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Additive average Schwarz methods for discretization of elliptic problems with highly discontinuous coefficients

Maksymilian Dryja [1]

Department of Mathematics, Warsaw University
Banacha 2, 00-097 Warsaw, Poland
E-mail: dryja@mimuw.edu.pl

Marcus Sarkis

Instituto Nacional de Matemática Pura e Aplicada
Estrada Dona Castorina 110, Rio de Janeiro 22460-320, Brazil
and
Department of Mathematical Sciences, Worcester Polytechnic Institute
Worcester, MA01609, USA
E-mail: msarkis@impa.br and msarkis@wpi.edu

Abstract — The second order elliptic problem with highly discontinuous coefficients is considered. The problem is discretized by two methods: 1) a continuous finite element method (FEM) and 2) a composite discretization given by a continuous FEM inside the substructures and a discontinuous Galerkin method (DG) across the boundaries of these substructures. The main goal of this paper is to design and analyze parallel algorithms for the resulting discretizations. These algorithms are additive Schwarz methods (ASMs) with special coarse spaces spanned by functions that are almost piecewise constant with respect to the substructures for the first discretization and by piecewise constant functions for the second discretization. It is established that the condition number of the preconditioned systems does not depend on the jumps of the coefficients across the substructure boundaries and outside of a thin layer along the substructure boundaries. The algorithms are very well suited for parallel computations.

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1. Introduction

In this paper a second order elliptic problem with highly discontinuous coefficient $\varrho(x)$ in a 2-D polygonal region Ω is considered. For the simplicity of the presentation we assume Dirichlet homogeneous boundary conditions. The region Ω is partitioned into disjoint polygonal substructures $\Omega_i, \overline{\Omega} = \cup_i \overline{\Omega}_i,\ i = 1, \cdots, N$, and denote by $\varrho_i(x)$ the restriction of $\varrho(x)$ to Ω_i. Associated to this partition, let us denote by Ω_i^h the layer around $\partial\Omega_i$ with width h_i

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and define $\overline{\alpha}_i = \sup_{x\in\Omega_i^h} \varrho_i(x)$ and $\underline{\alpha}_i = \inf_{x\in\Omega_i^h} \varrho_i(x)$. We say that the coefficient $\varrho_i(x)$ has moderate variations on Ω_i^h if $\overline{\alpha}_i/\underline{\alpha}_i = O(1)$. The coefficient ϱ can be highly discontinuous in $\Omega_i \backslash \Omega_i^h$ and across $\partial\Omega_i$.

We consider two discretization methods: a standard continuous finite element method (FEM), see [3], and a composite discretization FEM with discontinuous Galerkin (DG), see [1,10]. The latter means that in each Ω_i the problem is discretized by a continuous FEM inside Ω_i and a DG method across $\partial\Omega_i$, see [4,5]. This discretization is determined by the regularity of $\varrho(x)$ and the regularity of the solution.

The main goal of this paper is to design and analyze parallel algorithms for these two considered discretizations. They are additive Schwarz methods (ASMs) with coarse space functions which are piecewise constant on each $\Omega_i\backslash\Omega_i^h$ for the first discretization and piecewise constant on each Ω_i for the second one. The unknowns associated to these coarse spaces are related to the average values on $\partial\Omega_i$. These algorithms are called additive average Schwarz methods (AASMs) and they are generalizations of the algorithms considered in [2] for the case of the continuous FEMs and for regular coefficients.

In this paper it is proved that the condition number of the preconditioned systems obtained by AASMs for the first discretization is bounded by $C \max_i (\frac{H_i}{h_i})^2 \frac{\overline{\alpha}_i}{\underline{\alpha}_i}$, where C independents of the jumps of ϱ, the size of the substructures $H_i := diam(\Omega_i)$ and the parameters h_i of the triangulation in Ω_i, $i = 1, \cdots, N$. For the second discretization (the composite discretization) it is proved that the condition number is bounded by $C \max_i \max_{j\in\mathcal{I}_i} (\frac{H_i^2}{h_i h_{ij}}) \frac{\overline{\alpha}_i}{\underline{\alpha}_i}$ where $\mathcal{I}_i$ is the set of indices j such that $|\partial\Omega_i \cap \partial\Omega_j| \neq 0$ and $h_{ij} := 2h_i h_j/(h_i + h_j)$, as the harmonic average of h_i and h_j. These estimates can be improved when $\underline{\alpha}_i$ and $\overline{\alpha}_i$ are of the same order and $\overline{\alpha}_i \leqslant \varrho_i(x)$ on $\Omega_i\backslash\Omega_i^h$. In this case we get the estimates with $C \max_i (H_i/h_i)$ for the first discretization and $C \max_i \max_{j\in\mathcal{I}_i} (H_i/h_{ij})$ for the second one.

The discussed algorithms can be straightforwardly extended to the 3-D case. In this paper the 2-D case is considered only for the simplicity of the presentation.

Parallel algorithms for the considered discretizations in the case of piecewise constant coefficients with respect to Ω_i have been discussed in many papers, see [11] and references therein. The case of coefficients with highly discontinuous coefficients inside Ω_i and across $\partial\Omega_i$ has been discussed only in few papers. For the first discretization, the standard Schwarz method with overlap and FETI method were considered in [8] and [9], respectively. In [6] the FETI-DP is discussed where the estimate of condition number of the preconditioned system is better than in [9]. In the present paper we consider simpler coarse spaces and smaller local problems than in those papers mentioned above and with better condition number estimates. For the second discretization the parallel algorithms have not been discussed in literature to our knowledge, i.e., in the case when the coefficients are highly discontinuous inside of Ω_i and across $\partial\Omega_i$. In the literature is discussed only the case when $\varrho(x)$ is piecewise constant with respect Ω_i, see for example [7], [5] and references therein.

This paper is organized as follows. In Section 2 the differential problem and assumptions on the triangulations and on the coefficients are introduced. In Section 3 the continuous finite element discretization on matching triangulation is formulated, and in Section 4 an additive average Schwarz method (AASM) for the resulting discrete problem is designed and analyzed. The main result is Theorem 4.1, where we establish the estimate of the condition number of the preconditioned system. In Section 5 the original problem is discretized on nonmatching triangulation across $\partial\Omega_i$ by continuous FEM in each Ω_i and DG with interior penalty term across $\partial\Omega_i$, and in Section 6 we design and analyze the AASM for the resulting discrete problem. The main result is Theorem 6.1, where we estimate the condition number of the preconditioned system. In Section 7 we discuss the implementation of these preconditioned systems.

2. Differential problems and assumptions

In this section the differential problem with discontinuous coefficient is formulated and we describe some of the assumptions on the coefficients and triangulations.

2.1. Differential problem

Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \qquad v \in H_0^1(\Omega) \tag{1}$$

where

$$a(u, v) := (\varrho(\cdot)\nabla u, \nabla v)_{L^2(\Omega)}, \qquad f(v) := \int_\Omega f v dx. \tag{2}$$

We assume that $\varrho \in L^\infty(\Omega)$ and $\varrho(x) \geqslant \varrho_0 > 0, f \in L^2(\Omega)$, and Ω is a 2-D polygonal region. Under these assumptions the problem has a unique solution, see for example [3].

2.2. Assumptions

We suppose that Ω is decomposed into disjoint polygonals Ω_i, $\overline{\Omega} = \cup_i \overline{\Omega}_i, i = 1, \cdots, N$. Inside each Ω_i we introduce a shape regular and quasi-uniform triangulation $\mathcal{T}^h(\Omega_i)$ with mesh parameter h_i and $H_i := diam(\Omega_i)$. For the first discretization we assume that the global mesh is regular (no hanging nodes) while for the second discretization we allow nonmatching meshes across substructure boundaries. Denote Ω_i^h as the layer around $\partial\Omega_i$ which is the union of $e_k^{(i)}$ triangles of $\mathcal{T}^h(\Omega_i)$ which touch $\partial\Omega_i$, and we introduce

$$\overline{\alpha}_i := \sup_{x \in \overline{\Omega}_i^h} \varrho(x), \qquad \underline{\alpha}_i := \inf_{x \in \overline{\Omega}_i^h} \varrho(x). \tag{3}$$

3. Discrete continuous problem

To define the first discretization, the continuous finite element method for problem (1), we introduce the space of piecewise linear continuous functions as

$$V_h(\Omega) := \{v \in C_0(\Omega); v_{|e_k} \in P_1(x)\}$$

where e_k are the triangles of $\mathcal{T}^h(\Omega)$ and $P_1(x)$ is the set of linear polynomials.

The discrete problem is defined as: Find $u_h^* \in V_h(\Omega)$ such that

$$a(u_h^*, v) = f(v), \quad v \in V_h(\Omega). \tag{4}$$

4. Additive average Schwarz method for (4)

In this section we design and analyze an additive average Schwarz method for the discrete problem (4). For that we use the general theory of additive Schwarz methods (ASMs) described in [11].

4.1. Decomposition of $V_h(\Omega)$

Let us decompose

$$V_h(\Omega) = V_0(\Omega) + V_1(\Omega) + \cdots + V_N(\Omega) \tag{5}$$

where for $i = 1, \cdots, N$, we define $V_i(\Omega) = V_h(\Omega) \cap H_0^1(\Omega_i)$ on Ω_i and extended by zero outside of Ω_i. The coarse space $V_0(\Omega)$ is defined as the range of the following interpolation operator I_A. For $u \in V_h(\Omega)$, let $I_A u \in V_h(\Omega)$ be defined such that on $\overline{\Omega}_i$

$$I_A u := \begin{cases} u(x), & x \in \partial\Omega_{ih} \\ \bar{u}_i, & x \in \Omega_{ih} \end{cases} \tag{6}$$

where

$$\bar{u}_i := \frac{1}{n_i} \sum_{x \in \partial\Omega_{ih}} u(x). \tag{7}$$

Here Ω_{ih} and $\partial\Omega_{ih}$ are the sets of nodal points of Ω_i (interior) and $\partial\Omega_i$, respectively, and n_i is the number of nodal points of $\partial\Omega_{ih}$.

4.2. Inexact solvers

For $i = 1, \cdots, N$, let us introduce

$$b_i(u, v) := a_i(u, v), \quad u, v \in V_i(\Omega) \tag{8}$$

and $a_i(\cdot, \cdot)$ is the restriction of $a(\cdot, \cdot)$ to Ω_i.

For $i = 0$ let us introduce

$$b_0(u, v) := \sum_{i=1}^{N} \sum_{x \in \partial\Omega_{ih}} \overline{\alpha}_i (u(x) - \bar{u}_i)(v(x) - \bar{v}_i), \quad u, v \in V_0(\Omega). \tag{9}$$

Note that (9) reduces to

$$b_0(u, v) = \sum_{i=1}^{N} \overline{\alpha}_i \sum_{x \in \partial\Omega_{ih}} (u(x) - \bar{u}_i) v(x). \tag{10}$$

4.3. The operator equation

For $i = 0, \cdots, N$, we define the operators $T_i^{(A)} : V_h(\Omega) \to V_i(\Omega)$ by

$$b_i(T_i^{(A)} u, v) = a(u, v), \quad v \in V_i(\Omega). \tag{11}$$

Of course each of these problems have a unique solution. Let us introduce

$$T_A := T_0^{(A)} + T_1^{(A)} + \cdots + T_N^{(A)}. \tag{12}$$

We replace (4) by the operator equation

$$T_A u_h^* = g_h \tag{13}$$

where

$$g_h = \sum_{i=0}^{N} g_i, \qquad g_i = T_i^{(A)} u_h^* \tag{14}$$

and u_h^* is the solution of (4). Note that to compute g_i we do not need to know u_h^*, see (11). We note also that the solution of (4) and (13) are the same. This follows from the first main result of this paper:

Theorem 4.1. *For any $u \in V_h(\Omega)$ the following holds:*

$$C_1\beta_1^{-1}a(u,u) \leqslant a(T_A u, u) \leqslant C_2 a(u,u) \tag{15}$$

where $\beta_1 = \max_i(\overline{\alpha}_i/\underline{\alpha}_i)(H_i/h_i)^2$ and the positive constants C_1 and C_2 do not depend on ρ_i, $\overline{\alpha}_i/\underline{\alpha}_i$, H_i, and h_i, $i = 1,\cdots,N$.

Remark 4.1. The estimate (15) can be improved when $\overline{\alpha}_i$ and $\underline{\alpha}_i$ are of the same order and $\underline{\alpha}_i \leqslant \varrho_i(x)$ on $\Omega_i \backslash \Omega_i^h$. In this case $\beta_1 = \max_i(H_i/h_i)$.

Remark 4.2. The layer Ω_i^h can be replaced by Ω_i^δ, the layer around $\partial\Omega_i$ with width δ_i. In this case $\beta_1 = \max_i(\frac{\overline{\alpha}_i}{\underline{\alpha}_i}\frac{H_i^2}{h_i\delta_i})$ where $\overline{\alpha}_i$ and $\underline{\alpha}_i$ here are defined on Ω_i^δ, see [6].

Proof of Theorem 3.1 For that we need to check the three key assumptions of the general theory of ASMs; see Theorem 2.7 of [11].

Assumption(i) We need to show that $\eta(\varepsilon)$, the spectral radius of $\varepsilon = \{\varepsilon_{ij}\}_{i,j=1,\cdots,N}$, defined by

$$a(u_i,u_j) \leqslant \varepsilon_{ij}a^{1/2}(u_i,u_i)a^{1/2}(u_j,u_j) \quad \forall u_i \in V_i \quad \text{and} \quad \forall u_j \in V_j,$$

is bounded by a constant that does not depend on the jumps of $\varrho_i(x), H_i$ and h_i. In our case V_i and V_j are orthogonal for $i,j = 1,\cdots,N$ and $i \neq j$, therefore, $\eta(\varepsilon) = 1$.

Assumption (ii) We need to show that for $i = 0,\cdots,N$,

$$a(u,u) \leqslant \omega_i b_i(u,u), \qquad u \in V_i$$

with $\omega_i \leqslant C$ where C is independent of the jumps of $\varrho_i(x), H_i$ and h_i.

For $i = 1,\cdots,N$, it is obvious that $\omega_i = 1$. For $i = 0$ and $u \in V_h(\Omega)$ we have

$$a(I_A u, I_A u) = \sum_{i=1}^{N} a_i(I_A u, I_A u)$$

and, see (6),

$$\begin{aligned} a_i(I_A u, I_A u) &\equiv (\varrho_i(\cdot)\nabla I_A u, \nabla I_A u)_{L^2(\Omega_i)} = \\ &= (\varrho_i(\cdot)\nabla(I_A u - \bar{u}_i), \nabla(I_A u - \bar{u}_i))_{L^2(\Omega_i)} = \\ &= (\varrho_i(\cdot)\nabla(I_A u - \bar{u}_i), \nabla(I_A u - \bar{u}_i))_{L^2(\Omega_i^h)} \leqslant \\ &\leqslant C \sum_{x \in \partial\Omega_{ih}} \overline{\alpha}_i (u_i(x) - \bar{u}_i)^2 \end{aligned} \tag{16}$$

where $\overline{\alpha}_i$ is defined in (3). We have used the inverse inequality. Hence

$$a(I_A u, I_A u) \leqslant C b_0(u,u)$$

with $\omega_0 \leqslant C$. Thus $\max_{i=0}^N \omega_i \leqslant C$.

Assumption(iii) We prove that for $u \in V_h(\Omega)$ there exist $u_i \in V_i, i = 0.\cdots,N$, such that $u = \sum_{i=0}^N u_i$ and

$$\sum_{i=0}^{N} b_i(u_i,u_i) \leqslant C\beta_1 a(u,u). \tag{17}$$

Let $u_0 := I_A u$ for $u \in V_h(\Omega)$ and $u_i := u - u_0$ on $\overline{\Omega}_i$ and $u_i = 0$ outside of Ω_i. Of course $u_i \in V_i(\Omega)$ for $i = 0, \cdots, N$, and $u = \sum_{i=0}^N u_i$. We have

$$\begin{aligned} \sum_{i=1}^N b_i(u_i, u_i) &= \sum_{i=1}^N a_i(u - u_0, u - u_0) \leqslant \\ &\leqslant 2\sum_{i=1}^N \{a_i(u, u) + a_i(u_0, u_0)\} = 2\{a(u, u) + a(u_0, u_0)\}. \end{aligned} \tag{18}$$

To obtain β_1 in (17) we only need to estimate $a(u_0, u_0)$. We have

$$\begin{aligned} a_i(u_0, u_0) &\leqslant C \sum_{x \in \partial\Omega_{ih}} \overline{\alpha}_i (u(x) - \bar{u}_i)^2 \leqslant \\ &\leqslant C\frac{\overline{\alpha}_i}{h_i} \parallel u - \bar{u}_i \parallel^2_{L^2(\partial\Omega_i)} \leqslant C\frac{H_i^2}{h_i}\overline{\alpha}_i |u|^2_{H^1(\partial\Omega_i)} \end{aligned} \tag{19}$$

where we have used (16) and a Friedrich's inequality. Note that

$$\overline{\alpha}_i |u|^2_{H^1(\partial\Omega_i)} \leqslant \frac{\overline{\alpha}_i}{\underline{\alpha}_i h_i} (\varrho_i(\cdot)\nabla u, \nabla u)_{L^2(\Omega_i^h)}. \tag{20}$$

Using this in (19) we obtain

$$\sum_{i=1}^N a_i(u_0, u_0) \leqslant \sum_{i=1}^N C\frac{\overline{\alpha}_i}{\underline{\alpha}_i}\frac{H_i^2}{h_i^2} a_i(u, u) \leqslant C\beta_1 a(u, u). \tag{21}$$

Using this in (18) we obtain (17). The proof of Theorem 4.1 is complete.

5. Discrete discontinuous Galerkin problem

In this section the original problem (1) is discretized by a composite discretization. We decompose Ω into disjoint polygonals $\Omega_i, i = 1, \cdots, N$, so $\overline{\Omega} = \cup_i \overline{\Omega}_i$ as in Section 4 and we define $H_i = diam(\Omega_i)$. The problem (1) is discretized by a continuous FEM in each Ω_i and by a DG across $\partial\Omega_i$.

Let us introduce a triangulation $\mathcal{T}^h(\Omega_i)$ in each Ω_i with triangular elements $e_k^{(i)}$ and a mesh parameter h_i. We assume that this triangulation is shape-regular on $\overline{\Omega}_i$. The resulting triangulation is nonmatching across $\partial\Omega_i$. Let $X_i(\Omega_i)$ be the finite element space of piecewise linear continuous functions on Ω_i. We do not assume that functions of $X_i(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$. Let us introduce

$$X_h(\Omega) := X_1(\Omega_1) \times \cdots \times X_N(\Omega). \tag{22}$$

Functions v of $X_h(\Omega)$ are represented as $v = \{v_i\}_{i=1}^N$ with $v_i \in X_i(\Omega_i)$. Note that $X_h(\Omega) \not\subseteq H^1(\Omega)$ but $X_h(\Omega) \subset L_2(\Omega)$.

The coefficients $\varrho(x)$ on the introduced triangulation can be discontinuous. We assume that $\varrho(x)$ on each element $e_k^{(i)} \subset \overline{\Omega}_i$ is a constant $\varrho_k^{(i)}$, which can be defined, for example, by $|e_k^{(i)}|^{-1} \int_{e_k^{(i)}} \varrho(x) ds$. It means that this is done in the formulation of the original problem.

Let Ω_i^h, as in Section 2, denote a layer with width h_i around $\partial\Omega_i$ which is the union of $e_k^{(i)}$ triangles which touch $\partial\Omega_i$. We will use also $\overline{\alpha}_i$ and $\underline{\alpha}_i$ defined in (3). Note that this time $\varrho(x)$ is piecewise constant on triangles of Ω_i^h.

A discrete problem for (1) is obtained by a composite discretization, i.e., a regular continuous FEM in each Ω_i and a DG across of $\partial\Omega_i$, see [1,10,4,5]. The discretization is defined as follows: Find $u_h^* \in X_h(\Omega)$ such that

$$\hat{a}_h(u_h^*, v_h) = f(v_h), \qquad v_h \in X_h(\Omega) \tag{23}$$

where

$$\hat{a}_h(u,v) := \sum_{i=1}^N \hat{a}_i(u,v), \quad f(v) := \sum_{i=1}^N \int_{\Omega_i} f v_i dx. \tag{24}$$

Each bilinear form $\hat{a}_i$ is given as the sum of three bilinear forms:

$$\hat{a}_i(u,v) := a_i(u,v) + s_i(u,v) + p_i(u,v) \tag{25}$$

where

$$a_i(u,v) := \int_{\Omega_i} \varrho_i(x) \nabla u_i \nabla v_i dx, \tag{26}$$

$$s_i(u,v) := \sum_{E_{ij}\subset\partial\Omega_i} \frac{1}{l_{ij}} \int_{E_{ij}} \varrho_{ij}(x)\left(\frac{\partial u_i}{\partial n_i}(v_j - v_i) + \frac{\partial v_i}{\partial n_i}(u_j - u_i)\right)ds \tag{27}$$

and

$$p_i(u,v) := \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}h_{ij}} \int_{E_{ij}} \varrho_{ij}(x)(u_j - u_i)(v_j - v_i)ds. \tag{28}$$

Here, the bilinear form p_i is called the penalty term with a positive penalty parameter δ. In the above equations, we set $l_{ij} = 2$ when $E_{ij} := \partial\Omega_i \cap \partial\Omega_j$ is a common edge (or part of an edge) of $\partial\Omega_i$ and $\partial\Omega_j$. On E_{ij} we define $\varrho_{ij}(x) = 2\varrho_i(x)\varrho_j(x)/(\varrho_i(x) + \varrho_j(x))$, i.e., as the harmonic average of $\varrho_i(x)$ and $\varrho_j(x)$ on E_{ij}. Similarly, we define $h_{ij} = 2h_ih_j/(h_i + h_j)$. In order to simplify notation we include the index $j = \partial$ when $E_{i\partial} := \partial\Omega_i \cap \partial\Omega$ is an edge of $\partial\Omega$ and set $l_{i\partial} = 1$, $v_\partial = 0$ for all $v \in X_h(\Omega)$, $\varrho_{i\partial}(x) = \varrho_i(x)$ and $h_{i\delta} = h_i$. The outward normal derivative on $\partial\Omega_i$ is denoted by $\partial/\partial n_i$. Note that when $\varrho_{ij}(x)$ is given by the harmonic average then $\min\{\varrho_i, \varrho_j\} \leqslant \varrho_{ij} \leqslant 2\min\{\varrho_i, \varrho_j\}$.

We also define the positive local bilinear form d_i with weights $\varrho_i(x)$ and $\delta\varrho_{ij}(x)/(l_{ij}h_{ij})$ as

$$d_i(u,v) = a_i(u,v) + p_i(u,v) \tag{29}$$

and introduce the global bilinear form $d_h(\cdot,\cdot)$ on $X_h(\Omega)$ defined by

$$d_h(u,v) = \sum_{i=1}^N d_i(u,v). \tag{30}$$

For $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$ the associated broken norm is then defined by

$$\| u_h \|_h^2 := d_h(u,u) = \sum_{i=1}^N \{\| \varrho_i^{1/2} \nabla u_i \|_{L^2(\Omega_i)}^2 + \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}h_{ij}} \int_{E_{ij}} \varrho_{ij}(x)(u_i - u_j)^2 ds\}. \tag{31}$$

The discrete problem (23) has a unique solution for sufficiently large penalty parameter δ. This follows from the following lemma:

Lemma 5.1. *There exists $\delta_0 > 0$ such that for $\delta \geqslant \delta_0$ and for all $u \in X_h(\Omega)$, it holds*

$$\gamma_0 d_i(u,u) \leqslant \hat{a}_i(u,u) \leqslant \gamma_1 d_i(u,u) \tag{32}$$

and

$$\gamma_0 d_h(u,u) \leqslant \hat{a}_h(u,u) \leqslant \gamma_1 d_h(u,u) \tag{33}$$

where γ_0 and γ_1 are positive constants independent of the ϱ_i, h_i and H_i.

Proof It is a slight modification of the proof given in [4,5], therefore, it is omitted here.

We will assume below that $\delta \geqslant \delta_0$; i.e., that (32) and (33) are valid. A priori error estimates for the discussed method are optimal for regular coefficients and when h_i and h_j are of the same order, see for example [1], [10]. For piecewise constant coefficients ρ_i and/or when the mesh sizes h_i and h_j are not of the same order, the error estimates depend on the ratio h_i/h_j. There is also the question of regularity of the solution of (1). Assuming the regularity of solution we have the following result:

Lemma 5.2. *Let u^* and u_h^* be the solutions of (1) and (23). For $u^* \in H_0^1(\Omega)$ and $u^*_{|\Omega_i} \in H^{1+r}(\Omega_i)), i = 1, \cdots, N$, we have*

$$\| u^* - u_h^* \|_h^2 \leqslant C \sum_{i=1}^{N} (h_i^{1+r} + \frac{h_j^{2+r}}{h_i}) |u^*|^2_{H^{1+r}(\Omega_i)}$$

with $r \in (1/2, 1]$ and C which is independent of h_i, H_i and u^*.

For the proof see [1,10] and [4,5].

6. Additive average Schwarz method for (23)

In this section we design and analyze an additive average Schwarz method for the discrete problem (23). For that we use the general theory of additive Schwarz methods (ASMs) descroibed in [11].

6.1. Decomposition of $X_h(\Omega)$

Let us decompose

$$X_h(\Omega) = V^{(0)}(\Omega) + V^{(1)}(\Omega) + \cdots + V^N(\Omega) \tag{34}$$

where for $i = 1, \cdots, N$

$$V^{(i)}(\Omega) := \{v = \{v_k\}_{k=1}^N \in X_h(\Omega) : v_k = 0 \quad \text{for} \quad k \neq i\}. \tag{35}$$

This means that $V^{(i)}(\Omega)$ is the zero extension of $X_i(\Omega_i)$ to $\overline{\Omega}_j$ for $j \neq i$. The coarse space $V^{(0)}$ is defined as

$$V^{(0)}(\Omega) = span\{\phi^{(i)}\}_{i=1}^N \tag{36}$$

where $\phi^{(i)} = \{\phi_k^{(i)}\}_{k=1}^N \in X_h(\Omega)$ with $\phi_k^{(i)} = 1$ for $k = i$ and $\phi_k^{(i)} = 0$ for $k \neq i$. This is a space of piecewise constant functions with respect to $\Omega_i, i = 1, \cdots, N$. Note that the introduced spaces $V^{(i)}(\Omega)$ satisfy (34).

6.2. Inexact solver

For $u^{(i)} = \{u_k^{(i)}\}_{k=1}^N$ and $v^{(i)} = \{v_k^{(i)}\}_{k=1}^N$ belonging to $V^{(i)}(\Omega)$, $i = 1, \cdots N$, we set

$$b_i(u^{(i)}, v^{(i)}) = d_i(u^{(i)}, v^{(i)}) \tag{37}$$

where in this case, see (29),

$$d_i(u^{(i)}, v^{(i)}) = (\varrho_i(\cdot)\nabla u_i^{(i)}, \nabla v_i^{(i)})_{L^2(\Omega_i)} + \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}}\frac{1}{h_{ij}}(\varrho_{ij}(\cdot)u_i^{(i)}, v_i^{(i)})_{L^2(E_{ij})}. \tag{38}$$

For the coarse space $V^{(0)}$ and $u^{(0)} = \{u_i^{(0)}\}_{i=1}^N$ and $v^{(0)} = \{v_i^{(0)}\}_{i=1}^N$ belonging to $V^{(0)}(\Omega)$ we set

$$b_0(u^{(0)}, v^{(0)}) = d_h(u^{(0)}, v^{(0)}). \tag{39}$$

Note that in this case

$$b_0(u^{(0)}, v^{(0)}) = \sum_{i=1}^N \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}}\frac{1}{h_{ij}}(\varrho_{ij}(\cdot)(u_j^{(0)} - u_i^{(0)}), (u_j^{(0)} - u_i^{(0)}))_{L^2(E_{ij})} \tag{40}$$

since $u^{(0)}$ and $v^{(0)}$ are piecewise constant functions with respect to $\Omega_i, i = 1, \cdots, N$.

6.3. The operator equation

For $i = 0, \cdots, N$, let us define the operators $T_i^{(DG)} : X_h(\Omega) \to V^{(i)}(\Omega)$ by

$$b_i(T^{(DG)}u, v) = \hat{a}_h(u, v), \qquad v \in V^{(i)}(\Omega). \tag{41}$$

Of course each of these problems have a unique solution. Let us define

$$T_{DG} = T_0^{(DG)} + T_1^{(DG)} + \cdots + T_N^{(DG)}. \tag{42}$$

We replace (23) by the following operator equation:

$$T_{DG}u_h^* = g_h \tag{43}$$

where

$$g_h = \sum_{i=0}^N g_i, \qquad g_i = T_i^{(DG)}u_h^* \tag{44}$$

and u_h^* is the solution of (23). Note that to compute g_i we do not need to know u_h^*, see (41). The solutions (23) and (43) are the same. This follows from the following theorem, the second main result of this paper.

Theorem 6.1. *For any $u \in X_h(\Omega)$ the following holds:*

$$C_3\beta_2^{-1}\hat{a}_h(u, u) \leqslant \hat{a}_h(T_{DG}u, u) \leqslant C_4\hat{a}_h(u, u) \tag{45}$$

where $\beta_2 = \max_i \max_{j\in\mathcal{I}_i}(\overline{\alpha}_i/\underline{\alpha}_i)(\frac{H_i^2}{h_ih_{ij}})$ and the positive constants C_3 and C_4 do not depend on ρ_i, $\overline{\alpha}_i/\underline{\alpha}_i$, H_i, and h_i, $i = 1, \cdots, N$.

Proof We need to check three key assumptions of the general theory of ASMs, see Theorem 2.7 of [11].

Assumption(i) We check it in the same way as Assumption(i) in the proof of Theorem 4.1. Thus $\eta(\varepsilon) = 1$.

Assumption(ii) We need to prove that for $i = 0, 1, \cdots, N$

$$\hat{a}_h(u, u) \leqslant \omega_i b_i(u, u), \qquad u \in V^{(i)}(\Omega) \tag{46}$$

with $\omega_i \leqslant C$, where C is independent of the jumps of $\varrho_i(x), \varrho_{ij}(x), H_i$ and h_i. Using Lemma 5.1 it is enough to prove (46) for $d_h(\cdot,)$, see (30). For $i = 1, \cdots, N$ and $u^{(i)} \in V^{(i)}(\Omega)$ we have:

$$\begin{aligned} d_h(u^{(i)}, u^{(i)}) &= (\varrho_i(\cdot)\nabla u_i^{(i)}, \nabla v_i^{(i)})_{L^2(\Omega_i)} + \\ &+ \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}}\frac{1}{h_{ij}}(\varrho_{ij}(\cdot)u_i^{(i)}, u_i^{(i)})_{L^2(E_{ij})} = b_i(u^{(i)}, u^{(i)}). \end{aligned} \tag{47}$$

For the coarse space $V^{(0)}(\Omega)$ and $u^{(0)} \in V^{(0)}(\Omega)$

$$d_h(u^{(0)}, u^{(0)}) = \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}}\frac{1}{h_{ij}}(\varrho_{ij}(\cdot)(u_i^{(0)} - u_j^{(0)}), (u_i^{(0)} - u_j^{(0)}))_{L^2(E_{ij})} = b_0(u^{(0)}, u^{(0)}).$$

Thus $\omega_i \leqslant C$ for $i = 0, \cdots, N$ in view of Lemma 5.1.

Assumption(iii) We need to show that for $u \in X_h(\Omega)$ there exist $u^{(i)} \in V^{(i)}(\Omega), i = 0, \cdots, N$, such that $u = \sum_{i=0}^N u^{(i)}$ and

$$\sum_{i=0}^N b_i(u^{(i)}, u^{(i)}) \leqslant C\beta_2 \hat{a}_h(u, u). \tag{48}$$

Using Lemma 5.1, it is enough to prove (48) for $d_h(\cdot, \cdot)$.

For $u = \{u_i\}_{i=1}^N \in X_h(\Omega)$ let

$$u^{(0)} = \{\bar{u}_i\}_{i=1}^N, \qquad \bar{u}_i := \frac{1}{|\partial\Omega_i|}\int_{\partial\Omega_i} u_i(x)ds \tag{49}$$

and set

$$u = u^{(0)} + (u - u^{(0)}) = u^{(0)} + \sum_{i=1}^N u^{(i)}$$

where $u^{(i)} := \{u_k^{(i)}\}_{k=1}^N$ with $u_k^{(i)} := u_i - \bar{u}_i$ for $k = i$ and $u_k^{(i)} = 0$ for $k \neq i$. Of course $u^{(i)} \in V^{(i)}(\Omega)$ and $u = \sum_{i=0}^N u^{(i)}$.

We now check (48) for $d_h(.,.)$. For $i = 0$, see (40), we have

$$b_0(u^{(0)}, u^{(0)}) = \sum_{i=1}^N \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}}\frac{1}{h_{ij}}(\varrho_{ij}(\cdot)(\bar{u}_j - \bar{u}_i), \bar{u}_j - \bar{u}_i)_{L^2(E_{ij})}. \tag{50}$$

Note that

$$\begin{aligned} (\varrho_{ij}(\cdot)(\bar{u}_j - \bar{u}_i), \bar{u}_j - \bar{u}_i)_{L^2(E_{ij})} &\leqslant C\{\| \varrho_{ij}^{1/2}(\cdot)(\bar{u}_j - u_j) \|_{L^2(E_{ji})}^2 + \\ &+ \| \varrho_{ij}^{1/2}(\cdot)(\bar{u}_i - u_i) \|_{L^2(E_{ij})}^2 + \| \varrho_{ij}^{1/2}(\cdot)(u_i - u_j) \|_{L^2(E_{ij})}^2 \end{aligned} \tag{51}$$

where $E_{ij} = E_{ji}, E_{ij} \subset \partial\Omega_i, E_{ji} \subset \partial\Omega_j$. By a Friedrich's inequality we have

$$\begin{aligned} &\frac{1}{h_{ij}} \parallel \varrho_{ij}^{1/2}(\cdot)(\bar{u}_i - u_i) \parallel^2_{L^2(E_{ij})} \leqslant C\frac{\overline{\alpha}_i}{h_{ij}} \parallel u_i - \bar{u}_i \parallel^2_{L^2(\partial\Omega_i)} \leqslant \\ &\leqslant C\frac{\overline{\alpha}_i H_i^2}{h_{ij}} |u_i|^2_{H^1(\partial\Omega_i)} \leqslant C\frac{\overline{\alpha}_i}{\underline{\alpha}_i}\frac{H_i^2}{h_i h_{ij}} \parallel \varrho_i^{1/2}\nabla u_i \parallel^2_{L^2(\Omega_i^h)} \leqslant \\ &\leqslant C\frac{\overline{\alpha}_i}{\underline{\alpha}_i}(\frac{H_i^2}{h_i h_{ij}})(\varrho_i(\cdot)\nabla u_i, \nabla u_i)_{L^2(\Omega_i)} \end{aligned} \tag{52}$$

since $\varrho_{ij}(x) \leqslant 2\varrho_i(x) \leqslant 2\overline{\alpha}_i$ on $\partial\Omega_i$. In the same way we show that

$$\frac{1}{h_{ij}} \parallel \varrho_{ij}^{1/2}(\cdot)(\bar{u}_j - u_j) \parallel^2_{L^2(E_{ij})} \leqslant C\frac{\overline{\alpha}_j}{\underline{\alpha}_j}(\frac{H_j^2}{h_j h_{ji}})(\varrho_j(\cdot)\nabla u_j, \nabla u_j))_{L^2(\Omega_j)}. \tag{53}$$

Substituting (52), (53) into (51) and the resulting inequality into (50), we obtain

$$b_0(u^{(0)}, u^{(0)}) \leqslant C\beta_2 d_h(u, u) \leqslant C\beta_2 \hat{a}_h(u, u). \tag{54}$$

We have by (38) that

$$\begin{aligned} \sum_{i=1}^N b_i(u^{(i)}, u^{(i)}) = \sum_{i=1}^N (\varrho_i(\cdot)\nabla u_i, \nabla u_i)^2_{L^2(\Omega_i)} + \\ + \sum_{i=1}^N \sum_{E_{ij}\subset\partial\Omega_i} \frac{\delta}{l_{ij}}\frac{1}{h_{ij}}(\varrho_{ij}(\cdot)(u_i - \bar{u}_i), (u_i - \bar{u}_i)_{L^2(E_{ij})}. \end{aligned} \tag{55}$$

Using (52) and Lemma 5.1 we obtain

$$\sum_{i=1}^N b_i(u^{(i)}, u^{(i)}) \leqslant C\beta_2 d_h(u, u) \leqslant C\beta_2 \hat{a}_h(u, u). \tag{56}$$

Adding the estimates (54) and (56) we get (48). The proof of Theorem 6.1 is complete.

Remark 6.1. The estimate (45) can be improved when $\overline{\alpha}_i$ and $\underline{\alpha}_i$ are of the same order and $\overline{\alpha}_i \leqslant \varrho_i(x)$ on $\overline{\Omega}_i \backslash \Omega_i^h$. In this case $\beta_2 = \max_i \max_{j\in\mathcal{I}_i}(H_i/h_{ij})$.

Remark 6.2. The layer Ω_i^h can be replaced by Ω_i^δ, the layer around $\partial\Omega_i$ with width δ_i. In this case $\beta_2 = \max_i \max_{j\in\mathcal{I}_i}(\frac{\overline{\alpha}_i}{\underline{\alpha}_i}\frac{H_i^2}{h_{ij}\delta_i})$ where $\overline{\alpha}_i$ and $\underline{\alpha}_i$ here are defined on Ω_i^δ, see (3).

7. Implementation

To find the solution of (4), for the first discretization, and (23), for the second discretization, we need to solve the equations (13) and (43), respectively. The operators T_A and T_{DG} are symmetric positive definite and relatively well conditioned, see Theorem 4.1 and Theorem 6.1. To solve these equations a conjugate gradient method is used. We next discuss an implementation of the method for the equation (43) (for (13) is similar). For the simplicity of the presentation we discuss only the Richardson method.

The problem (43) is solved by the method

$$u^{n+1} = u^n - \tau(T_{DG}u^n - g_h) = u^n - \tau T_{DG}(u^n - u_h^*)$$

where the relaxation parameter τ can be chosen using the estimates in Theorem 6.1.

To compute

$$r^n := T_{DG}(u^n - u_h^*) = \sum_{i=0}^{n} T_i^{(DG)}(u^n - u_h^*)$$

we need to find $r_i^n := T_i^{(DG)}(u^n - u_h^*)$ by solving the following equations, see (41),

$$b_i(T_i^{(DG)} r_i^n, v) = \hat{a}_h(r_i^n, v) = \hat{a}_h(u^n, v) - f(v), \quad v \in V^{(i)}(\Omega)$$

for $i = 0, \cdots, N$. Note that these problems are independent to each other, therefore, they can be solved in parallel. The problems for $i = 1, \cdots, N$ are local and defined on $\overline{\Omega}_i$ and reduce to discrete problems with continuous FEM and piecewise linear functions. The problem for $i = 0$ has a local and a global component, where the local problem involves a diagonal preconditioning while the global problem has the number of unknowns equals to the number of subregions Ω_i and it reduces to a system with a mass matrix.

The above implementation shows that the proposed algorithm is very well suited for parallel computations.

References

[1] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal., 39 (2001/02), pp. 1749–1779.

[2] P. E. Bjorstad, M. Dryja, and E. Vainikko, *Parallel implementation of a Schwarz domain decomposition algorithms*, in *Applied Parallel Computing in Industrial Problems and Optimization* (J. Wasniewski, J. Dongara, K. Madsen, and D. Olsem, eds.) Lecture Notes in Computer Science, Springer, v. 1184 (1996), pp.141–157.

[3] P. G. Ciarlet, *The Finite Element Methods for Elliptic Problems*, North-Holland, Amsterdam, 1978.

[4] M. Dryja, *On discontinuous Galerkin methods for elliptic problems with discontinuous coefficients*, Comput. Methods App. Math., v. 3 (2003), pp. 76–85.

[5] M. Dryja, J. Galvis, and M. Sarkis, *BDDC methods for discontinuous Galerkin discretization of elliptic problems,* J. Complexity, 23 (2007), pp. 715–739.

[6] M. Dryja and M. Sarkis, *Technical tools for boundary layers and application on heterogeneous coefficients*, 2009, submitted.

[7] X. Feng and O. A. Karakashion, *Two-level additive Schwarz methods for discontinuous approximation of second order elliptic problems*, SIAM J. Numer. Anal. 39 (2001) pp. 1343–1365.

[8] I. G. Graham, P. O. Lechner, and R. Scheichl, *Domain decomposition for multiscale PDEs*, Numer. Math., 106 (2007), pp. 589–626.

[9] C. Pechstein and R. Scheichl, *Analysis of FETI methods for multiscale PDEs*, Numer. Math., 111 (2008), pp. 293–333.

[10] B. Riviere, *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations - Theory and Implementation*, Frontiers in Applied Mathematics, SIAM, 2008.

[11] A. Toselli and O. Widlund, *Domain Decomposition Methods - Algorithms and Theory*, v. 34, Springer Series in Computational Mathematics, Springer-Verlag, 2005.