On a tower of Ihara and its limit

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1 Introduction

Towers of function fields over a fixed finite field have attracted much attention, specially for the connections with Coding Theory and Cryptography (see [TV], [NX], [Z], [GS2] and [GS3]). Ihara was the first to realize that the so-called Hasse-Weil upper bound was weak if the genus of the function field is large with respect to the cardinality of the finite field (see [Iha]). The first explicit tower (i.e., a tower having the function fields in it given by explicit polynomial equations) with an optimal asymptotic behaviour was obtained over square finite fields (see [GS]). Zink has shown the existence of towers over cubic finite fields with an exceptional asymptotic behaviour (see [Z]). The first explicit tower with this exceptional behaviour was obtained by van der Geer and van der Vlugt over the finite field with eight elements (see [GV]). Generalizations of this tower in [GV] were obtained in [BeGS] and [BaGS]. Here we study another tower $\mathcal{F}_0$ over cubic finite fields, also generalizing the tower in [GV]. This tower $\mathcal{F}_0$ was introduced by Ihara in [Ih] as a subtower of the tower in [BeGS]. A detailed exposition of $\mathcal{F}_0$ can be seen in [C] where it is also determined the genera of the function fields of the tower in [BaGS]. Let k be a finite field. A tower $\mathcal{F}$ over k is an infinite sequence of function fields F_n over k such that:

$$\mathcal{F} = (F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots \subseteq F_n \subseteq F_{n+1} \subseteq \cdots)$$

a) k is algebraically closed in F_n, for all $n \in \mathbb{N}$.

b) $g(F_n) \to \infty$ as $n \to \infty$, where $g(F_n)$ denotes the genus.

We can assume that the extensions F_n/F_1 are finite and separable. The ramification locus $R(\mathcal{F})$ is the set of places P of the first field F_1 that are ramified in $\mathcal{F}$; i.e., for some $n \geq 2$ and some place Q of F_n above P we have $e(Q|P) > 1$, where $e(Q|P)$ denotes the ramification index. When $R(\mathcal{F})$ is a finite set we denote deg $R(\mathcal{F}) := \Sigma$ deg P, where we sum over the places $P \in R(\mathcal{F})$. The splitting locus $S(\mathcal{F})$ is the set of k-rational places P of the first field F_1 such that, for every $n \geq 2$, the number of places of F_n above P is equal to the degree $[F_n : F_1]$. In particular, any such place of F_n is again k-rational.

The tower $\mathcal{F}$ has a limit (see Lemma 7.2.3 of [Sti])

$$\lambda(\mathcal{F}) := \lim_{n\to\infty} \frac{N(F_n)}{g(F_n)},$$

where $N(F_n)$ is the number of k-rational places of F_n.

A tower $\mathcal{F}$ over k is said to be recursive if there is a polynomial $f(X, Y) \in k[X, Y]$ such that for each $n \in \mathbb{N}$:

$$F_n = k(x_1, x_2, \ldots, x_{n-1}, x_n) \quad \text{and}$$

$$f(x_i, x_{i+1}) = 0, \quad \text{for } i = 1, 2, \ldots, n-1.$$

In the particular case of a cubic finite field $k = \mathbb{F}_{q^3}$, it is shown in [BeGS] that the equation

$$\frac{1-Y}{Y^q} = \frac{X^q + X - 1}{X} \tag{0}$$

defines a recursive tower $\mathcal{F}_1$ over $\mathbb{F}_{q^3}$ with an exceptional asymptotic behaviour; i.e.,

$$\lambda(\mathcal{F}_1) \geq \frac{2(q^2-1)}{q+2}. \tag{1}$$

A tower $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$ is said to be a subtower of $\mathcal{E} = (E_m)_{m\in\mathbb{N}}$ if for each $n \in \mathbb{N}$ we have:

$$F_n \subseteq E_m \quad \text{for some} \quad m = m(n).$$

For a subtower $\mathcal{F}$ of $\mathcal{E}$ we have $\lambda(\mathcal{F}) \geq \lambda(\mathcal{E})$ (see Prop. 7.2.8 of [Sti]).

Ihara [Ih] shows that the equation

$$Y^{q+1} + Y = \frac{X+1}{X^{q+1}} \tag{2}$$

defines a subtower $\mathcal{F}_0$ of the tower $\mathcal{F}_1$ above. Actually Ihara used Eq. (11) below to define the recursive tower $\mathcal{F}_0$.

Hence we also have that

$$\lambda(\mathcal{F}_0) \geq \frac{2(q^2-1)}{q+2}. \tag{3}$$

The objective of this note is to give a direct proof of the inequality in (3), without using the fact that $\mathcal{F}_0$ is a subtower of $\mathcal{F}_1$. This direct proof is much simpler than the proof of Inequality (1) in [BeGS].

The novelty here is that although the extensions in the pyramid associated to the tower $\mathcal{F}_0$ are not Galois for $q \neq 2$, they become Galois after completion at certain places.

The paper is organized as follows:

Section 2 describes ramification indices and different exponents in the two basic extensions associated to Eq. (2):

$$k(X,Y)/k(X) \quad \text{and} \quad k(X,Y)/k(Y).$$

Section 3 introduces the concept of B-bounded towers and gives a formula for its limit (see Eq. (5) below). A basic reference in this section is [GS2].

Only in Section 4 we show that $\mathcal{F}_0$ is indeed a tower of function fields over the cubic finite field $k = \mathbb{F}_{q^3}$. In this Section 4 we give the strategy used here to show that the tower $\mathcal{F}_0$ is B-bounded with $B = q/(q-1)$.

Using completion at certain places in the tower $\mathcal{F}_0$ and by showing that after completion the bottom extensions become Galois, we finish in Section 5 the proof that $B = q/(q-1)$. Here we use in a fundamental way the Proposition 12 of [GS2]. We end up with a remark showing that a tower of Ihara (see [Ih]) given by Eq. (11) below is the same as the tower $\mathcal{F}_0$ given by Eq. (2) above.

2 The tower $\mathcal{F}_0$ over $\mathbb{F}_{q^3}$

We consider the tower $\mathcal{F}_0$ over $k = \mathbb{F}_\ell$ with $\ell = q^3$ given recursively by the equation below

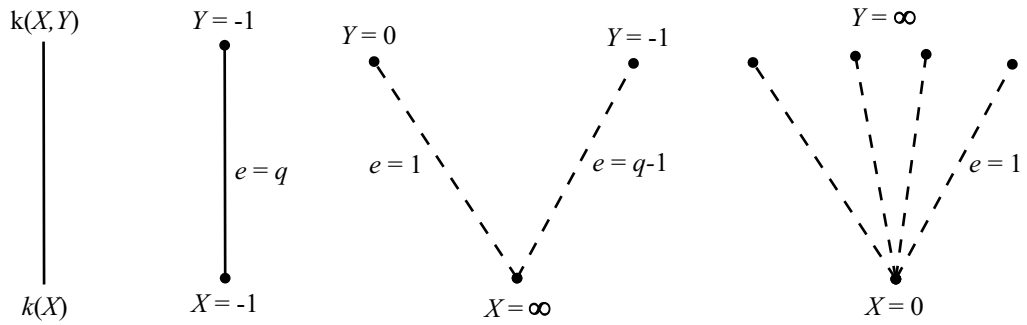
$$Y^{q+1} + Y = \frac{X+1}{X^{q+1}}. \tag{2}$$

We call the attention to the fact that we will show only in Section 4 that $\mathcal{F}_0$ is indeed a tower over k. Note that $T^{q+1} + T + 1 = 0$ is separable and has all roots in $\mathbb{F}_{q^3}$, and also that from Eq. (2):

$$X^{q+1} + X + 1 = 0 \;\Rightarrow\; Y^{q+1} + Y + 1 = 0.$$

This shows that $x_1^{q+1} + x_1 + 1 = 0$ is completely splitting over $\mathbb{F}_{q^3}$ and hence the splitting locus $S(\mathcal{F}_0)$ satisfies $\#S(\mathcal{F}_0) \geq q + 1$.

Note that Eq. (2) is not irreducible; in fact, one can easily see that $Y = -\frac{X+1}{X}$ is a root of it. Actually, we will see that Eq. (2) defines a tower $\mathcal{F}_0 = (F_1, F_2, F_3, \dots)$ over the cubic finite field $\mathbb{F}_{q^3}$ with $[F_{n+1} : F_n] = q$. We have the following pattern for the ramification in the basic extension $k(X, Y)/k(X)$ associated to Eq. (2):



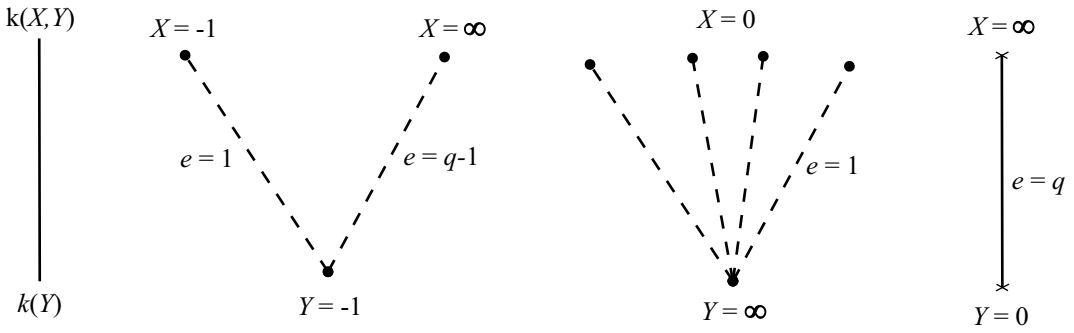
Above we have $[k(X, Y) : k(X)] = q$ and also that the place of $k(X)$ with $X = 0$ has above it q places P of $\bar{k}(X, Y)$ with $Y = \infty$ and ramification index $e = 1$. Here $\bar{k}$ denotes an algebraic closure of k. To see the last assertion, we substitute $Y = y - \frac{X+1}{X}$ into Eq. (2), and we get that the following equation holds true:

$$(Xy)^q = (X + 1)(Xy)^{q-1} + 1. \tag{4}$$

From Eq. (4) we get that at the places P of $\bar{k}(X, Y)$ above $X = 0$, we have:

$$(Xy)(P) = \alpha^{-1} \quad \text{with } \alpha \in \bar{k} \text{ such that } \alpha^q + \alpha = 1.$$

Similarly we get for the extension $k(X, Y)/k(Y)$:



For the different exponents we have (see Prop. 3.5.12 of [Sti]):

- $d(Q_1|P_1) = q$, where Q_1 is the unique place of $k(X,Y)$ above $P_1 := (X = -1)$ of $k(X)$.
- $d(R_1|S_1) = q$, where R_1 is the unique place of $k(X,Y)$ above $S_1 := (Y = 0)$ of $k(Y)$.

We show first that $d(Q_1|P_1) = q$. Eq. (4) can be written as below

$$Z^q = (X+1)\cdot(Z+1)^{q-1}, \quad \text{where } Z := Xy - 1.$$

The derivative is $(X+1)\cdot(Z+1)^{q-2}$ and its value at the place Q_1 is equal to q.

Similarly we have that $d(R_1|S_1) = q$. Again in this case we rewrite Eq. (4) as below

$$W^q = Y\cdot(W+1)^{q-1} + Y^q, \quad \text{where } W := \frac{1}{X} + Y.$$

The derivative is $Y\cdot(W+1)^{q-2}$ and its value at the place R_1 is equal to q.

3 Bounded towers

A place P_n of a field F_n in a tower $\mathcal{F} = (F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots)$ is said B-bounded if we have

$$d(P_n|P_1) \leq B\cdot(e(P_n|P_1) - 1),$$

where P_1 is the restriction of P_n to the first field F_1, $d(P_n|P_1)$ denotes the exponent of the different and $e(P_n|P_1)$ denotes the ramification index.

The tower $\mathcal{F}$ is said B-bounded if all places P_n of F_n, for every $n \in \mathbb{N}$, are B-bounded.

We have the following result for the limit of a B-bounded tower $\mathcal{F}$ with finite ramification locus $R(\mathcal{F})$ and nonempty splitting locus $S(\mathcal{F})$ (see [GS2]):

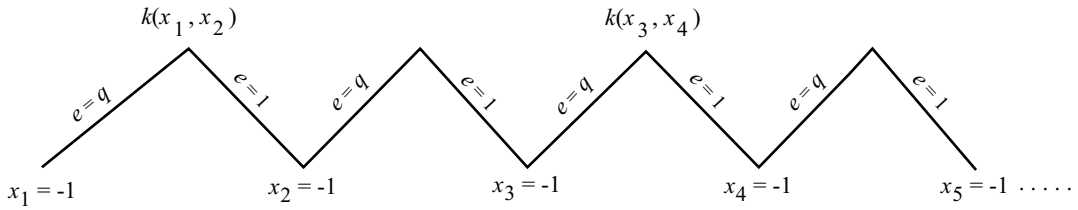
$$\lambda(\mathcal{F}) \geq \frac{\#S(\mathcal{F})}{g(F_1) - 1 + \frac{B}{2}\cdot \deg\, R(\mathcal{F})}. \tag{5}$$

In the case of the tower $\mathcal{F}_0$ given recursively by Eq. (2), the point here is to show that it is B-bounded with $B = q/(q-1)$. From Eq. (5) we then get the limit (using $\#S(\mathcal{F}_0) = q+1$ and $\deg\, R(\mathcal{F}_0) = 3$):

$$\lambda(\mathcal{F}_0) \geq \frac{q+1}{-1 + \frac{3}{2}\cdot\frac{q}{q-1}} = \frac{2(q^2-1)}{q+2}.$$

4 Strategy to show that B=q/(q-1)

We show first that the place $P_1 := (x_1 = -1)$ is fully ramified in the tower $\mathcal{F}_0$; i.e., it is fully ramified in all extensions F_n/F_1 for $n \geq 2$. Indeed we have the following pattern as below:

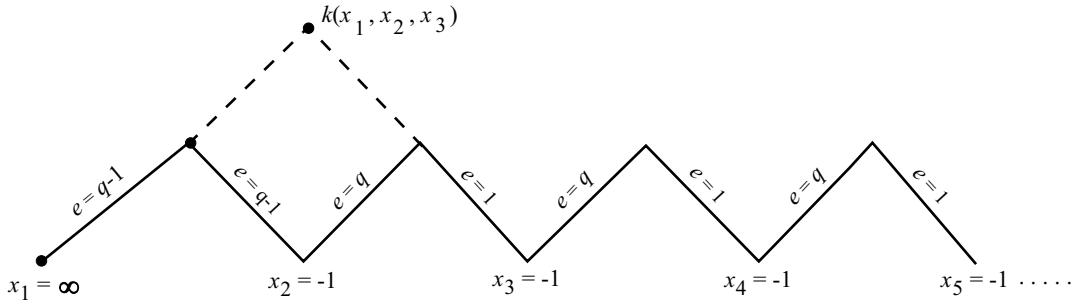


This shows that P_1 is fully ramified in F_n. At the same time it shows that $\mathcal{F}_0$ is a tower with $[F_{n+1} : F_n] = q$, and in particular, also that $k = \mathbb{F}_{q^3}$ is algebraically closed in each field F_n, for $n \geq 1$. Denoting by P_n the unique place of F_n above P_1 we have that

$$e(P_{n+1}|P_n) = d(P_{n+1}|P_n) = q, \qquad \text{for all } n \geq 1.$$

This shows that P_n is B-bounded with $B = q/(q-1)$, as follows from the transitivity of different exponents (see [GS2]).

Another possible pattern of ramification is as below:



We denote by Q_2 the unique place of F_2 with $x_1 = \infty$ and $x_2 = -1$. We have a unique place Q_n of F_n above Q_2, for each $n \geq 3$. For the place extension $Q_3|Q_2$, using Abhyankar's lemma and the transitivity of differents (see [Sti], Sections 3.5 and 3.9), we have:

$$d(Q_3|Q_2) + q \cdot (q-2) = q - 2 + (q-1) \cdot q. \tag{6}$$

Hence $Q_3|Q_2$ is 2-bounded; more precisely, $d(Q_3|Q_2) = 2\cdot(q-1)$. With similar arguments we have that $Q_n|Q_2$ is 2-bounded, for all $n \geq 3$; more precisely, we have that

$$d(Q_n|Q_2) = 2 \cdot (e(Q_n|Q_2) - 1), \qquad \text{for } n \geq 3.$$

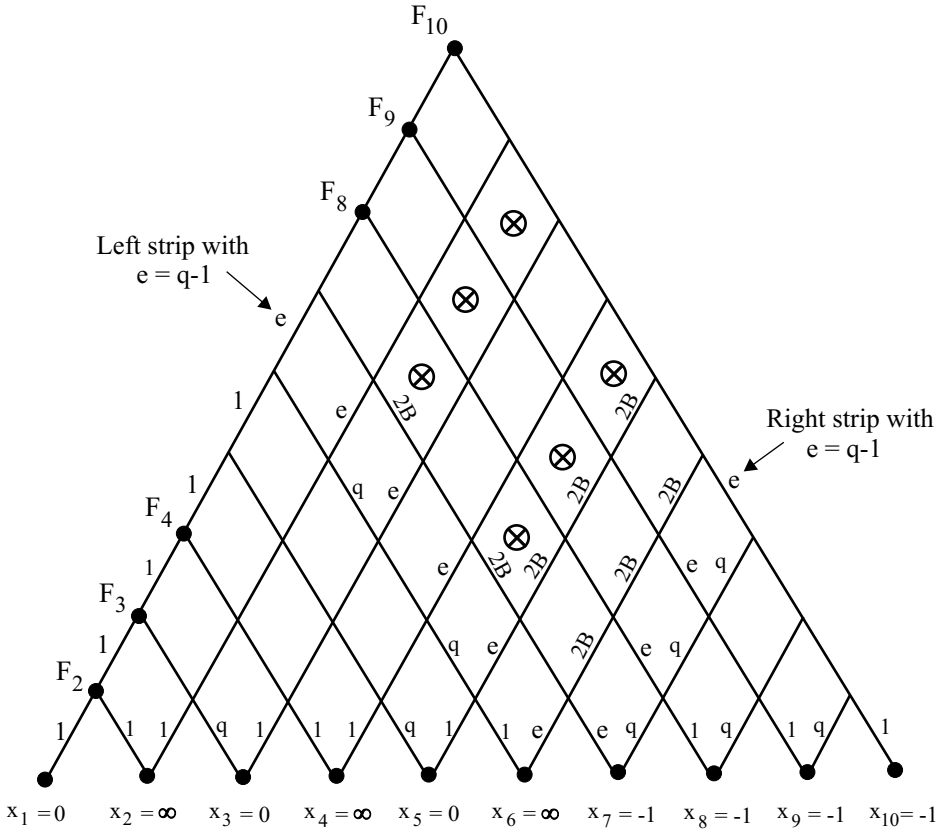
We then have a situation of ramification as follows:

In the extension F_2/F_1 the place Q_2 is tamely ramified with ramification index $e = q-1$ and in the extension F_n/F_2 the place Q_n of F_n above Q_2 is fully ramified and 2-bounded. From the transitivity and denoting $E = e(Q_n|Q_2)$, we get

$$\begin{aligned} d(Q_n|Q_2) + E \cdot (q-2) &= 2 \cdot (E-1) + E \cdot (q-2) \\ &= Eq - 2 \leq \frac{q}{q-1} \cdot \big((q-1)E - 1\big). \end{aligned} \tag{7}$$

Note that $(q-1)E$ is the ramification index of Q_n over the first field F_1 of the tower $\mathcal{F}_0$. This shows that the places Q_n above, for $n \geq 3$, are B-bounded with $B = q/(q-1)$.

Now we consider the following (Figure 1):

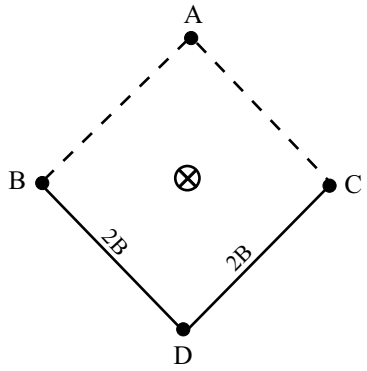


In Figure 1 above we write ramification indices over the edges of the pyramid, with the notation $e := q - 1$. We have also written $2B$ over an edge to indicate that the corresponding place extension is 2-bounded, more precisely to indicate that it holds:

$$\text{different exponent} = 2\ (\text{ramification index})\ - 2.$$

The argument for the equality above is the one given in Eq. (6). Of course the situation in Figure 1 is just a concrete instance that helps in understanding the strategy.

We are then left with the problem of deciding the behaviour of different exponents in a diamond as below (Figure 2):



where A denotes a place; B, C and D denote the restrictions of A to the corresponding subfields, and we know that it holds:

$$\begin{aligned} d(B|D) &= 2 \cdot (e(B|D) - 1) \\ d(C|D) &= 2 \cdot (e(C|D) - 1). \end{aligned}$$

This is the main difficulty of the tower $\mathcal{F}_0$: to show that in Figure 2 we always have

$$\begin{aligned} d(A|B) &= 2 \cdot (e(A|B) - 1) \\ d(A|C) &= 2 \cdot (e(A|C) - 1). \end{aligned} \tag{8}$$

If the field extensions at the bottom in Figure 2 were both Galois, then this would follow directly from Proposition 12 in [GS2]. But in our case of the tower $\mathcal{F}_0$, those extensions are Galois only if $q = 2$. What we are going to show in the next section is that after completion at the places in Figure 2, the bottom extensions become Galois allowing the use of Proposition 12 of [GS2]. After having this result that the completions

become Galois, the proof ends as follows (using here Figure 1 for clarity): we first deal with the diamonds (going upwards to the right) marked with $\otimes$ situated on the strip between $x_4 = \infty$ and $x_5 = 0$; having obtained the behaviour in Eq. (8) for the upper sides of those diamonds, we move upwards to the left and we deal with the diamonds marked with $\otimes$ situated on the strip between $x_2 = \infty$ and $x_3 = 0$, and so on... .

At the end (if $x_1 = 0$ or $x_1 = \infty$, and $x_n = -1$ for some $n \geq 2$) we will have in the tower $\mathcal{F}_0$ a situation where first occurs a ramification index $e = q - 1$ followed by the 2-bounded behaviour:

$$\text{different exponent} = 2\ (\text{ramification index})\ - 2.$$

Now the argument in Eq.(7) finishes the proof that $\mathcal{F}_0$ is B-bounded with $B = q/(q-1)$.

5 Galois after completion

In Figure 1 we have now to focus our attention on the diamonds marked with $\otimes$. Any such diamond for the tower $\mathcal{F}_0$ is represented in Figure 2. We are going to show that after completion at the corresponding places, the bottom extensions of the diamond become Galois. We will see that the right (left) strip with $e = q - 1$ in Figure 1 is responsible for the bottom extension at the right (left) in Figure 2 to become Galois.

We just prove here the right extension case, the left one is done similarly. We have an equation for the bottom extension at the right of the form (see Eq. (4)):

$$\left(\frac{1}{Xy}\right)^q + (X+1)\cdot\left(\frac{1}{Xy}\right) = 1,$$

where $X = x_n$ and $y = x_{n+1} + \frac{x_n+1}{x_n}$ for some n. For simpliciy we write $T = 1/Xy$ and we want to determine the Galois closure of the equation

$$T^q + (X+1)\cdot T - 1 = 0. \tag{9}$$

It $T + V$ is another root

$$(T+V)^q + (X+1)\cdot(T+V) - 1 = 0,$$

then we get the Kummer extension

$$V^{q-1} + (X+1) = 0. \tag{10}$$

So to move to the Galois closure of Equation (9) we have to add on the top a Kummer extension given by Eq. (10). At the places we are considering we have that $X+1$ has a zero and its zero-order is a multiple of $e = q-1$, as follows from the right strip with $e = q-1$ (see Figure 1). This shows that the places we are considering are unramified in the Kummer extension given by Eq. (10) and hence we finally conclude that the completion becomes Galois.

This finishes the proof that the tower $\mathcal{F}_0$ has the exceptional asymptotic behaviour of Eq. (3). The strategy used above to deal with $\mathcal{F}_0$ is inspired in [GS1] and [BS]. The novelty here is this phenomenon of becoming Galois after completion at certain places.

Remark: In [Ih] it is shown that the following equation over $k = \mathbb{F}_{q^3}$

$$\frac{y-1}{y^{q+1}} = \frac{-x^q}{(1-x)^{q+1}} \tag{11}$$

defines a subtower of the tower $\mathcal{F}_1$ considered in [BeGS]. This subtower is the same as the tower $\mathcal{F}_0$ considered in this paper; in fact the substitutions

$$x = \frac{1}{X+1} \quad \text{and} \quad y = \frac{1}{Y+1}$$

transform Equation (11) of Ihara into our Equation (2).

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