Embedded Curves and Foliations [1]

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We consider in this paper the problem of finding regular holomorphic foliations in neighborhoods of smooth, compact, holomorphic curves embedded in complex surfaces. Our primary motivation stems from a Linearization Theorem due to Grauert([3]): the curve possesses a neighborhood isomorphic to a neighborhood of the zero section of its normal bundle if the embedding is sufficiently negative. In ([1]) we proved a special result using techniques from holomorphic foliations; more precisely, let $C \hookrightarrow S$ be a negative embedding of the curve C into the surface S such that the self-intersection number satisfies $C \cdot C < 2-2g$, where g is the genus of C; therefore any transverse holomorphic foliation to C is isomorphic to the linear fibration of the normal bundle NC. The first step to prove Grauert's Theorem is to guarantee the existence of a foliation transverse to the curve; we need to assume the stronger condition $C \cdot C < 4-4g$. Once this is acomplished we just proceed as in ([1]), finding a holomorphic foliation in a neighborhood V of C which has C as a leaf; we use this foliation and the transverse one as a kind of system of coordinates for V in order to construct the desired isomorphism.

In fact we are able to prove a more general result, replacing transversal foliation by generically transverse foliation: we fix a divisor D of C, and show that that there exists a holomorphic foliation whose divisor of tangencies with the curve is exactly D; as before we assume the negativety of the self-intersection number of the curve. Let us state our main result:

Theorem. *Let $C \hookrightarrow S$ be an embedding of the curve C into the surface S such that $C \cdot C < 0$ and let $D = \sum_{k=1}^{l} n_k p_k,\ n_k \in \mathbb{N},\ p_k \in C$ be a*

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divisor in C. Assume

$$C \cdot C < 4 - 4g + \sum_{k=1}^{l} (n_k - 1).$$

Then there exists a regular foliation $\mathcal{F}$ defined in a neighborhood of C which is transversal to C except at the points $p_1, \cdots, p_l \in C$, where $\mathrm{tang}_{p_k}(\mathcal{F}, C) = n_k$ *for every $k = 1, \ldots, l$.*

As a consequence, if $C \cdot C < 0$ and $C \cdot C < 4 - 4g$, there exists a foliation transverse to C; as we said above, ([1]) can be applied to provide a proof of the Linearization Theorem.

Our method to prove this theorem consists in i) find a holomorphic line field defined *along* the curve C with the prescribed set of tangencies and the prescribed order of tangencies; for this purpose we have no need to assume that the curve is negatively embedded; ii) extend the line field to a neighborhood of the curve; here we must work under the hypothesis $C \cdot C < 0$ in order to assure the annihilation of some cohomology groups.

We discuss also how to produce examples of embeddings such that there are no foliations with a given divisor of tangencies when the negativity condition is violated. In particular, examples where linearization is not possible are presented. All these examples depend of properties of line fields defined along the curve.

1. **Line Fields and Embeddings**

Let us consider an embedding $C \hookrightarrow S$ of the compact, smooth, holomorphic curve C into the surface S. In this Section we study existence of line fields defined along C. Existence of a line field with a given divisor of tangencies is always granted when the degree of the divisor is sufficiently bigger then $C \cdot C$. On the other hand, uniqueness (but perhaps not the existence) follows when this degree is not too big, and we will see later how this leads to the construction of interesting examples.

A holomorphic subbundle $Y \hookrightarrow TS|C$ is a *holomorphic line field along* C. Equivalently we may say that a line field is a section of the $\mathbb{P}^1$-bundle $\mathbb{P}(TS|C)$ over C. Y has a *tangency* with C at the point $p \in C$ when the morphism of line bundles $Y \to NC = \dfrac{TS|C}{TC}$ has a zero at p; the order of the zero is the *order of tangency* between Y and C. We write the set of tangencies as an effective divisor $D = \sum_{k=1}^{l} n_k D_k$ of C; the point p_k is a point of tangency of order n_k.

In order to motivate the next Proposition, let us remark that when Y is a line field along C whose divisor of tangencies with C is D then $Y \simeq \mathcal{O}(-D) \otimes NC$ as line bundles. In fact, the morphism $Y \to NC$ seen as a section of $H^0(C, Y^* \otimes NC)$ has D as its divisor of zeroes; therefore $Y^* \otimes NC \simeq \mathcal{O}(D)$. This allows us to confound a line field along C having D as divisor of tangencies with an injective morphism $\mathcal{O}(-D) \otimes NC \to TS|C$.

Proposition 1. *Let D be an effective divisor of C, and assume*

$$C \cdot C < 4 - 4g + \sum_{k=1}^{l} (n_k - 1)$$

There exists an injective bundle morphism $Y : \mathcal{O}(-D) \otimes NC \to TS|_C$ which has D as divisor of tangencies with C.

Proof. Let us use $L := \mathcal{O}(-D) \otimes NC$ for simplicity. Firstly we construct Y locally: given i) an open subset U' of S with coordinates $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ such that $U = U' \cap C$ is $\{z_2 = 0\}$; ii) a holomorphic function f of U such $D|_U = \{f = 0\}$ and iii) trivialization coordinates (z_1, t) for $L|_U$, we may define

$$Y(z_1)(t) = (z_1, t, f(z_1)t).$$

We define then $Y_i : L|_{U_i} \to TS|_{U_i}$ with the desired property for an open covering $\{U_i\}_{i \in I}$ of C; we assume that the support of each $D|_{U_i}$ consists of a point at most and that there are no points of tangency in the intersections $U_i \cap U_j$ when $i \neq j$. Let $\tilde{Y}_i$ denote the composition $L|_{U_i} \to TS|_{U_i} \to NC|_{U_i}$. As $\tilde{Y}_i = a_{ij} \tilde{Y}_j$, where $\{a_{ij}\} \in H^1(C, \mathcal{O}^*(C))$ defines a line bundle J, $\{\tilde{Y}_i\}$ is a section of $J \otimes Hom(L, NC) \simeq J \otimes L^* \otimes NC$ having D as divisor of zeroes, so that $J \otimes L^* \otimes NC \simeq \mathcal{O}(D)$. Consequently J is the trivial line bundle and we may suppose $a_{ij} = 1$, or $\tilde{Y}_i = \tilde{Y}_j$.

Now we have that

$$\{Y_{ij}\} := \{Y_i - Y_j\} \in H^1(C, Hom(L, TC)) \simeq H^1(C, L^* \otimes TC);$$

Let $\tilde{D} = \sum_{k=1}^{l} p_k$ and $s = \{s_i\} \in H^0(C, \mathcal{O}(\tilde{D}))$ whose divisor of zeroes is $\tilde{D}$. Therefore

$$(Y_i - Y_j) \otimes s^{-1} \in H^1(C, \mathcal{O}(-\tilde{D}) \otimes L^* \otimes TC)$$

and by Serre's duality

$$H^1(C, \mathcal{O}(-\tilde{D}) \otimes L^* \otimes TC) \simeq H^0(C, KC^2 \otimes \mathcal{O}(\tilde{D}) \otimes \mathcal{O}(-D) \otimes NC)$$

(KC stands for the canonical bundle of C). By hypothesis the Chern class of the line bundle $KC^2 \otimes \mathcal{O}(\tilde{D}) \otimes \mathcal{O}(-D) \otimes NC$ is negative; we conclude that $(Y_i - Y_j) \otimes s^{-1} = X_i - X_j$ for $X_i \in H^0(U_i, \mathcal{O}(-\tilde{D}) \otimes L^* \otimes TC)$, and therefore $Y_i - Y_j = (X_i - X_j) \otimes s = s_i X_i - s_j X_j$. We define $Y := Y_i - s_i X_i$ in each U_i. Clearly Y is injective outside the support of $\tilde{D}$; at each p_i, it is equal to Y_i, so it is also injective. As for the order of tangency at a point p_i, it coincides with the order of tangency of Y_i, which is n_i by construction. □

Consequently, there exists always a holomorphic line field along any curve if we admit a number of tangencies sufficiently big. We see also that there exists always a holomorphic line field with any number of tangencies if $C \cdot C < 4 - 4g$.

In the next section we will analyse how to extend this holomorphic line field to a neighborhood of the curve. For the moment, let us state a general result concerning uniqueness.

Proposition 2. *Let D be an effective divisor of C and assume*

$$c(NC) > 2 - 2g + \sum n_i$$

There exists at most one line field along C having D as divisor of tangencies.

Proof. Let us consider two such line fiels Y_1 and Y_2 as bundle morphisms from $\mathcal{O}(-D) \otimes NC$ into $TS|_C$. The induced morphisms $\tilde{Y}_i : \mathcal{O}(-D) \otimes NC \to NC$ seen as sections of $\mathcal{O}(D) \otimes NC^* \otimes NC = \mathcal{O}(D)$ have the same divisor D of zeroes, so that $\tilde{Y}_1 = c\tilde{Y}_2$ for some $c \in \mathbb{C}^*$. It follows that $Y_1 - cY_2$ is a bundle morphism from $\mathcal{O}(-D) \otimes NC$ to TC; the hypothesis tells us that $\mathcal{O}(D) \otimes NC^* \otimes TC$ is a negative line bundle and so $Y_1 - cY_2 = 0$. □

2. Neighborhoods of Negatively Embedded Curves

Before proving the Theorem stated in the Introduction, we collect some properties that are verified in the case of a negatively embedded curve $C \hookrightarrow S$ ([2]).

- C has a fundamental system of strictly pseudoconvex neighborhoods in S.
- if $\mathcal{G}$ is a coherent sheaf defined in one of these neighborhoods, say V, and $\mathcal{I}_C$ is the ideal sheaf of C in V then

$$\exists k > 0 \text{ such that } H^i(V, \mathcal{I}_C^k \cdot \mathcal{G}) = 0, \quad i = 1, 2.$$

Lemma 1. *We have $H^2(V, \mathcal{I}_C \cdot \mathcal{G}) = 0$. Moreover if*

$$H^0(C, KC \otimes NC^\nu \otimes \mathcal{G}^*|_C) = 0$$

for all $\nu \geq 1$ then $H^1(V, \mathcal{I}_C \cdot \mathcal{G}) = 0$.

Proof. From $H^i(V, \mathcal{I}_C^\nu / \mathcal{I}_C^{\nu+1} \cdot \mathcal{G}) \simeq H^i(C, (NC^*)^\nu \otimes \mathcal{G}|_C)$ we get immediately $H^2(V, \mathcal{I}_C^\nu / \mathcal{I}_C^{\nu+1} \cdot \mathcal{G}) = 0$. As

$$H^1(C, (NC^*)^\nu \otimes \mathcal{G}|_C) \simeq H^0(C, KC \otimes NC^\nu \otimes \mathcal{G}^*|_C)$$

(by Serre's duality) we get $H^1(V, \mathcal{I}_C^\nu / \mathcal{I}_C^{\nu+1} \cdot \mathcal{G}) = 0$ as well.

Let us consider the short exact sequence

$$0 \to \mathcal{I}_C^{\nu+1} \cdot \mathcal{G} \to \mathcal{I}_C^\nu \cdot \mathcal{G} \to \mathcal{I}_C^\nu / \mathcal{I}_C^{\nu+1} \cdot \mathcal{G} \to 0$$

which leads to

$$\cdots \to H^i(V, \mathcal{I}_C^{\nu+1} \cdot \mathcal{G}) \to H^i(V, \mathcal{I}_C^\nu \cdot \mathcal{G}) \to H^i(V, \mathcal{I}_C^\nu / \mathcal{I}_C^{\nu+1} \cdot \mathcal{G}) \to \cdots$$

Therefore the maps $H^i(V, \mathcal{I}_C^{\nu+1} \cdot \mathcal{G}) \to H^i(V, \mathcal{I}_C^\nu \cdot \mathcal{G})$, $\quad i = 1, 2$, are always surjective. Consequently $H^i(V, \mathcal{I}_C^k \cdot \mathcal{G}) = 0$ for some $k > 0$ implies $H^i(V, \mathcal{I}_C \cdot \mathcal{G}) = 0$, $i = 1, 2$. □

The next Lemma allows us to extend any line bundle over C to a line bundle over V. Of course there are certain line bundles which are extendible regardless of the negativity of the embedding $C \hookrightarrow V$. For example, $KC = KV|_C \otimes NC = KV|_C \otimes [C]|_C$, so that KC always has an extension to V. Below in our Theorem we find this situation when no tangencies are present.

Lemma 2. *The restriction $H^1(V, \mathcal{O}_V^*) \to H^1(C, \mathcal{O}_C^*)$ is surjective.*

Proof. Let J be the subsheaf of $\mathcal{O}_V^*$ defined as

- $J_q = (\mathcal{O}_V^*)_q$ if $q \notin C$.
- $J_q = \{\phi \in (\mathcal{O}_V^*)_q; \phi|_C \simeq 1\}$ if $q \in C$.

We have then the short exact sequence

$$1 \to J \to \mathcal{O}_V^* \to \mathcal{O}_V^*/J \to 1;$$

we remark that $\mathcal{O}_V^*/J$ can be taken as $\mathcal{O}_C^*$.

In order to have the surjectivity stated above, we need $H^2(V, J) = 0$. Since the exponencial map gives an isomorphism between $\mathcal{I}_C$ and J, it is enough to have $H^2(V, \mathcal{I}_C) = 0$. □

3. **Constructing Foliations**

We are able now to prove the Theorem stated in the Introduction.

Let $Y : C \to TS|_C$ be the line field constructed in Corollary 1. Let $\{U_i\}$ be a covering of C and $\tilde{U}_i$ be an open set such that $\tilde{U}_i \cap C = U_i$. In each $\tilde{U}_i$ we choose a 1-form ω_i satisfying $ker(\omega_i(p)) = Y(p)$ when $p \in U_i$. We may take coordinates $(x_i, y_i) \in \tilde{U}_i$ as to have $U_i = \{y_i = 0\}$ and $\omega_i = dy_i - x_i^{n_i} dx_i$(remember that the possibility $n_i = 0$ is allowed). We remark that $\omega_i|_{U_i \cap U_j} = f_{ij}\, \omega_j|_{U_i \cap U_j}$ whenever $U_i \cap U_j \neq \emptyset$, $f_{ij} \in Z^1(\{U_i\}, \mathcal{O}_C^*)$. We denote by $L = \{F_{ij}\}$ the line bundle over V whose restriction to C is defined by the transition functions $\{f_{ij}\}$ (Lemma 2); we have

$$L|_C = \mathcal{O}(D) \otimes KC^*,$$

where $D = \sum_{i=1}^{l} n_i p_i$. The boundary $\delta\{\omega_i\}$ computed in $Z^1(S, \Omega_S^1 \otimes L)$ belongs effectively to $Z^1(S, \mathcal{I}_C \cdot \Omega_S^1 \otimes L)$, where Ω_S^1 is the sheaf of germs of holomorphic 1-forms of S.

We claim that $H^1(S, \mathcal{I}_C \cdot \Omega_S^1 \otimes L) = 0$. As discussed before, we need that $\forall \nu \geq 1$

$$H^0(C, KC \otimes NC^\nu \otimes (\Omega_S^1 \otimes L)^*|_C) = 0$$

which depends on

$$H^0(C, KC^2 \otimes NC^\nu \otimes \mathcal{O}(-D) \otimes TC) = 0 \;\; \forall \nu \geq 1$$

and

$$H^0(C, KC^2 \otimes NC^\nu \otimes \mathcal{O}(-D) \otimes NC) = 0 \;\; \forall \nu \geq 1;$$

these equalities follow from the hypothesis.

It follows that there exists a 0-cocycle $\{\eta_i\} \in H^0(\tilde{U}_i, \mathcal{I}_C \cdot \Omega_S^1 \otimes L) = 0$ such that

$$\omega_i - F_{ij}\, \omega_j = \eta_i - F_{ij}\, \eta_j$$

and the foliation we look for is defined by the 1-form

$$\{\omega_i - f_{ij}\, \eta_i\} \in H^0(V, \Omega_S^1 \otimes L).$$

□

Corollary 1. *Let $C \hookrightarrow S$ be an embedding of the curve C into the surface S such that $C \cdot C < 0$. Then there exists a regular holomorphic foliation defined in a neighborhood of C.*

4. **Examples**

Example 1. *A plane smooth projective curve C different from the projective line does not have a transverse holomorphic line field* (this is a particular case of a theorem of Van de Ven ([4])). In fact, suppose Y is a transverse holomorphic line field defined along C. We consider a holomorphic automorphism A of the plane close to the Identity which fixes some point $p \in C$ and such that $(A_*Y)(p) \neq Y(p)$; the line field $Y_A = A_*Y$ is of course transverse to $A(C)$. Given $q \in A(C)$, we denote as l_q the projective line tangent to $Y_A(q)$ at q. We may therefore induce along C a new holomorphic line field $Z \neq Y$ in the following way: given $q \in A(C)$ take $q' = l_q \cap A(C)$ (the intersection is taken in a small neighborhoood of C); then $Z(q')$ is the tangent line to l_q at the point q'. Since $Z(p) = Y_A(p) \neq Y(p)$ and Z is transverse to C, we get a contradiction with the Proposition 2 (notice that $c(NC) = d^2$ is greater than $3d - d^2 = 2 - 2g$ when $d = \text{degree}(C) > 1$).

A different, "foliated" argument goes as follows: we take some Riemannian metric in $\mathbb{P}^2$; for a small η the discs centered at the points of C, of radius η and contained in the projective lines $\{l_p\}_{p\in C}$ form a holomorphic fibration. We pick up a non-constant meromorphic function in C and extend it to a neighborhood of C as a constant along each fiber. This is a meromorphic function that can be extend to all of $\mathbb{P}^2$ since the complement of C is a Stein surface. We observe that the extension is constant along each projective line l_p. The only possibility is that these projective lines form a pencil issued from some point of the plane.

Example 2. The Proposition 2 is useful to get examples of non-existence of certain regular foliations when the self-intersection of C is not sufficiently negative. In order to see this, let us consider a pair $C \hookrightarrow S$ obtained by the following procedure:

(1) we blow up the origin 0 of the polydisc $\Delta \subset \mathbb{C}^2$, introducing an exceptional divisor; we choose the point in this divisor which belongs to the strict transform of $\{y = 0\}$ and blow up again. We keep doing this in order to get a chain of projective lines $E_1, ..., E_{m-1}$ of self-intersection -2 and a last projetive line E_m of self-intersection -1; there is a holomorphic projection π from the resulting surface $\tilde{\Delta}$ to Δ, which collapses $E_1 \cup \cdots \cup E_m$ to 0, and which is an isomorphism from the complement of this divisor to $\Delta \setminus \{(0,0)\}$. Denote by $q \in E_m$ the point which belongs to the strict transform of $\{y = 0\}$ and take the u-coordinate along E_m as to have $\pi(x, u) = (x, ux^m)$. We take

also a polydisc $V = \{x,u); |x| < 1, |u| < \epsilon\}$, for a small ϵ, around $(x,u) = (0,0) = q \in E_m$.

(2) let us consider a linear bundle over a compact, holomorphic, smooth curve $\tilde{C}$ whose self-intersection satisfies $\tilde{C} \cdot \tilde{C} > 2 - 2g$; we select some point in $\tilde{C}$ and introduce coordinates $(\tilde{x}, \tilde{u})$ in a neighborhood W of this point as to have $\{\tilde{x} = \text{const}\}$ contained in the linear fiber through $(\tilde{x}, 0) \in \tilde{C}$ for every $\tilde{x}$.

(3) finally we glue W to V by means of a holomorphic diffeomorphism $\Phi : W \to V$ in order to get a holomorphic surface $\tilde{S}$ containing $E_1 \cup \cdots \cup E_m \cup \tilde{C}$ as a divisor whose components have the self-intersection numbers described above; Φ must send $(\tilde{x}, \tilde{u}) = (0,0)$ to $(x,u) = (0,0) = q$, the $\tilde{x}$-axis into the x-axis and the $\tilde{u}$-axis transversely to the u-axis. We remark that $\tilde{C}$ has a unique field $\tilde{\mathcal{L}}$ of transversal lines because $\tilde{C} \cdot \tilde{C} > 2 - 2g$; by construction the line $\tilde{\mathcal{L}}_q$ is different from $T_q E_m$.

We blow down $E_1 \cup \cdots \cup E_m$ to $p = (0,0) \in \Delta$ and get a surface S with an embedded curve C such that $C \cdot C > m + 2 - 2g$ and $p \in C$.

We claim that there exists no regular foliation $\mathcal{F}$ in S transverse to $C \setminus \{p\}$ with order of tangency $0 \leq n \leq m-1$ at p . Otherwise after blowing up a times as explained before starting at p, we would get a foliation $\tilde{\mathcal{F}}$ transverse to $\tilde{C}$ and having E_m as a leaf. Each leaf $\tilde{F}_s$ through $s \in \tilde{C}$ has $\tilde{L}_s$ as tangent line at $s \in \tilde{C}$; but this property is not verified at the point $q \in \tilde{C} \cap E_m$.

We remark that the particular case $m = 1$ gives examples of embeddings $C \hookrightarrow S$ such that $C \cdot C > 3 - 2g$ without transversal foliations to C; in particular, there is no holomorphic tubular neighborhood.

5. **Plane curves and line fields**

We develop here Example 1 in order to understand the role of tangencies. Let us consider in $\mathbb{P}^2$ a smooth algebraic curve C of degree d and a holomorphic line field X along C. We have then a holomorphic map $\phi_X : C \longrightarrow \check{\mathbb{P}}^2$ of some degree $l \in \mathbb{N}$ defined as $\phi_X(p) = X(p) \in \check{\mathbb{P}}^2$; its image is an algebraic curve $\check{X} \subset \check{\mathbb{P}}^2$.

For instance, let us suppose that X is induced by a pencil of lines issued from some point $b \in \mathbb{P}^2$. Then $\check{X}$ is a line in $\check{\mathbb{P}}^2$ and ϕ_X has degree d or $d-1$ according to $b \in C$ or $b \notin C$ (in this last case, $X(b)$ is the tangent line to C at $b \in C$). We have then $tang(X,C) = d^2 - d$ or $tang(X,C) = d^2 - d - 1$.

Proposition 3. *$tang(X,C) = l.deg(\check{X}) + d^{\,2} - 2d$.*

Proof. We consider $\mathbb{P}(T\mathbb{P}^2|_C)$, which is a $\mathbb{P}^1$-bundle over C with the section $\mathbb{P}(TC)$. The vector bundle $T\mathbb{P}^2|_C$ may be described by the following transition maps:

$$x_\alpha = \xi_{\alpha\beta}(z_\beta)x_\beta + \eta_{\alpha\beta}(z_\beta)y_\beta, \quad y_\alpha = c_{\alpha\beta}(z_\beta)y_\beta$$

where (x_β, y_β) are coordinates for $T\mathbb{P}^2|_C$ at the point of C of coordinate z_β, $z_\alpha = g_{\alpha\beta}(z_\beta)$, $\xi_{\alpha\beta}(z_\beta) = g'_{\alpha\beta}(z_\alpha)$ and $\{c_{\alpha\beta}\}$ defines the normal bundle to C in $\mathbb{P}^2$.

In order to get the transition functions of $\mathbb{P}(T\mathbb{P}^2|_C)$, we put $u_\beta = x_\beta/y_\beta$ and $t_\beta = y_\beta/x_\beta$; then

$$u_\alpha = \frac{\xi_{\alpha\beta}(z_\beta)}{c_{\alpha\beta}(z_\beta)}u_\beta + \frac{\eta_{\alpha\beta}(z_\beta)}{c_{\alpha\beta}(z_\beta)}$$

and

$$t_\alpha = \frac{c_{\alpha\beta}(z_\beta)t_\beta}{\xi_{\alpha\beta}(z_\beta) + \eta_{\alpha\beta}(z_\beta)t_\beta}$$

Let us consider the line field X as a section of $\mathbb{P}(T\mathbb{P}^2|_C)$; we choose also a generic pencil of lines P. In the u-coodinates, we have

$$X_\alpha = \frac{\xi_{\alpha\beta}(z_\beta)}{c_{\alpha\beta}(z_\beta)}X_\beta + \frac{\eta_{\alpha\beta}(z_\beta)}{c_{\alpha\beta}(z_\beta)}$$

and

$$P_\alpha = \frac{\xi_{\alpha\beta}(z_\beta)}{c_{\alpha\beta}(z_\beta)}P_\beta + \frac{\eta_{\alpha\beta}(z_\beta)}{c_{\alpha\beta}(z_\beta)}$$

The intersection number of both sections X and P with $\mathbb{P}(TC)$ will be denoted by $Poles(X)$ and $Poles(P)$; of course $tang(X,C) = Poles(X)$ and $Poles(P) = d^2 - d$.

¿From the formulae above we see that $\{X_\alpha - P_\alpha\}$ is a section of the linear bundle given by the cocycle $\{\frac{\xi_{\alpha\beta}(z_\beta)}{c_{\alpha\beta}(z_\beta)}\}$, which is $TC \otimes NC^*$. Consequently:

$$Zeroes(X-P) - Poles(X-P) = -2d^2 + 3d$$

Therefore $Poles(X) = Zeroes(X-P) - Poles(P) + 2d^2 + 3d$. Now since $Poles(P) = d^2 - d$ and $Zeroes(X-P) = l.deg(\check{X})$, we get finally

$$tang(X,C) = l.deg(\check{X}) + d^2 - 2d.$$

□

Corollary 2. $tang(X, C) \geq (d-1)^2$

This Corollary gives another explanation why a a smooth, plane algebraic curve C of degree greater than one has no transversal holomorphic line field; consequently a neighborhood of C can not be linearized.

We see also that *if we blow up at $d^2 - 2d$ different points of C, the resulting curve $\hat{C}$ has not a linearizable neighborhood as well.* In fact, a tranversal holomorphic line field to $\hat{C}$ corresponds to a holomorphic line field along C with at most $d^2 - 2d$ points of ordinary tangency, which is not possible.

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