Technical Tools for Boundary Layers and Applications to Heterogeneous Coefficients

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1 Summary

We consider traces and discrete harmonic extensions on thin boundary layers. We introduce *sharp* estimates on how to control the $H^{1/2}-$ or $H_{00}^{1/2}-$ boundary norm of a finite element function by its energy in a thin layer and vice versa, how to control the energy of a discrete harmonic function in a layer by the $H^{1/2}$ or $H_{00}^{1/2}$ norm on the boundary. Such results play an important role in the analysis of domain decomposition methods in the presence of high-contrast media inclusions, small overlap and/or inexact solvers.

2 Introduction and Assumptions

Let Ω be a well-shaped polygonal domain of diameter $O(1)$ in $\Re^2$. We assume that the substructures $\Omega_i, 1 \leq i \leq N$, are well-shaped polygonal domains of diameters $O(H_i)$, and also assume that the Ω_i form a geometrically conforming nonoverlapping partitioning of Ω. Let $\mathcal{T}^{h_i}(\Omega_i)$ be a conforming shape-regular simplicial triangulation of Ω_i where h_i denotes the smallest diameter of the simplices of $\mathcal{T}^{h_i}(\Omega_i)$. We assume that the union of the triangulations $\mathcal{T}^{h_i}(\Omega_i)$, which we denote by $\mathcal{T}^h(\Omega)$, forms a conforming triangulation for Ω.

For purpose of analysis, let us introduce an auxiliary conforming shape-regular simplicial triangulation $\mathcal{T}^{\eta_i}(\Omega_i)$ of Ω_i where η_i denotes the smallest diameter of its simplices of $\mathcal{T}^{\eta_i}(\Omega_i)$. We do not assume that the triangulations $\mathcal{T}^{\eta_i}(\Omega_i)$ and $\mathcal{T}^{h_i}(\Omega_i)$ are nested. Let us introduce the boundary layer $\Omega_{i,\eta_i} \subset \Omega_i$ of width $O(\eta_i)$ as the union of all simplices of $\mathcal{T}^{\eta_i}(\Omega_i)$ that touch $\partial\Omega_i$ in at least one point. We assume that the mesh parameter η_i is large enough compared to h_i in the sense that all simplices of $\mathcal{T}^{h_i}(\Omega_i)$ that touch

$\partial\Omega_i$ must be contained in Ω_{i,η_i}. We also introduce the boundary layer Ω'_{i,η_i} of width $O(\eta_i)$ as the union of all simplices of $\mathcal{T}^{h_i}(\Omega_i)$ which intersect Ω_{i,η_i}, hence, it is easy to see that $\Omega_{i,\eta_i} \subset \Omega'_{i,\eta_i}$. We denote by $\mathcal{T}^{\eta_i}(\Omega_{i,\eta_i})$ the triangulation of $\mathcal{T}^{\eta_i}(\Omega_i)$ restricted to Ω_{i,η_i}, and by $\mathcal{T}^{h_i}(\Omega'_{i,\eta_i})$ the triangulation of $\mathcal{T}^{h_i}(\Omega_i)$ restricted to Ω'_{i,η_i}. Throughout the paper, the notation $c \preceq d$ (for quantities c and d) means that c/d is bounded from above by a positive constant independently of h_i, H_i, η_i and ρ_i. Moreover, $c \asymp d$ means $c \preceq d$ and $d \preceq c$. We also use $c \leq d$ to stress that $c/d \leq 1$.

We study the following selfadjoint second order elliptic problem:

Find $u^* \in H_0^1(\Omega)$ such that

$$a_\rho(u^*, v) = f(v), \quad \forall v \in H_0^1(\Omega) \tag{1}$$

where

$$a_\rho(u^*, v) := \sum_{i=1}^{N} \int_{\Omega_i} \rho_i(x) \nabla u^* \cdot \nabla v \, dx \quad \text{and} \quad f(v) := \int_{\Omega_i} f v \, dx \quad \text{for} \quad f \in L^2(\Omega).$$

We assume that $0 < c_i \leq \rho_i(x) \leq C_i$ for any $x \in \Omega_i$. We note that the condition number estimates of the preconditioned systems considered in this paper do not depend on the constants c_i and C_i.

Definition: We say that a coefficient ρ_i satisfies the *Boundary Layer Assumption* on Ω_i if $\rho_i(x)$ is equal to a constant $\bar{\rho}_i$ for any $x \in \Omega'_{i,\eta_i}$.

Definition: We say that a coefficient ρ_i associated to a subdomain Ω_i is of the *Inclusion Hard* type or *Inclusion Soft* type if the *Boundary Layer Assumption* holds with $\rho_i(x) = \bar{\rho}_i$ on Ω'_{i,η_i}, and

- *Inclusion Hard* type: $\rho_i(x) \succeq \bar{\rho}_i$ for all $x \in \Omega_i \backslash \Omega'_{i,\eta_i}$,
- *Inclusion Soft* type: $\rho_i(x) \preceq \bar{\rho}_i$ for all $x \in \Omega_i \backslash \Omega'_{i,\eta_i}$.

We allow the coefficients $\{\bar{\rho}_i\}_{i=1}^N$ to have large jumps across the interface of the subdomains $\Gamma := (\cup_{i=1}^N \partial\Omega_i) \backslash \partial\Omega$. The results to be presented in this paper can be extended easily to moderated variations of the coefficients ρ_i on Ω'_{i,η_i}.

We point out that the extension of our results to problems where the coefficient ρ_i has large jumps inside Ω'_{i,η_i} is not trivial. We point out, however, that for certain distributions of coefficients ρ_i where weighted Poincaré type inequalities are explicitly given (see [7]), the technical tools introduced here can be applied to derive sharper analysis. For instance, in the case where a hard inclusion G crosses an edge $E_{ij} := \partial\Omega_i \cap \partial\Omega_j$, we can impose primal constraints to guarantee average continuity on each connected component of $G \cap E_{ij}$; see numerical experiments on [3]. See also the related work on energy minimizing coarse spaces [5] and on expensive and robust methods based on enhanced partition of unity coarse spaces based on eigenvalue problems [1, 8] on the diagonally scaled system, see Remark 4.1 of [1], or equivalently, using generalized eigenvalue problems on the original system [4].

3 Technical Tools for Layers

We now introduce technical tools that are essential for obtaining sharp bounds for certain domain decomposition methods. The next lemma shows how $|w|_{H^{1/2}(\partial\Omega_i)}$ can be controlled by the energy of w on Ω_{i,η_i}.

Lemma 1. *Let* $w \in H^1(\Omega_{i,\eta_i})$*. Then*

$$|w|^2_{H^{1/2}(\partial\Omega_i)} \preceq \frac{H_i}{\eta_i}|w|^2_{H^1(\Omega_{i,\eta_i})}. \tag{2}$$

Proof. Let $V^{\eta_i}(\Omega_{i,\eta_i}) \subset H^1(\Omega_{i,\eta_i})$ be the space of piecewise linear and continuous functions associated to $\mathcal{T}_{\eta_i}(\Omega_{i,\eta_i})$. Let Π^{η_i} be the Zhang-Scott-Clemént interpolation operator from $H^1(\Omega_{i,\eta_i})$ to $V^{\eta_i}(\Omega_{i,\eta_i})$. Using a triangular inequality we obtain

$$|w|^2_{H^{1/2}(\partial\Omega_i)} \leq 2\left(|w - \Pi^{\eta_i}w|^2_{H^{1/2}(\partial\Omega_i)} + |\Pi^{\eta_i}w|^2_{H^{1/2}(\partial\Omega_i)}\right). \tag{3}$$

We now estimate the first term of the right-hand side of (3). Let us first define the cut-off function θ_i on Ω_i equals to one on $\partial\Omega_i$, equals to zero at all interior nodes of $\mathcal{T}^{\eta_i}(\Omega_i)$ and linear in each element of $\mathcal{T}^{\eta_i}(\Omega_i)$. Note that $0 \leq \theta_i(x) \leq 1$ for $x \in \Omega_i$, $\theta_i(x) = 1$ for $x \in \partial\Omega_i$, $\theta_i(x) = 0$ for $x \in \Omega_i \backslash \Omega_{i,\eta_i}$, and $\|\theta_i\|_{W^{1,\infty}(\Omega_{i,\eta_i})} \preceq 1/\eta_i$. Denoting by $z = w - \Pi^{\eta_i}w$ on Ω_{i,η_i} and using trace and minimal energy arguments plus standard calculations we obtain

$$|z|^2_{H^{1/2}(\partial\Omega_i)} \preceq |\theta_i z|^2_{H^1(\Omega_{i,\eta_i})} \preceq |z|^2_{H^1(\Omega_{i,\eta_i})} + \frac{1}{\eta_i^2}\|z\|^2_{L^2(\Omega_{i,\eta_i})}. \tag{4}$$

The right-hand side of (4) can be bounded by $|w|^2_{H^1(\Omega_{i,\eta_i})}$ by using the $H^1(\Omega_{i,\eta_i})$-stability and the $L_2(\Omega_{i,\eta_i})$-approximation properties of the Zhang-Scott-Clemént interpolation operator Π^{η_i}. We note that the proofs of these properties are based only on local arguments, therefore, they hold also for domains like Ω_{i,η_i}.

We now estimate the second term of the right-hand side of (3). We first use scaling and embedding arguments to obtain

$$|\Pi^{\eta_i}w|^2_{H^{1/2}(\partial\Omega_i)} \preceq H_i|\Pi^{\eta_i}w|^2_{H^1(\partial\Omega_i)}. \tag{5}$$

To bound the right-hand side of (5), let us first introduce the subregion $\hat{\Omega}_{i,\eta_i} \subset \Omega_{i,\eta_i}$ as the union of elements of $\mathcal{T}^{\eta_i}(\Omega_{i,\eta_i})$ which have an edge on $\partial\Omega_i$. Using only properties of linear elements of $V^{\eta_i}(\Omega_{i,\eta_i})$ we have

$$H_i|\Pi^{\eta_i}w|^2_{H^1(\partial\Omega_i)} \preceq \frac{H_i}{\eta_i}|\Pi^{\eta_i}w|^2_{H^1(\hat{\Omega}_{i,\eta_i})} \leq \frac{H_i}{\eta_i}|\Pi^{\eta_i}w|^2_{H^1(\Omega_{i,\eta_i})}. \tag{6}$$

The lemma follows by using the $H^1(\Omega_{i,\eta_i})$-stability of the Zhang-Scott-Clemént interpolation operator.

3.1 Technical Tools for DDMs

In this section we present the technical tools necessary to establish sharp analysis for exact and inexact two-dimensional FETI-DP with vertex constraints. More general technical tools can also be extended to obtain sharp analysis for non-overlapping Schwarz methods such as FETI and FETI-DP with edge and vertex primal constraints [9], Additive average Schwarz methods [2], inexact iterative substructuring methods and for three-dimensional problems; see [3].

Let $w \in V^{h_i}(\partial\Omega_i)$. Define the following discrete harmonic extensions:

1. The $\mathcal{H}^{(i)}_{\rho_i} w \in V^{h_i}(\Omega_i)$ as the ρ_i-discrete harmonic extension of w inside Ω_i, i.e., $\mathcal{H}^{(i)}_{\rho_i} w = w$ on $\partial\Omega_i$ and
$$\int_{\Omega_i} \rho_i(x) \nabla \mathcal{H}^{(i)}_{\rho_i} w \cdot \nabla v \; dx = 0 \text{ for any } \; v \in V_0^{h_i}(\Omega_i). \tag{7}$$
Here, $V_0^{h_i}(\Omega_i)$ is the space of functions of $V^{h_i}(\Omega_i)$ which vanish on $\partial\Omega_i$.
2. The $\mathcal{H}^{(i)}_{\rho_i,\mathcal{D}} w \in V^{h_i}(\Omega'_{i,\eta_i})$ as the zero Dirichlet boundary layer harmonic extension of w inside Ω'_{i,η_i}, i.e., $\mathcal{H}^{(i)}_{\rho_i,\mathcal{D}} w = w$ on $\partial\Omega_i$ and $\mathcal{H}^{(i)}_{\rho_i,\mathcal{D}} w = 0$ on $\partial\Omega'_{i,\eta_i} \backslash \partial\Omega_i$, and
$$\int_{\Omega'_{i,\eta_i}} \rho_i(x) \nabla \mathcal{H}^{(i)}_{\rho_i,\mathcal{D}} w \cdot \nabla v \; dx = 0 \text{ for any } \; v \in V_{0,\mathcal{D}}^{h_i}(\Omega'_{i,\eta_i}).$$
Here, $V^{h_i}(\Omega'_{i,\eta_i})$ is the space of continuous piecewise linear finite elements on $\mathcal{T}^{h_i}(\Omega'_{i,\eta_i})$, and $V_{0,\mathcal{D}}^{h_i}(\Omega'_{i,\eta_i})$ is the space of functions of $V^{h_i}(\Omega'_{i,\eta_i})$ which vanish on $\partial\Omega'_{i,\eta_i}$.
3. The $\mathcal{H}^{(i)}_{\rho_i,\mathcal{N}} w \in V^{h_i}(\Omega'_{i,\eta_i})$ as the zero Neumann boundary layer harmonic extension of w inside Ω'_{i,η_i}, i.e.,$\mathcal{H}^{(i)}_{\rho_i,\mathcal{N}} w = w$ only on $\partial\Omega_i$ and
$$\int_{\Omega'_{i,\eta_i}} \rho_i(x) \nabla \mathcal{H}^{(i)}_{\rho_i,\mathcal{N}} w \cdot \nabla v \; dx = 0 \text{ for any } \; v \in V_{0,\mathcal{N}}^{h_i}(\Omega'_{i,\eta_i}).$$
Here, $V_{0,\mathcal{N}}^{h_i}(\Omega'_{i,\eta_i})$ is the space of functions of $V^{h_i}(\Omega'_{i,\eta_i})$ which vanish on $\partial\Omega_i$.

Lemma 2. *Let us assume that the Boundary Layer Assumption holds on Ω_i and let $w \in V^{h_i}(\partial\Omega_i)$. Then*
$$|\mathcal{H}^{(i)}_{\rho_i} w|^2_{H^1_{\rho_i}(\Omega_i)} \leq |\mathcal{H}^{(i)}_{\rho_i,\mathcal{D}} w|^2_{H^1_{\rho_i}(\Omega'_{i,\eta_i})} \preceq |\mathcal{H}^{(i)}_{\rho_i,\mathcal{N}} w|^2_{H^1_{\rho_i}(\Omega'_{i,\eta_i})} + \frac{\bar{\rho}_i}{\eta_i} \|w\|^2_{L^2(\partial\Omega_i)}. \tag{8}$$
When $\rho_i(x) \preceq \bar{\rho}_i$ (Inclusion Soft type) on Ω_i, then
$$|\mathcal{H}^{(i)}_{\rho_i} w|^2_{H^1_{\rho_i}(\Omega_i)} \preceq \bar{\rho}_i |w|^2_{H^{1/2}(\partial\Omega_i)} \preceq \frac{H_i}{\eta_i} |\mathcal{H}^{(i)}_{\rho_i,\mathcal{N}} w|^2_{H^1_{\rho_i}(\Omega'_{i,\eta_i})}. \tag{9}$$

Proof. The result (8) follows from [6]; see also [3] for an alternative proof. The result (9) follows from Lemma 1 and the fact that $\Omega_{i,\eta_i} \subset \Omega'_{i,\eta_i}$.

Let E be an edge of $\partial\Omega_i$ and $I^{H_i}w : V^{h_i}(\partial\Omega_i) \to V^{H_i}(E)$ be the linear interpolation of w on E defined by the values of w on ∂E. Using some of the ideas shown in the proof of Lemma 1 (see [3] for details), it is possible to prove the following lemma:

Lemma 3. *Let us assume that the Boundary Layer Assumption holds on Ω_i and let $w \in V^{h_i}(\partial\Omega_i)$, $v_E := w - I^{H_i}w$ on E and $v_E := 0$ on $\partial\Omega_i \backslash E$. Then*

$$\bar{\rho}_i \|v_E\|^2_{H^{1/2}_{00}(E)} \preceq \left(\frac{H_i}{\eta_i}\,(1+\log\frac{\eta_i}{h_i}) + (1+\log\frac{\eta_i}{h_i})^2 \right) |\mathcal{H}^{(i)}_{\rho_i,\mathcal{N}} w|^2_{H^1_{\rho_i}(\Omega'_{i,\eta_i})}, \tag{10}$$

$$|\mathcal{H}^{(i)}_{\rho_i,\mathcal{N}} v_E|^2_{H^1_{\rho_i}(\Omega'_{i,\eta_i})} \preceq (1+\log\frac{\eta_i}{h_i})^2\; |\mathcal{H}^{(i)}_{\rho_i,\mathcal{N}} w|^2_{H^1_{\rho_i}(\Omega'_{i,\eta_i})}, \tag{11}$$

and

$$\frac{\bar{\rho}_i}{\eta_i}\|v_E\|^2_{L^2(E)} \preceq \frac{H_i^2}{\eta_i^2}\,(1+\log\frac{\eta_i}{h_i})\; |\mathcal{H}^{(i)}_{\rho_i,\mathcal{N}} w|_{H^1_{\rho_i}(\Omega'_{i,\eta_i})}. \tag{12}$$

When $\bar{\rho}_i \preceq \rho_i(x)$ (Inclusion Hard type) on Ω_i, then

$$\frac{\bar{\rho}_i}{\eta_i}\|v_E\|^2_{L^2(E)} \preceq \frac{H_i}{\eta_i}\,(1+\log\frac{\eta_i}{h_i})\; |\mathcal{H}^{(i)}_{\rho_i} w|^2_{H^1_{\rho_i}(\Omega_i)}. \tag{13}$$

4 Dual-Primal Formulation

The discrete problem associated to (1) will be formulated below in (17) as a saddle-point problem. We follow [9] for the description of the FETI-DP method.

Let $V^{h_i}(\Omega_i)$ be the space of continuous piecewise linear functions on $\mathcal{T}^{h_i}(\Omega_i)$ which vanish on $\partial\Omega_i \cap \partial\Omega$. The associated subdomain stiffness matrices $A^{(i)}$ and the load vectors $f^{(i)}$ from the contribution of the individual elements are given by

$$v^{(i)^T} A^{(i)} u^{(i)} := a_{\rho_i}(u^{(i)}, v^{(i)}) := \int_{\Omega_i} \rho_i\, \nabla u^{(i)} \cdot \nabla v^{(i)}\, dx, \quad \forall\, u^{(i)}, v^{(i)} \in V^{h_i}(\Omega_i)$$

and

$$v^{(i)^T} f^{(i)} := \int_{\Omega_i} f v^{(i)}\, dx, \quad \forall\, v^{(i)} \in V^{h_i}(\Omega_i).$$

Here and below we use the same notation to denote both finite element functions and their vector representations. We denote by $V^h(\Omega)$ the product space of the $V^{h_i}(\Omega_i)$ and represent a vector (or function) $u \in V^h(\Omega)$ as $u = \{u^{(i)}\}_{i=1}^N$ where $u^{(i)} \in V^{h_i}(\Omega_i)$.

Let the interface $\Gamma := (\cup_{i=1}^N \partial\Omega_i)\backslash\partial\Omega$ be the union of interior edges and vertices. The nodes of an edge are shared by exactly two subdomains, and the edges are open subsets of Γ. The vertices are endpoints of the edges. For each subdomain $\overline{\Omega}_i$, let us partition the vector $u^{(i)}$ into a vector of primal variables $u_\Pi^{(i)}$ and a vector of nonprimal variables $u_\Sigma^{(i)}$. We choose only vertices as primal nodes since we are considering only two dimensional problems. Let us partition the nonprimal variables $u_\Sigma^{(i)}$ into a vector of interior variables $u_I^{(i)}$ and a vector of edge variables $u_\triangle^{(i)}$. We will enforce continuity of the solution in the primal unknowns of $u_\Pi^{(i)}$ by making them global; we subassemble the subdomain stiffness matrix $A^{(i)}$ with respect to this set of variables and denote the resulting matrix by $\tilde{A}$. For the remaining interfaces variables, i.e., the edge variables $u_\triangle := \{u_\triangle^{(i)}\}_{i=1}^N$, we will introduce Lagrange multipliers to enforce continuity. We also refer to the edge variables as dual variables.

Here we include more details: we partition the stiffness matrices according to the different sets of unknowns and obtain

$$A^{(i)} = \begin{bmatrix} A_{\Sigma\Sigma}^{(i)} & {A_{\Pi\Sigma}^{(i)}}^T \\ A_{\Pi\Sigma}^{(i)} & A_{\Pi\Pi}^{(i)} \end{bmatrix}, \quad A_{\Sigma\Sigma}^{(i)} = \begin{bmatrix} A_{II}^{(i)} & {A_{\triangle I}^{(i)}}^T \\ A_{\triangle I}^{(i)} & A_{\triangle\triangle}^{(i)} \end{bmatrix}, \tag{14}$$

and

$$f^{(i)} = [{f_\Sigma^{(i)}}^T \, {f_\Pi^{(i)}}^T]^T, \quad f_\Sigma^{(i)} = [f_I^{(i)} f_\triangle^{(i)}]^T.$$

Next we define the block diagonal matrices

$$A_{\Sigma\Sigma} = \mathrm{diag}_{i=1}^N(A_{\Sigma\Sigma}^{(i)}), \quad A_{\Pi\Sigma} = \mathrm{diag}_{i=1}^N(A_{\Pi\Sigma}^{(i)}), \quad A_{\Pi\Pi} = \mathrm{diag}_{i=1}^N(A_{\Pi\Pi}^{(i)}),$$

and load vectors

$$f_\Sigma = \{f_\Sigma^{(i)}\}_{i=1}^N, \quad f_\Pi = \{f_\Pi^{(i)}\}_{i=1}^N.$$

Assembling the local subdomain matrices and load vectors with respect to the primal variables, we obtain the partially assembled global stiffness matrix $\tilde{A}$ and the load vector $\tilde{f}$,

$$\tilde{A} = \begin{bmatrix} A_{\Sigma\Sigma} & \tilde{A}_{\Pi\Sigma}^T \\ \tilde{A}_{\Pi\Sigma} & \tilde{A}_{\Pi\Pi} \end{bmatrix}, \quad \tilde{f} = \begin{bmatrix} f_\Sigma \\ \tilde{f}_\Pi \end{bmatrix}, \tag{15}$$

where a tilde refers an assembled quantity. It is easy to see that the matrix $\tilde{A}$ is positive definite.

To enforce the continuity on the dual variables $u_\triangle$, we introduce a jump matrix $B_\triangle$ with entries 0, -1 and 1 given by

$$B_\triangle = [B_\triangle^{(1)}, \cdots, B_\triangle^{(N)}], \tag{16}$$

where $B_\triangle^{(i)}$ consists of columns of $B_\triangle$ attributed to the i-th component of the dual variables. The space $\Lambda := range(B_\triangle)$ is used as the space for the Lagrange multipliers λ. The Dual-Primal saddle point problem is given by

$$\begin{bmatrix} A_{II} & A_{\triangle I}^T & \tilde{A}_{\Pi I}^T & 0 \\ A_{\triangle I} & A_{\triangle\triangle} & \tilde{A}_{\Pi\triangle}^T & B_{\triangle}^T \\ \tilde{A}_{\Pi I} & \tilde{A}_{\Pi\triangle} & \tilde{A}_{\Pi\Pi} & 0 \\ 0 & B_{\triangle} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_I \\ u_{\triangle} \\ \tilde{u}_{\Pi} \\ \lambda \end{bmatrix} = \begin{bmatrix} f_I \\ f_{\triangle} \\ \tilde{f}_{\Pi} \\ \lambda \end{bmatrix} \tag{17}$$

where $A_{II} := \operatorname{diag}_{i=1}^N(A_{II}^{(i)})$ and $\tilde{u}_{\Pi}$ means the primal unknowns at the vertices of the substructures Ω_i. By eliminating $u_I := \{u_I^{(i)}\}_{i=1}^N$, $u_{\triangle} := \{u_{\triangle}^{(i)}\}_{i=1}^N$ and $\tilde{u}_{\Pi}$ from (17), we obtain a system on the form

$$F\lambda = d \tag{18}$$

where

$$F = B_{\Sigma}\tilde{A}^{-1}B_{\Sigma}^T, \quad d = B_{\Sigma}\tilde{A}^{-1}[f_{\Sigma}^T \ \tilde{f}_{\Pi}^T]^T \quad \text{with} \quad B_{\Sigma} = (0, B_{\Delta}).$$

5 FETI-DP Preconditioner

To define the FETI-DP preconditioner M for F, we need to introduce a scaled variant of the jump matrix $B_{\triangle}$, which we denote by

$$B_{D,\triangle} = [D_{\triangle}^{(1)}B_{\triangle}^{(1)}, \cdots, D_{\triangle}^{(N)}B_{\triangle}^{(N)}].$$

The diagonal scaling matrices $D_{\triangle}^{(i)}$ operates on the dual variables $u_{\triangle}^{(i)}$ and they are defined as follows. Let $\mathcal{J}_i$ be the indices of the substructures which share an edge with Ω_i. An edge shared by Ω_i and Ω_j is denoted by E_{ij}, and the set of dual nodes on $\mathcal{T}^{h_i}(\partial\Omega_i)$ on E_{ij} is denoted by $E_{ij,h}$. The diagonal matrix $D_{\triangle}^{(i)}$ is defined via $\delta_i^{\dagger}(x)$ where

$$\delta_i^{\dagger}(x) := \frac{\bar{\rho}_i}{\bar{\rho}_i + \bar{\rho}_j}(x) \quad x \in E_{ij,h} \ \text{ and } j \in \mathcal{J}_i,$$

and let

$$P_{\triangle} := B_{D,\triangle}^T B_{\triangle}. \tag{19}$$

The FETI-DP preconditioner is defined by

$$M^{-1} = P_{\triangle}S_{\triangle\triangle}P_{\triangle}^T \quad \text{ where}$$

$$S_{\triangle\triangle} := \operatorname{diag}_{i=1}^N\langle S_{\triangle\triangle}^{(i)}\rangle, \quad \langle S_{\triangle\triangle}^{(i)}w_{\triangle}^{(i)}, w_{\triangle}^{(i)}\rangle := \int_{\Omega_i} \rho_i \nabla\mathcal{H}_{\rho_i}^{(i)}w_{\triangle}^{(i)} \cdot \nabla\mathcal{H}_{\rho_i}^{(i)}w_{\triangle}^{(i)}\, dx,$$

where $w_{\triangle}^{(i)}$ is identified with a function on $V^{h_i}(\partial\Omega_i)$ which vanishes at the vertices of Ω_i. Using Lemma 2 and Lemma 3, it is possible to prove (see [3] for details) the following theorem:

Theorem 1. *Let us assume that the Boundary Layer Assumption holds for any substructures Ω_i. Then, for any $\lambda \in \Lambda$ we have:*

$$\langle M\lambda, \lambda\rangle \ \leq \ \langle F\lambda, \lambda\rangle \leq \lambda_{\max}\langle M\lambda, \lambda\rangle$$

where

$$\lambda_{\max} \preceq \max_{i=1}^{N} \frac{H_i^2}{\eta_i^2}\ (1 + \log\frac{\eta_i}{h_i}).$$

When the coefficients ρ_i, $1 \leq i \leq N$, are simultaneously of the Inclusion Hard type, or are simultaneously of the Inclusion Soft type, then:

$$\lambda_{\max} \preceq \max_{i=1}^{N} \left\{ \frac{H_i}{\eta_i}\ (1 + \log\frac{\eta_i}{h_i}) + (1 + \log\frac{\eta_i}{h_i})^2 \right\}.$$

The linear dependence result on H_i/η_i for Inclusion Soft type coefficients is the first one given in the literature. The bounds in Theorem 1 hold also for the FETI method and are sharper than $O(\frac{H_i}{\eta_i}\ (1 + \log\frac{H_i}{h_i})^2)$ obtained in [6] for Inclusion Hard type coefficients.

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