THE GENERIC RANK OF THE BAUM-BOTT MAP FOR FOLIATIONS OF THE PROJECTIVE PLANE

A. LINS NETO AND J. V. PEREIRA

Abstract. Our main result says that the generic rank of the Baum-Bott Map for foliations of degreee d, $d \geq 2$, of the projective plane is $d^2 + d$. This answers a question of Gomez-Mont and Luengo and shows that are no other universal relation between the Baum-Bott indexes of a foliation of $\mathbb{P}^2$ besides the Baum-Bott formula. We also define the *Camacho-Sad Field* for foliations on surfaces and prove its invariance under meromorphic maps. In an appendix we show that the monodromy of the singular set of the universal foliation with *very ample* cotangent bundle is the full symmetric group

1. Introduction and Statement of Results

1.1. **The Baum-Bott Map.** One of the most basic invariant for singularities of holomorphic foliations of surfaces is the **Baum-Bott index**: if $\mathcal{F}$ is a germ of holomorphic foliation of $(\mathbb{C}^2, 0)$ induced by a holomorphic 1-form $\omega = A(x,y)dy - B(x,y)dx$ with an isolated singularity at 0 then the Baum-Bott index of $\mathcal{F}$ at 0 is defined as

$$BB(\mathcal{F}, 0) = \frac{1}{(2\pi i)^2} \int_{\Gamma} \eta \wedge d\eta$$

where η is any $(1,0)$-form (C^∞ on a punctured neighborhood of $0 \in \mathbb{C}^2$) satisfying $d\omega = \eta \wedge \omega$ and Γ is the boundary of a small ball around 0 (see for instance [3]). When the dual vector field $X = A(x,y)\partial_x + B(x,y)\partial_y$ has invertible linear part, i.e., $\det(DX(0)) \neq 0$, a simple computation shows that

$$BB(\mathcal{F}, 0) = \frac{\operatorname{tr}^2(DX(0))}{\det(DX(0))}.$$

Singularities with invertible linear part are usually called **simple singularities**.

Let S be a compact complex surface S. A singular foliation by curves $\mathcal{F}$ on S can be defined by a global holomorphic section of $TS \otimes \mathcal{L}$, for a suitable line bundle $\mathcal{L}$. This line bundle $\mathcal{L}$ is the cotangent bundle of $\mathcal{F}$ and is usually denoted by $T^*_{\mathcal{F}}$. We will denote by $\mathbb{F}\mathrm{ol}(\mathcal{L})$ the space of foliations on S with cotangent bundle $\mathcal{L}$, i.e.,

$$\mathbb{F}\mathrm{ol}(\mathcal{L}) = \mathbb{P}\mathrm{H}^0(S, TS \otimes \mathcal{L}).$$

For any $\mathcal{F} \in \mathbb{F}\mathrm{ol}(L)$ with isolated singularities $\mathrm{sing}(\mathcal{F})$, the **singular set** of $\mathcal{F}$, contains $N(\mathcal{L}) = c_2(TS \otimes \mathcal{L})$ singularities counted with multiplicities.

When there exists a foliation $\mathcal{F}_0 \in \mathbb{F}\mathrm{ol}(\mathcal{L})$ with only simple singularities then the set $U \subset \mathbb{F}\mathrm{ol}(\mathcal{L})$, of foliations with only simple singularities is an open Zariski set. In this case any foliation $\mathcal{F} \in \mathbb{F}\mathrm{ol}(\mathcal{L})$ has exactly $N(\mathcal{L}) = N$ singularities. If $\mathrm{sing}(\mathcal{F}_0) = \{p_1, \ldots, p_N\}$, then there exist a neighborhood $V \subset U$ and holomorphic maps $\gamma_1, \ldots, \gamma_N \colon V \to S$ such that $\gamma_j(\mathcal{F}_0) = p_j$ and, for any $\mathcal{F} \in V$, we have

$\text{sing}(\mathcal{F}) = \{\gamma_1(\mathcal{F}), \ldots, \gamma_N(\mathcal{F})\}$. In this case, we can define a holomorphic map $BB \colon V \to \mathbb{C}^N$ by

$$BB(\mathcal{F}) = (BB(\mathcal{F}, \gamma_1(\mathcal{F})), \ldots, BB(\mathcal{F}, \gamma_N(\mathcal{F}))).$$

We will call the map BB, the **local Baum-Bott** map. We observe that it is possible to extend the domain of BB to U, if we symetrize the coordinates in $\mathbb{C}^N$. More precisely, if we denote by $\mathbb{C}^N/S_N$ the quotient of $\mathbb{C}^N$ by the equivalence relation which identifies two points $(z_1, \ldots, z_N)$ and $(z_{\sigma(1)}, \ldots, z_{\sigma(N)})$, where $\sigma \in S_N$(the symmetric group in N elements), then we define $\mathbb{BB} \colon U \to \mathbb{C}^N/S_N$ by

$$\mathbb{BB}(\mathcal{F}) = [BB(\mathcal{F}, p_1), \ldots, BB(\mathcal{F}, p_N)],$$

where $\text{sing}(\mathcal{F}) = \{p_1, \ldots, p_N\}$ and $[\lambda_1, \ldots, \lambda_N]$ denotes the class of $(\lambda_1, \ldots, \lambda_N)$ in $\mathbb{C}^N/S_N$. Of course, this map can be extended to a rational map

$$\mathbb{BB} \colon \mathbb{F}\text{ol}(\mathcal{L}) \dashrightarrow (\mathbb{P}^1)^N/S_N \cong \mathbb{P}^N$$

which we will call the **global Baum-Bott map**.

The well-known Baum-Bott Index Theorem [2] (first proved by Chern [5] in the case of foliations with only simple singularities) says that for a foliation $\mathcal{F}$ with isolated singularities of compact surface S,

$$N_{\mathcal{F}} \cdot N_{\mathcal{F}} = \sum_{p \in \text{sing}(\mathcal{F})} BB(\mathcal{F}, p),$$

where $N_{\mathcal{F}}$ is the normal bundle of $\mathcal{F}$, i.e., $N_{\mathcal{F}} = T^*_{\mathcal{F}} \otimes KS^*$ with KS being the canonical bundle of S. In particular the maximal rank of $\mathbb{BB}$ on $\mathbb{F}\text{ol}(\mathcal{L})$ is always less than $N(\mathcal{L})$ and the Baum-Bott map is never dominant: the closure of its image has codimension at least one.

In this paper we are interested on the generic rank of the Baum-Bott map just defined for foliations of the projective plane. Of course the generic rank of the local and global Baum-Bott maps coincide. Recall that the degree of a foliation $\mathcal{F}$ of $\mathbb{P}^2$, denoted by $\deg(\mathcal{F})$, is defined as the number d of tangencies of a generic line with $\mathcal{F}$ and that $\mathcal{F}$ has $N(d) := N(T^*_{\mathcal{F}}) = d^2 + d + 1$ singularities counted with multiplicities.

For foliations of degree 0 of $\mathbb{P}^2$ we have just one singularity and its index is determined by Baum-Bott's Theorem. For foliations of degree 1 we have three singularities(counted with multiplicities) and every foliation admits an invariant line. Camacho-Sad index Theorem imposes an extra condition on the Baum-Bott indexes and thus the rank of the Baum-Bott map is one, see [6]. A natural problem, proposed by Gomez-Mont and Luengo in loc. cit., is the following:

Question 1. *When $d \geq 2$, are there other hidden relations between the Baum-Bott indexes of a degree d foliation of the projective plane? In other terms, what is the generic rank of the Baum-Bott map for foliations of projective plane?*

Our first result says that the only *universal* relation among the Baum-Bott indexes is Baum-Bott's formula.

Theorem 1. *If $d \geq 2$ then the maximal rank of the Baum-Bott map for degree d foliations of $\mathbb{P}^2$ is $N(d) - 1 = d^2 + d$.*

An immediate consequence of Theorem 1 is the following:

Corollary 1. *If $d \geq 2$ then the dimension of the generic fiber of the map $\mathbb{BB} \colon \mathbb{F}\text{ol}(d) \dashrightarrow \mathbb{P}^N$ is $3d + 2$.*

In fact one has just to remark that $\dim \mathbb{F}\text{ol}(d) = (d+1)(d+3) - 1$. We do not know if the generic fiber of the Baum-Bott map is irreducible or not.

1.2. **The rank at Jouanolou's Foliations.** In general it does not seems to be an easy problem to compute the rank of the Baum-Bott map at an specific foliation. For $\mathcal{J}_d$, the degree d Jouanolou foliation(cf. §3 for the definition), we are able to determine the rank: this is the content of our next result.

Theorem 2. *For any $d \geq 2$, the rank of the local Baum-Bott map at $\mathcal{J}_d$ is*

$$\frac{d^2 + 7d - 6}{2}.$$

In particular, if $d = 2, 3$ then $\mathrm{rk}(\mathrm{BB}, \mathcal{J}_d) = d^2 + d$ and if $d \geq 4$ then $\mathrm{rk}(\mathrm{BB}, \mathcal{J}_d) < d^2 + d$.

Note that at these points the rank of the global Baum-Bott map is strictly less then the rank of the local Baum-Bott map: since all the singularities of $\mathcal{J}_d$ have the same Baum-Bott indexes then $\mathrm{BB}(\mathcal{J}_d) \in (\mathbb{P}^1)^{N(d)}$ is on the critical set of the symmetrization

$$(\mathbb{P}^1)^{N(d)} \to \mathbb{P}^{N(d)}.$$

1.3. **The Camacho-Sad Field.** Another local index often considered in the theory of holomorphic foliations is the so called *Camacho-Sad index* of a foliation $\mathcal{F}$ with respect to a separatrix C through a singular point p. Suppose that the germ of $\mathcal{F}$ at $p \in C$ is represented by a germ of holomorphic 1-form ω and that $(f = 0)$ is a reduced equation of the germ of C at p. Then there exist germs $g, h \in \mathcal{O}_p$ and a germ of holomorphic 1-form η at p such that $g\omega = h \cdot df + f \cdot \eta$ and $g, h|_C \not\equiv 0$ (cf. [4], [10] and [3]). The Camacho-Sad index of $\mathcal{F}$ at p with respect to C, is defined as

$$\mathrm{CS}(\mathcal{F}, C, p) = Res_p\left(-\frac{\eta}{h}\right) = \frac{1}{2\pi i}\int_\gamma -\frac{\eta}{h},$$

where γ is a union of small circles positively oriented around p, one for each local irreducible branch of the germ of C at p.

If p is a reduced and simple singularity of $\mathcal{F}$, i.e., we have two distinct non-zero eigenvalues at p, say λ_1 and $\lambda_2 \neq 0$, such that $\lambda_1/\lambda_2 \notin \mathbb{Q}_+$, then it is known that $\mathcal{F}$ has exactly two local separatrices, say $\Sigma_j, j = 1, 2$, tangent to the eigenspace associated to λ_j. In this case, we have

$$\begin{array}{lcl} \mathrm{CS}(\mathcal{F}, \Sigma_1, p) & = & \lambda_2/\lambda_1, \\ \mathrm{CS}(\mathcal{F}, \Sigma_2, p) & = & \lambda_1/\lambda_2, \\ \mathrm{BB}(\mathcal{F}, p) & = & \mathrm{CS}(\mathcal{F}, \Sigma_1, p) + \mathrm{CS}(\mathcal{F}, \Sigma_2, p) + 2. \end{array} \tag{1}$$

If p is reduced and non-simple singularity, i.e., p is a saddle-node singularity then, in general, one has just one analytic local separatrix, which is tangent to the eigenspace of the non-zero eigenvalue. The Camacho-Sad index with respect to this separatrix is zero(cf. [3] or [4]). In the direction of the zero eigenvalue there is always an unique formal separatrix(which sometimes is convergent). This follows from the formal normal form of the saddle-node (cf. [11]): the foliation is formally equivalent to the one induced by

$$\omega = x^{k+1}dy - y(1 + \lambda \cdot x^k)dx,$$

where $k \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. When there exists an analytic separatrix tangent to the eigendirection of the eigenvalue zero, then its Camacho-Sad index is λ. Even if this separatrix is formal, it can be proved that the number λ is invariant by formal diffeomorphisms (cf. [11]). Therefore, we can define its Camacho-Sad index as λ.

On the other hand, Seidenberg's resolution theorem asserts that for any foliation $\mathcal{F}$ on a surface S there exists finite composition of pontual blow-ups, say $\pi\colon M \to S$, such that the foliation $\tilde{\mathcal{F}} := \Pi^*(\mathcal{F})$ (the strict transform) on M, has only reduced singularities. The foliation $\tilde{\mathcal{F}}$ is usually called a *resolution* of $\mathcal{F}$.

Definition 1. *Let $\mathcal{F}$ be a foliation on a compact surface S. We define its Camacho-Sad field, denoted by $\mathbb{K}(\mathcal{F})$, as follows:*

- **Reduced case.** *All singularities of $\mathcal{F}$ are either reduced or saddle-nodes. Let $\operatorname{sing}(\mathcal{F}) = \{p_1, \ldots, p_k\}$ and let Σ_j^i, $i = 1, 2$, be the two separatrices of $\mathcal{F}$ through p_j (formal or not), $j = 1, \ldots, k$. Then we define*
$$\mathbb{K}(\mathcal{F}) = \mathbb{Q}(\mathrm{CS}(\mathcal{F}, \Sigma_1^1, p_1), \mathrm{CS}(\mathcal{F}, \Sigma_1^2, p_1), \ldots, \mathrm{CS}(\mathcal{F}, \Sigma_k^2, p_k))$$
- **General case.** *We take any resolution $\tilde{\mathcal{F}}$ of $\mathcal{F}$ and define $\mathbb{K}(\mathcal{F}) = \mathbb{K}(\tilde{\mathcal{F}})$.*

We invite the reader to verify that the definition above does not depend on the choosen resolution using the following facts:

(1) There exists a minimal resolution, that is a resolution with the minimal number of blowing-ups.
(2) When we blow-up in a reduced and simple singularity with Camacho-Sad indexes with respect to the separatrixes λ and λ^{-1} then two new simple and reduced singularities appears and theirs Camacho-Sad indexes are $\lambda - 1$, $1/(\lambda - 1)$, $\lambda^{-1} - 1$ and $\lambda/(1 - \lambda)$.
(3) When we blow-up at a saddle node with Camacho-Sad indexes 0 and λ then two new singularities appears, one saddle-node with Camacho-Sad indexes 0 and $\lambda - 1$, and a simple singularity with both Camacho-Sad indexes equal to -1.

The next corollary is in fact a reformulation of Theorem 1 in terms of the concept just defined.

Corollary 2. *If $d \geq 2$ then there exists a dense subset $G(d) \subset \mathbb{F}\mathrm{ol}(d)$ such that for any $\mathcal{F} \in G(d)$ the transcendence degree of $\mathbb{K}(\mathcal{F})$ over $\mathbb{Q}$ is $d^2 + d$.*

Our main result concerning the Camacho-Sad field is the following

Theorem 3. *Let M and S be two complex compact and connected surfaces, $\mathcal{F}$ be a foliation on S and $\phi : M \dashrightarrow S$ be a meromorphic map. Suppose that ϕ has generic rank two. Then $\mathbb{K}(\phi^*(\mathcal{F})) = \mathbb{K}(\mathcal{F})$.*

One of our motivations to introduce the Camacho-Sad Field was to prove the

Corollary 3. *The generic foliation of degree $d \geq 2$ is not the pull-back of a foliation of smaller degree.*

1.4. **Monodromy.** In an appendix we prove that the monodromy of the singular set of a generic family of holomorphic foliations is the full symmetric group. An immediate corollary is that the functions $\gamma_1, \ldots, \gamma_N : V \subset \mathbb{F}\mathrm{ol}(d) \to \mathbb{P}^2$ used to parametrize the singularities in the proof of Theorem 1 although algebraic is not solvable by radicals when $d \geq 2$, i.e., it cannot be expressed in terms of combinations of radicals of rational functions in the $\mathbb{F}\mathrm{ol}(d)$.

Contents

2. The Generic Rank of Baum-Bott's Map

2.1. **Some words about the notation.** Let $\mathbb{F}\text{ol}(d)$ be the space of foliations of degree d on $\mathbb{P}^2$, $d \geq 0$. A foliation of degree d on $\mathbb{P}^2$, can be expressed in an affine coordinate system $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$, by a polynomial vector field on $\mathbb{C}^2$ of the form $X = P(x, y)\partial_x + Q(x, y)\partial_y$, where

$$\begin{cases} P(x,y) = p(x,y) + x \cdot g(x,y) \\ Q(x,y) = q(x,y) + y \cdot g(x,y) \end{cases} \tag{2}$$

with $\max(\deg(p), \deg(q)) \leq d$ and g is a homogeneous polynomial of degree d.

We will denote by $\mathbb{R}(d) \subset \mathbb{F}\text{ol}(d)$ the Zariski dense subset of foliations $\mathcal{F}$ of degree d with all singularities simple. If $\mathcal{F} \in \mathbb{F}\text{ol}(d)$ then $N\mathcal{F} = \mathcal{O}(d + 2)$. Thus the Baum-Bott Theorem mentioned on the introduction says that

$$\sum_{p \in \operatorname{sing}\mathcal{F}} BB(\mathcal{F}, p) = (d + 2)^2,$$

for every $\mathcal{F} \in \mathbb{F}\text{ol}(d)$ with isolated singularities. We recall that $\mathbb{R}(d)$ is open and dense in $\mathbb{F}\text{ol}(d)$, cf. for instance [10]. Recall that for any $\mathcal{F}_0 \in \mathbb{R}(d)$, $\#(\operatorname{sing}(\mathcal{F}_0)) = d^2+d+1$.

2.2. **The Key Lemma.** The proof of Theorem 1 will be by induction on $d \geq 2$. The result for $d = 2$ is due to A. Guillot (cf. [7]). Note that Theorem 2 contains, in particular, a new proof of Guillot's result. The induction step will be reduced to the following lemma:

Lemma 2.1. *Let $F = (G, H) : \mathbb{D}^* \times \mathbb{D}^{k-1} \times \mathbb{D}^\ell \to \mathbb{C}^k \times \mathbb{C}^\ell$ be a holomorphic map. Denote the variables in $\mathbb{D} \times \mathbb{D}^{k-1} \times \mathbb{D}^\ell$ by $(s, Z, T) = (s, z_1, \ldots, z_{k-1}, t_1, \ldots, t_\ell)$. Suppose that:*

(a). *H extends to a holomorphic function on $\mathbb{D} \times \mathbb{D}^{k-1} \times \mathbb{D}^\ell$ and*

$$\frac{\partial H}{\partial z_j}(0, Z, T) = 0, \forall j = 1, \ldots, k-1.$$

(b). *G is of the form:*

$$G(s, Z, T) = \frac{1}{s}[A(Z, T) + s \cdot R(s, X, T)],$$

where $A = (A_1, \ldots, A_k)\colon \mathbb{D}^{k-1} \times \mathbb{D}^\ell \to \mathbb{C}^k$ and $R\colon \mathbb{D}^k \times \mathbb{D}^\ell \to \mathbb{C}^k$ are holomorphic.

(c). *There exists $Z_0 \in \mathbb{D}^{k-1}$ satisfying:* $\det(M(Z_0,0)) \neq 0$, *where $M(Z,T)$ is the $k \times k$ matrix*

$$\begin{bmatrix} A(Z,T) \\ \frac{\partial A}{\partial z_1}(Z,T) \\ \vdots \\ \frac{\partial A}{\partial z_{k-1}}(Z,T) \end{bmatrix} := \begin{bmatrix} A_1(Z,T) & A_2(Z,T) & \vdots & A_k(Z,T) \\ \frac{\partial A_1}{\partial z_1}(Z,T) & \frac{\partial A_2}{\partial z_1}(Z,T) & \vdots & \frac{\partial A_k}{\partial z_1}(Z,T) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial A_1}{\partial z_{k-1}}(Z,T) & \frac{\partial A_2}{\partial z_{k-1}}(Z,T) & \vdots & \frac{\partial A_k}{\partial z_{k-1}}(Z,T) \end{bmatrix}$$

(d). *For $Z_0 \in \mathbb{D}^{k-1}$ we have that* $\mathrm{rk}(H_{Z_0},0) = \ell$, *where* $H_{Z_0}(T) = H(0,Z_0,T)$.

Then there exists $r > 0$ such that $\mathrm{rk}(F,(s_0,Z_0,0)) = k+\ell$ *for every s_0 with $0 < |s_0| < r$.*

Proof. Let $\Delta(s,X,T)$ be given by

$$\Delta(s,X,T) = \det \begin{bmatrix} \frac{\partial G}{\partial s} & \frac{\partial H}{\partial s} \\ \frac{\partial G}{\partial z_1} & \frac{\partial H}{\partial z_1} \\ \vdots & \vdots \\ \frac{\partial G}{\partial z_{k-1}} & \frac{\partial H}{\partial z_{k-1}} \\ \frac{\partial G}{\partial t_1} & \frac{\partial H}{\partial t_1} \\ \vdots & \vdots \\ \frac{\partial G}{\partial t_\ell} & \frac{\partial H}{\partial t_\ell} \end{bmatrix}.$$

Using (b), we get the following relations:

$$\begin{aligned} \frac{\partial G}{\partial s}(s,Z,T) &= -\frac{1}{s^2}A(Z,T) + C(s,Z,T), \\ \frac{\partial G}{\partial z_j}(s,Z,T) &= \frac{1}{s}\frac{\partial A}{\partial z_j}(Z,T) + D_j(s,X,T) \\ \frac{\partial G}{\partial t_i}(s,X,T) &= \frac{1}{s}\frac{\partial A}{\partial t_i}(Z,T) + E_i(s,Z,T), \end{aligned}$$

where $C = \partial R/\partial s$ and $D_j = \partial R/\partial x_j$.

These relations imply that

$$\Delta(s,Z,T) = \det \begin{bmatrix} -\frac{1}{s^2}A(Z,T) + C(s,Z,T) & \frac{\partial H}{\partial s} \\ \frac{1}{s}\frac{\partial A}{\partial z_1}(Z,T) + D_1(s,Z,T) & \frac{\partial H}{\partial z_1}(s,Z,T) \\ \vdots & \vdots \\ \frac{1}{s}\frac{\partial A}{\partial z_{k-1}}(Z,T) + D_{k-1}(s,X,T) & \frac{\partial H}{\partial z_{k-1}}(s,Z,T) \\ \frac{1}{s}\frac{\partial A}{\partial t_1}(Z,T) + E_1(s,Z,T) & \frac{\partial H}{\partial t_1}(s,Z,T) \\ \vdots & \vdots \\ \frac{1}{s}\frac{\partial A}{\partial t_\ell}(Z,T) + E_\ell(s,Z,T) & \frac{\partial H}{\partial t_\ell}(s,Z,T) \end{bmatrix} =$$

$$= \frac{1}{s^k} \det \begin{bmatrix} -\frac{1}{s}A(Z,T) + s\cdot C(s,Z,T) & \frac{\partial H}{\partial s} \\ \frac{\partial A}{\partial z_1}(Z,T) + s\cdot D_1(s,Z,T) & \frac{\partial H}{\partial z_1}(s,Z,T) \\ \vdots & \vdots \\ \frac{\partial A}{\partial z_{k-1}}(Z,T) + s\cdot D_{k-1}(s,X,T) & \frac{\partial H}{\partial z_{k-1}}(s,Z,T) \\ \frac{\partial A}{\partial t_1}(Z,T) + s\cdot E_1(s,Z,T) & \frac{\partial H}{\partial t_1}(s,Z,T) \\ \vdots & \vdots \\ \frac{\partial A}{\partial t_\ell}(Z,T) + s\cdot E_\ell(s,Z,T) & \frac{\partial H}{\partial t_\ell}(s,Z,T) \end{bmatrix} =$$

$$= \frac{1}{s^{k+1}} \det \begin{bmatrix} -A(Z,T) + s^2\cdot C(s,Z,T) & s\cdot\frac{\partial H}{\partial s} \\ \frac{\partial A}{\partial z_1}(Z,T) + s\cdot D_1(s,Z,T) & \frac{\partial H}{\partial z_1}(s,Z,T) \\ \vdots & \vdots \\ \frac{\partial A}{\partial z_{k-1}}(Z,T) + s\cdot D_{k-1}(s,X,T) & \frac{\partial H}{\partial z_{k-1}}(s,Z,T) \\ \frac{\partial A}{\partial t_1}(Z,T) + s\cdot E_1(s,Z,T) & \frac{\partial H}{\partial t_1}(s,Z,T) \\ \vdots & \vdots \\ \frac{\partial A}{\partial t_\ell}(Z,T) + s\cdot E_\ell(s,Z,T) & \frac{\partial H}{\partial t_\ell}(s,Z,T) \end{bmatrix}.$$

Hence, using **(a)**, we deduce that $\lim_{s\to 0} s^{k+1}\cdot\Delta(s,Z,T)$ is equal to

$$\det \begin{bmatrix} -A(Z,T) & 0 \\ \frac{\partial A}{\partial z_1}(Z,T) & 0 \\ \vdots & \vdots \\ \frac{\partial A}{\partial z_{k-1}}(Z,T) & 0 \\ \frac{\partial A}{\partial t_1}(Z,T) & \frac{\partial H}{\partial t_1}(0,Z,T) \\ \vdots & \vdots \\ \frac{\partial A}{\partial t_\ell}(Z,T) & \frac{\partial H}{\partial t_\ell}(0,Z,T) \end{bmatrix} = -\det(M(Z,T))\cdot\det\left(\frac{\partial H_i}{\partial t_j}(0,Z,T)\right).$$

In other words, if we set $\phi(s,Z,T) = -s^{k+1}\cdot\Delta(s,Z,T)$ then ϕ extends continuously to $s=0$ as

$$\phi(0,Z,T) = \det(M(Z,T))\cdot\det\left(\frac{\partial H_i}{\partial t_j}(0,Z,T)\right)_{1\le i,j\le\ell}.$$

It follows from **(c)** and **(d)** that $\phi(0,Z_0,0)\neq 0$. Thus there exists $r>0$ such that, if $0<|s|\le r$ then $\Delta(s,Z_0,0)\neq 0$. $\square$

Now we will work to construct a family of foliations with Baum-Bott map fitting in the above setup.

2.3. **Construction of the family.** Let us consider the following situation; let $\mathcal{F}_0 \in \mathbb{R}(d-1)$ be a foliation of degree $d-1 \geq 2$, L be a line on $\mathbb{P}^2$ and $E = (\mathbb{C}^2, (x,y))$ be an affine coordinate system in $\mathbb{P}^2$, such that:

(I). $\mathrm{rk}(BB, \mathcal{F}_0) = (d-1)^2 + d - 1 = d^2 - d := \ell$.

(II). $\mathrm{sing}(\mathcal{F}_0) \cap L = \emptyset$ and $\mathrm{sing}(\mathcal{F}_0) = \{q_1^0, \ldots, q_{\ell+1}^0\} \subset \mathbb{C}^2 \subset \mathbb{P}^2$.

(III). $\mathcal{F}_0$ is defined on E by the polynomial vector field

$$X_0 := P_0(x,y)\partial_x + Q_0(x,y)\partial_y,$$

where $P_0(x,y) = P^0(x,y) + x \cdot g(x,y)$, $Q_0(x,y) = Q^0(x,y) + y \cdot g(x,y)$, $\deg(P^0) = \deg(Q^0) = d-1$ and $g(x,y)$ is a homogeneous polynomial of degree $d-1$. We will assume that $g(x,0) \not\equiv 0$, i.e., the line at infinite of this affine coordinate system is not invariant for $\mathcal{F}_0$.

(IV). $L = (y = 0)$. In particular the polynomials $P(x) := P_0(x,0)$ and $Q(x) := Q_0(x,0)$ are relatively primes, that is $gcd(P(x), Q(x)) = 1$.

(V). $\deg(P(x)) = d$ and $\deg(Q(x)) = d-1$. This condition is generic and it implies that all tangencies of $\mathcal{F}_0$ with the line L are contained in $\mathbb{C}^2 \cap L$, because these tangencies are given by $(y = P(x) = 0)$.

Let V be a neighborhood of $\mathcal{F}_0$ in $\mathbb{R}(d-1)$ such that there exist holomorphic maps $q_1^0, \ldots, q_{\ell+1}^0 \colon V \to \mathbb{C}^2$ with $q_j^0(\mathcal{F}_0) = q_j^0$, $j = 1, \ldots, \ell+1$, and $\mathrm{sing}(\mathcal{F}) = \{q_1^0(\mathcal{F}), \ldots, q_{\ell+1}^0(\mathcal{F})\}$. We can take V sufficiently small in order to assure that that $q_j^0(\mathcal{F}) \cap (y=0) = \emptyset$ for all $j = 1, \ldots, \ell+1$ and all $\mathcal{F} \in V$.

Since, by hypothesis, $\mathrm{rk}(BB, \mathcal{F}_0) = d^2 - d = \ell$, there exist polynomials vector fields of the form (2), $X_1, \ldots, X_\ell$, $X_i = P_i\partial_x + Q_i\partial_y$, with the following additional properties:

(VI). For any $T = (t_1, \ldots, t_\ell) \in \mathbb{D}^\ell$ then $X_T := X_0 + \sum_{i=1}^{\ell} t_i \cdot X_i \in V$.

In this situation, we can define $H_1 \colon \mathbb{D}^\ell \to \mathbb{C}^\ell$, by

$$H_1(T) = (BB(X_T, q_1^0(X_T)), \ldots, BB(X_T, q_\ell^0(X_T))).$$

It follows from **(I)** that we can assume:

(VII). $\mathrm{rk}(H_1, 0) = d^2 - d = \ell$.

Next, we will see how to obtain foliations $\mathcal{F} \in \mathbb{R}(d)$ such that $\mathrm{rk}(BB, \mathcal{F}) = d^2 + d$. We will consider the vector field $y \cdot X_0$ as a foliation, say $\tilde{\mathcal{F}}_0$, of degree d, with a line of singularities.

Let $p(x), q(x) \in \mathbb{C}[x]$ be polynomials with the following properties:

(VIII). $p(x)$ in monic of degree $d+1$ and $q(x)$ has degre $\leq d$.

We will set $Z(x,y) = p(x)\partial_x + (q(x) + y \cdot x^d)\partial_y$. Note that this vector field defines an element in $\mathbb{F}\mathrm{ol}(d)$. Moreover, the space of such vector fields has dimension $2d$. Consider the family of foliations $(\mathcal{F}(s,Z,T))_{s,Z,T}$ of degree d on $\mathbb{P}^2$, which are defined on E by the polynomial vector field

$$X(s,Z,T) = y \cdot \left(X_0 + \sum_{i=1}^{\ell} t_i \cdot X_i \right) + s \cdot Z$$

Note that the components of $X(s,Z,T)$ are

$$\begin{cases} W_1 & := & y\left(P_0(x,y) + \sum_i t_i \cdot P_i(x,y)\right) + s \cdot p(x) \\ W_2 & := & y\left(Q_0(x,y) + \sum_i t_i \cdot Q_i(x,y) + s \cdot (q(x) + y \cdot x^d\right). \end{cases}$$

For $s \neq 0$ and Z, T fixed, the singularities of $\mathcal{F}(s, Z, T)$ are contained in the affine curve $\{F_{(Z,T)}(x, y) = 0\} \subset \mathbb{C}^2$, where $F_{(Z,T)}(x, y)$ is equal to

$$p(x) \cdot \left[Q_0(x, y) + \sum_i t_i \cdot Q_i(x, y) \right] - (q(x) + y \cdot x^d) \cdot \left[P_0(x, y) + \sum_i t_i \cdot P_i(x, y) \right].$$

Since P and Q are relatively prime we have the

Lemma 2.2. *Given a polynomial $f(x) \in \mathbb{C}[x]$ of degree $2d$ there exist unique polynomials $p(x), q(x) \in \mathbb{C}[x]$ such that*

$$\deg(p) = d + 1, \deg(q) \leq d - 2 and f(x) = p(x)Q(x) - q(x)P(x).$$

Proof. In fact, since $\gcd(P(x), Q(x)) = 1$, there exist $a(x), b(x) \in \mathbb{C}[x]$ such that

$$a(x) \cdot Q(x) - b(x) \cdot P(x) = 1 \implies (f \cdot a)(x) \cdot Q(x) - (f \cdot b)(x) \cdot P(x) = f(x).$$

Dividing $f \cdot b(x)$ by $Q(x)$ we get $f \cdot b = g \cdot Q + q$, where $\deg(q) \leq d - 2$. Thus

$$f = (f \cdot a - g \cdot P)Q - qP =: pQ - qP \implies p \cdot Q = f + q \cdot P.$$

Since $\deg(q \cdot P) = \deg(q) + \deg(P) \leq 2d - 1$, we have $\deg(f + q \cdot P) = 2d$. This implies that $2d = \deg(p \cdot Q) = \deg(p) + d - 1$, and so $\deg(p) = d + 1$. If we have another solution $p_1 \cdot Q - q_1 \cdot P = f$, with $\deg(p_1) = d + 1$ and $\deg(q_1) \leq d - 2$, then

$$(p - p_1)Q = (q - q_1)P \implies Q | q - q_1 \text{and} \deg(Q) > \deg(q - q_1),$$

which implies that $q = q_1 \text{and} p = p_1$. $\square$

Similar arguments also prove the:

Lemma 2.3. *Let $P_k = \{g \in \mathbb{C}[x] | \deg(g) \leq k\}$ and consider the linear map $\Phi \colon P_{d+1} \times P_{d-2} \to P_{2d}$ given by $\Phi(p, q) = p \cdot Q - q \cdot P$. Then Φ is an isomorphism.*

After setting $f_{(Z,T)}(x) = F_{(Z,T)}(x, 0)$ we can take Z_0 in such a way that

(IX). The polynomial $f_{(Z_0,0)}(x)$ has simple roots and has degree $2d$.

Let $(p(x), q(x)) \in P_{d+1} \times P_{d-2}$ be such that $p(x)$ is monic and $Z = p(x)\partial_x + (q(x) + y \cdot x^d)\partial_y$. Then, we can write, $p(x) = x^{d+1} + \sum_{j=0}^{d} z_{j+1} \cdot x^j$ and $q(x) = \sum_{j=0}^{d-2} z_{d+2+j} \cdot x^j$. Consider the space of vector fields Z as above, parametrized by $(z_1, \ldots, z_{2d}) \in \mathbb{C}^{2d}$. In what follows, we will use this parametrization and the notation $Z = (z_1, \ldots, z_{2d})$.

2.4. **Applying the Key Lemma I: First Properties.** Next we will describe how to apply lemma 2.1 to the family $(s, Z, T) \mapsto X(s, Z, T)$. The first step is the

Lemma 2.4. *Let $Z_0 = p_0(x)\partial_x + (q_0(x) + y \cdot x^d)\partial_y$ be such that* **(IX)** *is satisfied and let $\{x_1^0, \ldots, x_{2d}^0\}$ be the roots of $f_{(Z_0,0)}(x) = 0$. Then there exist neighborhoods $D = D(0, r)$ of $0 \in \mathbb{C}$, U of Z_0, D^ℓ of $0 \in \mathbb{C}^\ell$ and holomorphic functions*

$$\begin{aligned} q_i &: D \times U \times D^\ell \to \mathbb{C}^2, \quad i = 1, \ldots, d^2 - d + 1 = \ell + 1 \\ p_j &: D \times U \times D^\ell \to \mathbb{C}^2, \quad j = 1, \ldots, 2d, \end{aligned}$$

with the following properties:

- **(a).** *For any $(Z, T) \in U \times D^\ell$ the equation $f_{(Z,T)}(x) = 0$ has $2d$ simple roots, say $x_1(Z, T), \ldots, x_{2d}(Z, T)$, such that $x_i \colon U \times D^\ell \to \mathbb{C}$ is holomorphic and $x_i(Z_0, 0) = x_i^0$ for all $i = 1, \ldots, 2d$.*

(b). *$p_j(0,Z,T) = (x_j(Z,T),0)$ for every $j = 1,\dots,2d$ and for every $(Z,T) \in U \times D^\ell$.*

(c). *$q_i(0,0,T) = q_i^0(T)$ for all $T \in \mathbb{D}^\ell$ and all $i = 1,\dots,\ell+1$. In particular, $q_i(0,0,0) = q_i^0$ for all $i = 1,\dots,\ell+1$ and*

$$\operatorname{sing}(X_T) = \{q_1(0,0,T),\dots,q_{\ell+1}(0,0,T)\},$$

for all $T \in U$.

(d). *For $(s,Z,T) \in D \times U \times \mathbb{D}^\ell$, $s \neq 0$, we have that $\operatorname{sing}(\mathcal{F}(s,Z,T))$ is equal to $\{p_1(s,Z,T),\dots,p_{2d}(s,Z,T), q_1(s,Z,T),\dots,q_{\ell+1}(s,Z,T)\}$.*

(e). *If $H_i(s,Z,T)$ denotes the Baum-Bott index of $\mathcal{F}(s,Z,T)$ at the point $q_i(s,Z,T)$, $i = 1,\dots,\ell+1$, then*

$$\frac{\partial H_i}{\partial z_r}(0,Z,T) \equiv 0, \forall 1 \le i \le \ell+1 and 1 \le r \le 2d.$$

(f). *For every $(s,T) \in D \times \mathbb{D}^\ell$, with $s \neq 0$, then $p_j(s,Z,T)$ is a non-degenerate singularity of $\mathcal{F}(s,Z,T)$. Furthermore, if $G_j(s,Z,T)$ denotes the Baum-Bott index of $\mathcal{F}(s,Z,T)$ at the singularity $p_j(s,Z_0,T)$ then*

$$\lim s \cdot G_j(s,Z,T) = \frac{Q_T^2(x_j(Z,T),0)}{f'_{(Z,T)}(x_j(Z,T))} := A_j(Z,T). \tag{3}$$

Proof. The Lemma is a consequence of the implicit function theorem (IFT) applied in several cases. In part **(a)** we apply the IFT to the function

$$(x,Z,T) \in \mathbb{C} \times P_{d+1} \times P_{d-2} \times \mathbb{C}^d \mapsto f_{(Z,T)}(x) \in \mathbb{C}$$

at the points $(x_{i0}, Z_0, 0)$, $i = 1,\dots,2d$, where x_{i0}, $i = 1,\dots,2d$, are the roots of $f_{(Z_0,0)}(x) = 0$. We leave the details for the reader.

For the existence of the functions $q_1,\dots,q_{\ell+1}$, defined in a neighborhood of $(0,Z_0,0)$ in $\mathbb{C} \times P_{d+1} \times P_{d-2} \times \mathbb{C}^{\ell+1}$, we apply the IFT at the points $(x_i^0, y_i^0, 0, Z_0, 0)$ where $q_i^0 := (x_i^0, y_i^0) \in \mathbb{C}^2$, $1 \le i \le \ell+1$, are the singularities of $\mathcal{F}_0$, to the function $W(x,y,s,Z,T) = (W_1(x,y,s,Z,T), W_2(x,y,s,Z,T))$ defined as

$$\left(y(P_0(x,y) + \sum_i t_i P_i(x,y)) + sp(x),\ y(Q_0(x,y) + \sum_i t_i Q_i(x,y) + s(q(x) + yx^d) \right).$$

In order to prove that $\det(\partial W/\partial x, \partial W/\partial y)(x_i^0, y_i^0, 0, Z_0, 0) \neq 0$ just observe that $W(x,y,0,Z_0,0) = (y \cdot P_0(x,y), y \cdot Q_0(x,y))$, q_i^0 is a non-degenerate singularity of $\mathcal{F}_0$ and that $y_i^0 \neq 0$ (see **(II)**). We leave the details for the reader. Note that we can choose the neighborhood $V := D \times U \times D^\ell$ of $(0,Z_0,0)$ in such a way that $q_i(s,Z,T) \notin (y=0)$ for all $(s,Z,T) \in V$.

Let us prove **(e)**. Since $W_1(x,y,s,Z,T)$ and $W_2(x,y,s,Z,T)$ are the components of $X(s,Z,T)$, we have to compute $H_i(0,Z,T) = BB(X(0,Z,T), q_i(0,Z,T))$. Note that $W_1(x,y,0,Z,T) = y \cdot P_T(x,y)$ and $W_2(x,y,0,Z,T) = y \cdot Q_T(x,y)$. This implies that $q_i(0,Z,T) = q_i(0,0,T)$ and , since $q_i(0,Z,T) \notin (y=0)$ then

$$H_i(0,Z,T) = BB(y(P_T\partial_x + Q_T\partial_y), q_i(0,0,T)) = BB(P_T\partial_x + Q_T\partial_y, q_i(0,0,T)).$$

This proves **(e)**.

Let us prove the existence of the functions $p_1,\dots,p_{2d}$. As we have observed before, if $s \neq 0$ then $\operatorname{sing}(\mathcal{F}(s,Z,T)) \cap \mathbb{C}^2 \subset (F_{(Z,T)} = 0)$. Let $W = (W_1, W_2)$ be as above. If we set $P_T = P_0 + \sum_i t_i \cdot P_i$ and $Q_T = Q_0 + \sum_i t_i \cdot Q_i$, then we can write

$$W = (W_1, W_2) = (y \cdot P_T + s \cdot p(x), y \cdot Q_T + s \cdot (q(x) + y \cdot x^d)).$$

As the reader can check

$$(W = 0) = (W_1 = F_{(Z,T)} = 0) = (W_2 = F_{(Z,T)} = 0).$$

Therefore, we have to apply the IFT at the points $(x_{i0}, 0, 0, Z_0, 0)$ to one of the functions

$$(x, y, s, Z, T) \mapsto (W_j(x, y, s, Z, T), F_{(Z,T)}(x, y)) = \Phi_j(x, y, s, Z, T), j = 1 \text{or} j = 2.$$

Note that

$$\Phi_1(x, y, 0, Z, T) = (y \cdot P_T(x, y), F_{(Z,T)}(x, y)).$$

Therefore $\det(\partial\Phi_1/\partial x, \partial\Phi_1/\partial y)(x, 0, 0, Z_0, 0)$ is equal to

$$\det \begin{pmatrix} 0 & P_0(x, 0) \\ f'_{(Z_0,0)}(x) & * \end{pmatrix} = -P(x) \cdot f'_{(Z_0,0)}(x).$$

Similarly,

$$\det(\partial\Phi_2/\partial x, \partial\Phi_2/\partial y)(x, 0, 0, Z_0, 0) = -Q(x) \cdot f'_{(Z_0,0)}(x).$$

It follows from **(IV)** that, either $P(x_i^0) \neq 0$, or $Q(x_i^0) \neq 0$. Since $f_{(Z_0,0)}$ has simple roots, we can apply the IFT to obtain the function p_i.

Set $p_i(s, Z, T) = (x_i(s, Z, T), y_i(s, Z, T))$.

Assertion 2.1. *For every $i \in \{1, \ldots, 2d\}$ we have $y_i(s, Z, T) = s \cdot u_i(s, Z, T)$, where u_i is holomorphic and $F_{Z,T}(x_i(0, Z, T), 0) = f_{(Z,T)}(x_i(0, Z, T)) = 0$. In particular, $x_i(s, Z, T) = x_i(Z, T)$ (in the notation of* **(a)***). Moreover, if $P_0(x_i^0, 0) = P(x_i^0) \neq 0$ and we take the neighborhood V small then*

$$u_i(0, Z, T) = -\frac{p(x_i(Z, T))}{P_T(x_i(Z, T), 0)}. \tag{4}$$

Similarly, if $Q_0(x_{i0}, 0) \neq 0$ and we take V small then

$$u_i(0, Z, T) = -\frac{q(x_i(Z, T))}{Q_T(x_i(Z, T), 0)}. \tag{5}$$

In any case, we have that

$$\begin{cases} u_i(0, Z, T) \cdot Q_T(x_i(Z, T)) + q(x_i(Z, T)) & = & 0 \\ u_i(0, Z, T) \cdot P_T(x_i(Z, T), 0) + p(x_i(Z, T)) & = & 0 \end{cases} \tag{6}$$

for all $(0, Z, T) \in V$.

Proof of the assertion. Let us suppose that $P(x_i^0) \neq 0$. If we take V small then $P_T(x_i(s, Z, T), y_i(s, Z, T)) \neq 0$ for all $(s, Z, T) \in V$. It follows that

$$y_i \cdot P_T(x_i, y_i) + s \cdot p(x_i) \equiv 0 \implies y_i(0, Z, T) = 0$$

and

$$\frac{\partial y_i}{\partial s}(0, Z, T) \cdot P_T(x_i(Z, T), 0) + p(x_i(Z, T)) \equiv 0.$$

Since $u_i(0, Z, T) = \frac{\partial y_i}{\partial s}(0, Z, T)$, this implies (4). The proofs of (5) and (6) are left for the reader. □

Let's continue the proof of Lemma 2.4 by proving **(f)**. We will prove first that the singularities $p_i(s, Z, T)$ are non-degenerate for $s \neq 0$. Denote by J the Jacobian matrix

$$J = \begin{pmatrix} \frac{\partial W_1}{\partial x} & \frac{\partial W_1}{\partial y} \\ \frac{\partial W_2}{\partial x} & \frac{\partial W_2}{\partial y} \end{pmatrix}.$$

First we prove, for all $i = 1, \ldots, 2d$, that $\det(J(p_i(s,Z,T), s, Z, T)) \neq 0$ whenever $s \neq 0$ and $(s, Z - Z_0, T)$ have a small norm. Since $W_1 = y \cdot P_T + s \cdot p$ and $W_2 = y \cdot Q_T + s \cdot (q + y \cdot x^d)$, by a direct computation, we get that $\det(J(p_i, s, Z, T)$ is equal to

$$\begin{aligned} & W_{1x} \cdot W_{2y} - W_{1y} \cdot W_{2x} = \\ = \quad & (yP_{Tx} + sp')(Q_T + yQ_{Tx} + sx^d) - (P_T + y \cdot P_{Ty})(yQ_{Tx} + sq' + dsyx^{d-1})\ (p_i(s,Z,T)) \\ = \quad & s\ (u_i P_{Tx} + p')(Q_T + su_i Q_{Tx} + sx^d) - (P_T + su_i P_{Ty})(u_i Q_{Tx} + q' + dsu_i x^{d-1})\ (x_i, y_i). \end{aligned}$$

Therefore if we define $\Delta(Z,T) := \lim \frac{1}{s} \det(J(p_i(s,Z,T), s, Z, T)$, then

$$\Delta(Z,T) = [(u_i \cdot P_{Tx} + p') \cdot Q_T - P_T \cdot (u_i \cdot Q_{Tx} + q')](p_i(0,Z,T)).$$

On the other hand, (6) implies that $\Delta(Z,T)$ is equal to

$$\begin{aligned} & [(p' \cdot Q_T - u_i \cdot P_T \cdot Q_{Tx}) - (P_T \cdot q' - u_i \cdot P_{Tx} \cdot Q_T)](p_i(0,Z,T)) \\ = \quad & [(p' \cdot Q_T + p \cdot Q_{Tx}) - (P_T \cdot q' + P_{Tx} \cdot q)](p_i(0,Z,T)) \\ = \quad & \frac{\partial}{\partial x}[p \cdot Q_T - q \cdot P_T](p_i(0,Z,T)) \\ = \quad & f'_{(Z,T)}(x_i(Z,T)). \end{aligned}$$

If we take the neighborhood V of $(0, Z_0, 0)$ small then the polynomial $f_{(Z,T)}$ has simple roots, for every $(0, Z, T) \in V$. Since $x_i(0,Z,T) = x_i(Z,T)$ is a root of $f_{(Z,T)}$, we get that $\Delta(Z,T) = f'_{(Z,T)}(x_i(Z,T)) \neq 0$. Hence, $\det(J(p_i(s,Z,T), s, Z, T) \neq 0$ for small $|s| > 0$. It remains to prove (3) in **(f)**. Since

$$G_i(s,Z,T) = \frac{\operatorname{tr}^2(J(p_i(s,Z,T), s, Z, T))}{\det(J(p_i(s,Z,T), s, Z, T))}$$

and

$$\operatorname{tr}(J(p_i(s,Z,T), s, Z, T)) = [s \cdot u_i \cdot P_{Tx} + s \cdot p' + Q_T + s \cdot u_i \cdot Q_{Ty} + s \cdot x^d](p_i(s,Z,T))$$

we get

$$\lim \operatorname{tr}^2(J(p_i(s,Z,T), s, Z, T)) = Q_T^2(x_i(Z,T))$$

and

$$\begin{aligned} \lim \frac{1}{s} G_i(s,Z,T) = \lim \frac{\operatorname{tr}^2(J(p_i(s,Z,T), s, Z, T))}{s \cdot \det(J(p_i(s,Z,T), s, Z, T))} = \\ = \frac{Q_T^2(x_i(Z,T), 0)}{f'_{(Z,T)}(x_i(Z,T))}. \end{aligned}$$

This finishes the proof of the lemma. □

To apply Lemma 2.1 we set $BB(s,Z,T)$ equal to $(G(s,Z,T), H(s,Z,T))$, i.e.,

$$BB(s,Z,T) = (G_1(s,Z,T), \ldots, G_{2d}(s,Z,T), H_1(s,Z,T), \ldots, H_{d^2-d}(s,Z,T)).$$

We are going to prove that we can choose Z_0 in such a way that, for $|s| > 0$ small, $\operatorname{rk}(BB, (s, Z_0, 0)) = d^2 + d$.

It follows from **(VII)** and from **(e)** of Lemma 2.4 that H satisfies the hypothesis **(a)** and **(d)** of Lemma 2.1. We have seen also that

$$G(s,Z,T) = \frac{1}{s}\left[A(Z,T) + s \cdot R(s,Z,T)\right],$$

where R is holomorphic,

$$A(Z,T) = \lim s \cdot G(s,Z,T) = (A_1(Z,T), \ldots, A_{2d}(Z,T))$$

and

$$A_j(Z,T) = \frac{Q_T^2(x_j(Z,T),0)}{f'_{(Z,T)}(x_j(Z,T))}.$$

In order to finish the proof, it is sufficient to prove that there exists Z_0 and $j \in \{1,\dots,2d\}$ such that $\det(M_j(Z_0)) \neq 0$, where

$$M_j(Z) = \left[A^T(Z,0), \frac{\partial A^T}{\partial z_1}(Z,0), \dots, \frac{\partial A^T}{\partial z_{j-1}}(Z,0), \frac{\partial A^T}{\partial z_{j+1}}(Z,0), \dots, \frac{\partial A^T}{\partial z_{2d}}(Z,0)\right].$$

In the above expression, for $C \in \mathbb{C}^{2d}$, we are denoting by C^T the transpose of C, that is, we are considering the transpose of the matrix given in **(c)** of Lemma 2.1.

2.5. **Applying the Key Lemma II: Fine Tuning.** According to Lemma 2.3, the map $\Phi\colon P_{d+1} \times P_{d-2} \to P_{2d}$ defined by $\Phi(Z) = \Phi(p,q) = p \cdot Q - q \cdot P := f$ is an isomorphism. On the other hand, observe that

$$A_j(Z,0) = \frac{Q_0^2(x_j(Z),0)}{f'_Z(x_j(Z))} = \frac{Q^2(x_j(Z))}{f'_Z(x_j(Z))},$$

where $x_1(Z) := x_1(Z,0), \dots, x_{2d}(Z) := x_{2d}(Z,0)$ are the roots of $f_Z := f_{(Z,0)}$.

The idea is to use Lemma 2.3 to parametrize the space P_{2d} by the roots of f_Z instead of the coefficients $(z_1,\dots,z_{2d})$ of $Z = (p,q)$. We have seen before that $\deg(p\cdot Q - q \cdot P) = \deg(p \cdot Q) = 2d$. Since we are free to choose one of the coefficients of Q, we will suppose that it is monic of degree $d-1$. This implies that $f_Z = p \cdot Q - q \cdot P$ is monic (see **(VIII)**). Therefore, we can write

$$f_Z(x) = (x - x_1(Z)) \cdots (x - x_{2d}(Z))$$

and the map $\rho(Z) = (x_1(Z), \dots, x_{2d}(Z))$ is a biholomorphism in a neighborhood of Z_0. Let ζ be the local inverse of ρ, defined in a neighborhood W of $(x_1(Z_0),\dots,x_{2d}(Z_0))$. Set $C = A \circ \zeta\colon W \to \mathbb{C}^{2d}$. If $X = (x_1,\dots,x_{2d})$ then

$$f_{\zeta(X)}(x) := f_X(x) = (x - x_1) \cdots (x - x_{2d}).$$

Therefore, $C(X) = (C_1(X), \dots, C_{2d}(X))$, where

$$C_j(X) = A_j(\zeta(X)) = \frac{Q^2(x_j)}{f'_X(x_j)}.$$

Let $N(X)$ be the $2d \times 2d$ matrix defined by

$$N(X) = [C^T(X), \frac{\partial C^T}{\partial x_2}(X), \dots, \frac{\partial C^T}{\partial x_{2d}}(X)].$$

We assert that it is enough to prove that $\det(N(X)) \not\equiv 0$. In fact, since $C(X) = A \circ \zeta(X)$ we get

$$\frac{\partial C}{\partial x_j} = \sum_{i=1}^{2d} \frac{\partial A}{\partial z_i} \circ \zeta \frac{\partial \zeta_i}{\partial x_j} = \sum_{i=1}^{2d} \frac{\partial A}{\partial z_i} \frac{\partial \zeta_i}{\partial x_j},$$

where in the third expression we have omitted the composition with ζ. This implies that

$$\begin{aligned}\det(N) &= \det\left[A, \sum_{i_2=1}^{2d}\frac{\partial A}{\partial z_{i_2}}\frac{\partial \zeta_{i_2}}{\partial x_2}, \ldots, \sum_{i_{2d}=1}^{2d}\frac{\partial A}{\partial z_{i_{2d}}}\frac{\partial \zeta_{i_{2d}}}{\partial x_{2d}}\right] = \\ &= \sum_{i_2,\ldots,i_{2d}}\frac{\partial \zeta_{i_2}}{\partial x_2}\cdots\frac{\partial \zeta_{i_{2d}}}{\partial x_{2d}}\det\left[A, \frac{\partial A}{\partial z_{i_2}}, \cdots, \frac{\partial A}{\partial z_{i_{2d}}}\right] \\ &= \sum_{j=1}^{2d}\Phi_j \cdot \det(M_j \circ \zeta),\end{aligned}$$

where,

$$\Phi_j = \pm\det\left(\frac{\partial \zeta_i}{\partial x_k}\right)_{1\le i\le 2d, i\ne j, 2\le k\le 2d}.$$

In particular, if $\det(N(X)) \not\equiv 0$ then $\det(M_j(Z)) \not\equiv 0$, for some $j \in \{1, \ldots, 2d\}$.

To conclude the proof of the Theorem it remains to show that $\det(N(X)) \not\equiv 0$. Recall that $Q(x)$ is a monic polynomial of degree $d-1$ and $C(X) = (C_1(X), \ldots, C_{2d}(X))$, where

$$C_j(X) = C_j(x_1, \ldots, x_{2d}) = \frac{Q^2(x_j)}{f'_X(x_j)} = \frac{Q^2(x_j)}{\Pi_{i\ne j}(x_j - x_i)} \tag{7}$$

because $f_X(x) = \Pi_{i=1}^{2d}(x - x_i)$. Fix $x_0 \in \mathbb{C}$ which is not a root of $Q(x) = 0$ and a neighborhood $D := D(x_0, r)$ such that $Q(x) \ne 0$ for all $x \in D$. We will work in the open set $U \subset \mathbb{C}^{2d}$ defined by

$$U = \{(x_1, \ldots, x_{2d}) | x_i \ne x_j \text{if} i \ne j\}.$$

If $X \in U$ then $C_j(X) \ne 0$ and

$$\det(N(X)) = C_1(X) \cdots C_{2d}(X) \cdot \det(K(X)),$$

where K is the matrix

$$K = \begin{bmatrix} 1 & \ldots & 1 \\ \frac{\partial C_1}{\partial x_2}/C_1 & \ldots & \frac{\partial C_{2d}}{\partial x_2}/C_{2d} \\ \ldots & \ldots & \ldots \\ \frac{\partial C_1}{\partial x_{2d}}/C_1 & \ldots & \frac{\partial C_{2d}}{\partial x_{2d}}/C_{2d} \end{bmatrix}$$

It follows from (7) that

$$\frac{\frac{\partial C_j}{\partial x_i}}{C_j}(X) = \begin{cases} \frac{1}{x_i - x_j} & ,\text{if} i \ne j. \\ \frac{2Q'(x_j)}{Q(x_j)} + \sum_{i\ne j}\frac{1}{x_i - x_j} & ,\text{if} i = j. \end{cases}$$

In particular, if we denote $\phi_j = \frac{2Q'(x_j)}{Q(x_j)}$, $j = 2, \ldots, 2d$, then, for any $X \in U$ we have the following expression for $K(X)$

$$\begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ \frac{1}{x_2 - x_1} & \phi_2 + \sum\limits_{i\ne 2}\frac{1}{x_i - x_2} & \ldots & \frac{1}{x_2 - x_{2d-1}} & \frac{1}{x_2 - x_{2d}} \\ \ldots & \ldots & \ldots & \ldots & \\ \frac{1}{x_{2d-1} - x_1} & \frac{1}{x_{2d-1} - x_2} & \ldots & \phi_{2d-1} + \sum\limits_{i\ne 2d-1}\frac{1}{x_i - x_{2d-1}} & \frac{1}{x_{2d-1} - x_{2d}} \\ \frac{1}{x_{2d} - x_1} & \frac{1}{x_{2d} - x_2} & \ldots & \frac{1}{x_{2d} - x_{2d-1}} & \phi_{2d} + \sum\limits_{i\ne 2d}\frac{1}{x_i - x_{2d}} \end{bmatrix}.$$

Now, define

$$\Delta_1(x_1,\dots,x_{2d-1}) := \lim_{x_{2d}\to x_1}(x_1 - x_{2d})\cdot\det(K(X))$$

and inductively

$$\Delta_j(x_1,\dots,x_{2d-j}) := \lim_{x_{2d-j+1}\to x_1}(x_1 - x_{2d-j+1})\cdot\Delta_{j+1}(x_1,\dots,x_{2d-j+1}).$$

We will prove that $\Delta_{2d-1}(x_1) = (2d)! \neq 0$ and this fact will imply that $\det(N(X)) \not\equiv 0$. As the reader can check, $\Delta_1(x_1,\dots,x_{2d-1})$ is equal to

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ \frac{1}{x_2-x_1} & \phi_2+\sum_{i=3}^{2d-1}\frac{1}{x_i-x_2}+\frac{2}{x_1-x_2} & \cdots & \frac{1}{x_2-x_{2d-1}} & \frac{1}{x_2-x_1} \\ \cdots & \cdots & \cdots & \cdots & \\ \frac{1}{x_{2d-1}-x_1} & \frac{1}{x_{2d-1}-x_2} & \cdots & \phi_{2d-1}+\sum_{i=2}^{2d-2}\frac{1}{x_i-x_{2d-1}}+\frac{2}{x_1-x_{2d-1}} & \frac{1}{x_{2d-1}-x_1} \\ -1 & 0 & \cdots & 0 & 1 \end{vmatrix},$$

where $|\cdot|$ denotes the determinant. If we sum the first column with the last in the above determinant, we get

$$\begin{vmatrix} 2 & 1 & \cdots & 1 \\ \frac{2}{x_2-x_1} & \phi_2+\sum_{i=3}^{2d-1}\frac{1}{x_i-x_2}+\frac{2}{x_1-x_2} & \cdots & \frac{1}{x_2-x_{2d-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{2}{x_{2d-1}-x_1} & \frac{1}{x_{2d-1}-x_2} & \cdots & \phi_{2d-1}+\sum_{i=2}^{2d-2}\frac{1}{x_i-x_{2d-1}}+\frac{2}{x_1-x_{2d-1}} \end{vmatrix}.$$

By a similar argument, we have that $\Delta_2(x_1,\dots,x_{2d-2})$ is equal to

$$\begin{vmatrix} 2 & 1 & \cdots & 1 & 1 \\ \frac{2}{x_2-x_1} & \phi_2+\sum_{i=3}^{2d-2}\frac{1}{x_i-x_2}+\frac{3}{x_1-x_2} & \cdots & \frac{1}{x_2-x_{2d-2}} & \frac{1}{x_2-x_1} \\ \cdots & \cdots & \cdots & \cdots & \\ \frac{2}{x_{2d-2}-x_1} & \frac{1}{x_{2d-2}-x_2} & \cdots & \phi_{2d-2}+\sum_{i=2}^{2d-3}\frac{1}{x_i-x_{2d-2}}+\frac{3}{x_1-x_{2d-2}} & \frac{1}{x_{2d-2}-x_1} \\ -2 & 0 & \cdots & 0 & 2 \end{vmatrix},$$

or, more succinctly,

$$2\cdot\begin{vmatrix} 3 & 1 & \cdots & 1 \\ \frac{3}{x_2-x_1} & \phi_2+\sum_{i=3}^{2d-2}\frac{1}{x_i-x_2}+\frac{3}{x_1-x_2} & \cdots & \frac{1}{x_2-x_{2d-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{3}{x_{2d-2}-x_1} & \frac{1}{x_{2d-2}-x_2} & \cdots & \phi_{2d-2}+\sum_{i=2}^{2d-3}\frac{1}{x_i-x_{2d-2}}+\frac{3}{x_1-x_{2d-2}} \end{vmatrix}.$$

Similarlly, $\Delta_3(x_1,\dots,x_{2d-3})$ is equal to

$$6\cdot\begin{vmatrix} 4 & 1 & \cdots & 1 \\ \frac{4}{x_2-x_1} & \phi_2+\sum_{i=3}^{2d-3}\frac{1}{x_i-x_2}+\frac{4}{x_1-x_2} & \cdots & \frac{1}{x_2-x_{2d-3}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{4}{x_{2d-3}-x_1} & \frac{1}{x_{2d-3}-x_2} & \cdots & \phi_{2d-3}+\sum_{i=2}^{2d-4}\frac{1}{x_i-x_{2d-3}}+\frac{4}{x_1-x_{2d-3}} \end{vmatrix}.$$

Proceeding in this way we see that $\Delta_j(x_1,\dots,x_{2d-j})$ is given by

$$j!\cdot\begin{vmatrix} j+1 & 1 & \cdots & 1 \\ \frac{j+1}{x_2-x_1} & \phi_2+\sum_{i=3}^{2d-j}\frac{1}{x_i-x_2}+\frac{j+1}{x_1-x_2} & \cdots & \frac{1}{x_2-x_{2d-j}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{j+1}{x_{2d-j}-x_1} & \frac{1}{x_{2d-j}-x_2} & \cdots & \phi_{2d-j}+\sum_{i=2}^{2d-j-1}\frac{1}{x_i-x_{2d-j}}+\frac{j+1}{x_1-x_{2d-j}} \end{vmatrix}.$$

In particular,

$$\Delta_{2d-2}(x_1,x_2) = (2d-2)!\cdot\begin{vmatrix} 2d-1 & 1 \\ \frac{2d-1}{x_2-x_1} & \phi_2+\frac{2d-1}{x_1-x_2} \end{vmatrix}.$$

Hence,

$$\Delta_{2d-1}(x_1) = \lim_{x_2 \to x_1} (x_1 - x_2) \cdot \Delta_{2d-2}(x_1, x_2) = (2d-2)! \begin{vmatrix} 2d-1 & 1 \\ 1-2d & 2d-1 \end{vmatrix} = (2d)!.$$

This finishes the proof of Theorem 1.

3. The Rank at Jouanolou's Foliations

Jouanolou's foliations are the first examples of foliations of $\mathbb{P}^2$ without invariant algebraic curves, (cf. [9]). They can be defined as follows: for every integer d, $d \geq 2$, the degree d Jouanolou foliation, $\mathcal{J}_d$, is induced in affine coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2)$ by the vector field

$$X_d(x, y) = (1 - x \cdot y^d)\partial_x + (x^d - y^{d+1})\partial_y = \partial_x + x^d \partial_y - y^d \cdot R,$$

where $R = x\partial_x + y\partial_y$ is the radial vector field on $\mathbb{C}^2$.

Most of arguments proving that $\mathcal{J}_d$ has no invariant algebraic curves take advantage of the highly symmetrical character of $\mathcal{J}_d$: $\mathrm{A}ut(\mathcal{J}_d)$, the automorphism group of $\mathcal{J}_d$, is a semi-direct product of a cyclic group of order 3 and a cyclic group of order $d^2 + d + 1$. If β is a primitive $(d^2 + d + 1)^{th}$ root of the unity then generators of $\mathrm{A}ut(\mathcal{J}_d)$, in the affine coordinates $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2$, are

$$\begin{array}{rcl} A : (x, y) & \mapsto & (\beta^{-d}x, \beta y), \\ B : (x, y) & \mapsto & (y^{-1}, xy^{-1}). \end{array}$$

The singular set of $\mathcal{J}_d$ is equal to

$$\mathrm{sing}(\mathcal{J}_d) = \{p_j \mid p_j = A^{j-1}(1, 1), 1 \leq j \leq d^2 + d + 1\},$$

i.e., it is the orbit of the point $p_1 = (1, 1)$ under the action on $\mathbb{P}^2$ of the subgroup of $\mathrm{A}ut(\mathcal{J}_d)$ generated by A. It follows that all the singularities of $\mathcal{J}_d$ are isomorphic simple singularities with Baum-Bott index

$$\frac{(d+2)^2}{d^2 + d + 1}.$$

We will also take advantage of $\mathrm{A}ut(\mathcal{J}_d)$ to determine the rank of the Baum-Bott map at $\mathcal{J}_d$. Instead of considering the Baum-Bott map as defined from $\mathbb{F}\mathrm{ol}(d)$ to $\mathbb{P}^{d^2+d+1}$ we will consider it defined from $V_d = \mathrm{H}^0(\mathbb{P}^2, T\mathbb{P}^2(d-1))$ to the same target. Our problem translates to compute the rank at X_d.

It will be convenient to consider V_d as the $\mathbb{C}$-vector space generated by the set

$$\mathcal{P}_d = \{x^i \cdot y^j \partial_x, x^k \cdot y^\ell \partial_y, x^m \cdot y^n \cdot R \,|\, 0 \leq i + j, k + \ell \leq d \text{ and } m + n = d\}.$$

Note that all the elements in $\mathcal{P}_d$ are eigenvectors of $A^* : V_d \to V_d$, where $A^*(X) = DA^{-1} \cdot X \circ A$. Explicitly, we have

$$\begin{array}{rcl} A^*(x^i \cdot y^j \partial_x) & = & \beta^{j-d(i-1)} \cdot x^i \cdot y^j \partial_x \\ A^*(x^k \cdot y^\ell \partial_y) & = & \beta^{\ell-1-dk} \cdot x^k \cdot y^\ell \partial_y \\ A^*(x^m \cdot y^n \cdot R) & = & \beta^{n-dm} \cdot x^m \cdot y^n \cdot R \end{array}$$

The invariance of $\mathcal{J}_d$ under A is expressed in the formula

$$A^*(X_d) = \beta^d \cdot X_d.$$

Since β is a primitive $(d^2+d+1)^{th}$ root of unity, A^* has at most d^2+d+1 maximal eigenspaces. If we denote by E_j, $1 \le j \le d^2+d+1$, the maximal eigenspace associated to the eigenvalue β^j then

$$V_d = \bigoplus_{j=1}^{d^2+d+1} E_j.$$

Now, let U be a neighborhood of X_d in V_d and $\gamma_j \colon U \to \mathbb{P}^2, j = 1 \dots d^2+d+1$, be holomorphic maps such that $\gamma_j(X_d) = p_j$ and

$$\mathrm{sing}(\mathcal{F}(X)) = \{\gamma_1(X), \dots, \gamma_{d^2+d+1}(X)\},$$

for every $X \in U$. Compute the rank of the Baum-Bott map is equivalent to compute the rank of $B = (B_1, \dots, B_{d^2+d+1}) : U \to \mathbb{C}^{d^2+d+1}$ given by

$$B_j(X) = BB(X, \gamma_j(X)) = \frac{\mathrm{tr}^2}{\det}(DX(\gamma_j(X))).$$

3.1. **The rank of B at X_d.** By definition the rank of B at X_d is the rank of the liner map $T := DB(X_d) : V_d \to \mathbb{C}^{d^2+d+1}$, the derivative of B at X_d. If we denote by $T_j := DB_j(X_d)$, $1 \le j \le d^2+d+1$, then the next lemma describes some useful relations between A^* and T_j.

Lemma 3.1. *For any $Y \in V_d$*

$$T_j(A^*(Y)) = \beta^d \cdot T_{j+1}(Y), \tag{8}$$

where $1 \le j \le d^2+d+1$, and $T_{d^2+d+1} = T_0$. In particular,

(a). $A^*(\ker(T)) = \ker(T)$.

(b). *If we set $K_j := E_j \cap \ker(T)$, $j = 1, \dots, d^2+d+1$, then*

$$\ker(T) = \bigoplus_{j=1}^{d^2+d+1} K_j.$$

(c). *$E_j \cap \ker(T_1) = K_j$, for all $j = 1, \dots, d^2+d+1$.*

(d). *Let $k = \#\{j | T_1|_{E_j} \not\equiv 0\}$. Then $\mathrm{rk}(T) = \mathrm{rk}(BB, \mathcal{J}_d) = k$.*

Proof. Observe first that for any $Y \in V$, we have that the foliations induced by $A^*(X_d+Y)$ and $X_d + \beta^{-d} \cdot A^*Y$ are equal, i.e.,

$$\mathcal{F}(A^*(X_d+Y)) = \mathcal{F}(X_d + \beta^{-d} \cdot A^*(Y)).$$

Moreover, since $A^*(X) = DA^{-1} \cdot X \circ A$,

$$p \in \mathrm{sing}(\mathcal{F}(A^*(X_d+Y))) \iff A(p) \in \mathrm{sing}(X_d+Y).$$

If we set $P_j(Y) = A^{-1}(\gamma_j(X_d+Y))$ then $P_j(0) = A^{-1}(p_j) = p_{j-1}$ and $P_j(Y) = \gamma_{j-1}(X_d + \beta^{-d} \cdot A^*(Y))$. Thus

$$\gamma_j(X_d+Y) = A(\gamma_{j-1}(X_d + \beta^{-d} \cdot A^*(Y))),$$

for all $Y \in V_d$ sufficiently small where, by convention, we set $\gamma_0 = \gamma_{d^2+d+1}$. Now we can easily verify that

$$\begin{aligned} B_j(X_d+Y) &= BB(X_d+Y, \gamma_j(X_d+Y)) \\ &= BB(X_d + \beta^{-d} \cdot A^*(Y), \gamma_{j-1}(X_d + \beta^{-d} \cdot A^*(Y)) \\ &= B_{j-1}(X_d + \beta^{-d} \cdot A^*(Y)). \end{aligned}$$

Hence,

$$T_j(Y) = DB_j(X_d)\cdot Y = DB_{j-1}(X_d)\cdot(\beta^{-d}\cdot A^*(Y)) = \beta^{-d}\cdot T_{j-1}(A^*(Y)).$$

This proves (8). Observe that (8) implies **(a)** and **(b)**.

Relation (8) also implies that $T_1((A^*)^k(Y)) = \beta^{kd}\cdot T_{1+k}(Y)$. Thus $Y \in E_j \cap \ker(T_1)$ if, and only if,

$$A^*(Y) = \beta^j \cdot Y \text{ and } 0 = T_1(\beta^{kj}\cdot Y) = T_1((A^*)^k(Y)) = \beta^{kd}\cdot T_{1+k}(Y),$$

or, equivalently, $T_n(Y) = 0$ for all $n \in \{1, \ldots, d^2+d+1\}$ and $A^*(Y) = \beta^j \cdot Y$. Thus we can conclude that

$$E_j \cap \ker(T_1) = E_j \cap K,$$

proving in this way **(c)**.

Let us prove **(d)**. Note that $\operatorname{rk}(B(X_d)) = \dim(Im(T))$. Let $k = \#\{j | T_1|_{E_j} \not\equiv 0\}$ and $\{j|T_1|_{E_j} \not\equiv 0\} = \{j_1, \ldots, j_k\}$, where $j_1 < \ldots < j_k$. Choose $Y_1, \ldots, Y_k \in V_d$ such that $Y_i \in E_{j_i}$ and $T_1(Y_i) \neq 0$ for all $i = 1, \ldots, k$. It follows from (8) that

$$T_j(Y_i) = \beta^{-d}\cdot T_{j-1}(A^*(Y_i)) = \beta^{j_i - d}\cdot T_{j-1}(Y_i)$$

and by induction, that

$$T_j(Y_i) = \beta^{(j_i-d)(j-1)}\cdot T_1(Y_i) \implies T(Y_i) = T_1(Y_i)\cdot(1, \beta^{(j_i-d)}, \ldots, \beta^{(N-1)(j_i-d)}).$$

We want to prove that the vectors $T(Y_1), \ldots, T(Y_k) \in \mathbb{C}^N$ are linearly independent. Since $T_1(Y_i) \neq 0$ for all $i = 1, \ldots, k$, this is equivalent to prove that the vectors $(1, \beta^{(j_i-d)}, \beta^{2(j_i-d)}, \ldots, \beta^{(N-1)(j_i-d)}) \in \mathbb{C}^N$ are linearly independent. Observe that

$$\det\begin{vmatrix} 1 & \beta^{(j_1-d)} & \beta^{2(j_1-d)} & \ldots & \beta^{(k-1)(j_1-d)} \\ 1 & \beta^{(j_2-d)} & \beta^{2(j_2-d)} & \ldots & \beta^{(k-1)(j_2-d)} \\ .. & .. & .. & \ldots & .. \\ 1 & \beta^{(j_k-d)} & \beta^{2(j_k-d)} & \ldots & \beta^{(k-1)(j_k-d)} \end{vmatrix} = \Pi_{r<s}(\beta^{j_s-d} - \beta^{j_r-d}) \neq 0,$$

because $\beta^{j_s-d} \neq \beta^{j_r-d}$ for $r < s$. This finishes the proof of the lemma. □

3.2. **Maximal Eigenspaces of A^*.** Recall that $\mathcal{P}_d$ is a basis for V_d. We will denote by $\mathcal{P}_d(Y)$ the subset of $\mathcal{P}_d$ of the form

$$\mathcal{P}_d(Y) = \{x^i\cdot y^j\cdot Y | x^i\cdot y^j\cdot Y \in V_d, 0 \le i+j \le d\}.$$

In these notations we have that $\mathcal{P}_d$ is the disjoint union of $\mathcal{P}_d(\partial_x)$, $\mathcal{P}_d(\partial_y)$ and $\mathcal{P}_d(x\partial_x + y\partial_y)$.

Lemma 3.2. *Let $i, j \ge 0$ be such that $0 \le i+j \le d$ and $A^*(x^i\cdot y^j) = x^i\cdot y^j$. Then $i = j = 0$. In particular, given $Y \in V_d$ then the eigenvalues of $Y_1, Y_2 \in \mathcal{P}_d(Y)$ are distinct for $Y_1 \neq Y_2$.*

Proof. Note that $A^*(x^i\cdot y^j) = \beta^{j-i\cdot d}\cdot x^i\cdot y^j$. In particular $A^*(x^i\cdot y^j) = x^i\cdot y^j$ if, and only if,

$$j - i\cdot d = 0 \mod N \iff (d+1)\cdot j + i = 0 \mod N \iff i = j = 0.$$

In the first equivalence we have used that $-d(d+1) = 1 \mod N$ and in the second that

$$0 \le (d+1)\cdot j + i = d\cdot j + i + j \le d(j+1) \le d(d+1) = N - 1 < N.$$

We leave the proof of the second part for the reader. □

In the next result we describe the dimensions of the maximal eigenspaces of A^*.

Lemma 3.3. *For any $j = 1, \dots, d^2 + d + 1$ we have*

$$0 \leq dim(E_j) \leq 3.$$

Moreover,

(a). $dim(E_d) = 3$ *and* $E_d \subset \ker(T)$.
(b). $dim(E_j) = 3$ *if, and only if,* $j = d$.
(c). $\#\{j | E_j \neq \{0\}\} = \frac{d^2+7d-4}{2}$.

Proof. Note that $\mathcal{P}_d(\partial_x) \cup \mathcal{P}_d(\partial_y) \cup \mathcal{P}_d(R)$, $(R = x\partial_x + y\partial_y)$, is a basis of V_d formed by eigenvectors of A^*. From Lemma 3.2, it follows that the vectors in $\mathcal{P}_d(\partial_x)$ have distinct eigenvalues. Analogously, the vectors in $\mathcal{P}_d(\partial_y)$ (resp. in $\mathcal{P}_d(R)$) have different eingenvalues. This implies that $0 \leq dim(E_j) \leq 3$.

If $dim(E_j) = 3$, then E_j must contain one vector in each part of the basis; $\mathcal{P}_d(\partial_x)$, $\mathcal{P}_d(\partial_y)$ and $\mathcal{P}_d(R)$.

Note that $E_d =< \partial_x, x^d \cdot \partial_y, y^d \cdot R >$. Let us prove that $E_d \subset \ker(T)$. Let $C_{(s,t)}(x, y) = (s \cdot x, t \cdot y)$ and consider the family $X(r, s, t) \in V_d$ given by

$$X(r, s, t) = r \cdot C^*_{(s,t)}(X_d) = r \cdot s^{-1}\partial_x + r \cdot s^d \cdot t^{-1} \cdot x^d \cdot \partial_y + r \cdot t^d \cdot y^d \cdot R.$$

Of course, for $r, s, t \neq 0$ we have

$$B(X(r, s, t)) = B(X_d).$$

This implies that the vectors $\frac{\partial}{\partial x}$, $x^d \frac{\partial}{\partial y}$ and $y^d \cdot R$ belong to $\ker(T)$. This proves **(a)**.

Let us prove **(b)**. Suppose that $dim(E_r) = 3$ for some $r \in \{1, \dots, d^2 + d + 1\}$. Then, we must have $E_r =< x^i \cdot y^j \cdot \partial_x, x^k \cdot y^\ell \cdot \partial_y, x^m \cdot y^n \cdot R >$, where $0 \leq i+j, k+\ell \leq d$ and $m + n = d$. This implies that

$$-d(i-1) + j = -d \cdot k + \ell - 1 = -d \cdot m + n = r \mod N. \tag{9}$$

Since $-d(d + 1) = 1 \mod N$, this implies that

$$i - 1 + (d + 1)j = m + (d + 1)n = d \cdot n + m + n = d(n + 1) \mod N \implies$$
$$\implies d \cdot j + i + j - 1 = d(n + 1) \mod N.$$

Let us suppose by contradiction that $r \neq d$. In the case $i = j = 0$ we have $r = d$, and so we must have $1 \leq i + j \leq d$. This implies that

$$0 \leq d \cdot j + i + j - 1 \leq d \cdot j + d - 1 = d(j + 1) - 1 \leq d(d + 1) - 1 < N \implies$$
$$d \cdot j + i + j - 1 = d(n + 1),$$

because $0 < d(n + 1) \leq d(d + 1) < N$. Therefore, d divides $i + j - 1$. Since $0 \leq i + j - 1 \leq d - 1$, we get $i + j = 1$ and $j = n + 1 > 0$. Hence, $i = 0$, $j = n + 1$ and $r = n - d \cdot m = n + 1 + d \mod N$. It follows that $d(m + 1) + 1 = 0 \mod N$, which implies that $i = 0$, $j = 1$, $m = d$, $n = 0$ and $r = d + 1$.

On the other hand this, together with (9), implies that

$$r = d + 1 = -d \cdot k + \ell - 1 \mod N \implies d(k + 1) + 2 = \ell \mod N.$$

We assert that this is impossible, if $0 \leq k + \ell \leq d$. In fact, if $0 \leq k \leq d - 1$ then we would get

$$0 < d(k + 1) + 2 \leq d^2 + 2 < N \implies \ell = d(k + 1) + 2 \implies \ell > d,$$

which is impossible. If $k = d$, then $\ell = 0$ and we would get $d(d+1)+2 = 0 \mod N$, which is a contradiction. Therefore, $r = d$, which proves **(b)**.

It remains to prove **(c)**. Set $M = \#\{j|E_j \neq \{0\}\}$. It is clear that M is the number of different eigenvalues of A^*. Lemma 3.2 implies that all vectors in $P(\partial_x)$ have different eigenvalues. Since $\#(P(\partial_x)) = (d+1)(d+2)/2$, we get this number of eigenvalues, such that the correspondent eigenvectors are in $P(\partial_x)$. Consider the function $\phi\colon \mathcal{P}_d(x\cdot\partial_x) \to \mathcal{P}_d(y\cdot\partial_y)$ defined by

$$\phi(x^i\cdot y^j\cdot\partial_x) = x^{i-1}\cdot y^{j+1}\cdot\partial_y.$$

A straightforward computation shows that, if $Y \in P(x\cdot\partial_x)$ is such that $A^*(Y) = \lambda\cdot Y$ then $A^*(\phi(Y)) = \lambda\cdot\phi(Y)$. This proves that the eingenvectors of A^* in $\mathcal{P}_d(\partial_y)$ which correspond to new eigenvalues (not found in the previous set) must be in $\mathcal{P}_d(\partial_y)\setminus\mathcal{P}_d(y\cdot\partial_y)$. Therefore, they are of the form $x^k\cdot\partial_y$, where $0\le k\le d$. We assert that there are $d-1$ new eigenvalues in this set.

In fact, if $x^i\cdot y^j\cdot\partial_x$ and $x^k\cdot\partial_y$ have the same eigenvalue then $-d(i-1)+j = -k\cdot d-1 \mod N$. Thus

$$i-1+(d+1)j = k-(d+1) \mod N,$$

which implies that

$$k = d(j+1)+i+j \mod N.$$

Of course, we have the known solution, $k=d$, $i=j=0$, which corresponds to vectors in E_d. Another solution is $k=d-1$, $i=0$ and $j=d$, as the reader can check. On the other hand, if $0\le j\le d-1$ then

$$0 < d(j+1)+i+j \le d^2+d < N \implies d(j+1)+i+j = k,$$

implying that

$$i=j=0 \text{ and } k=d.$$

Therefore, there are only two repeated eigenvalues and $d-1$ new in this set. The repeated eigenvalues correspond to E_d and E_{2d}.

It remains to find how many new eigenvalues we can find in the set $\mathcal{P}_d(R)$. Suppose first that we have a vector $x^m\cdot y^n\cdot R$ in $\mathcal{P}_d(R)$ with the same eigenvalue of a vector $x^i\cdot y^j\cdot\partial_x\in\mathcal{P}_d(\partial_x)$. This case, was already considered in the proof of **(b)**. We have found two possibilities: $(i,j)=(0,0)$, $(m,n)=(0,d)$ (which corresponds to vectors in E_d) and $(i,j)=(0,1)$, $(m,n)=(d,0)$ (which corresponds to E_{d+1}). Suppose now that we have a vector $x^m\cdot y^n\cdot R$ in $\mathcal{P}_d(R)$ and a vector $x^k\cdot\partial_y$ in $\mathcal{P}_d(\partial_y)$ with the same eigenvalue. Then

$$-k\cdot d-1 = -d\cdot m+n \mod N \implies k-(d+1) = m+n(d+1) = d(n+1) \mod N$$

which implies that

$$k = d\cdot n+2d+1 \mod N.$$

We have the following two solutions of the above relation: $k=d$, $(m,n)=(0,d)$ (which corresponds to E_d) and $k=0$, $(m,n)=(1,d-1)$. On the other hand, if $0\le n\le d-2$ then

$$2d+1 \le d\cdot n+2d+1 \le d^2+1 < N \implies k = d\cdot n+2d+1 > d,$$

which contradicts $0\le k\le d$. Therefore, there are two repeated solutions, which correspond to E_d and E_{d^2+d}. This implies that there is a total of 3 eigenvalues in $\mathcal{P}_d(R)$ which were already found in the previous sets. Since $\#(\mathcal{P}_d(R)) = d+1$, we find $d-2$ new eigenvalues corresponding to eigenvectors in the set $\mathcal{P}_d(R)$. Hence, the total number of eigenvalues of A^* is

$$M = \frac{(d+1)(d+2)}{2} + d-1+d-2 = \frac{d^2+7d-4}{2},$$

which proves the lemma. □

In order to finish the proof of theorem 2, it is sufficient to verify the following fact: For any $j \in \{0, \dots, N-1\}$ such that $j \neq d$ and $E_j \neq \{0\}$ then $T_1|_{E_j} \not\equiv 0$. To do this will need first to carry on a study of the local variation of the Baum-Bott index.

3.3. **Local Variation of the Baum-Bott Index.** We will consider the following situation: let X be a polynomial vector field in V_d and $p_0 \in \mathbb{C}^2$ be a non-degenerate singularity of X. Denote by X_1 the 1-jet of X at p_0, that is $X_1 = DX(p_0)$. Let $U \subset V_d$ be a neighborhood of X such that there exists a holomorphic map $p\colon U \to \mathbb{C}^2$ with $p(X) = p_0$ and for any $Y \in U$ then $p(Y)$ is a non-degenerate singularity of Y. Let $B\colon U \to \mathbb{C}$ be defined by $B(Y) = BB(Y, p(Y))$. We will prove the following result:

Lemma 3.4. *Suppose that the eigenvalues of X_1 are in the Poincaré domain and have no ressonances. Let $Z \in V_d \cap \ker(DB(X))$, that is $dB(X) \cdot Z = 0$. Then there exists $\lambda \in \mathbb{C}$ and a germ of holomorphic vector field Y at p_0, such that*

$$Z_{p_0} = \lambda \cdot X_{p_0} + [X_{p_0}, Y],$$

where in the above relation, X_{p_0} and Z_{p_0} denote the germs of the respective vector fields at p_0. In particular, if $Z(p_0) = 0$ then $Y(p_0) = 0$ and

$$Z_1 = \lambda \cdot X_1 + [X_1, Y_1],$$

where $Z_1 = DZ(p_0)$ and $Y_1 = DY(p_0)$.

Proof. Let $B\colon U \to \mathbb{C}$ be as before. Set $B(X) = b_0$ and let $S := B^{-1}(b_0)$. We will prove first that $DB(X) \not\equiv 0$. This will imply that we can suppose (by taking a smaller U) that S is a smooth codimension one sub-variety of U.

To simplify the notations, we will suppose that $p_0 = 0 \in \mathbb{C}^2$. In this case, we have $X = X_1 + h.o.t.$, where in a suitable affine coordinate system,

$$X_1 = \lambda_1 \cdot x\partial_x + \lambda_2 \cdot y\partial_y, \lambda_1, \lambda_2 \notin \mathbb{R}_- \text{and} \lambda_2/\lambda_1, \lambda_1/\lambda_2 \notin \mathbb{N} (\text{Poincaré condictions}).$$

Consider the curve $X(t)$ in V_d defined by

$$X(t) = X + t \cdot x\partial_x.$$

Then $X(0) = X$, $X(t)(0) \equiv 0 \in \mathbb{C}^2$ and $X(t)_1 = X_1 + t \cdot x\partial_x$, which implies that

$$B(X(t)) = \frac{(\lambda_1 + \lambda_2 + t)^2}{(\lambda_1 + t)\lambda_2}$$

and, consequently,

$$DB(X) \cdot (x\partial_x) = \frac{d}{dt} B(X(t))_{|t=0} = \frac{1 - (\lambda_2/\lambda_1)^2}{\lambda_2} \neq 0,$$

because $\lambda_2/\lambda_1 \neq \pm 1$. Therefore, we will suppose that S is smooth of codimension one.

Now, let $Z \in \ker(DB(X))$. Since S is smooth, there exists a real analytic curve $Y(t) \subset S$, $t \in (-\epsilon, \epsilon)$, such that $Y(0) = X$ and $\frac{d}{dt} Y(t)|_{t=0} = Z$. Therefore, we can write

$$Y(t) = X + t \cdot Z + \sum_{n=2}^{\infty} t^n \cdot Y_n, Y_n \in V_d, \forall n \geq 2.$$

Set $p(t) := p(Y(t))$, so that $p(0) = p_0$ and $p(t)$ is a non-degenerate singularity of $Y(t)$. Let $\lambda_1(t)$ and $\lambda_2(t)$ be eigenvalues of $DY(t)(p(t))$, where we can suppose that

$t \mapsto \lambda_j(t)$ is real analytic and $\lambda_j(0) = \lambda_j$ for $j = 1, 2$. Since $B(Y(t)) = b_o$ for all $t \in (-\epsilon, \epsilon)$, we get

$$b_o \equiv \frac{(\lambda_1(t) + \lambda_2(t))^2}{\lambda_1(t) \cdot \lambda_2(t)} \equiv \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1 \cdot \lambda_2} \implies \lambda_2(t)/\lambda_1(t) \equiv \lambda_2/\lambda_1, \forall t \in (-\epsilon, \epsilon),$$

as the reader can check, by using the condition $\lambda_j(0) = \lambda_j$, $j = 1, 2$. This implies that,

$$\lambda_2(t)/\lambda_2 \equiv \lambda_1(t)/\lambda_1 := \phi(t),$$

where ϕ is real analytic and $\phi(0) = 1$. Now, we use the Poincaré conditions. It follows from Poincaré's linearization theorem that, there exist $0 < \delta \leq \epsilon$, a neighborhood V of $0 \in \mathbb{C}^2$ and a real analytic map $\Psi \colon (-\delta, \delta) \times V \to \mathbb{C}^2$, with the following properties:

(i). $\Psi(t, 0) = p(t)$ for all $t \in (-\delta, \delta)$.

(ii). For all $t \in (-\delta, \delta)$, $\Psi_t(x, y) := \Psi(t, x, y)$ is a biholomorphism from V to $V(t) := \Psi_t(V)$ and $\Psi_0 = id_V$ (the identity map).

(iii). For all $t \in (-\delta, \delta)$ we have $\Psi_t^*(Y(t)) = \phi(t) \cdot Y(0) = \phi(t) \cdot X$.

Writing explicitly the last relation, we have

$$D\Psi_t^{-1} \cdot Y(t) \circ \Psi_t = \phi(t) \cdot X \implies Y(t) \circ \Psi_t = \phi(t) \cdot D\Psi_t \cdot X. \tag{10}$$

Let $\Psi_t(x, y) = (\Psi_t^1(x, y), \Psi_t^2(x, y))$ and consider the vector field $W = P_1 \frac{\partial}{\partial x} + P_2 \frac{\partial}{\partial y}$, where

$$P_j(x, y) = \frac{\partial \Psi^j}{\partial t}(0, x, y), j = 1, 2.$$

Note that the components of W and $\frac{\partial}{\partial t}\Psi|_{t=0}$ coincide. Taking the partial derivative of both members of (10) with respect to t at $t = 0$, we get

$$\begin{aligned} Z + DX \cdot W &= Z + DY(0) \cdot W = \\ &= Y'(0) \circ \Psi_0 + DY(0) \circ \Psi_0 \cdot \frac{\partial \Psi_t}{\partial t}|_{t=0} = \\ &= \phi'(0) \cdot D\Psi_0 \cdot X + \phi(0) \cdot D\left(\frac{\partial}{\partial t}\Psi_{|t=0}\right) \cdot X = \\ &= \phi'(0) \cdot X + DW \cdot X. \end{aligned}$$

If we set $\lambda = \phi'(0)$ then we get

$$Z = \lambda \cdot X + DW \cdot X - DX \cdot W = \lambda \cdot X + [W, X].$$

This proves the first part of the lemma. We leave the proof of the second part for the reader. $\square$

3.4. **Conclusion of the proof of Theorem 2.** Back to our original problem it remains to show that: For any $j \in \{0, \ldots, N-1\}$ such that $j \neq d$ and $E_j \neq \{0\}$ then $T_1|_{E_j} \not\equiv 0$. This will be achivied in the next result.

Lemma 3.5. *Let $W \in \mathcal{P}_d$ be such that $W \in \ker(T_1)$. Then $W \in E_d$.*

Proof. Let W be in $\mathcal{P}_d \cap \ker(T_1)$. We have three possible cases.
1^{st} case: $W = x^i \cdot y^j \partial_x$, where $0 \leq i + j \leq d$. Recall that $\partial_x \in \ker(T_1)$. We assert that, if $1 \leq i + j \leq d$ then $W \notin \ker(T_1)$.

In fact, set $Z = W - \partial_x = (x^i \cdot y^j - 1)\partial_x$. Since $T_1(\partial_x) = 0$, we have

$$T_1(W) = 0 \iff T_1(Z) = 0.$$

Recall that $T_1 = DB_1(X_d)$, $B_1(X) = BB(\mathcal{F}(X), \gamma_1(X))$ and $\gamma_1(X_d) = (1,1) = p_1$. Since $Z(1,1) = 0$, it follows from lemma 3.4 that it is enough to verify if $Z_1 = DZ(1,1)$ belongs or not to the image of the linear map $\Psi\colon \mathbb{C} \times L_1 \to L_1$ defined by

$$\Psi(\lambda, Y_1) = \lambda \cdot X_1 + [X_1, Y_1],$$

where L_1 is the set of 1-jets of germs of holomorphic vector fields Y at $(1,1)$ such that $Y(1,1) = 0$. Note that L_1 is isomorphic to the set M_2, of 2×2 matrices, via the linear map $\Phi\colon L_1 \to M_2$ defined by

$$Y = P\partial_x + Q\partial_y \overset{\Phi}{\mapsto} DY(1,1) = \begin{bmatrix} \frac{\partial P}{\partial x}(1,1) & \frac{\partial P}{\partial y}(1,1) \\ \frac{\partial Q}{\partial x}(1,1) & \frac{\partial Q}{\partial x}(1,1) \end{bmatrix}$$

The map Φ is an isomorphism of Lie algebras. We will call $\Phi(Y_1)$ the matrix form of Y_1 and, to simplify, we will keep the notation Y_1 instead of $\Phi(Y_1)$. Note that,

$$X_1 = \begin{bmatrix} -1 & -d \\ d & -(d+1) \end{bmatrix} \text{and} Z_1 = \begin{bmatrix} i & j \\ 0 & 0 \end{bmatrix}.$$

Let $Y_1 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. Then, $\Psi(\lambda, Y_1) = \lambda \cdot X_1 + [X_1, Y_1]$ and

$$[X_1, Y_1] = Y_1X_1 - X_1Y_1 = \begin{bmatrix} d(\beta+\gamma) & d(\delta-\alpha-\beta) \\ d(\gamma+\delta-\alpha) & -d(\beta+\gamma) \end{bmatrix} := \begin{bmatrix} x & y \\ z & -x \end{bmatrix},$$

as the reader can check. In particular, we get $\mathrm{tr}([X_1, Y_1]) = 0$ and the following relation between the entries of $[X_1, Y_1]$

$$x = z - y. \tag{11}$$

Let us suppose that $Z_1 = \Psi(\lambda, Y_1)$. Since $\mathrm{tr}([X_1, Y_1]) = 0$, we get

$$i = \mathrm{tr}(Z_1) = \lambda \cdot \mathrm{tr}(X_1) = -\lambda \cdot (d+2) \implies \lambda = -\frac{i}{d+2}.$$

This implies that the matrix $Z_1 + \frac{i}{d+2}X_1$ must satisfy (11). On the other hand, we have,

$$Z_1 + \frac{i}{d+2}X_1 = \begin{bmatrix} \frac{(d+1)i}{d+2} & \frac{(d+2)j-d\cdot i}{d+2} \\ \frac{d\cdot i}{d+2} & -\frac{(d+1)i}{d+2} \end{bmatrix}$$

Hence, $Z \in \ker(T_1)$ if, and only if,

$$\frac{(d+1)i}{d+2} = \frac{d\cdot i}{d+2} - \frac{(d+2)j - d\cdot i}{d+2}$$

if, and only if,

$$(d-1)i = (d+2)j.$$

The last relation implies that $d+2|i$, which implies that $i = 0$ and $j = 0$, which contradicts the assumption $i + j \geq 1$.

2^{nd} case: $W = x^k \cdot y^\ell \partial_y$, where $0 \leq k + \ell \leq d$. Recall that $x^d\partial_y \in \ker(T_1)$. We assert that, if $0 \leq k \leq d-1$ and $0 \leq k+\ell \leq d$ then $W \notin \ker(T_1)$.

The idea is the same as in the 1^{st} case. Let $Z = W - x^d\partial_y = (x^k \cdot y^\ell - x^d)\partial_y$. Since $x^d\partial_y \in \ker(T_1)$, then $W \in \ker(T_1) \iff Z \in \ker(T_1)$. In this case, we have $Z(1,1) = 0$ and

$$Z_1 = \begin{bmatrix} 0 & 0 \\ k-d & \ell \end{bmatrix} \implies \lambda = -\frac{\ell}{d+2} \implies Z_1 - \lambda X_1 = \begin{bmatrix} -\frac{\ell}{d+2} & -\frac{d\cdot\ell}{d+2} \\ \frac{d\cdot\ell+(k-d)(d+2)}{d+2} & \frac{\ell}{d+2} \end{bmatrix}.$$

Hence, $Z \in \ker(T_1)$ if, and only if,

$$-\frac{\ell}{d+2} = \frac{d\cdot\ell + (k-d)(d+2)}{d+2} + \frac{d\cdot\ell}{d+2} \iff (d-k)(d+2) = (2d+1)\ell.$$

As the reader can check, if $0 \le k+\ell \le d$ then the last relation is possible only for $k=d$ and $\ell = 0$, which proves the assertion.

3^{rd} case: $W = x^m y^n R$, where $m+n=d$. Recall that $y^d \cdot R \in \ker(T_1)$. We assert that, if $0 \le n \le d-1$ then $W \notin \ker(T_1)$.

In this case, if $Z = W - y^d \cdot R = (x^m \cdot y^n - y^d)\cdot R$ then $Z(1,1) = 0$ and

$$Z_1 = \begin{bmatrix} m & n-d \\ m & n-d \end{bmatrix} \implies \operatorname{tr}(Z_1) = 0 \text{ and } \lambda = 0 \implies m = m-(n-d) \implies$$

$$\implies n = d \text{ and } m = 0.$$

This finishes the proof of the lemma and of Theorem 2. □

4. The Camacho-Sad Field

4.1. **Preliminaries.** Let M and S be two complex compact surfaces, $\phi\colon M \dashrightarrow S$ be a meromorphic map and $\mathcal{F}$ be a foliation on S. We want to prove that $\mathbb{K}(\phi^*(\mathcal{F})) = \mathbb{K}(\mathcal{F})$. We will use the notation $\mathcal{G} := \phi^*(\mathcal{F})$. As it was sketched in the introduction, the theorem is true when ϕ consists of a sequence of blowing-ups. This fact allow us to reduce the problem to the case where $\mathcal{F}$ and $\mathcal{G}$ are reduced and ϕ is holomorphic. Thus, from now on, we will suppose that the foliations $\mathcal{F}$ and $\mathcal{G} = \phi^*(\mathcal{F})$ are reduced and that $\phi\colon M \to S$ is holomorphic. Before going on, let us fix some notations.

Let $\mathcal{H}$ be a reduced foliation on a compact surface V. Given $p \in V$ we will associate a field, $\mathbb{K}(\mathcal{H}, p)$, as follows: let X be a holomorphic vector field which represents $\mathcal{H}$ in a neighborhood of p. When $p \in \operatorname{sing}(\mathcal{H})$, we will denote by λ_1, λ_2 the eigenvalues of $DX(p)$. We have three possibilities:

- **(I).** $p \in \operatorname{sing}(\mathcal{F})$, $\lambda_1, \lambda_2 \neq 0$ and $\lambda_2/\lambda_1 \notin \mathbb{Q}_+$. In this case, $\mathcal{H}$ has two local separatrixes Σ_1 and Σ_2 through p and $\mathrm{CS}(\mathcal{H}, \Sigma_1, p) = \lambda_2/\lambda_1$, $\mathrm{CS}(\mathcal{H}, \Sigma_2, p) = \lambda_1/\lambda_2$. In this case, we set: $\mathbb{K}(\mathcal{H}, p) = \mathbb{Q}(\lambda_2/\lambda_1) = \mathbb{Q}(\lambda_1/\lambda_2)$.
- **(II).** $\lambda_1 = 0$ and $\lambda_2 \neq 0$. We will suppose $\lambda_2 = 1$. In this case, $\mathcal{H}$ has one local analytic separatrix Σ_2 through p, tangent to the eigenspace of $\lambda_2 = 1$ and $\mathrm{CS}(\mathcal{H}, \Sigma_2, p) = 0$. The separatrix Σ_1, tangent to the eigenspace of $\lambda_1 = 0$ is formal, in general, but X is formally equivalent to the vector field $Y := x^{k+1}\partial_x + y(1+\lambda x^k)\partial_y$. We have $\mathrm{CS}(\mathcal{H}, \Sigma_1, p) = \lambda$ (by definition) and we set $\mathbb{K}(\mathcal{H}, p) = \mathbb{Q}(\lambda)$.
- **(III).** $p \notin \operatorname{sing}(\mathcal{F})$. In this case, we set $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}$.

In general, if $\emptyset \neq A \subset V$, and $A \cap \operatorname{sing}(\mathcal{H}) = \{p_1, \ldots, p_k\}$, we set

$$\mathbb{K}(\mathcal{H}, A) = \mathbb{Q}(\mathbb{K}(\mathcal{H}, p_1), \ldots, \mathbb{K}(\mathcal{H}, p_k)).$$

When $A \cap \operatorname{sing}(\mathcal{H}) = \emptyset$ we set $\mathbb{K}(\mathcal{H}, A) = \mathbb{Q}$.

With the above notation, we have

- **(IV).** $\mathbb{K}(\mathcal{H}) = \mathbb{K}(\mathcal{H}, V)$.
- **(V).** If $A, B \subset V$ then $\mathbb{K}(\mathcal{H}, A \cup B) = \mathbb{Q}(\mathbb{K}(\mathcal{H}, A), \mathbb{K}(\mathcal{H}, B))$.

The next result implies Theorem 3.

Lemma 4.1. *For any $p \in S$ we have*

$$\mathbb{K}(\phi^*(\mathcal{F}), \phi^{-1}(p)) = \mathbb{K}(\mathcal{F}, p).$$

We first note that $\phi^{-1}(p) \neq \emptyset$, because the generic rank of ϕ is two, which implies that ϕ is surjective. Moreover, $\phi^{-1}(p)$ is an analytic subset whose connected components have dimension zero (points) or one (curves). In fact, we will prove that for any connected component C of $\phi^{-1}(p)$ we have

$$\mathbb{K}(\phi^*(\mathcal{F}), C) = \mathbb{K}(\mathcal{F}, p).$$

Clearly this fact implies the lemma. Before going on, we will state some remarks and preliminary results.

Remark 4.1. Let Z be vector field representing $\mathcal{F}$ in a sufficiently small neighborhood U of a point $p \in S$. Locally, and up to an analytic change of coordinates, we have three possibilities:
1st. p is not a singularity of $\mathcal{F}$. In this case, $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}$. We can suppose that $Z = \partial_y$. In particular, $\mathcal{F}$ has a local holomorphic first integral (y) and has just one local separatrix through p: the curve $y = 0$.
2nd. p is a reduced and simple singularity of $\mathcal{F}$ and the eigenvalues of $DZ(p)$ are $\lambda_1, \lambda_2 \neq 0$. In this case, $\lambda_2/\lambda_1 \notin \mathbb{Q}_+$ and $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}(\lambda_2/\lambda_1)$. The foliation $\mathcal{F}$ has two local separatrices through p, which are smooth and transversal at p. We can suppose that they are $(x = 0)$ and $(y = 0)$ and that

$$Z = \lambda_1 \cdot x\partial_x + \lambda_2 \cdot y(1 + R(x, y))\partial_y \,. \tag{12}$$

where $R(0, 0) = 0$.
3rd. p is a saddle-node of $\mathcal{F}$ and we can suppose that the eigenvalues of $DZ(p)$ are 0 and 1. In this case, Z is formally equivalent at p to the vector field $\hat{Z} = x^{k+1}\partial_x + \hat{y}(1+\lambda\cdot x^k)\partial_{\hat{y}}$, where $k \geq 1$, and $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}(\lambda)$. Here, we will use Dulac's normal form (cf. [11]). For every $m \geq k + 1$ there exists a holomorphic coordinate system $(U, (x, y))$ such that $x(p) = y(p) = 0$ and $\mathcal{F}$ is defined by

$$Z = x^{k+1}\partial_x + [y(1 + \lambda \cdot x^k) + R(x, y)]\partial_y \,. \tag{13}$$

where the m jet of R is zero at $0 \in \mathbb{C}^2$. When $\mathcal{F}$ has two local analytic separatrices through p, we can suppose that y divides R. When it has just one analytic separatrix, then it has also a formal one, given by $\hat{y} = 0$, where $\hat{y}$ is a divergent series of the form (cf. [11]):

$$\hat{y} = y - \sum_{j=r+1}^{\infty} a_j x^j. \tag{14}$$

We will break down the proof of Lemma 4.1 in three cases.

Proof of Lemma 4.1, 1st Case: p is not a singularity of $\mathcal{F}$. Here $\mathcal{F}$ admits a holomorphic first integral in a neighborhood of p. If $g \in \mathcal{O}_p$ is such holomorphic first integral then $\phi^* g$ is an holomorphic first integral for $\mathcal{G} = \phi^*\mathcal{F}$ in a neighborhood of $\phi^{-1}(p)$. Thus $\mathbb{K}(\mathcal{G}, \phi^{-1}(p)) = \mathbb{Q}$. □

¿From now on, we will suppose that $p \in \operatorname{sing}(\mathcal{F})$. In the next results, we will consider the following situation: let $q \in \phi^{-1}(p) \cap \operatorname{sing}(\mathcal{G})$. Suppose that $\mathcal{G}$ has a local analytic separatrix $\tilde{\Sigma}$ through q such that $\phi(\tilde{\Sigma}) \neq \{p\}$. In this case, $\phi(\tilde{\Sigma}) := \Sigma$ is a local analytic separatrix of $\mathcal{F}$ through p.

Lemma 4.2. *In the above situation, we have*
(a). $\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}, q) \in \mathbb{Q}(\mathrm{CS}(\mathcal{F}, \Sigma, p))$.
(b). *If* $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}(\mathrm{CS}(\mathcal{F}, \Sigma, p))$ *then* $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}(\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}, q))$.

Proof. Let $(f = 0)$ be a reduced equation Σ and write

$$g \cdot \omega = h \cdot df + f \cdot \mu, \tag{15}$$

where $g, h|_\Sigma \neq 0$. From the definition, we have

$$\mathrm{CS}(\mathcal{F}, \Sigma, p) = \frac{1}{2\pi i} \int_\gamma -\frac{\mu}{h},$$

where γ is a small circle in Σ around p, positively oriented. Note that $\phi^*(\omega) = \tilde{k} \cdot \theta_q$, where $\tilde{k} \in \mathcal{O}_q$ and θ_q represents the germ of $\mathcal{G}$ at q. Let $\tilde{f} = 0$ be a reduced equation of $\tilde{\Sigma}$. Since $\phi(\tilde{\Sigma}) = \Sigma = (f = 0)$, we get

$$\phi^*(f) = f \circ \phi = \tilde{g} \cdot \tilde{f}^m,$$

where $m \geq 1$ and $\tilde{g}|_{\tilde{\Sigma}} \neq 0$. It follows from (15) that

$$\begin{aligned}
&\phi^*(g) \cdot \tilde{k} \cdot \theta_q = \phi^*(h) \cdot d(\tilde{g} \cdot \tilde{f}^m) + \tilde{g} \cdot \tilde{f}^m \cdot \phi^*(\mu) \implies \\
\implies \quad & \frac{\phi^*(g) \cdot \tilde{k}}{m \cdot \phi^*(h) \cdot \tilde{g} \cdot \tilde{f}^m} \cdot \theta_q = \frac{d\tilde{f}}{\tilde{f}} + \frac{1}{m}[\frac{d\tilde{g}}{\tilde{g}} + \phi^*(\frac{\mu}{h})] \implies \\
\implies \quad & \mathrm{CS}(\mathcal{G}, \tilde{\Sigma}, q) = -\frac{1}{2\pi i} \int_{\tilde{\gamma}} \frac{1}{m} \left[\frac{d\tilde{g}}{\tilde{g}} + \phi^*(\frac{\mu}{h})\right],
\end{aligned}$$

where $\tilde{\gamma}$ is a small circle in $\tilde{\Sigma}$ around q. Note that $\phi(\tilde{\gamma}) = \gamma^n$, where $n \geq 1$. Observe also that $\int_{\tilde{\gamma}} \frac{d\tilde{g}}{\tilde{g}} = \ell \in \mathbb{Z}$. Hence,

$$\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}, q) = -\frac{\ell}{m} + \frac{1}{m}\frac{1}{2\pi i} \int_{\gamma^n} -\frac{\mu}{h} = \frac{1}{m}\left(-\ell + n \cdot \mathrm{CS}(\mathcal{F}, \Sigma, p)\right) \in \mathbb{Q}(\mathrm{CS}(\mathcal{F}, \Sigma, p)).$$

Since $n \neq 0$, we get also that

$$\mathbb{Q}(\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}, q)) = \mathbb{Q}(\mathrm{CS}(\mathcal{F}, \Sigma, p)),$$

which implies **(b)**. □

Remark 4.2. The above result is true in the general case, that is, even if the map ϕ is meromorphic and the separatrices $\tilde{\Sigma}$ and Σ are singular.

Remark 4.3. If the connected component C of $\phi^{-1}(p)$ is a curve, then all irreducible components of C are invariant for the foliation $\mathcal{G}$. Moreover, all the singular points of C are nodes.

Proof of Lemma 4.1, 2^{nd} Case: p is a singularity with two analytic separatrices. We will prove that every connected component C of $\phi^{-1}(p)$ is such that

$$\mathbb{K}(\mathcal{G}, C) = \mathbb{K}(\mathcal{F}, p).$$

First of all, observe for one of the two separatrices, say Σ, we have

$$\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}(\mathrm{CS}(\mathcal{F}, \Sigma, p)).$$

Let W be a neighborhood of C. Note that $\phi^{-1}(\Sigma) \cap W$ is a germ of analytic set around $\phi^{-1}(p)$, different from $\phi^{-1}(p)$. Each component of $\phi^{-1}(\Sigma) \setminus \phi^{-1}(p)$ is a curve biholomorphic to $\mathbb{D}^*$, whose closure contains an unique point in $\phi^{-1}(p)$. Let $\tilde{\Sigma}$ be a closure of some of these components and set $\tilde{\Sigma} \cap \phi^{-1}(p) = \{q\}$. It follows from **(b)** of lemma 4.2 that

$$\mathbb{K}(\mathcal{G}, q) = \mathbb{K}(\mathcal{F}, p).$$

This implies that

$$\mathbb{K}(\mathcal{G}, C) \subset \mathbb{K}(\mathcal{G}, q) = \mathbb{K}(\mathcal{F}, p).$$

It remains to prove that, for any $q \in \operatorname{sing}(\mathcal{G}) \cap C$, then $\mathbb{K}(\mathcal{G}, q) \subset \mathbb{K}(\mathcal{F}, p)$. If C has dimension zero, that is $C = q$, the above argument shows that $\mathbb{K}(\mathcal{G}, C) = \mathbb{K}(\mathcal{F}, p)$.

¿From now on, we will suppose that C is a curve. The next result implies the second case of lemma 4.1.

□

Lemma 4.3. *Let $q \in C \cap \operatorname{sing}(\mathcal{G})$ and $\tilde{\Sigma}_1$ be a separatrix of $\mathcal{G}$ through q. Then $\tilde{\Sigma}_1$ is analytic and*

$$\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}_1, q) \in \mathbb{K}(\mathcal{F}, p).$$

Proof. Suppose first that q is a smooth point of C and that $\tilde{\Sigma}_1 \not\subset C$. If $\tilde{\Sigma}_1$ is a formal separatrix of $\mathcal{G}$ which is non convergent then $\mathcal{F}$ would have a formal non-convergent separatrix at p contrary to our assumptions. This $\tilde{\Sigma}_1$ is analytic. Thus lemma 4.2 implies that

$$\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}_1, q) = \mathrm{CS}(\mathcal{G}, \phi(\tilde{\Sigma}_1), p) \in \mathbb{K}(\mathcal{F}, p)$$

and we are done in this case.

Let us suppose now that $\tilde{\Sigma}_1 \subset C$. In this case, $\tilde{\Sigma}_1$ is analytic and smooth, but $\phi(\tilde{\Sigma}_1) = \{p\}$ and we cannot use directly lemma 4.2. The result will follow from the lemma below. □

Lemma 4.4. *In the above situation, there is a bimeromorphism $\psi\colon \hat{S} \to S$ (a sequence of blowing-ups) such that, if we set $\hat{\phi} := \psi^{-1} \circ \phi\colon M \dashrightarrow \hat{S}$, $\hat{\mathcal{F}} = \psi^*(\mathcal{F})$ and $D = \psi^{-1}(p)$ then:*

- **(a).** *There exists $\hat{p} \in D \cap \operatorname{sing}(\hat{\mathcal{F}})$ and a separatrix $\Sigma_1 \subset D$ of $\hat{\mathcal{F}}$ through $\hat{p}$ such that $\hat{\phi}(\tilde{\Sigma}_1) = \Sigma_1$.*
- **(b).** $\mathrm{CS}(\mathcal{G}, \tilde{\Sigma}_1, q) \in \mathbb{K}(\hat{\mathcal{F}}, \hat{p}) \subset \mathbb{K}(\mathcal{F}, p)$.

Proof. Let $\tilde{\Sigma}_2$ be the other separatrix of $\mathcal{G}$ through q and $(V, (u, v))$ be a local coordinate system around q such that $u(q) = v(q) = 0$, $\operatorname{sing}(\mathcal{G}) \cap V = \{q\}$, $\tilde{\Sigma}_1 = (u = 0)$, $\tilde{\Sigma}_2 = (v = 0)$, $V = \{(u, v) | |u|, |v| < \epsilon\}$ and $\phi(V) \subset U$. As before, we have $X_q(u, v) = u^m \cdot f(u, v)$ and $Y_q(u, v) = u^n \cdot g(u, v)$, where $m, n \geq 1$, $f, g \in \mathcal{O}_q$ and $f(0, v), g(0, v) \not\equiv 0$. For $|c| < \epsilon$, let γ_c be the germ at p of the curve $u \mapsto \phi(u, c)$. Note that, maybe γ_0 is a point (if $\phi(\tilde{\Sigma}_2) = \{p\}$), however if we take a smaller $\epsilon > 0$, then we can suppose that γ_c is a curve, for all $0 < |c| < \epsilon$. Moreover, there is a sequence of blowing-ups $\psi\colon \hat{S} \to S$ such that, if $D = \psi^{-1}(p)$ and ϵ is small enough then:

- (i). $\psi\colon \hat{S} \setminus D \to S \setminus \{p\}$ is a bimeromorphism.
- (ii). There is a divisor $D_1 \subset D$ such that, for all $0 < |c| < \epsilon$, the strict transform $\hat{\gamma}_c$ of γ_c meets D_1 in a unique point, say $p(c)$.
- (iii). If $c_1 \neq c_2$ and $0 \neq c_1, c_2$ then $p(c_1) \neq p(c_2)$. In particular, the map $c \in \{z | 0 < |z| < \epsilon\} \simeq \mathbb{D}^* \mapsto p(c) \in D_1$ is a holomorphic embedding.

The sequence of blowing-ups ψ, is a simultaneous resolution of the germs γ_c, $0 < |c| < \epsilon$. We leave the details for the reader. In this case, it follows from Picard's theorem that there exist $\lim_{c \to 0} p(c) = \hat{p} \in D_1$. Moreover, if $\hat{\mathcal{F}} = \psi^*(\mathcal{F})$ then the germ Σ_1 of D_1 at $\hat{p}$, is a separatrix of $\hat{\mathcal{F}}$ through $\hat{p}$ and $\psi^{-1} \circ \phi(\tilde{\Sigma}_1) = \Sigma_1$. This proves **(a)**.

Let us prove **(b)**. Note first that

$$\mathrm{CS}(\hat{\mathcal{F}},\Sigma_1,\hat{p}) \in \mathbb{K}(\mathcal{F},p) \implies \mathbb{Q}(\mathrm{CS}(\hat{\mathcal{F}},\Sigma_1,\hat{p})) \subset \mathbb{K}(\mathcal{F},p),$$

because ψ is a sequence of blowing-ups (see the introduction). On the other hand, lemma 4.2 implies that

$$\mathrm{CS}(\mathcal{G},\tilde{\Sigma}_1,q) \in \mathbb{Q}(\mathrm{CS}(\hat{\mathcal{F}},\Sigma_1,\hat{p})).$$

This finishes the proof. □

To finish the prove of Lemma 4.1 it remains to treat just one case:

Proof of Lemma 4.1, 3^{rd} Case: p is singular with just one analytic separatrix. In this case, $\mathcal{F}$ has a normal form like in (13) of remark 4.1: for every $r \geq k+1$ there exists a local coordinate system $(U,(x,y))$ where $\mathcal{F}$ is represented by

$$\omega = x^{k+1}dy - [y(1+\lambda\cdot x^k) + R(x,y)]dx, \tag{16}$$

where $k \geq 1$ and $j_0^r(R) = 0$. Let C be a connected component of $\phi^{-1}(p)$ and consider a sufficiently small neighborhood W of C. We will denote by Σ_1 the non-convergent separatrix and by Σ_2 the convergent one. In the coordinate system $(U,(x,y))$ we have $\Sigma_2 = (x=0)$ and Σ_1 is given by the divergent series

$$y = \sum_{j=r+1}^{\infty} a_j x^j.$$

As before, the proof consists in proving that

(I): For any $q \in C \cap \mathrm{sing}(\mathcal{G})$ we have $\mathbb{K}(\mathcal{G},q) \subset \mathbb{K}(\mathcal{F},p)$, and;

(II): There exists $q_0 \in C \cap \mathrm{sing}(\mathcal{G})$ such that $\mathbb{K}(\mathcal{G},q_0) = \mathbb{K}(\mathcal{F},p)$.

Proof of (I). Let us consider first the case where the two separatrices through q are analytic. Let $\tilde{\Sigma}$ be one of these separatrices. It is sufficient to prove that $\mathrm{CS}(\mathcal{G},\tilde{\Sigma},q) \in \mathbb{K}(\mathcal{F},p)$.

In fact, if $\phi(\tilde{\Sigma}) \neq \{p\}$ then $\phi(\tilde{\Sigma})$ is a curve and $\phi(\tilde{\Sigma}) \subset \Sigma_2$. Since $\mathrm{CS}(\mathcal{F},\Sigma_2,p) = 0$, we get from lemma 4.2 that $\mathrm{CS}(\mathcal{G},\tilde{\Sigma},q) \in \mathbb{Q}$, as asserted. On the other hand, if $\phi(\tilde{\Sigma}) = \{p\}$ then the assertion follows from **(b)** of lemma 4.4.

Let us suppose now that there is a non-convergent separatrix, say $\tilde{\Sigma}_1$, and a convergent one, say $\tilde{\Sigma}_2$, through q. We assert that there is a coordinate system $(V,(u,v))$ around q such that $u(q) = v(q) = 0$, $\phi(V) \subset W$ and $\phi|_V(u,v) = (X(u,v),Y(u,v))$, where

(i). $X(u,v) = u^m$, $m \geq 1$.

(ii). $Y(u,v) = u^n \cdot v$, where $n = 0$ if $C = \{q\}$ and $n \geq 1$ if C is a curve.

In fact, we can write $\phi|_W = (X,Y)$, where $X,Y\colon W \to \mathbb{C}$ and $X(q) = Y(q) = 0$ ($\phi(W) \subset U$ as in 4.3). Let X_q and Y_q be the germs of X and Y at q. Since $\Sigma_2 = (X=0)$ is invariant for $\mathcal{F}$, the irreducible components of $(X_q = 0)$ are local analytic separatrices of $\mathcal{G}$ through q. This implies that $(X_q = 0) = \tilde{\Sigma}_2$. Choose a local coordinate system (u,v) such that $\tilde{\Sigma}_2 = (u=0)$. In this case, we get $X_q = u^m \cdot g$, where $m \geq 1$ and $g \in \mathcal{O}_q^*$. If we consider the local change of variables $u_1 = u \cdot g^{1/m}$, then $X_q = u_1^m$, and so we can suppose $X_q = u^m$. In this coordinate system we must have $Y_q = u^n \cdot Y_1$, where $Y_1 \in \mathcal{O}_q$. If C is a curve then $\tilde{\Sigma}_2 \subset C$ (by remark 4.3) and $n \geq 1$. If $C = \{q\}$ then $n = 0$ and $Y(0,v) \not\equiv 0$. We assert that $Y_v(0,0) \neq 0$. Note that this implies that, after a holomorphic change of variables, we can suppose $Y_1(u,v) = v$.

In fact, to say that the formal separatrix $\hat{y} := y - \sum_j a_j x^j$ is invariant for $\mathcal{F}$ is equivalent to

$$d\hat{y} \wedge \omega = \hat{f} \cdot \hat{y} \cdot dx \wedge dy, \tag{17}$$

where $\hat{f} \in \hat{\mathcal{O}}_p$ and $\hat{\mathcal{O}}_p$ denotes the ring of formal power series at p. Consider the formal power series

$$u^n \cdot \hat{Y}_1 := \hat{Y}(u,v) := \phi^*(\hat{y}) = u^n \left(Y_1(u,v) - \sum_{j \geq r+1} a_j u^{mj-n} \right), \tag{18}$$

where $\hat{Y}_1 \in \hat{\mathcal{O}}_q$ if we take r big enough. Let $\hat{Y}_1 = g_1^{n_1} \cdots , g_s^{n_s}$ be the decomposition of $\hat{Y}_1$ into irreducible factors of $\hat{\mathcal{O}}_q$. Write $\phi^*(\omega) = h \cdot \theta_q$, where θ_q represents the germ of $\mathcal{G}$ at q. It follows from (17) that

$$\begin{aligned} & h \left[n \cdot g_1 \cdots g_s du + u \left(\sum_j n_j \cdot g_1 \cdots g_{j-1} \cdot g_{j+1} \cdots g_s \cdot dg_j \right) \right] \wedge \theta_q = \\ = \quad & \Delta \cdot \tilde{f} \circ \phi \cdot u \cdot g_1 \cdots g_s du \wedge dv, \end{aligned}$$

where $\Delta = X_u \cdot Y_v - X_v \cdot Y_u = u^{m+n-1} \cdot Y_{1v}$. We assert that h divides Δ in the $\mathcal{O}_q$.

In fact, as the reader can check, we have $\phi^*(\omega) = u^{m+n-1}(Adv - Bdu)$, where

$$\begin{aligned} A &= u^{km+1} \cdot Y_{1v} \\ B &= m \cdot Y_1(1 + (\lambda - \tfrac{n}{m} \cdot u^{km}) + u^{km+1} \cdot T(u,v) \end{aligned}$$

and $T \in \mathcal{O}_q$. This implies that $h = u^{m+n-1} \cdot h_1$, where any factor of h_1 is also a factor Y_{1v}, because u does not divides B. Therefore, h divides Δ.

It follows that

$$\left[n \cdot g_1 \cdots g_s du + u \left(\sum_j n_j g_1 \cdots g_{j-1} \cdot g_{j+1} \cdots g_s dg_j \right) \right] \wedge \theta_q = \hat{f} \cdot u \cdot g_1 \cdots g_s du \wedge dv,$$

where $\hat{f} \in \hat{\mathcal{O}}_q$. Hence, all factors $g_1, \ldots, g_s$ and $(u = 0)$ are invariant for $\mathcal{G}$. Since $\mathcal{G}$ has only two separatrices through q, we get that $s = 1$ and g_1 is the formal separatrix of $\mathcal{G}$ through q. Since $\mathcal{G}$ is reduced, we get $g_{1v}(0) \neq 0$ and $\hat{Y}_1 = g^s$, where $g = g_1$ and $s = n_1$. It follows from (18) that

$$Y_{1v} = \hat{Y}_{1v} = s g^{s-1} g_v$$

Therefore, $Y_{1v}(0) = 0$ if, and only if, $s > 1$. Suppose by contradiction that $s > 1$. Since $g_v(0) \neq 0$, by the formal Weierstrass' theorem we can write $g = f \cdot (v - h(u))$, where $f \in \hat{\mathcal{O}}_q$, $f(0) \neq 0$ and $h(u)$ is a power series. Therefore, if we set $k = s \cdot f^{s-1} \cdot g_v$, then we have $k \in \hat{\mathcal{O}}_q$, $k(0) \neq 0$ and $Y_{1v} = k \cdot (v - h(u))^{s-1}$. This implies that the germ of analytic set $(Y_{1v} = 0)$ (which is not empty), is also given by $(v - h(u) = 0)$, and so, $h(u)$ is convergent. But this is a contradiction, because $\phi(v - h(u) = 0) = (\hat{y} = 0)$, which is divergent. Hence $s = 1$ and $Y_{1v}(0) \neq 0$.

Let us finish the proof of **(I)**. Since $X(u,v) = u^m$ and $Y(u,v) = u^n \cdot v$, we get from (16) that $\phi^*(\omega) = u^{m+n-1} \cdot \theta_q$, where, given $\ell > mk + 1$ then

$$\theta_q = u^{km+1} dv - m[v\big(1 + (\lambda - \frac{n}{m}) \cdot u^{km}\big) + \tilde{R}(u,v)]du$$

and $\tilde{R}(u,v) = u^{-n} \cdot R(u^m, u^n \cdot v) \in u^\ell \cdot \mathcal{O}_q$, if r is big enough. This implies that the formal normal form of $\mathcal{G}$ at q is given by

$$u^{km+1}dv - m\left[v\big(1+\Big(\lambda - \frac{n}{m}\Big)\,u^{km}\big)\right]du \implies \mathbb{K}(\mathcal{G},q) = \mathbb{Q}(m\lambda - n) = \mathbb{Q}(\lambda) = \mathbb{K}(\mathcal{F},p).$$

Proof of (II). We will suppose that C is a curve. The case where C is a point will be left for the reader. It follows from the proof of **(I)** that it is sufficient to find a point $q \in C \cap \mathrm{sing}(\mathcal{G})$ with a non-convergent separatrix. Let W be a sufficiently small neighborhood of C. Consider the curve $C_1 := \phi^{-1}(y=0) \cap W$. Since $\phi(C_1) = (y = 0) \neq \{p\}$, it follows that $C_1 \setminus C \neq \emptyset$. Moreover, if δ is a component of $C_1 \setminus C$ then δ is biholomorphic to $\mathbb{D}^*$ and $\overline{\delta} \cap C$ is a point, say q. We will denote by δ_q the germ of δ at q. We assert that $\mathcal{G}$ has a non-convergent separatrix through q.

We will see at the end that q is smooth point of C. Let us suppose this fact for a moment. Since $\phi(C) = \{p\}$, there exists a coordinate system $(V,(u,v))$ such that $V \subset W$, $u(q) = v(q) = 0$ and $C \cap V = (u = 0)$. In this case, the germ of ϕ at q can be written as

$$\phi_q(u,v) = (X_q(u,v), Y_q(u,v)) = (u^m X_1(u,v), u^n Y_1(u,v)),$$

where $X_1, Y_1 \in \mathcal{O}_q$ and $X_1(0,v), Y_1(0,v) \not\equiv 0$. Note that $Y_1(0,0) = 0$ and $\delta_q \subset (Y_1 = 0)$. On the other hand, since $(x = 0)$ is an analytic separatrix of $\mathcal{F}$ through p, $X_1(0,0) \neq 0$, because otherwise q would be a node of C. This implies that, after a holomorphic change of variables, we can suppose that $X(u,v) = u^m$. It follows that the formal series

$$\hat{Y}_1 = \frac{1}{u^n}\left(Y - \sum_{j \geq r+1} a_j X^j\right) = Y_1 - \sum_j a_j u^{jm-n}$$

defines a formal separatrix of $\mathcal{G}$ through q (see the proof of **(I)**).

It remains to prove that q is a smooth point of C. Suppose by contradiction that q is a node of C. The idea is to prove that in this case $\mathcal{G}$ has more than two separatrixes through q, which is not possible for a reduced foliation. Let $(V,(u,v))$ be a coordinate system such that $C \cap V = (u \cdot v = 0)$. In this case, we can write $X_q(u,v) = u^m \cdot v^\ell \cdot X_1(u,v)$ and $Y_q(u,v) = u^n \cdot v^s \cdot Y_1(u,v)$, where $X_1(0,v), Y_1(0,v), X_1(u,0), Y_1(u,0) \not\equiv 0$ and $m,n,\ell,s \in \mathbb{N}$. As before, we must have $X_1(0,0) \neq 0$, because $(x = 0)$ is an analytic separatrix through p. Hence, after a holomorphic change of variables, we can suppose that $X(u,v) = u^m \cdot v^\ell$. If $r \gg 1$, then we get the formal power series

$$\hat{Y}_1 = \frac{1}{u^n \cdot v^s}\left(Y - \sum_{j \geq r+1} a_j u^{jm} \cdot v^{j\ell}\right) = Y_1 - \sum_{j \geq r+1} a_j u^{jm-n} \cdot v^{j\ell - s} \in \hat{\mathcal{O}}_q.$$

Note that $\hat{Y}_1(0,0) = 0$. This implies that all irreducible components of $\hat{Y}_1$ in the ring $\hat{\mathcal{O}}_q$ are invariant for $\mathcal{G}$ (see the proof of **(I)**). Since u and v do not divide $\hat{Y}_1$ in $\hat{\mathcal{O}}_q$, $\mathcal{G}$ has more than two separatrices through q: $(u = 0)$, $(v = 0)$ and the irreducible components of $\hat{Y}_1$. This finishes the proof of the thrid case of Lemma 4.1 and of Theorem 3. □

4.2. **Proof of Corolary 2.** If $\mathbb{BB} : \mathbb{F}\mathrm{ol}(d) \dashrightarrow \mathbb{P}^{d^2+d+1}$ is the global Baumm-Bott then by Theorem 1 it follows that the closure of its image is an hypersurface H. Clearly this hypersurface is defined over $\mathbb{Q}$. This is sufficient to assure that there exists a dense set $U \subset H$, such that the field generated by the quotients of the coordinates of $p = [p_0 : \ldots : p_{d^2+d+2}]$ has transcendence degree $d^2 + d = \dim H$ for every $p \in U$.

Since the Camacho-Sad index and the Baum-Bott index of a simple singularity are algebraically dependent, if we take $G(d) = \mathbb{BB}^{-1}(U) \cap \mathbb{R}(d)$ then, for every $\mathcal{F} \in G(d)$, the transcendence degree of $\mathbb{K}(\mathcal{F}) = d^2 + d$. Moreover since U is dense in the image of $\mathbb{BB}$ we have that $G(d)$ is also dense. □

4.3. **A Basic property of the CS-Field and the Proof of Corolary 3.** We will derive corollary 3 from corollary 2 and the basic property of the Camacho-Sad Field is describe in the next proposition. Here we will use the terminology and notation of [3, Chapter 1].

Proposition 4.1. *Let $\mathcal{F}$ be foliation of compact surface S with isolated singularities and cotangent bundle isomorphic to $\mathcal{L}$. The transcendence degree of $\mathbb{K}(\mathcal{F})$ over $\mathbb{Q}$ is at most $c_2(TS \otimes \mathcal{L}) - 1$.*

Proof. If all the singularities are simple, i.e., they all have Milnor number one, then the result is and immediate consequence of Baum-Bott's Formula.

Suppose now that there is a singularity p of $\mathcal{F}$ with Milnor number $\mu(p) \geq 2$. We have three possibilities:

(1). p is a saddle-node;
(2). p is a singularity without linear part;
(3). p is a nilpotent singularity.

In case (1) we have already seen that the transcendence degree of $\mathbb{K}(\mathcal{F}, p)$ is at most 1.

In case (2) we can apply Van den Essen formula(cf. [3, page 13]) to see that after blowing up the sum of the Milnor numbers over the singularities on the exceptional divisor is strictly less then $\mu(p)$.

In case (3) the argument is more involved. After blowing-up a nilpotent singularity only one singularity q appears at the exceptional divisor. We have two possibilities

(3.1). q is a singularity without linear part: after blowing up q it appears 2 or 3 singularities at the exceptional divisor. The important fact is that the sum of its Milnor numbers is equal to $\mu(p)$. Thus here without further ado we have that the transcendence degree of $\mathbb{K}(\mathcal{F}, p)$ is at most $\mu(p)$;

(3.2). q is (again) a nilpotent singularity: blowing up q we obtain a singularity without linear part and after blowin-up again we obtain 3 singularities with non-nilpotent linear part. It follows from Camacho-Sad index Theorem that in this case $\mathbb{K}(\mathcal{F}, p) = \mathbb{Q}$.

An induction argument shows that the transcendence degree of $\mathbb{K}(\mathcal{F})$ is at most the sum of Milnor numbers of singularities of $\mathcal{F}$ which is equal to $c_2(TS \otimes \mathcal{L})$.

To conclude we will analyse two cases independently. In the first one saddle-nodes do not appear in $\tilde{\mathcal{F}}$ the resolution of $\mathcal{F}$. So at the end all the singularities of $\tilde{\mathcal{F}}$ are simple and from (1) and Baum-Bott's formula we have that the transcendence degree of $\mathbb{K}(\mathcal{F})$ is at most $c_2(TS \otimes \mathcal{L}) - 1$. In the second case at least one saddle-node appears at the resolution. Since they have Milnor number at least 2 and contributes to the transcendence degree with at most 1 the result also follows in this case. □

Proof of Corollary 3. Corollary 3 follows immediately combining Theorem 3 and Corollary 2 with the proposition above. □

5. An example

As already noted in the introduction the dimension of the generic fiber is given by $\dim \mathbb{F}\mathrm{ol}(d) - (d^2+d) = 3d+2$. It would be interesting to *classify* the exceptional fibers of the Baum-Bott map, i.e., fibers with dimension at least $3d+3$.

Example 5.1. Let $\mathcal{F}_0$ be a foliation on $\mathbb{P}^2$ with a meromorphic first integral of the type F/L^{d+1}, where F and L are homogeneous, F of degree $d+1$ and L of degree one. In an affine coordinate system $\mathbb{C}^2$ where L is the line at infinity, the foliation is defined by $dF = 0$ and so, it is of degree d. If F is generic then $\mathcal{F}_0$ has d^2 simple singularities on $\mathbb{C}^2$, all of them with Baum-Bott index zero, and $d+1$ singularities at the line L, all of them with Baum-Bott index $(d+2)^2/(d+1)$. In fact, we will see in the next result that the fiber of BB containing $\mathcal{F}_0$ has dimension greater than $3d+2$.

Proposition 5.1. *Let $\mathcal{F}$ be a degree d foliation of $\mathbb{P}^2$ with at least d^2 simple singularities with Baum-Bott index zero. Then $\mathcal{F}$ is a pencil generated by C and $(d+1)L$, where C has degree $d+1$ and L is a line. In particular the fiber of the Baum-Bott map containing $\mathcal{F}$ has dimension*

$$\binom{d+3}{2} + 2.$$

Proof. We will start by proving that $\mathcal{F}$ has an invariant line. Consider an affine coordinate system $(x,y) \in \mathbb{C}^2 \subset \mathbb{P}^2$, such that all singularities of $\mathcal{F}$ are contained in $\mathbb{C}^2$. In particular, the line at infinity is not invariant for $\mathcal{F}$. Recall that $\mathcal{F}$ is induced by a vector field X of the form,

$$X = (a + xg)\partial_x + (b + yg)\partial_y,$$

where a, b are polynomials with $\deg(a), \deg(b) \le d$ and g is a non-identically zero degree d homogeneous polynomial.

Let I be the ideal generated by $a + xg$ and $\mathrm{div}(X)$, where

$$\mathrm{div}(X) = \frac{\partial}{\partial x}(a + xg) + \frac{\partial}{\partial y}(b + yg) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} + (d+2)g$$

Note that, for any singularity p of $\mathcal{F}$ with Baum-Bott index zero, we have $\mathrm{div}(X)(p) = 0$. By Bezout's Theorem we have that $V(I) = \{p \in \mathbb{P}^2 | f(p) = 0 \forall f \in I\}$ has degree $\deg(\mathrm{div}(X))\deg(a + xg) = d(d+1)$, i.e., $V(I)$ has $d^2 + d$ points(counted with multiplicity): d of these points are at infinity they correspond to the intersection of the curve $\{g = 0\}$(which is a union of lines) with the line at infinity; the other d^2 correspond to the singularities of X in $\mathbb{C}^2$ with vanishing trace, i.e., with Baum-Bott index zero. In particular, the closure of the curves $a + x.g = 0$ and $\mathrm{div}(X) = 0$ intersect transversely in $\mathbb{P}^2$.

Since $b + y \cdot g$ vanishes on all points of $V(I)$ it must belong to I. Keeping in mind that $\deg(b + y \cdot g) = \deg(a + x \cdot g) = \deg(\mathrm{div}(X)) + 1$ we can apply Noether's Lemma to see that there exists $\ell_1, \ell_2 \in \mathbb{C}[x, y]$ such that $\deg(\ell_1) = \deg(\ell_2) = 1$ and

$$X(\ell_1) = \ell_2 \cdot \mathrm{div}(X)$$

Note that the left-hand side of the equation above vanishes at all singularities of X. This implies that all the singularities of $\mathcal{F}$ with Baum-Bott index distinct from zero have to be in ℓ_2. Comparing the homogeneous terms of degree $d+1$ of the equation one obtains that

$$g\left(\frac{\partial \ell_1}{\partial x}x+\frac{\partial \ell_1}{\partial y}y\right)=(d+2)g\left(\frac{\partial \ell_2}{\partial x}x+\frac{\partial \ell_2}{\partial y}y\right).$$

Thus $\ell_1-(d+2)\ell_2 \in \mathbb{C}$, and consequently

$$X(\ell_2)=\frac{1}{d+2}\cdot \operatorname{div}(X)\cdot \ell_2,$$

proving that ℓ_2 is invariant.

Let us choose an affine coordinate system where the line at infinity is invariant and

$$X=a\partial_x+b\partial_y,$$

with $\deg(a)=\deg(b)=d$. We claim that $\operatorname{div}(X)\equiv 0$. Let I be the ideal generated by $\operatorname{div}(X)$ and a. If $\operatorname{div}(X)\not\equiv 0$, then $\operatorname{div}(X)$ has degree $\leq d-1$ and $V(I)$ in this case has degree $\leq d(d-1)$. Since $V(I)$ has to vanish at d^2 points we get $\operatorname{div}(X)\equiv 0$.

The condition $\operatorname{div}(X)=0$ is equivalent to the closedness of the polynomial 1-form $\omega=bdx-ady$. So $\omega=dF$ for some polynomial F of degree $d+1$, i.e., $\mathcal{F}$ is a pencil generated by F and L^{d+1}, where F has degree $d+1$ and L is the line at infinity.

We conclude that the fiber of the Baum-Bott map that contains $\mathcal{F}$ can be parametrized as

$$(F,L)\in \mathcal{P}_{d+1}\times\mathcal{P}_1\mapsto \mathcal{F}(F/L^{d+1}),$$

where $\mathcal{P}_j$ denotes the set of homogeneous polynomials on $\mathbb{C}^3$ of degree j and $\mathcal{F}(G)$ the foliation with first integral G. Note that $\mathcal{F}(F/L^{d+1})$ is defined in homogeneous coordinates by the 1-form

$$\omega(F,L)=L\cdot dF-(d+1)\cdot F\cdot dL.$$

On the other hand, the reader can check that $\omega(F,L)=\omega(F_1,L_1)$ if, and only if, $(F_1,L_1)=\lambda\cdot(F,L)$, where $\lambda\in\mathbb{C}^*$. This implies that the dimension of the fiber of the Baum-Bott map that contains $\mathcal{F}$ has dimension $\dim(\mathbb{P}(\mathcal{P}_{d+1}\times\mathcal{P}_1))=\binom{d+3}{2}+2$. □

6. Some Remarks and Problems

6.1. **The image of the Baum-Bott Map.** If F and L are generic, then the singularities of $\mathcal{F}(F/L^{d+1})$ are all simple. Moreover, there are two kinds of singularities: the d^2 singularities with Baum-Bott index zero and the $d+1$ in the line L, all of them with Baum-Bott index $(d+2)^2/(d+1)$. In particular, we see that $\mathbb{BB}(\mathbb{R}(d))$ is not the whole hyperplane given by the Baum-Bott theorem. In fact, any point of the form $(0,\dots,0,\lambda_1,\dots,\lambda_{d+1})$, where $\sum_j \lambda_j=(d+2)^2$ and $\lambda_1\neq(d+2)^2/(d+1)$ is not in $\mathrm{BB}(\mathbb{R}(d))$.

It would be interesting to describe $\mathrm{BB}(\mathbb{R}(d))$, or more specifically, give a criterion to decide if a point $[b_1,\dots,b_N]$ belongs or not to $\mathrm{BB}(\mathbb{R}(d))$.

6.2. **Affine versions of Theorem 1.** Let $L\subset\mathbb{P}^2$ be a line and $\mathbb{F}\mathrm{ol}_L(d)$ be the space of foliations of degree d which leave L invariant. If $\mathcal{F}\in\mathbb{F}\mathrm{ol}_L(d)$ has only simple singularities, it is known (cf. [3]) that L contains $(d+1)$ singularities and that

$$\sum_{p\in\operatorname{sing}(\mathcal{F})\cap L}\mathrm{CS}(\mathcal{F},L,p)=C\cdot C=1.$$

This implies in particular, that the maximal rank of $\mathrm{BB}|_{\mathbb{F}\mathrm{ol}_L(d)}$ is less than d^2+d. When $d\geq 2$, is the maximal rank of $\mathrm{BB}|_{\mathbb{F}\mathrm{ol}_L(d)}$ equal to d^2+d-1? If C is an smooth curve, what can be said about the generic rank of $\mathrm{BB}|_{\mathbb{F}\mathrm{ol}_C(d)}$ for $d \gg 0$? We believe that our strategy of proof should work on these situations.

6.3. **The Fibers of the Baum-Bott Map.** Recall that the dimension of the generic fiber of the global Baum-Bott for degree d foliations of $\mathbb{P}^2$ is $3d+2$. How many irreducible components it has and which is its degree as an algebraic subset of $\mathbb{F}\mathrm{ol}(d)$?

6.4. **Other Surfaces.** For an arbitrary compact complex surface S and an arbitrary non-negative integer k we have that the number of singularities(counted with multiplicities) of a foliation in $\mathbb{F}\mathrm{ol}(S,\mathcal{L})$ with isolated singularities is given by

$$c_2(TS \otimes \mathcal{L}^{\otimes k}) = k^2 \cdot c_1(\mathcal{L})^2 + k \cdot c_1(\mathcal{L}) \cdot c_1(S) + c_2(S)\,.$$

On the other hand if $\mathcal{L}$ is an ample line-bundle and $k \gg 0$ then, combining Hirzebruch-Riemann-Roch Theorem with Serre's Vanishing Theorem(see [1]), we have that $\dim \mathbb{F}\mathrm{ol}(S,\mathcal{L}^{\otimes k}) = h^0(TS \otimes \mathcal{L}^{\otimes k}) - 1$ is equal to

$$\frac{1}{2}\left(c_1^2(TS \otimes \mathcal{L}^{\otimes k}) - 2c_2(TS \otimes \mathcal{L}^{\otimes k})\right) + \frac{1}{2} c_1(TS \otimes \mathcal{L}^{\otimes k}) \cdot c_1(S) + 2\chi(S) - 1\,.$$

Straight-forward manipulations shows that the dimension $\mathbb{F}\mathrm{ol}(S,\mathcal{L}^{\otimes k})$

$$k^2 c_1(\mathcal{L})^2 + 2k c_1(\mathcal{L}) \cdot c_1(S) + c_1^2(S) - c_2(S) + 2\chi(S) - c_2(S) - 1\,.$$

Thus we have that $\dim \mathbb{F}\mathrm{ol}(S,\mathcal{L}^{\otimes k}) - c_2(TS \otimes \mathcal{L}^{\otimes k})$ is equal to

$$k c_1(\mathcal{L}) \cdot c_1(S) + (c_1^2(S) - c_2(S) + 2\chi(S) - 1)\,.$$

If $c_1(\mathcal{L}) \cdot c_1(S) < 0$(this happens,for example, when S is of general type) then

$$\dim \mathbb{F}\mathrm{ol}(S,\mathcal{L}^{\otimes k}) - c_2(TS \otimes \mathcal{L}^{\otimes k}) < 0,$$

for $k \gg 0$, i.e., we have more singularities then foliations. In particular we have other relations between the Baum-Bott indexes besides the Baum-Bott's formula. It would be really interesting to understand the nature of these relations. For instance one could ask how they change when S and $\mathcal{L}$ are deformed. Another natural problem is to know if the Baum-Bott map in this situation is generically finite or not.

6.5. **Endomorphisms and Foliations on $\mathbb{P}^n$.** In [8] Baum-Bott-like formulas are worked out for endomorphisms of projective spaces. There, by a dimension counting, it is shown the existence of extra unknown relations among such multipliers. An analogous phenomena happens also with one-dimensional foliations of $\mathbb{P}^n$, $n \geq 3$. Can these extra relations be produced by some *index* formula? We refer to [8] for a more complete discussion.

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Appendix: On the monodromy of the singular set

Let M be a projective manifold of dimension m, Θ_M be the tangent sheaf of M and $\mathcal{L}$ be a line-bundle over M. The space of foliations by curves on M with cotangent bundle isomorphic to $\mathcal{L}$, denoted by $\mathrm{Fol}(M, \mathcal{L}) = \mathrm{Fol}(\mathcal{L})$, can be identified with the projectivization of the global sections of the bundle $\Theta_M \otimes \mathcal{L}$, i.e.,

$$\mathrm{Fol}(\mathcal{L}) = \mathbb{P}\mathrm{H}^0(M, \Theta_M \otimes \mathcal{L}) .$$

Over the product of $\mathrm{Fol}(\mathcal{L})$ with M we consider the natural foliation $\mathcal{F}(\mathcal{L})$ characterized by the property that the restriction of $\mathcal{F}(\mathcal{L})$ to the fiber over $\mathcal{F}$ under the natural projection $\pi : \mathrm{Fol}(\mathcal{L}) \times M \to \mathrm{Fol}(\mathcal{L})$ coincides with $\mathcal{F}$, i.e.,

$$\mathcal{F}(\mathcal{L})_{|\pi^{-1}(\mathcal{F})} = \mathcal{F} .$$

We will denote by $\mathcal{S}(\mathcal{L})$ the singular set of $\mathcal{F}(\mathcal{L})$.

Suppose that all the irreducible components of $S(\mathcal{L})$ are of the same dimension and that $\pi = \pi_{|\mathcal{S}(\mathcal{L})} : \mathcal{S}(\mathcal{L}) \to \mathrm{Fol}(\mathcal{L})$ is generically finite. If we denote by $\Delta(\mathcal{L})$ the discriminant of the π then for every foliation $\mathcal{F} \in \mathcal{F}(\mathcal{L}) \setminus \Delta(\mathcal{L})$ we can lift closed paths contained in $\mathcal{F}(\mathcal{L}) \setminus \Delta(\mathcal{L})$ to $\mathcal{S}(\mathcal{L})$ inducing a representation

$$\Phi(\mathcal{F}) : \pi_1(\mathcal{F}(\mathcal{L}) \setminus \Delta(\mathcal{L}), \mathcal{F}) \to \mathrm{Perm}(\mathrm{sing}(\mathcal{F})) .$$

Of course if we choose another foliation $\mathcal{F}' \in \mathcal{F}(\mathcal{L}) \setminus \Delta(\mathcal{L})$ as a base point for the lifting of paths we obtain $\Phi(\mathcal{F}')$ which is conjugated to $\Phi(\mathcal{F})$. Therefore we will say the the *monodromy of the singular set of* $\mathcal{F}(\mathcal{L})$ is a subgroup of the symmetric group on k elements, where k is the cardinality of $\mathrm{sing}(\mathcal{F})$, given by the image of $\Phi(\mathcal{F})$ up to conjugacy.

The aim of the appendix is to prove the

Theorem 4. *Let $\mathcal{L}$ be an ample line-bundle over a projective manifold M of dimension m. For $k \gg 0$ the monodromy of the singular set of Fol$(\mathcal{L}^{\otimes k})$ is the full symmetric group in $c_m(\Theta_M \otimes \mathcal{L})$ elements.*

We remark that the strategy of the proof is very similar to the ones presented in [1] and [2]. The carefull reader will note that over $\mathbb{P}^n$ the result is valid for foliations of degree at least 2.

Proof of Theorem 4. Let $S \subset M \times \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ be the singular set, i.e.,

$$S = \{(p, \mathcal{F}) | p \in \mathrm{sing}(\mathcal{F})\}.$$

The set S can also be described as the projectivization of the kernel of the map of vector bundles

$$\begin{array}{rcl} M \times \mathrm{H}^0(M, \Theta_M \otimes \mathcal{L}^{\otimes k}) & \to & TM \otimes \mathcal{L}^{\otimes k} \\ (p, X) & \mapsto & X(p) . \end{array}$$

Since $k \gg 0$ and $\mathcal{L}$ is ample it follows from Serre's vanishing theorem that $\Theta_M \otimes \mathcal{L}^{\otimes k}$ is generated by global sections. In particular the above map has constant rank and its kernel is a sub-bundle of $M \times \mathrm{H}^0(M, \Theta_M \otimes \mathcal{L}^{\otimes k})$ of codimension equal to $\dim M$. It follows that $S \subset M \times \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ is a smooth irreducible subvariety and that the projection $\pi : S \to \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ is surjective and generically finite. The irreducibility of S implies that the monodromy of π is 1-transitive.

First Step: The monodromy group is 2-transitive. Let p be an arbitrary point in M and let $\mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})_p \subset \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ be the set of foliations having p as a singularity. If

$$S_p = \{(q,\mathcal{F}) \in M \setminus \{p\} \times \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})_p \mid q \in \mathrm{sing}(\mathcal{F})\}.$$

then as before S_p is the projectivization of the kernel of Φ,

$$\begin{array}{rcl}\Phi : U \times V & \to & TU \otimes \mathcal{L}^{\otimes k} \\ (z,X) & \mapsto & X(z)\end{array}$$

where $U = M \setminus \{p\}$, $V = \mathrm{H}^0(M, \Theta_{M,p} \otimes \mathcal{L}^{\otimes k})$ and $\Theta_{M,p}$ is the subsheaf of Θ_M generated by vector fields vanishing at p. Clearly $\Theta_{M,p}$ is a coherent sheaf and hence we can apply again Serre's vanishing theorem to assure that S_p is a smooth irreducible subvariety of $M \setminus \{p\} \times \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})_p$ and that $\pi_p : S_p \to F\mathrm{ol}(\mathcal{L}^{\otimes k})_p$ is surjective and generically finite. As before the monodromy of π_p is thus transitive.

Let G be the monodromy group of π and (p_1,q_1) and (p_2,q_2) be two pairs of the points in $M \times M$. Then, from the 1-transivity of G, there exists $\alpha \in G$ such that $\alpha(p_1) = p_2$. From the discussion above on the monodromy of π_p it follows that there exists $\beta \in G$ such that $\beta(p_2) = p_2$ and $\beta(q_1) = q_2$.

We have just proved that G, the monodromy group of π, is 2-transitive.

Sencond Step: The monodromy group contains a transposition. First consider the local situation. Let X and Y be germs of holomorphic vector fields on a neighborhood of $0 \in \mathbb{C}^2$. Suppose that 0 is a singularity of multiplicity 2 of X and that $Y(0) \neq 0$. Consider the equation

$$(X + tY)(s(t)) = 0$$

with boundary value $s(0) = 0$ where $s \in \mathbb{C}[[t]]$ is a formal power series. Deriving with respect to t we obtain that

$$DX(s(0)) \cdot s'(0) + Y(0) = 0\,.$$

When $Y(0)$ is not contained in the image of $DX(0)$ then the above equation has no solutions and in particular the local monodromy is generated by the transposition. As an example of this situation one can take $X = x\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + \dots$ and $Y = \frac{\partial}{\partial y}$, where

$$\mathrm{sing}(X + tY) = (0, \pm\sqrt{-t}).$$

Back to the global situation suppose first that there exists $\mathcal{F} \in \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ with one singularity with the 2-jet equal to the 2-jet of X and all other singularities with multiplicity one. Since $\Theta_M \otimes \mathcal{L}^{\otimes k}$ is generated by global sections there exists $Y \in \mathrm{H}^0(M, \Theta_M \otimes \mathcal{L}^{\otimes k})$ such that $Y(p)$ is not in the image of $DX(p)$. The local discussion above shows that G, the monodromy group of π, contains a transposition.

Let p be a point of M and m_p its ideal sheaf. If we consider the inclusion of $\Theta_M \otimes m_p^3$ into Θ_M then we will define $J_p^2\Theta_M$ as the cokernel of this inclusion. More succinctly the sequence

$$0 \to \Theta_M \otimes m_p^3 \to \Theta_M \to J_p^2\Theta_M \to 0$$

is exact. It is clear from the definition that $J_p^2\Theta_M$ is supported on p and its sections are 2-jets of vector fields at p. Again from Serre's vanishing Theorem $\mathrm{H}^1(M, \Theta_M \otimes m_p^3 \otimes \mathcal{L}^{\otimes k}) = 0$ and consequently the map

$$\mathrm{H}^0(M, \Theta_M \otimes \mathcal{L}^{\otimes k}) \to \mathrm{H}^0(M, J_p^2\Theta_M)$$

is surjective. Thus there are foliations in $\mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ with arbitrary 2-jet. One can use the arguments applied in §6.5 to assure that there exists $\mathcal{F} \in \mathbb{F}\mathrm{ol}(\mathcal{L}^{\otimes k})$ with one singularity with the 2-jet equal to the 2-jet of X and all other singularities with multiplicity one.

Conclusion. To conclude the argument is well-known. Let (p_1, q_1) and (p_2, q_2) be pairs of singularities in $\mathrm{sing}(\mathcal{F})$. Suppose that G contains the transposition $\tau = (p_1\ q_1)$. Since G is 2-transitive there exists $\alpha \in G$ such that $\alpha(p_1) = p_2$ and $\alpha(q_1) = q_2$. Since $\alpha\tau\alpha^{-1} = (p_2\ q_2)$ we conclude that G contains all the transpositions in the full symmetric group. This is sufficient to prove Theorem 4. □

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Alcides Lins Neto
email: alcides@impa.br
IMPA
Estrada Dona Castorina,110
22460-320 Jardim Botânico
Rio de Janeiro
Brasil

Jorge Vitório Pereira
email: jvp@impa.br
IMPA
Estrada Dona Castorina,110
22460-320 Jardim Botânico
Rio de Janeiro
Brasil