Geodesic conic subdivision curves on surfaces

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Abstract

In this paper we present a nonlinear curve subdivision scheme, suitable for designing curves on surfaces. The scheme is inspired in the concept of geodesic Bézier curves introduced in [MCV07]. Starting with a geodesic control polygon with vertices on a surface S, the scheme generates a sequence of geodesic polygons that converges to a continuous curve on S. In the planar case, the limit curve is a conic Bézier spline curve. Each section of the subdivision curve, corresponding to three consecutive points of the control polygon, depends on a free parameter which can can be used to obtain a local control of the shape of the curve. Furthermore, it has the convex hull property. Results are extended to triangulated surfaces showing that the scheme is suitable for designing curves on these surfaces.

1 Introduction

Designing free-form curves is a basic operation in Geometric Modeling. In the Euclidean space it is a widely studied problem, nevertheless it becomes much harder if we wish to design on a curved geometry, such as a triangulated surface. The problem has been addressed on smooth manifolds as well as on triangulations, see for instance [HP04], [CKS99], [BH04].
Subdivision methods are currently very popular as a design tool, since subdivision curves can be easily computed in the Euclidean space. Nevertheless, their counterpart on curved surfaces are more involved and expensive. A first step on this sense are linear subdivision schemes on smooth and discrete manifolds [Kap99],[KYA05], [KS95], [MVC08]. Nonlinear schemes, which arise as perturbations of linear schemes on smooth manifolds, are the next step. They have been described by Wallner and Pottmann in [WP06]. Several examples where nonlinear subdivision schemes are useful in Computer Graphics are also presented in [WP06]. The convergence and smoothness analysis of these subdivision schemes can be found in the work of Wallner and Dyn [WD05]. They generalize the linear schemes to manifolds in two different ways. The first approach substitutes linear average by geodesic average. This method is very good because it is completely intrinsic,

although for some schemes it requires to compute many geodesics. The second method performs each subdivision step in the ambient space, projecting the new points into the manifold. This approach is more efficient, but depending on the complexity of the geometry it could conduce to wrong or unexpected results. Some variants of de Casteljau's Algorithm have been also used to define curves on Riemannian manifolds [RSJ05] and Lie groups [CKS99].
In [MCV05] a new algorithm to compute a geodesic path over a triangulated surface is proposed. This algorithm is used to define geodesic Bézier curves [MCV07]. They are a natural extension of Bézier curves in the sense that linear interpolation is substituted by geodesic interpolation. In [MVC08] a simple method to define subdivision schemes on triangulations is proposed. Using both, shortest and straightest geodesics, a perturbation of a planar binary subdivision is translated on the triangulation. This method allows to extend to a triangulated surface any binary subdivision scheme, regardless whether it is linear or not.
Inspired in these ideas we introduce in the present paper a natural extension of geodesic Bézier curves [MCV07] for the rational quadratic case: geodesic conic Bézier curves. They are defined as subdivision curves on a surface. More precisely, starting with a set of points on a surface S, a control polygon composed by geodesic arcs joining two consecutive points is defined. In each step of the subdivision a new geodesic polygon is computed defining a subdivision scheme that converges to a continuous curve living on S. Each section of the subdivision curve corresponding to three consecutive points of the control polygon, depends on a free parameter which can be used to obtain a local control of the shape of the curve. In the planar case the subdivision curve is a conic Bézier spline curve. Furthermore, we show that the limit curve has the convex hull property. Results are extended to triangulated surfaces showing that the scheme is suitable for designing curves on these surfaces and may be useful for trimming and segmentation.
The rest of the paper is organized as follows. In section 2 we introduce the notation and the classical planar subdivision scheme for conics. In section 3 we define the geodesic conic subdivision scheme on smooth surfaces and prove its convergence to a continuous curve. Section 4 is devoted to geodesic conic curves on triangulated surfaces. We include in this section details of the interaction with the user and several examples. Finally, in section 5 we give concluding remarks.

2 Basic theory: the subdivision scheme for conics

A rational Bézier curve of degree n is a parametric curve which is described by $n+1$ control points, $b_i \in R^m,\ m = 2, 3$ and $n+1$ weights ω_i. For $t \in [0, 1]$ the curve has the form [Far02]

$$c(t) = \frac{\sum_{i=0}^{n}\ \omega_i b_i B_i^n(t)}{\sum_{i=0}^{n}\ \omega_i B_i^n(t)}$$

where $B_i^n(t),\ i = 0, 1, ..., n$ are the Bernstein Bézier basis functions of degree n.

Conics are rational Bezier curves of degree $n = 2$. It has been shown [Patt86] that without loss of generality we may assume that any nondegenerate conic is written in *standard representation*, where $\omega_0 = \omega_2 = 1$. Since in what follows all Bezier conics are in standard representation, we will not mention explicitly the whole set of homogeneous weights $\omega_i,\ i = 0, 1, 2$ and we will denote the weight $\omega_1 > 0$ by $\omega > 0$.
Rational Bezier curves may be evaluated by de Casteljau recursive algorithm [Far83]. In the case of conics this algorithms is described as follows.

Algorithm 1 *de Casteljau rational algorithm*

Input: Control points b_i and weighs $\omega_i, i = 0, 1, 2$, parameter value $t \in [0, 1]$
Output: $c(t)$
step 1. **for** $i = 0, 1, 2$ set $b_i^0(t) = b_i$ and $\omega_i^0(t) = \omega_i$
step 2. **for** $j = 1, 2$
for $i = 0, \ldots, 2 - j$
$\omega_i^j(t) = (1 - t)\omega_i^{j-1}(t) + t\omega_{i+1}^{j-1}(t)$
$b_i^j(t) = (1 - t)\frac{\omega_i^{j-1}(t)}{\omega_i^j(t)}\, b_i^{j-1}(t) + t\frac{\omega_{i+1}^{j-1}(t)}{\omega_i^j(t)}\, b_{i+1}^{j-1}(t)$
step 3. $c(t) = b_0^2(t)$

The intermediate Bezier points $b_i^j(t)$ of the above recursive algorithm may be used to subdivide the curve c at parameter value $t \in (0, 1)$. More precisely, the left segment of c corresponding to the parameter values in the interval $[0, t]$ is a quadratic rational Bezier curve $c_0^1(u),\ u \in [0, 1]$ with control polygon $b_0, b_0^1(t), b_0^2(t)$ and weights $1, \omega_0^1(t), \omega_0^2(t)$. Similarly, the right segment of c corresponding to the parameter values in $(t, 1)$ is a quadratic rational Bezier curve $c_1^1(u),\ u \in [0, 1]$ with control polygon $b_0^2(t), b_1^1(t), b_2$, and weights $\omega_0^2(t), \omega_1^1(t), 1$. Algorithm 2 describes the basic subdivision.

Algorithm 2 *Basic classic conic subdivision*

Input: Control points b_i and weighs $\omega_i, i = 0, 1, 2$, parameter value $t \in [0, 1]$
Output: Control points and weights of the segments $c_0^1(u)$ and $c_1^1(u)$
step 1. $[b_0, b_0^1(t), b_1^1(t), b_0^2(t)]$ = **deCasteljau**($b_0, b_1, b_2, \omega_0, \omega_1, \omega_2, t$)
step 2. Control points: $b_0, b_0^1(t), b_0^2(t)$, weights: $1, \omega_0^1(t), \omega_0^2(t)$
Control points: $b_0^2(t), b_1^1(t), b_2$, weights: $\omega_0^2(t), \omega_1^1(t), 1$

This process may be repeated, subdividing each conic segment $c_0^1(u), c_1^1(u)$ in a parameter value $u \in (0, 1)$, for instance $u = \frac{1}{2}$. If we use this subdivision, after j steps we obtain 2^j control polygons (and the corresponding weights) that allow to represent a segment of the (unique) conic curve $c(t), t \in [0, 1]$ as a Bezier rational quadratic curve. When

$j \to \infty$, this sequence of control polygons tends to the conic curve. In this paper, we will refer to this subdivision scheme, based on the dyadic parameters, as the *classic subdivision scheme*. Recall that even if we start with the standard representation of c, if we subdivide it in $t = \frac{1}{2}$ using the classic scheme, then $c_0^1(\frac{1}{2})$ is not necessarily neither the shoulder point of $c_0^1(u)$ nor the point $c(\frac{1}{4})$ (by the same reason $c_1^1(\frac{1}{2})$ is not necessarily neither the shoulder point of $c_1^1(u)$ nor the point $c(\frac{3}{4})$).
A different scheme, converging to the same curve, may be obtained if we make an standardization of the conics in each step. In fact, since the weight $\omega_0^2(t)$ in Algorithm 2 is not necessarily equal to 1, to write the left and the right segment of the conic in the standard form we have to introduce the following substitutions [Far89],

$$\omega_0^1(t) \leftarrow \frac{\omega_0^1(t)}{\sqrt{\omega_0^2(t)}}, \;\; \omega_1^1(t) \leftarrow \frac{\omega_1^1(t)}{\sqrt{\omega_0^2(t)}}, \;\; \omega_0^2(t) \leftarrow 1 \tag{1}$$

For a rational Bezier conic in *standard representation* the Farin points q_0, q_1 are characterized by the fact that $\omega = ratio(b_i, q_i, b_{i+1}),\; i = 0, 1$. In terms of the control points b_i, $i = 0, 1, 2$, they can be expressed as

$$q_0 = \frac{b_0 + \omega b_1}{1 + \omega} \;,\; q_1 = \frac{b_2 + \omega b_1}{1 + \omega} \tag{2}$$

From the step 2 of Algorithm 1 it is easy to check that $q_0 = b_0^1(\frac{1}{2})$ and $q_1 = b_1^1(\frac{1}{2})$. Moreover, $\omega_0^1 = \omega_1^1 = \omega_0^2 = \frac{1+\omega}{2}$ and after the standardization (1) we obtain,

$$\omega_0^1 = \omega_1^1 = \sqrt{\frac{1+\omega}{2}} \tag{3}$$

Hence, if we subdivide a rational Bezier conic curve c in the *standard representation* at the *shoulder point* $s = c(\frac{1}{2})$, then we obtain two arcs of the same conic that can be written in the standard Bezier representation. The left arc $\overline{c}_0^1(u), u \in [0, 1]$ corresponding to the interval $t \in [0, \frac{1}{2}]$ has control points b_0, q_0, s and weights $1, \sqrt{\frac{1+\omega}{2}}, 1$, while the right arc $\overline{c}_1^1(u), u \in [0, 1]$ corresponding to the interval $t \in [\frac{1}{2}, 1]$ has control points s, q_1, b_2, and weights $1, \sqrt{\frac{1+\omega}{2}}, 1$. Observe that the weighs of both segments are the same. Algorithm 3 describe this subdivision step.

Algorithm 3 *Basic shoulder point subdivision*

Input: Control points $b_i, i = 0, 1, 2$ and the weighs $1, \omega, 1$
Output: Control points and weights of the segments $\overline{c}_0^1(u)$ and $\overline{c}_1^1(u),\; u \in [0, 1]$

step 1. $q_0 = \frac{b_0+\omega b_1}{1+\omega}$, $q_1 = \frac{b_2+\omega b_1}{1+\omega}$, $s = \frac{q_0+q_1}{2}$, $\omega^1 = \sqrt{\frac{1+\omega}{2}}$
step 2. Control points: b_0, q_0, s, weights: $1, \omega^1, 1$
Control points: s, q_1, b_2, weights: $1, \omega^1, 1$

This process may be repeated, subdividing $\overline{c}_0^1(u)$ and $\overline{c}_1^1(u)$ in its shoulder points by means of the Algorithms 3. We call this scheme *Basic shoulder point subdivision scheme*. For $j \to \infty$, the sequence of control polygons obtained tends to the conic curve.
Summarizing, if we apply Algorithm 2 with $t = \frac{1}{2}$ and Algorithm 3 to the standard Bezier representation of a conic, we obtain the same control polygons but with *different weights*. Hence, if we repeat the process and subdivide in $u = \frac{1}{2}$ with Algorithm 2 the control polygons of the segments $c_0^1(u)$ and $c_1^1(u)$ previously obtained, then the results are different from those obtained subdividing at the shoulder point with Algorithm 3, the control polygon of the curves $\overline{c}_0^1(u)$ and $\overline{c}_1^1(u)$. In other words, the sequence of control polygons generated by the *classic subdivision scheme* and the *shoulder subdivision scheme* are different, as shown Figure 1. This observation is not significant in the planar case, but it is important when we work with curves on a surface (see section 3).

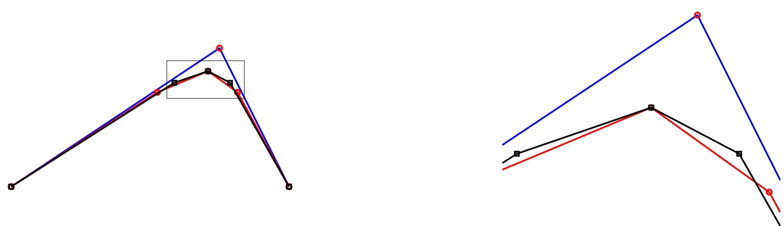

Figure 1: Control polygon after two subdivision steps. Left: the polygon with red circles corresponds to the shoulder point scheme, the polygon with black squares corresponds to the classic scheme. Right: zoom of the rectangular region.

Applying recursively the *shoulder point subdivision*, we obtain the following subdivision scheme that generates in the limit a *a piecewise conic curve*.
Given a sequence of points on the plane

$$P^0 = \{P_0^0, P_1^0, P_2^0, ..., P_{2n-1}^0, P_{2n}^0\}$$

and a local tension parameter $\omega_{2i}^0 > 0$ associated to the subsequence $P_{2i}^0, P_{2i+1}^0, P_{2i+2}^0$, $i = 0, 2, ..., 2n-2$, the subdivision rule is based on the recurrences (2) and (3). More precisely, for the $P_i^0, P_{i+1}^0, P_{i+2}^0$, with i even and $w_i^0 > 0$, it is given by (see Figure 2),

Shoulder point conic subdivision

$$P_{2i}^{j+1} = P_i^j \quad (4)$$
$$P_{2i+1}^{j+1} = (1-\gamma_{2i}^{j+1})\, P_i^j + \gamma_{2i}^{j+1}\, P_{i+1}^j \quad (5)$$
$$P_{2i+3}^{j+1} = \gamma_{2i}^{j+1}\, P_{i+1}^j + (1-\gamma_{2i}^{j+1})\, P_{i+2}^j \quad (6)$$
$$P_{2i+2}^{j+1} = \frac{1}{2}(P_{2i+1}^{j+1} + P_{2i+3}^{j+1}) \quad (7)$$

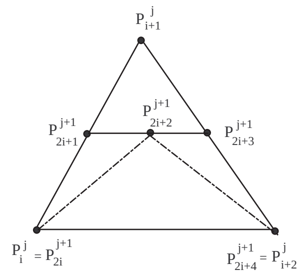

Figure 2: Control polygons of two consecutive steps

where the tension parameters of the step $j+1$ are computed as follows,

$$\omega_{2i}^{j+1} = \omega_{2i+2}^{j+1} = \sqrt{\frac{1+\omega_i^j}{2}} \tag{8}$$

and

$$\gamma_{2i}^{j+1} = \gamma_{2i+2}^{j+1} = \frac{\omega_{2i}^{j+1}}{1+\omega_{2i}^{j+1}} \tag{9}$$

Observe that P_{2i+1}^{j+1} and P_{2i+3}^{j+1} play the role of the Farin points for the subsequence $P_i^j, P_{i+1}^j, P_{i+2}^j$. Moreover, if the points of the subsequences $P_{2i-1}^0, P_{2i}^0, P_{2i+1}^0, i = 1, ..., n-1$ are collinear, then the subdivision curve is a G^1-continuous conic Bezier spline.

3 The conic subdivision scheme on surfaces

In this section we introduce *geodesic conic curves on surfaces* as the limit of a subdivision scheme, which can be considered as a natural generalization of the *shoulder point conic subdivision scheme* (4)-(7). Observe that the shoulder point scheme it is also well defined if the points of the initial polygon P^0 are in $\mathbb{R}^3$. Nevertheless, if they are on a surface S and we apply directly the shoulder point subdivision, the new points P^1 are not necessarily on S. A way of solving this problem is substituting straight lines in affine space by geodesic lines on the surface.

3.1 Definition of the scheme

Assume that S is an smooth surface and Q_0, Q_1 two points in S. We denote by $c_g(Q_0, Q_1)$ the shortest geodesic curve with initial point Q_0 and final point Q_1 and denote by $d_g(Q_0, Q_1)$ the arc-length of $c_g(Q_0, Q_1)$.

Definition 1 *Geodesic polygon.*
The geodesic polygon with vertices $Q_0, Q_1, ..., Q_n$ on a surface S is the piecewise curve composed by the geodesic shortest curves $c_g(Q_i, Q_{i+1}),\ i = 0, ..., n-1$.

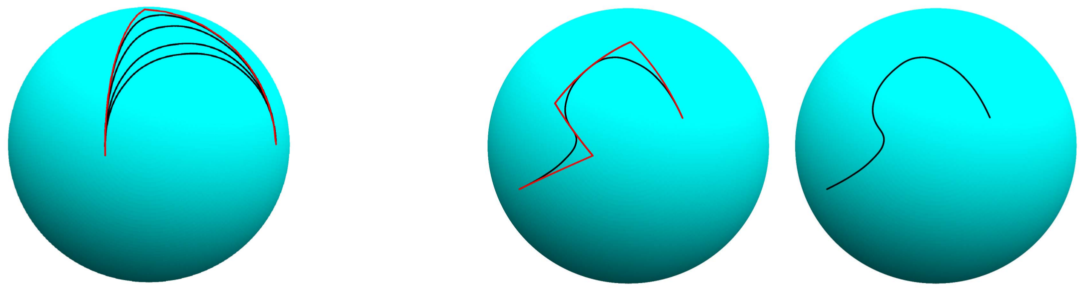

Figure 3: Left: 3 points of on a sphere, the control polygon and the geodesic subdivision conic curve after 10 steps using $w^0 = \{0.75, 1, 2, 5\}$. Middle: The control polygon and the conic geodesic spline composed by 3 segments computed by 10 geodesic subdivision steps using $w_i^0 = 1$ for $i = 1, 2, 3$. Right: the conic geodesic spline on the sphere.

Let

$$P^0 = \{P_0^0, P_1^0, P_2^0, ..., P_{2n-1}^0, P_{2n}^0\} \tag{10}$$

be a sequence of points on a surface S and denote by $\omega_i^0 > 0$ a local tension parameter associated to the subsequence $P_i^0, P_{i+1}^0, P_{i+2}^0,\ i = 0, 2, ..., 2n-2$. Moreover for $0 \leq t \leq 1$, denote by

$$(1-t)Q_0 \oplus tQ_1$$

the point $R \in c_g(Q_0, Q_1)$, such that

$$d_g(Q_0, R) = t\ d_g(Q_0, Q_1)$$

The *geodesic conic subdivision scheme* on the surface S is defined as follows.

Geodesic conic subdivision

$$\begin{aligned}
P_{2i}^{j+1} &= P_i^j && (11)\\
P_{2i+1}^{j+1} &= (1-\gamma_{2i}^{j+1})P_i^j \oplus \gamma_{2i}^{j+1} P_{i+1}^j && (12)\\
P_{2i+3}^{j+1} &= \gamma_{2i}^{j+1} P_{i+1}^j \oplus (1-\gamma_{2i}^{j+1})P_{i+2}^j && (13)\\
P_{2i+2}^{j+1} &= \frac{1}{2}P_{2i+1}^{j+1} \oplus \frac{1}{2}P_{2i+3}^{j+1} && (14)
\end{aligned}$$

where the parameters $\omega_{2i}^{j+1}, \gamma_{2i}^{j+1}$ are computed using the recurrences (8) and (9) respectively.

Given an affine invariant linear scheme M expressed in term of averages, the *geodesic analogue* of M is defined in [WD05] as the subdivision scheme obtained replacing the linear interpolation operator $a_t(Q_0, Q_1) := (1-t)Q_0 + tQ_1$ by the geodesic interpolation operator $ga_t(Q_0, Q_1) := (1-t)Q_0 \oplus tQ_1$. It is clear from the previous definition that the *geodesic conic subdivision scheme* (11)-(14) is the geodesic analogue of the shoulder point conic subdivision scheme (4)-(7).

Remark
The geodesic analogue of the classic conic scheme depends on the subdivision parameter t. Since geodesics curves are strongly dependent on the geometry of the surface, the limit curve generated by the geodesic analogue of the classic conic scheme is different for each value of t. Defining the geodesic conic subdivision scheme as the geodesic analogue of the shoulder point scheme has the advantage that we remove the dependence on the parameter t. Moreover, if the curvature of the surface in the region containing the control polygon P^0 does not vary very much, then the arc length of the geodesic curves that we have to compute using the the geodesic analogue of the shoulder point conic subdivision scheme is in general smaller than the arc length of the geodesic curves necessary for the geodesic analogue of the classic conic scheme. This is an important issue to take into account since the computational cost of computing geodesic curves increases with the arc-length of the geodesic curves.

3.2 Convergence analysis

Without loss of generality we restrict the analysis of the convergence to a subpolygon $P_i^0, P_{i+1}^0, P_{i+2}^0$, $i = 0, 2, ..., 2n-2$, of the initial polygon (10). To prove the convergence of the *geodesic conic subdivision scheme* we will use the strategy introduced in [WD05]. According to the results in [WD05], if T is a geodesic scheme analogue to an affine invariant linear scheme M, to prove the convergence of T and the continuity of its limit curve it is enough to show that M is a $0-admisible$.

Definition 2 0-*admisible scheme [WD05]*
A linear subdivision scheme M is $0-admisible$, if it is affinely invariant and fulfills the following convergence condition with a factor $\mu_0 < 1$

$$d(M^j P^0) \leq (\mu_0)^j d(P^0) \tag{15}$$

where $d(p)$ is defined for a vector $p = (p_i)$ as

$$d(p) = \max_i \|\Delta p_i\|$$

with $\Delta p_i = p_{i+1} - p_i$.

Since our geodesic conic subdivision scheme is the geodesic analogue of the shoulder point subdivision scheme, which is linear and invariant by affine transformations, to prove the convergence of the scheme (11)-(14) and the continuity of its limit curve, it is sufficient to show that condition (15) holds for the scheme (4)-(7). In Lemma 1 we show that Euclidean distance between two consecutive points in the polygon of the step $j+1$ is strongly related with the Euclidean distance between two consecutive points in

the polygon of the previous step. This relation is used in Proposition 1 to prove that the scheme (4)-(7) satisfies a condition like (15).
Denote by $P^j = \{P^j_{2^j i}, ..., P^j_{2^j(i+2)}\}$ the set of points on the surface S obtained applying j-times the *shoulder point conic subdivision algorithm* (4)-(7) to the subpolygon $P^0_i, P^0_{i+1}, P^0_{i+2}$, with i even. Let $\Delta^j_r = P^j_{r+1} - P^j_r, r = 2^j i, ..., 2^j(i+2) - 1$ be the difference of two consecutive points on P^j and denote by $d^j_r = \|\Delta^j_r\|$ the Euclidean distance between P^j_r and P^j_{r+1}. Denote by d^j the maximum distance between two consecutive points in P^j

$$d^j = \max_{r \in I^j_i} d^j_r$$

with

$$I^j_i = [2^j i \ , \ 2^j(i+2) - 1] \tag{16}$$

Lemma 1 *The Euclidean distance between two consecutive points of the polygons P^j and P^{j+1} generated by the shoulder point subdivision scheme (4)-(7) are related by*

$$\begin{aligned} d^{j+1}_{2i} &= \gamma^{j+1}_{2i} \, d^j_i && (17)\\ d^{j+1}_{2i+1} = d^{j+1}_{2i+2} &\leq \Big(\frac{1-\gamma^{j+1}_{2i}}{2}\Big)(d^j_i + d^j_{i+1}) && (18)\\ d^{j+1}_{2i+3} &= \gamma^{j+1}_{2i} \, d^j_{i+1} && (19) \end{aligned}$$

Proof
The inequalities (17) and (19) hold immediately from the subdivision rules (5) and (6) respectively. Applying the triangle inequality to the triangle with vertices $P^{j+1}_{2i+1}, P^j_{i+1}, P^{j+1}_{2i+3}$ (see Figure 2) and recalling that P^{j+1}_{2i+2} is the midpoint of the segment $P^{j+1}_{2i+1}, P^{j+1}_{2i+3}$ we obtain,

$$\begin{aligned} 2d^{j+1}_{2i+1} = 2\|\Delta^{j+1}_{2i+1}\| &\leq \|P^{j+1}_{2i+1} - P^j_{i+1}\| + \|P^j_{i+1} - P^{j+1}_{2i+3}\| \\ &= (\|\Delta^j_i\| - \|\Delta^{j+1}_{2i}\|) + (\|\Delta^j_{i+1}\| - \|\Delta^{j+1}_{2i+3}\|) \\ &= (d^j_i - d^{j+1}_{2i}) + (d^j_{i+1} - d^{j+1}_{2i+3}) \\ &= (d^j_i - \gamma^{j+1}_{2i} d^j_i) + (d^j_{i+1} - \gamma^{j+1}_{2i} d^j_{i+1}) \\ &= (1 - \gamma^{j+1}_{2i})(d^j_i + d^j_{i+1}) \end{aligned}$$

■

Proposition 1 *Applying j-times the subdivision rules (4)-(7) of the shoulder point conic subdivision scheme to the initial polygon $P^0_i, P^0_{i+1}, P^0_{i+2}$, with local tension parameters $\omega^0_i > 0$, it holds that there exists $\mu_0 \in (0, 1)$ such that*

$$d^{j+1} \leq (\mu_0)^j d^0 \tag{20}$$

Proof
Let us denote $\max\{\gamma_{2i}^{j+1}, 1-\gamma_{2i}^{j+1}\}$ by α_{2i}^{j+1}. Since $\omega_i^0 > 0$, we have $0 < \gamma_{2i}^{j+1} < 1$ and this implies $0 < \alpha_{2i}^{j+1} < 1$ for $j \geq 0$.
Using the recurrence (8) and the expression (9) for γ_{2i}^{j+1} it not difficult to check that, if $\omega_i^0 \geq 1$, then holds

$$\begin{aligned} 1 \leq \omega_{2i}^{j+1} = \omega_{2i+2}^{j+1} &\leq \omega_i^j \\ 0 < \gamma_{2i}^{j+1} = \gamma_{2i+2}^{j+1} &\leq \gamma_i^j < 1 \\ \gamma_{2i}^{j+1} &= \alpha_{2i}^{j+1} < 1 \end{aligned} \tag{21}$$

and if $0 < \omega_i^0 \leq 1$, then holds

$$\begin{aligned} \omega_i^j &\leq \omega_{2i}^{j+1} = \omega_{2i+2}^{j+1} \leq 1 \\ 0 < 1-\gamma_{2i}^{j+1} = 1-\gamma_{2i+2}^{j+1} &\leq 1-\gamma_i^j < 1 \\ 1-\gamma_{2i}^{j+1} &= \alpha_{2i}^{j+1} < 1 \end{aligned} \tag{22}$$

Thus, for $\omega_i^0 \geq 0$, from (21)-(22), we get

$$0 < \alpha_{2i}^{j+1} \leq \alpha_i^j < 1 \tag{23}$$

For any $j \geq 0$, using the relations (17),(18), (19) and (23) with I_i^j given by (16), we obtain,

$$\begin{aligned} d^{j+1} = \max_{r \in I_i^{j+1}} \{d_r^{j+1}\} &\leq \alpha_{2^{j+1}i}^{j+1} \max_{r \in I_i^j} \{d_r^j\} \\ &\leq \alpha_{2^{j+1}i}^{j+1} \alpha_{2^j i}^j \max_{r \in I_i^{j-1}} \{d_r^{j-1}\} \\ &\dots \\ &\leq \alpha_{2^{j+1}i}^{j+1} \alpha_{2^j i}^j \dots \alpha_{2i}^1 \max\{d_i^0, d_{i+1}^0\} \\ &\leq (\alpha_{2i}^1)^{j+1} \max\{d_i^0, d_{i+1}^0\} \\ &= (\alpha_{2i}^1)^j d^0 \end{aligned} \tag{24}$$

Thus condition (20) holds with $\mu_0 = \alpha_{2i}^1 < 1$. ∎

Theorem 1 *The geodesic conic subdivision scheme (11)-(14) applied to the initial polygon $P^0 = \{P_i^0, i = 0, ..., 2n\}$ with vertices on a surface S and local tension parameters $\omega_i^0 > 0$ converges to a continuous limit curve for sufficiently small $d(P^0)$.*

Proof
The geodesic conic subdivision scheme is the geodesic analogue of the shoulder point conic subdivision scheme. Moreover, the invariance by affine transformations and the inequality (24) means that shoulder point scheme is $0-$ *admisible* (and therefore it converges to a continuous curve [Dyn92]). Hence, the geodesic conic subdivision scheme also converges to a continuous curve for polygons P^0 such that $d(P^0)$ sufficiently small, see Theorem 7 in [WD05].

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4 The subdivision scheme on triangulated surfaces

Geodesic Bezier polynomial curves on *triangulated surfaces* were introduced in [MCV07] by means of a subdivision algorithm which is the *geodesic analogue* of the classical de Casteljau algorithm. More precisely, for a value of $t \in [0,1]$ previously selected and a control polygon $P^0 = \{P_0^0, P_1^0, ..., P_n^0\}$ with vertices in a triangulated surfaces S, the geodesic Bezier curve of degree n is defined in [MCV07] as the limit curve of the classic Bezier subdivision applied to P^0, substituting linear interpolation by geodesic interpolation. Since geodesic curves depend on the geometry of the surface, changing the subdivision parameter t may lead to a different curve. In [MCV07] authors select a mid-point subdivision scheme, i.e in the step j the Bezier control polygons for the intervals $[\frac{i}{2^j}, \frac{i+1}{2^j}],\ i = 0, ..., 2^j - 1$ are computed.
In this section we use a similar approach to compute *geodesic conic curves* on triangulated surfaces, extending the method proposed in the previous section for a smooth surface to a triangulated surface. As we previously saw, unlike the geodesic Bezier polynomial curves, the geodesic conic curves don't depend on the parameter t, since the subdivision algorithm (11)-(14) is the geodesic analogue of the shoulder point subdivision scheme (4)-(7).

4.1 Discrete geodesic curves

The key for the implementation of the geodesic conic subdivision algorithm when S is a triangulated surface is to compute geodesic curves on S. Due to the increasing development of discrete surface models different definitions of geodesic curves on polyhedral surfaces have been introduced. Such curves are called *discrete geodesics* and we are particulary interested in *shortest geodesic curves* passing through two prescribed points. The problem of computing shortest geodesic curves on meshes have been extensively treated, see for instance [KS98] and reference therein. We implemented the geodesic conic subdivision scheme (11)-(14) using the method proposed in [MCV05] to compute shortest geodesic curves passing through two prescribed points. This method is an iterative algorithm that performs the geodesic computation in two steps. The first step uses the Fast Marching Method to compute an initial approximation to the shortest geodesic. The initial approximation is a polygonal curve with nodes on the edges or vertices of the triangulation. In the second step, the position of the node with the largest error is corrected and the error at neighboring nodes is updated. The process is repeated until a small error is obtained. The error at a node is computed taking into account the discrete geodesic curvature, see [MCV08]. The position of a node on the initial approximation is corrected by unfolding a subset of faces adjacent to it.

4.2 Convex hull property

The following definitions were introduced in [MCV07].

Definition 3 *Convex set in a triangulated surface.*
Let M be a connected subset of a triangulated surface S. We say that M is convex *if its boundary ∂M can be parametrized by a closed curve $\alpha(t)$, such that the discrete geodesic curvature of $\alpha(t)$ does not change of sign and the interior of M is always situated in the same side of $\alpha(t)$.*

Definition 4 *Convex hull.*
The convex hull $\overline{M}$ of $M \subset S$ is the intersection of all convex sets of S containing M.

Moreover, in [MCV07] it is shown that the intersection of two convex sets is a collection of convex sets. Using the previous definitions and results it is easy to see that the i-th section of the curve generated by the *geodesic conic subdivision scheme* (11)-(14) is contained in the convex hull of the corresponding section $P_i^0, P_{i+1}^0, P_{i+2}^0$, i even, of the initial geodesic control polygon P^0. More precisely, it is contained in the convex hull of the geodesic polygon with vertices $P_{2i}^1, P_{2i+1}^1, P_{2i+2}^1, P_{2i+3}^1, P_{2i+4}^1$. Consequently, *each section of the subdivision curve corresponding to an edge of the initial geodesic control polygon has the convex hull property.*
Furthermore, the convex hull of the geodesic polygon with vertices $P_r^j, P_{r+1}^j, P_{r+2}^j$ for $r = 2^j i, 2^j i + 2, ..., 2^j(i + 2) - 2$ obtained applying j-times the subdivision scheme (11)-(14) to the control point $P_i^0, P_{i+1}^0, P_{i+2}^0$, contains the convex hull of the geodesic polygons with vertices $P_{2r}^{j+1}, P_{2r+1}^{j+1}, P_{2r+2}^{j+1}$ and $P_{2r+2}^{j+1}, P_{2r+3}^{j+1}, P_{2r+4}^{j+1}$ for $r = 2^j i, 2^j i+2, ..., 2^j(i+2)-2$, obtained applying $(j + 1)$-times the same subdivision rule to the points $P_i^0, P_{i+1}^0, P_{i+2}^0$. Therefore, the geodesic discrete curvature of the section of the subdivision curve with end points P_i^0, P_{i+2}^0 doesn't change its sign. The last observation means that any point on the conic subdivision curve where the geodesic curvature changes of sign has to be a vertex of even index of the initial polygon.

4.3 User interaction and results

The geodesic conic subdivision scheme is very useful to design curves on a surface. In this section we describe how to perform the interaction with the user in an intuitive and friendly way. To obtain an smooth conic spline curve the points $P_{2i-1}^0, P_{2i}^0, P_{2i+1}^0$, $i = 0, 1, ..., n-1$ of the initial control polygon have to lie on the same geodesic curve. Since this kind of "collinearity" is not natural for the user, we introduce a simple preprocessing step. Denote by $Q_0, Q_1, ..., Q_n$ the points selected by the user on the surface S. Then, we construct the geodesic control polygon P^0 as follows,

$$\begin{aligned} P_0^0 &= Q_0 \\ P_{2i-1}^0 &= Q_i, \; i = 1, ...n-1 \\ P_{2i}^0 &= (1-\beta_i)Q_i \oplus \beta_i Q_{i+1}, \; i = 1, ...n-2 \\ P_{2n-2}^0 &= Q_n \end{aligned}$$

where $0 < \beta_i < 1$. In other words, the vertices P_{2i}^0 are on the geodesic curve passing through P_{2i-1}^0 and P_{2i+1}^0 for $i = 1, ..., n-1$. Initially we set $\beta_i = 0.5$ for $i = 1, ..., n-2$ and

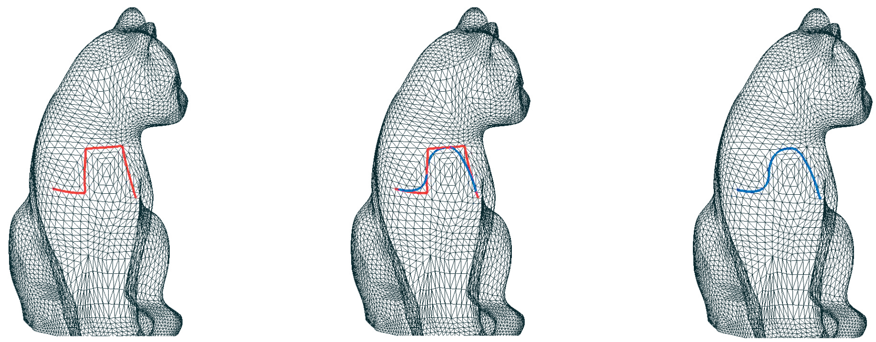

Figure 4: Left: Initial control polygon on a triangulated surface, middle: the control polygon and the geodesic conic subdivision curve after 3 steps, right: the subdivision spline curve composed by 3 segments with all weights equal to 1. method (11)-(14)

also $w_{2i}^0 = 1$ for each segment with control polygon $P_{2i}^0, P_{2i+1}^0, P_{2i+2}^0,\ i = 0, 1, ..., n-2$. We apply the geodesic conic subdivision rules (11)-(14) and stop at some prescribed level of subdivision or when control polygons can be considered as geodesic segments. In terms of the algorithm proposed in [MCV08] the last condition means that all the control vertices have an error smaller than a prescribed tolerance.
Figure 4 and Figure 5 show the performance of the geodesic conic subdivision scheme on a triangulated surface and the advantages of using this kind of curves:

- **local control**: changing the position of any vertex of the control polygon only affects at most two segments of the geodesic conic spline. This local control can not be obtained with geodesic Bezier polynomial curves.

- **geometric handles**: the weight $w_i^0 > 0$ is a geometric handle that allows to control the geometry of the section of the spline with control polygon $P_i^0, P_{i+1}^0, P_{i+2}^0$. A value of w_i^0 close to 0 generates a conic subdivision segment close to the curve $c_g(P_i^0, P_{i+2}^0)$. On the other hand, a large value of $w_i^0 > 0$ produces a subdivision segment close to the geodesic polygon with vertices $P_i^0, P_{i+1}^0, P_{i+2}^0$.

5 Conclusions

A new subdivision scheme for designing curves on surfaces has been proposed. The limit curve of this scheme is a continuous curve that can be considered as a natural generalization of conic Bezier curves. The scheme depends on free parameters that are very useful to control the shape of the subdivision curve, which also enjoys the convex hull property. These geometric handles make the curves generated for the proposed scheme a suitable tool for designing, editing and trimming on surfaces. We are currently working in the proof of the G^1 continuity of the subdivision curve.

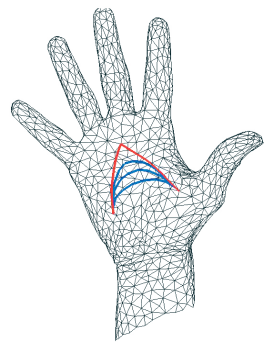

Figure 5: Initial polygon on a triangulated surface and the geodesic conic subdivision curves obtained with three values of the weight, $w_0^0 = 0.5, 1, 4$.

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