [30, 10]);

3. Polynomial and rational maps on $\mathbb{C}$ (quotient of polynomials acting on $\mathbb{C}$) with parabolic fixed points (points where the derivative has modulus one) in the Julia set and no critical points there, where the invariant measure is a $h-$conformal measure concentrated in the Julia set and h is the Hausdorff dimension of the Julia set (see Ref. [2]);

4. Some quadratic unimodal maps (or logistic type maps) where the invariant measure is absolute continuous and giving infinite measure to the domain (see Ref. [21, 9, 5]);

5. Horocycle flows on infinite regular covers of compact hyperbolic surfaces, where the invariant measure is the classical volume measure (see Ref. [25]);

6. Two dimensional version of the Boole's map, $T : \mathbb{R}^2 \setminus \{0\} \times \mathbb{R} \to \mathbb{R}^2$, $T(x,y) = (x - \frac{1}{y}, x + y - \frac{1}{y})$, where the invariant measure is the Lebesgue measure in the whole two dimensional plane (see Ref. [12]);

7. Special flows (see Ref. [28] and [32]) which are volume preserving flows of the two dimensional torus $\mathbb{T}^2$ given by solution of the equation

$$\dot{x} = f(x,y)$$
$$\dot{y} = f(x,y)\alpha$$

where α is irrational, and $f : \mathbb{T}^2 \to \mathbb{R}$ verifies that $f(0,0) = 0$ and $f(x,y) > 0 \, \forall \, (x,y) \neq (0,0)$.

The first two systems are conjugated to the classical doubling function acting on the interval $[0,1]$ and so their dynamics can be described by the symbolic shift acting on the space of sequences of two symbols. In particular, they are topologically mixing and have infinitely many periodic orbits. The Pomeau-Manneville maps were introduced to model intermittent behavior in fluid dynamics (see Ref. [30]). For those type of system, the resulting behavior is an alternation of chaotic (when the orbit stays far from the parabolic point and where the map is similar to the bakers map) and regular when the orbit is trapped near the parabolic point. On the other hand, the presence of a parabolic fixed point makes those systems to have zero Lyapunov exponent and to display an infinite invariant measure. Moreover, for those type of systems it had been calculated rigorously the information content (Kolmogorov complexity) of the symbolic orbits generated by these systems (see Ref. [6, 17]), showing a behavior for the information that is between a positive entropy system (the information grows linearly with time) and an integrable system (the information grows as the logarithm of time).

Polynomial and rational maps are the typical dynamics acting on the complex plane; the classical example is given by the family of quadratical polynomials $P_\mu(z) = z^2 + \mu$. Recall that the Julia set (in the case of polynomials) is defined as the boundary of the set of points that its trajectory does not escape to infinite and its concentrate all the dynamics complexity (see for instance Ref. [27]).

The quadratic family is the classical example of one-dimensional real dynamics exhibiting chaotic dynamics. In Ref. [21, 9, 5] is shown that for certain parameters, the associated map has an infinite absolute continuous invariant measure.

The two dimensional version of the Boole's map introduced by Henon to model certain problem in celestial mechanics (see Ref. [19, 20), are conjugated to the Baker map acting on a two dimensional rectangle (see Ref. 12]).

The Horocyclic flow on compact hyperbolic surfaces is the most classical example of minimal and ergodic dynamic respect to finite volume measure (fact proved by Hedlund in 1930). They are associated to the classical geodesic flow on hyperbolic surfaces (free motion on hyperbolic surfaces). However, as proved in Ref. [25], when is considered infinite covers, the measure become infinite.

The last example, is a topologically mixing conservative system on the torus and having only one periodic point (the point $(0,0)$ which is fixed). Those systems are semiconjugated to quasi-periodic ones. The fact that the fixed point is parabolic, allows to find an infinite invariant measure.

Some of those systems were treated in the context of non-extensive entropies, see for instance Ref. [31, 11] for the case of the quadratic family. We want to point out that there are many other systems having zero Lyapunov exponent but exhibiting weak form of mixing and which are not covered by the above list (see for instance Ref. [13, 13, 16, 3, 15, 24]). However, in some cases it is unknown if they have infinite invariant measure

Even though many of the above described systems could seem very restrictive, it is important to remark that for one-dimensional dynamics, whenever a parabolic periodic point appear, they present anomalous statistical behavior and for this reason they have been used as models of many interesting physical systems.

The first ergodic theorem for recurrent ergodic measure transformation (i.e., almost every point is recurrent and any invariant set or its complement has measure zero) in the context of infinite measure is the following:

Theorem 1 *(Birkhoff's Pointwise Ergodic Theorem). Let T be a recurrent ergodic measure transformation on the infinite measure space $(X; A; \mu)$, then*

$$\frac{1}{n} S_n(f) \to 0. \tag{3}$$

This theorem shows that smooth systems preserving an infinite measure has *zero Lyapunov exponent*. More precisely:

Remark 1 *Observe that if T is a smooth one dimensional map on the line and $\log T' \in L^1(\mu)$ then it follows that*

$$\frac{1}{n}\log {T^n}'(x) = \frac{1}{n}S_n(\log T'(x)) \to 0, \ \ x \ a.e \qquad (4)$$

meaning that for almost every point the Lyapunov exponent is zero.

It is natural to ask if it is possible to find a sequence $\{a_n\}$ of positive normalizing constants, such that for all $A \in \mathcal{A}$, follows that $\frac{1}{a_n}S_n(1_A) \to \mu(A)$ and for any $f \in L^1(\mu)$ follows that $\frac{1}{a_n}S_n(f) \to \int f d\mu$. This could be regarded as an appropriate version of the ergodic theorem for spaces with infinite measure. The following shows that this is not possible:

Theorem 2 *Let T be a recurrent ergodic measure transformation on the infinite measure space $(X; A; \mu)$, and let any sequence $\{a_n\}_{n\in}$. Then for all $f \in L^1(\mu)$ either*

$$\limsup \frac{1}{a_n}S_n(f) = 0 \qquad (5)$$

or

$$\limsup \frac{1}{a_n}S_n(f) = \infty. \qquad (6)$$

The previous theorem shows that any potential normalizing sequence either over or underestimates the actual size of ergodic sums. Moreover, this points out that in infinite measure systems, the time average of an observation function fluctuates.

To find the appropriate normalizing sequences a_n (or time rescaling), it is needed to define the dual operator $\hat{T} : L^1(\mu) \to L^1(\mu)$ given by $\hat{T}(f) = f \circ T^{-1}$, which describes the evolution of measures under the action of T in the level of densities:

Definition 1 *It is said that the system is pointwise dual ergodic, if there exist constants $a_n = a_n(T)$, $n \in \mathbb{N}$, such that for any $L^1(\mu)$ it follows that*

$$\lim \frac{1}{a_n}\Sigma_{k=0}^{n-1}\hat{T}^k(f) = \int f\, du \qquad (7)$$

The sequence $a_n = a_n(T)$ is uniquely determined up to asymptotic equality, and is called the return sequence of T. Moreover, when the map is pointwise dual ergodic, there exists sets $A \in \mathcal{A}$ with $\mu(A) < +\infty$ such that

$$\lim \frac{1}{a_n}\Sigma_{k=0}^{n-1}\hat{T}^k(1_A) = \mu(A). \qquad (8)$$

This type of sets are called *Darling-Kac (DK) sets*. This means that the measure of the set A is recovered by pulling back the Lebesgue measure in A and averaged by the sequences $\{a_n\}$. The next result shows that the proper normalized time rescaling of an observation function converges in distribution, provided that the return sequence has certain properties.

If the return sequence is regularly varying with index $\alpha \in [0, 1]$ (i.e. $a_n(T) = n^\alpha h(n)$ and for any $c > 0$ $\frac{h(cn)}{h(n)} \to c^\alpha$), the asymptotic behavior of $S_n(f)$ can be described almost surely in distribution as follows, whenever it is assumed that the map is a recurrent ergodic measure transformation:

Theorem 3 *(Aaronson's Darling-Kac Theorem) Let T be a recurrent ergodic measure transformation on the infinite measure space $(X; A; \mu)$. Assume there is some DK-set $A \in \mathcal{A}$. If $a_n = a_n(T)$ is regularly varying of index $1-\alpha$ (for some $\alpha \in [0; 1]$), then for all $f \in L^1(\mu)$ and all $t > 0$*

$$\mu_A(\frac{1}{a_n}S_n(f) < t) \to Pr[M_\alpha(t)\int_X f d\mu] \ \ as\ n \to \infty. \qquad (9)$$

In previous theorem, μ_A denotes the measure μ restricted to the set A (and can be replaced by any probability absolutely continuous respect to μ), and $M_\alpha, \alpha \in [0; 1]$ denotes a non-negative real random variable distributed according to the (normalized) Mittag-Leffler distribution of order α, which can be characterized by its moments

$$\mathbb{E}[M_\alpha^r] = r!\frac{\Gamma(1+\alpha)^r}{\Gamma(1+r\alpha)}, \ \ r \geq 0. \qquad (10)$$

Going back to the problem of finding the sub-exponential Lyapunov exponents, the previous theorem shows that in certain cases it is not possible to find a unique quantity for almost every point even in the case of ergodic systems. Actually, the "Lyapunov exponent" behaves as a random variable. In particular, assuming that the transformation T is one-dimensional and smooth, if $f = \log(T')$ then the hypothesis of theorem 3 hold and $\alpha \neq 0$, it follows that given three points $t_1 < t_2 < t_3$ there is a set of positive measure of initial conditions such that $\exp(a_n t_1) < (T^n)'(x) < \exp(a_n t_2)$ (provided n large) and a set of positive measure of initial conditions such that $\exp(a_n t_2) < (T^n)'(x) < \exp(a_n t_3)$.

However, theorem 3 gives the range of fluctuation of the Lyapunov exponent and the sequences a_n provides up to a constant that depends on set of initial conditions, the rate of separation of trajectories. In fact for almost every point x, there exists $t(x)$ such that for any $\epsilon > 0$ if n is large enough then

$$\exp(a_n(t(x)-\epsilon)) < (T^n)'(x) < \exp(a_n(t(x)+\epsilon)). \qquad (11)$$

So, for some of the maps described before, we are going to explicit the normalizing sequences $a_n = a_n(T)$ and we are going to apply theorem 3:

1. For the Pomeau-Mannivelle maps, $a_n = n^{\frac{1}{p}}$ if $p > 1$, $a_n = \frac{n}{\log n}$ if $p = 1$;

2. For the Boole map, $a_n = \sqrt{n}$;
3. For rational maps on on $\mathbb{C}$ with parabolic points in the Julia set, $a_n = n^{\beta-1}$ for $1 < \beta < 4$ and $a_n = \frac{n}{\log n}$ if $\beta = 4$ with $\beta = \frac{p}{p+1}h$ where p is the first integer larger than one such that the derivative at the parabolic fixed point does not vanish and h is the Hausdorff dimension of the Julia set J;
4. Horocycle flows on periodic hyperbolic surfaces, $a(t) = \frac{t}{\ln(t)^k}$ with $k \in \frac{1}{2}$ depending on the surface (see Ref. [26]).

Observe that in same cases, the rescaling of time has a polynomial fashion, but the situation for the asymptotic growth of the derivative is quite different. So now, using that in the one-dimensional $\log(T^{n\prime}(x)) = \Sigma_{j=0}^{n-1}\log(T'(T^j(x)))$, theorem 3 and the explicit calculation of the sequences a_n one can provide the asymptotic growth of the derivative and therefore the rate of sub-exponential instability:

1. *Sub-exponential Lyapunov exponents for Pomeau-Manneville maps:* Given any pair of positive numbers $t_1 < t_2$, in the case that $p > 1$ it follows that for large n there is a positive set of initial condition that $\exp(t_1 n^{\frac{1}{p}}) < (T^n)'(x) < \exp(t_2 n^{\frac{1}{p}})$. If $p = 1$ it follows that for large n there is a positive set of initial condition that $\exp(t_1 \frac{n}{\ln(n)}) < (T^n)'(x) < \exp(t_2 \frac{n}{\ln(n)})$.
2. *Sub-exponential Lyapunov exponents for maps on $\mathbb{C}$ with parabolic points in the Julia set:* it is similar to the Pomeau-Manneville maps.
3. *Sub-exponential Lyapunov exponents for Boole maps:* There is a positive set of initial conditions for which $\exp(t_1\sqrt{n}) < (T^n)'(x) < \exp(t_2\sqrt{n})$.

In any case, it follows that *the sub-exponential growth of the derivative by iteration is larger than the growth of any polynomial and therefore it can not be described by any* $\exp_q$ *for any value of* q.

For the case of quadratic family, no explicit calculation have been performed, however a vast range of different type of sub-instability can be expected. This is discussed latter at the end of the present section.

Even theorem 3 shows that normalized time averages only converge to distributions, it is natural to wonder if the double average of the weighted Birkhoff sums converge. In fact, the following theorem shows how the expected value of normalized time averages also has a limit:

Theorem 4 *Assume that $a_n = n^\alpha h(n)$ is regularly varying with index $\alpha \in (0,1]$ or $\alpha = 0$ and with $h(n) \approx \exp(\int_1^t \frac{\eta(t)}{t}dt)$, where η is monotonic, $\eta(t) \to 0$ as $t \to \infty$ and $\frac{\eta(t)}{\log t} \to \infty$ as $t \to \infty$. Then for any function $f \in L^1(u)$*

$$\lim \frac{1}{N}\Sigma\frac{1}{a_n}S_n(f) = \int f\, du \tag{12}$$

in measure.

So, previous theorem applied to the case that T is a one-dimensional smooth maps (recall remark 1) gives the expected value of the random fluctuation of the sub-exponential rate of separation. More precissely,

$$\lim \frac{1}{N}\Sigma\frac{1}{a_n}(T^n)'(x) = \int \ln(T')(x)\, du. \tag{13}$$

It is important also to point out, that in certain cases (for instance for the Boole maps and Pomeau-Mannivelle maps), the rates a_n are related to the *induced map (or return map)*: Given a set A with $\mu(A) < +\infty$ and assuming that almost every return point (which is the case in the examples considered and in the hypothesis of the theorems), then for almost every point x it can be defined $n(x) = \min\{n \geq 1 : T^n(x) \in A\}$ and then, is defined the map $x \to T^{n(x)}$. It turns out that the measure μ restricted to A is ergodic and finite. Moreover, the examples of unimodal maps (quadratic-type maps) with infinite measure is obtained through a return map construction (usually called towers) and showing a type of non-integrability condition for the return times (see Ref. [5] for details). These analysis would provide a precise description of the sub-exponential instability. Precise studies of quantitative recurrence in systems having an infinite invariant measure have been performed also in Ref. [18]. Moreover, in Ref. [7] is done, through a series of examples, a detailed investigation of the relationships between quantitative recurrence indicators and algorithmic complexity of orbits in weakly chaotic dynamical systems.

The problem of finding a Pesin's type formula that relates the "sub-exponential Lyapunov exponent" with some generalized entropy, was answered positively in Ref. [23] for the case of the Pomeau-Manneville maps. In this paper, is related the quantity (13) with the entropy introduced by Krengel in Ref. [22], which is the normalized Kolmorgorov-Sinai entropy for the first return map defined above.

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