ISOLATED SINGULARITIES OF SOLUTIONS TO THE YAMABE EQUATION

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Abstract. In this paper we study the asymptotic behavior of local solutions to the Yamabe equation near an isolated singularity, when the metric is not necessarily conformally flat. We are able to prove, when the dimension is less than or equal to 5, that any solution is asymptotic to a rotationally symmetric Fowler solution. We also prove refined asymptotics if deformed Fowler solutions are allowed in the expansion.

1. Introduction

Let g be a smooth Riemannian metric on the unit ball $B_1^n(0) \subset \mathbb{R}^n$, where $n \geq 3$. In this paper we will consider positive solutions to the Yamabe equation

$$\Delta_g u - \frac{n-2}{4(n-1)} R_g u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \tag{1.1}$$

in the punctured ball $\Omega = B_1^n(0) \setminus \{0\}$. Here Δ_g denotes the Laplace-Beltrami operator of the metric g, and R_g denotes its scalar curvature. Our primary interest will be to describe the asymptotic behavior of such a solution near the isolated singularity.

The geometric motivation comes from the fact that a solution u to the equation (1.1) gives rise to the metric $\tilde{g} = u^{\frac{4}{n-2}} g$ of constant scalar curvature $R_{\tilde{g}} = n(n-1)$. Therefore the asymptotics of these local solutions is related to the global problem known as the *Singular Yamabe Problem.* Given a compact Riemannian manifold (M^n, g), with $R_g > 0$, and a finite set $\Gamma \subset M$, it consists in studying the conformal deformations of g which are complete in $M \setminus \Gamma$, and have constant positive scalar curvature. The existence of these conformal metrics was proved by R. Schoen when the manifold M is the standard sphere $\mathbb{S}^n$ and the set Γ has at least two points (see [14]). In [12], Mazzeo and Pacard developed a different construction for this problem. Other related works are [11], [13], [15].

The issue of deriving asymptotics for local solutions to equation (1.1) was considered, in the case of a flat background metric, by L. Caffarelli,

B. Gidas and J. Spruck in [2](see also [3]). In this case, they prove that the local models, when 0 is a nonremovable singularity, are given by the radial solutions of

$$\Delta u_0 + \frac{n(n-2)}{4} u_0^{\frac{n+2}{n-2}} = 0 \text{ in } \mathbb{R}^n \setminus \{0\} \tag{1.2}$$

which blow-up at 0, referred to as the Fowler solutions (or Delaunay-type solutions). Here Δ denotes the Euclidean Laplacian. More precisely, their result states that, given any solution $u > 0$ to

$$\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \text{ in } B_1^n(0) \setminus \{0\}, \tag{1.3}$$

either u can be smoothly extended to the origin or there is a Fowler solution u_0 such that

$$u(x) = (1 + o(1))u_0(x) \text{ as } x \to 0.$$

Their proof relies on a complicated version of the Alexandrov reflection method, and it was later simplified by N. Korevaar, R. Mazzeo, F. Pacard and R. Schoen in [7], where they also improve the $o(1)$ remainder term to a $O(|x|^\alpha)$, for some $\alpha > 0$. See also [1] for a related result on the subcritical equation.

Another interesting problem consists in studying local singular solutions to the prescribed scalar curvature equation

$$\Delta u + K(x) u^{\frac{n+2}{n-2}} = 0, \tag{1.4}$$

where K is a positive C^1 function defined on a neighborhood around 0. The equation (1.4) can be seen as a perturbation of the equation (1.3), and so one can ask under what conditions on K the Fowler solutions still serve as asymptotic models. This question has been studied by C. C. Chen and C. S. Lin (see [4], [5], [6], [9]), whose work has inspired some of the techniques employed in our paper.

The main motivation of the present work was to determine whether these asymptotic results could be extended to a more general setting, namely, for an arbitrary background metric. The following theorem gives an affirmative answer in low dimensions.

Theorem 1.1. *Assume $3 \leq n \leq 5$ and let $u > 0$ be a solution to the equation (1.1) in $B_1^n(0) \setminus \{0\}$. If u has a nonremovable singularity at 0, then there exists a Fowler solution u_0 such that*

$$u(x) = (1 + O(|x|^\alpha))u_0(x)$$

as $x \to 0$, for some $\alpha > 0$.

There is a certain analogy between equations (1.1) and (1.4), in the sense that the first one can also be considered as a perturbation of the Euclidean equation (1.3). In this spirit, Theorem 1.1 is saying that, at least in low dimensions, the asymptotic behavior of local solutions to equation (1.1) is still described by the standard radial solutions of equation (1.2). It remains an interesting and open question to determine whether this result is true in higher dimensions.

Once we have established the convergence to a radial Fowler solution, we can use the arguments of Section 5 of [7] to improve the decay of the remainder term by allowing deformed Fowler solutions. This family of solutions is parametrized by a vector $a \in \mathbb{R}^n$:

$$u_{0,a}(x) = \left| \frac{x}{|x|} - a|x| \right|^{2-n} u_0 \left(|x| \left| \frac{x}{|x|} - a|x| \right|^{-1} \right).$$

The precise statement is:

Theorem 1.2. *Suppose $u > 0$ is a solution to the equation (2.1) in $B_1^n(0) \setminus \{0\}$. If $3 \leq n \leq 5$, then there exists a deformed Fowler solution $u_{0,a}$ such that*

$$u(x) = (1 + O(|x|^\gamma))u_{0,a}(x)$$

as $x \to 0$, for some $\gamma > 1$.

Let us now briefly describe the strategy used in the paper. First we need to establish the fundamental upper bound

$$u(x) \leq c d_g(x,0)^{\frac{2-n}{2}}. \tag{1.5}$$

In order to prove that, we will apply the Moving Planes Method. The difficulty relies on the fact that our equation (1.1) has no symmetries. It turns out that, when $3 \leq n \leq 5$, we can overcome that by constructing appropriate auxiliary functions (see [8] for a similar technique in dimensions 3 and 4). An important consequence of the bound (1.5) is that solutions have to satisfy a uniform spherical Harnack inequality around the singularity (see Corollary 4 in Section 2 below).

Then we study the Pohozaev integrals

$$P(r,u) = \int_{\partial B_r} \left(\frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{1}{2} r |\nabla u|^2 + r \left|\frac{\partial u}{\partial r}\right|^2 + \frac{(n-2)^2}{8} r u^{\frac{2n}{n-2}}\right) d\sigma_r,$$

and use a Pohozaev-type identity to show the invariant

$$P(u) = \lim_{r \to 0} P(r,u)$$

is well-defined. The fundamental result here is the following removable singularity theorem:

Theorem 1.3. *Assume* $3 \le n \le 5$ *and let* $u > 0$ *be a solution to the equation (1.1) in* $B_1^n(0) \setminus \{0\}$*. Then* $P(u) \le 0$*. Moreover,* $P(u) = 0$ *if and only if 0 is a removable singularity.*

As a consequence of Theorem 1.3, we can apply elliptic theory to establish the lower bound

$$u(x) \ge c_1 d_g(x,0)^{\frac{2-n}{2}}, \tag{1.6}$$

where $c_1 > 0$. Using the bounds (1.5), (1.6), we can use a scaling argument to prove that solutions are asymptotically symmetric. The precise statement of the Theorem 1.1 will follow from somewhat delicate arguments, originally due to Leon Simon in a different context, relying on the growth properties of Jacobi fields.

This paper is organized as follows. In Section 2 we apply the Moving Planes Method to prove the upper bound (1.5). In Section 3 we define the Pohozaev invariant of a solution, proving Theorem 1.3 and the lower bound (1.6). In Section 4 we prove Theorem 1.1. In Section 5 we describe how to achieve the refined asymptotics of Theorem 1.2.

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2. Upper bound near a singularity

Let g be a smooth Riemannian metric in geodesic normal coordinates on the unit ball $B_1^n(0) \subset \mathbb{R}^n$, $n \ge 3$. We are interested in studying positive solutions to the Yamabe equation

$$\Delta_g u - \frac{n-2}{4(n-1)} R_g u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \tag{2.1}$$

in the punctured ball $\Omega = B_1^n(0) \setminus \{0\}$. Here Δ_g denotes the Laplace-Beltrami operator of the metric g and R_g denotes its scalar curvature. The linear operator $L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g$ is called the *conformal Laplacian* of the metric g. The equation (2.1) has geometrical meaning, namely, the metric $\tilde{g} = u^{\frac{4}{n-2}} g$ has constant scalar curvature equal to $n(n-1)$. This is due to the formula $R_{\tilde{g}} = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} L_g u$.

Since a punctured ball is conformally diffeomorphic to a half cylinder, sometimes it will be convenient to use the cylindrical background. In other words, consider the conformal diffemorphism

$$\Phi : (\mathbb{R} \times \mathbb{S}^{n-1}, dt^2 + d\theta^2) \to (\mathbb{R}^n \setminus \{0\}, \delta)$$

defined by $\Phi(t,\theta) = e^{-t}\theta$. Then $\Phi^*\delta = e^{-2t}(dt^2 + d\theta^2)$.

Define $\hat{g} = e^{2t}\Phi^* g$ and $v(t,\theta) = e^{\frac{2-n}{2}t}u(e^{-t}\theta) = |x|^{\frac{n-2}{2}}u(x)$, where $t = -\log|x|$ and $\theta = \frac{x}{|x|}$. Since the scalar curvature of the metric $\Phi^*\tilde{g} = (e^{\frac{2-n}{2}t}u \circ \Phi)^{\frac{4}{n-2}}\hat{g}$ is constant equal to $n(n-1)$, we obtain the equation

$$L_{\hat{g}}v + \frac{n(n-2)}{4}v^{\frac{n+2}{n-2}} = 0, \tag{2.2}$$

for $t > 0$. Observe that if u is defined for every $|x| < r_0$, then v is defined for every $t > -\log r_0$.

It will be useful to compute $R_{\hat{g}}$. Using that $\hat{g} = (e^{\frac{n-2}{2}t})^{\frac{4}{n-2}}\Phi^* g$, we get

$$\begin{aligned} R_{\hat{g}} &= -\frac{4(n-1)}{n-2}(e^{\frac{n-2}{2}t})^{-\frac{n+2}{n-2}}L_{\Phi^* g}(e^{\frac{n-2}{2}t}) \\ &= -\frac{4(n-1)}{n-2}(e^{-\frac{n+2}{2}t})(\Delta_{\Phi^* g}(e^{\frac{n-2}{2}t}) - \frac{n-2}{4(n-1)}R_{\Phi^* g}e^{\frac{n-2}{2}t}) \\ &= -\frac{4(n-1)}{n-2}(e^{-\frac{n+2}{2}t})(\Delta_g(|x|^{\frac{2-n}{2}}) \circ \Phi - \frac{n-2}{4(n-1)}R_{\Phi^* g}e^{\frac{n-2}{2}t}), \end{aligned}$$

so

$$R_{\hat{g}} = (n-2)(n-1) + 2(n-1)e^{-t}\frac{\partial_r(\sqrt{g})}{\sqrt{g}} \circ \Phi + e^{-2t}R_g \circ \Phi. \tag{2.3}$$

The main result of this section is the following theorem, which establishes an upper bound near the isolated singularity.

Theorem 2.1. *Assume that u is a positive smooth solution of (2.1) in $\Omega = B_1^n(0) \setminus \{0\}$. If $3 \leq n \leq 5$, then there exists a constant $c > 0$ such that*

$$u(x) \leq c d_g(x,0)^{\frac{2-n}{2}} \tag{2.4}$$

for $0 < d_g(x,0) < \frac{1}{2}$.

Proof. Given $x_0 \in \Omega$, $|x_0| < \frac{1}{2}$ and $0 < s < \frac{1}{4}$ so that $\overline{B_s(x_0)} \subset \Omega$, define

$$f(x) = (s - d_g(x,x_0))^{\frac{n-2}{2}}u(x)$$

for $x \in B_s(x_0)$. Here $B_s(x_0)$ denotes the metric ball with respect to the backgrond metric g. It suffices to show that there exists a positive constant C such that any such f satisfies $f(x) \leq C$ in $B_s(x_0)$. This is because $f(x_0) = s^{\frac{n-2}{2}}u(x_0)$ and we can choose $s = \frac{|x_0|}{2}$.

The proof will be by contradiction so assume there is no such constant C. Then we can find a sequence of points $x_{0,i}$ and positive numbers s_i so that, if $x_{1,i}$ denotes the maximum point of the corresponding

f_i, we have

$$f_i(x_{1,i}) \to \infty.$$

Since $2^{n-2} f_i(x) \le u(x)$ we conclude that $u(x_{1,i}) \to \infty$ as well. In particular we also get that $x_{1,i} \to 0$ as $i \to \infty$.

Set $\varepsilon_i = u(x_{1,i})^{-\frac{2}{n-2}}$ and define

$$\tilde{u}_i(y) = \varepsilon_i^{\frac{n-2}{2}} u(\exp_{x_{1,i}}(\varepsilon_i y)),$$

so that $\tilde{u}_i(0) = 1$. If $d_g(x, x_{1,i}) \le r_i = \frac{1}{2}(s_i - d_g(x_{1,i}, x_{0,i}))$ then it follows from $f_i(x) \le f_i(x_{1,i}) = (2r_i\varepsilon_i^{-1})^{\frac{n-2}{2}}$ that $u(x) \le 2^{\frac{n-2}{2}} u(x_{1,i})$. Therefore we get that $\tilde{u}_i(y) \le 2^{\frac{n-2}{2}}$ on the ball $|y| < r_i\varepsilon_i^{-1} \to \infty$.

It is not difficult to check that

$$L_{\tilde{g}_i}\tilde{u}_i + \frac{n(n-2)}{4}\tilde{u}_i^{\frac{n+2}{n-2}} = 0,$$

for every $|y| < r_i\varepsilon_i^{-1}$, where $(\tilde{g}_i)_{kl}(y) = g_{kl}(\varepsilon_i y)$. Here g_{kl} denote the components of the metric g when written in normal coordiantes around $x_{1,i}$. Standard elliptic theory then implies that, after passing to a subsequence, the $\tilde{u}_i$ converge in the C^2 norm on compact subsets of $\mathbb{R}^n$ to a positive solution $\tilde{u}_0$ to

$$\Delta\tilde{u}_0 + \frac{n(n-2)}{4}\tilde{u}_0^{\frac{n+2}{n-2}} = 0,$$

which satisfies $\tilde{u}_0(0) = 1$ and $\tilde{u}_0(y) \le 2^{\frac{n-2}{2}}$ for every $y \in \mathbb{R}^n$. Here Δ denotes the Euclidean Laplacian. By a well-known theorem due to Caffarelli, Gidas and Spruck [2] we can conclude that there exists $\eta > 0$ and $y_0 \in \mathbb{R}^n$ such that

$$\tilde{u}_0(y) = \left(\frac{2\eta}{1 + \eta^2|y - y_0|^2}\right)^{\frac{n-2}{2}}.$$

Because of the conditions on $\tilde{u}_0$ we also get that $\frac{1}{2} \le \eta \le 1$ and $|y_0| \le 1$.

Since $\tilde{u}_0$ has a nondegenerate maximum point at y_0 there will be a sequence $y_i \to y_0$ such that y_i is a nondegenerate maximum point of $\tilde{u}_i$. We can assume $|y_i| \le 2$ and therefore there will be a corresponding local maximum point $x_{2,i}$ of u satisfying $d_g(x_{2,i}, x_{1,i}) \le 2\varepsilon_i$. If we redefine the functions $\tilde{u}_i$ replacing $x_{1,i}$ by $x_{2,i}$ we get as before that a subsequence $\tilde{u}_i$ converges in the C^2 norm on compact subsets of $\mathbb{R}^n$ to

$$\tilde{u}_0(y) = \left(\frac{1}{1 + \frac{1}{4}|y|^2}\right)^{\frac{n-2}{2}}.$$

Note that, by construction, we have that $|x_{2,i}| < \frac{7}{8}$ so we can consider $\tilde{u}_i$ as defined for $|y| \le \frac{1}{16}\varepsilon_i^{-1}$, with a possible singularity at some point on the sphere of radius $|x_{2,i}|\varepsilon_i^{-1} \to \infty$, where now $\varepsilon_i = u(x_{2,i})^{-\frac{2}{n-2}}$.

Now it is convenient to shift to the cylindrical background. Let us introduce

$$v_i(t,\theta) = |y|^{\frac{n-2}{2}}\tilde{u}_i(y),$$

where $t = -\log|y|$ and $\theta = \frac{y}{|y|}$.

This function is defined for $t > -\log(\frac{1}{16}\varepsilon_i^{-1})$, with a singularity at some point (t_i', θ_i'), $t_i' = -\log(|x_{2,i}|\varepsilon_i^{-1})$. We can also define $v_0(t) = |y|^{\frac{n-2}{2}}u_0(y)$, and it is not difficult to check that

$$v_0(t) = (e^t + \frac{1}{4}e^{-t})^{\frac{2-n}{2}}.$$

Since $\tilde{u}_i \to \tilde{u}_0$ in the C^2_{loc} topology, we know that given any $R > 0$ the inequalities

$$\begin{aligned}
|v_i(t,\theta) - v_0(t)| &\le R^{-1}e^{\frac{2-n}{2}t},\\
|\partial_t v_i(t,\theta) - v_0'(t)| &\le R^{-1}e^{\frac{2-n}{2}t},\\
|\partial_t^2 v_i(t,\theta) - v_0''(t)| &\le R^{-1}e^{\frac{2-n}{2}t},\\
|\partial_{\theta_k} v_i(t,\theta)| &\le R^{-1}e^{-\frac{n}{2}t},\\
|\partial^2_{\theta_k\theta_l} v_i(t,\theta)| &\le R^{-1}e^{-\frac{n}{2}t},
\end{aligned}$$

are satisfied for $t \ge -\log R$ and sufficiently large i.

In particular $\partial_t v_i(-\log 3, \theta) > 0$ for all $\theta \in \mathbb{S}^{n-1}$.

Let $\delta > 0$ be a small number, independent of i, to be chosen later.

We will apply the Alexandrov technique to v_i on the region

$$\Gamma_i = [-\log(\delta\varepsilon_i^{-1}), \infty) \times \mathbb{S}^{n-1},$$

reflecting across the spheres $\{\lambda\} \times \mathbb{S}^{n-1}$.

It is not difficult to check, from the definition, that

$$v_i(-\log(\delta\varepsilon_i^{-1}), \theta) \ge c(\delta) > 0$$

for every $\theta \in \mathbb{S}^{n-1}$.

In what follows we will occasionally drop the subscript i to simplify the notation.

Define

$$v_\lambda(t,\theta) = v(2\lambda - t, \theta).$$

If

$$b_\lambda = \frac{n(n-2)}{4}\left(\frac{v^{\frac{n+2}{n-2}} - v_\lambda^{\frac{n+2}{n-2}}}{v - v_\lambda}\right),$$

and

$$Q_\lambda = (L_{\hat{g}_\lambda} - L_{\hat{g}})(v_\lambda),$$

we obtain

$$L_{\hat{g}}(v - v_\lambda) + b_\lambda(v - v_\lambda) = Q_\lambda$$

on $\Gamma_\lambda \backslash \{(t_i', \theta_i')\}$, where $\Gamma_\lambda = [-\log(\delta\varepsilon^{-1}), \lambda] \times \mathbb{S}^{n-1}$. Here $\hat{g}_\lambda$ denotes the pull-back of the metric $\hat{g}$ by the reflection across the sphere $\{\lambda\} \times \mathbb{S}^{n-1}$. It is important to note also that $b_\lambda \geq 0$.

We now wish to construct an auxiliary family of functions $h_\lambda = h_\lambda(t)$, defined on Γ_λ, satisfying the following properties:

$$h_\lambda(\lambda) = 0; \tag{2.5}$$

$$h_\lambda \geq 0; \tag{2.6}$$

$$L_{\hat{g}} h_\lambda \geq Q_\lambda; \tag{2.7}$$

$$h_\lambda \leq v - v_\lambda \text{ if } \lambda \text{ is sufficiently large.} \tag{2.8}$$

When such a family exists, we define

$$w_\lambda = v - v_\lambda - h_\lambda.$$

Therefore

$$L_{\hat{g}} w_\lambda + b_\lambda w_\lambda = Q_\lambda - L_{\hat{g}} h_\lambda - b_\lambda h_\lambda \leq 0,$$

so we can apply the Maximum Principle where $w_\lambda \geq 0$.

Note also that $w_\lambda(\lambda, \theta) = 0$ for every $\theta \in \mathbb{S}^{n-1}$.

Claim 1. *(Moving Planes Method)* If there exists h_λ satisfying the properties (2.5)-(2.8) for every $\lambda \geq -\log 3$, then there exist $\lambda_0 > -\log 3$, and $\theta_0 \in \mathbb{S}^{n-1}$ such that

$$w_{\lambda_0}(-\log(\delta\varepsilon^{-1}), \theta_0) = 0.$$

In order to prove Claim 1, define

$$\lambda_0 = \inf\{\lambda_1 : w_\lambda(t, \theta) \geq 0 \text{ in } \Gamma_\lambda, \forall \lambda \geq \lambda_1\}.$$

Note that condition (2.8) guarantees this set is nonempty.

Since $\partial_t v_i(-\log 3, \theta) > 0$ for all $\theta \in \mathbb{S}^{n-1}$, we know that $\lambda_0 > -\log 3$.

Suppose the claim is false. Since, by continuity, $w_{\lambda_0} \geq 0$ in Γ_{λ_0}, we would have that $w_{\lambda_0}(-\log(\delta\varepsilon_i^{-1}), \theta) > 0$ for every $\theta \in \mathbb{S}^{n-1}$. By the Maximum Principle, we also know that $w_{\lambda_0}(t, \theta) > 0$ for every $-\log(\delta\varepsilon_i^{-1}) < t < \lambda_0$ and $\theta \in \mathbb{S}^{n-1}$. It is important to note that since

$$L_{\hat{g}} w_{\lambda_0} \leq -b_{\lambda_0} w_{\lambda_0} \leq 0,$$

the function w_{λ_0} has a positive lower bound near the singularity (t_i', θ_i').

Hence, from the definition of λ_0, we know that there exist sequences $\lambda_j \uparrow \lambda_0$, $t_j \to t^*$, $\theta_j \to \theta^*$ such that (t_j, θ_j) is an interior minimum point of w_{λ_j} with $w_{\lambda_j}(t_j, \theta_j) < 0$. Taking the limit we get $w_{\lambda_0}(t^*, \theta^*) = 0$ and $\nabla w_{\lambda_0}(t^*, \theta^*) = 0$. Therefore $t^* = \lambda_0$, but this is a contradiction to the Hopf's lemma. This proves Claim 1.

Now we need to estimate Q_λ.

Claim 2. There exists a constant $c_1 > 0$, not depending on δ, such that $|Q_\lambda(t,\theta)| \le q_\lambda(t) = c_1 \varepsilon^2 e^{\frac{n-6}{2}t} e^{(2-n)\lambda}$.

First, from equation (2.3),

$$\begin{aligned} R_{\hat{g}}(t,\theta) &= (n-2)(n-1) + 2(n-1)e^{-t}\frac{\partial_r(\sqrt{\tilde{g}})}{\sqrt{\tilde{g}}}(e^{-t}\theta) \\ &\qquad + e^{-2t} R_{\tilde{g}}(e^{-t}\theta) \\ &= (n-2)(n-1) + 2(n-1)\varepsilon e^{-t}\frac{\partial_r(\sqrt{g})}{\sqrt{g}}(\varepsilon e^{-t}\theta) \\ &\qquad + \varepsilon^2 e^{-2t} R_g(\varepsilon e^{-t}\theta). \end{aligned}$$

Using $R_{\hat{g}_\lambda}(t,\theta) = R_{\hat{g}}(2\lambda - t, \theta)$, $v_\lambda(t,\theta) = O(e^{\frac{2-n}{2}(2\lambda - t)})$, and $\frac{\partial_r(\sqrt{g})}{\sqrt{g}} = O(r)$, we have

$$\begin{aligned} |R_{\hat{g}_\lambda} - R_{\hat{g}}| v_\lambda(t,\theta) &\le C\varepsilon^2 e^{-2t} e^{\frac{2-n}{2}(2\lambda - t)} \\ &= C\varepsilon^2 e^{\frac{n-6}{2}t} e^{(2-n)\lambda}. \end{aligned}$$

Now one needs to observe that $\hat{g} = dt^2 + d\theta^2 + O(\varepsilon^2 e^{-2t})$, since $\hat{g} = e^{2t}\Phi^*\tilde{g}$ and $\tilde{g}_{ij} = \delta_{ij} + O(\varepsilon^2 |y|^2)$ in normal coordinates. It follows that

$$|(\Delta_{\hat{g}_\lambda} - \Delta_{\hat{g}})(v_\lambda)|(t,\theta) \le C\varepsilon^2 e^{\frac{n-6}{2}t} e^{(2-n)\lambda},$$

proving Claim 2.

Now we will turn to the construction of h_λ.

Claim 3. Suppose $3 \le n \le 5$, and let $\gamma > 0$ be a small number. Then there exists a family $h_\lambda = h_\lambda(t)$ satisfying the properties (2.5)-(2.8) and such that

$$h_\lambda(-\log(\delta\varepsilon^{-1})) \le c_3 \max\{\varepsilon^{\frac{n-2}{2}}, \varepsilon^{\frac{6-n}{2} - \gamma}\}, \tag{2.9}$$

for some $c_3 = c_3(\delta) > 0$.

Given a small $\gamma > 0$, let $\overline{L}$ be the linear operator:

$$\overline{L}(f) := f'' + \gamma f' - ((\frac{n-2}{2})^2 + \gamma) f.$$

Let $\gamma_1 = \frac{8-n}{2}\gamma > 0$ and $a(n) = \frac{1}{2(4-n) - \gamma_1}$.

Now define

$$h_\lambda(t) = a(n)c_1\varepsilon^2 e^{(2-n)\lambda} e^{\frac{n-6}{2}t}(1 - e^{(4-n-\gamma_2)(t-\lambda)}), \tag{2.10}$$

where $\gamma_2 > 0$ is chosen so that the function $e^{(\frac{2-n}{2}-\gamma_2)t}$ is in the kernel of $\overline{L}$.

Note that γ_1, γ_2 are also small.

It is clear that $h_\lambda(\lambda) = 0$, and

$$\overline{L}(h_\lambda) = q_\lambda.$$

It is also possible to check that

$$h_\lambda \geq 0 \text{ and } h'_\lambda \leq 0 \text{ in } (-\infty, \lambda],$$

from definition (2.10).

Now

$$L_{\hat{g}} h_\lambda = h''_\lambda + O(\varepsilon^2 e^{-2t})h'_\lambda - ((\frac{n-2}{2})^2 + O(\varepsilon^2 e^{-2t}))h_\lambda,$$

since $\hat{g} = dt^2 + d\theta^2 + O(\varepsilon^2 e^{-2t})$.

If $\delta^2 \leq \gamma$, then

$$L_{\hat{g}} h_\lambda \geq \overline{L}(h_\lambda) = q_\lambda \geq |Q_\lambda|$$

for $t \geq -\log(\delta\varepsilon^{-1})$.

The estimate (2.9) follows from the definition (2.10), since

$$\begin{aligned} h_\lambda(-\log(\delta\varepsilon^{-1})) = a(n)c_1(&\delta^{\frac{6-n}{2}} e^{(2-n)\lambda}\varepsilon^{\frac{n-2}{2}} \\ &-\delta^{\frac{n-2}{2}+\gamma_2} e^{(-2+\gamma_2)\lambda}\varepsilon^{\frac{6-n}{2}-\gamma_2}). \end{aligned} \tag{2.11}$$

It remains to prove the condition (2.8): $w_\lambda \geq 0$ in Γ_λ for sufficiently large λ.

The computation (2.11) and the fact that $h'_\lambda \leq 0$ imply

$$\max_{\Gamma_\lambda} h_\lambda \to 0 \tag{2.12}$$

as $\lambda \to \infty$, where δ and ε are fixed.

Suppose δ, ε, and t_0 are fixed, and assume t_0 is large. The function v has a positive lower bound on $\Gamma_{t_0} = [-\log(\delta\varepsilon_i^{-1}), t_0] \times \mathbb{S}^{n-1}$, and we know that

$$v_\lambda(t) \leq Ce^{\frac{2-n}{2}(2\lambda - t)}.$$

Therefore it follows, from the fact (2.12), that $w_\lambda \geq 0$ in Γ_{t_0} for sufficiently large λ.

Hence we now have to show that $w_\lambda \geq 0$ for $t \in [t_0, \lambda]$.

Let us estimate h'_λ.

First

$$h'_\lambda = \frac{n-6}{2} da(n) e^{\frac{n-6}{2}t}(1 - \frac{2}{n-6}(\frac{2-n}{2} - \gamma_2)e^{(4-n-\gamma_2)(t-\lambda)}),$$

where $d = c_1\varepsilon^2 e^{(2-n)\lambda}$. Hence

$$|h'_\lambda(t)| \leq C(\delta, \varepsilon, t_0)(e^{(2-n)\lambda} + e^{(-2+\gamma_2)\lambda})$$

for $t \in [t_0, \lambda]$.

If $3 \leq n \leq 5$ and $t \in [t_0, \lambda]$, then we get

$$|h'_\lambda(t)| = e^{\frac{2-n}{2}\lambda} o(\lambda)$$

as $\lambda \to \infty$, for fixed δ, ε, t_0.

Now

$$\frac{\partial w_\lambda}{\partial t}(t,\theta) = \frac{\partial v}{\partial t}(t,\theta) + \frac{\partial v}{\partial t}(2\lambda - t, \theta) - h'_\lambda(t),$$

so that, when $t \in [t_0, \lambda]$,

$$\frac{\partial w_\lambda}{\partial t}(t,\theta) \leq -Ce^{\frac{2-n}{2}\lambda} - h'_\lambda(t) \leq -\frac{C}{2} e^{\frac{2-n}{2}\lambda} < 0.$$

Since $w_\lambda(\lambda, \theta) = 0$ for every $\theta \in \mathbb{S}^{n-1}$, we see that $w_\lambda \geq 0$ if $t \in [t_0, \lambda]$, when λ is sufficiently large. This finishes the proof of the claim.

We will now derive a contradiction, if $3 \leq n \leq 5$.

From Claim 3 (existence of h_λ), and Claim 1 (Moving Planes Method), we know

$$w_{\lambda_0}(-\log(\delta\varepsilon^{-1}), \theta_0) = 0$$

for some $\lambda_0 > -\log 3$ and $\theta_0 \in \mathbb{S}^{n-1}$.

Then

$$0 < c(\delta) \leq v(-\log(\delta\varepsilon^{-1}), \theta_0) = (v_{\lambda_0} + h_{\lambda_0})(-\log(\delta\varepsilon^{-1}), \theta_0),$$

and so

$$0 < c(\delta) \leq \tilde{c}(\delta)\varepsilon^{\frac{n-2}{2}} + h_{\lambda_0}(-\log(\delta\varepsilon^{-1})). \tag{2.13}$$

But, from inequality (2.9),

$$h_{\lambda_0}(-\log(\delta\varepsilon^{-1})) \leq c\varepsilon^{\frac{1}{2}}$$

when $n = 3$,

$$h_{\lambda_0}(-\log(\delta\varepsilon^{-1})) \leq c\varepsilon^{1-\gamma_2}$$

when $n = 4$, and

$$h_{\lambda_0}(-\log(\delta\varepsilon^{-1})) \leq c\varepsilon^{\frac{1}{2}-\gamma_2}$$

when $n = 5$.

We get a contradiction from inequality (2.13), after $\varepsilon \to 0$. This completes the proof of the theorem.

□

As a consequence of the upper bound we get the following estimates.

Corollary 2.2. *Suppose u is a positive smooth solution of (2.1) in $\Omega = B_1^n(0) \setminus \{0\}$, $3 \leq n \leq 5$. Then there exists a constant $c_1 > 0$ such that*

$$\max_{|x|=r} u \leq c_1 \min_{|x|=r} u \tag{2.14}$$

for every $0 < r < \frac{1}{4}$. Moreover, $|\nabla u| \leq c_1 |x|^{-1} u$ and $|\nabla^2 u| \leq c_1 r^{-2} u$.

The inequality (2.14) is usually referred to as the *spherical Harnack inequality.*

Proof. Define $u_r(y) = r^{\frac{n-2}{2}} u(ry)$, for every $0 < r < \frac{1}{4}$, and $|y| < r^{-1}$. The Theorem 2.1 then implies that $u_r(y) \leq c|y|^{\frac{2-n}{2}}$ for $|y| < \frac{1}{2} r^{-1}$. In particular, if $\frac{1}{2} \leq |y| \leq \frac{3}{2}$, we have that $u_r(y) \leq 2^{\frac{n-2}{2}} c$.

Moreover $L_{g_r} u_r + \frac{n(n-2)}{4} u_r^{\frac{n+2}{n-2}} = 0$, where $(g_r)_{ij}(y) = g_{ij}(ry)$. The Harnack inequality for linear elliptic equations and standard elliptic theory imply there exists $c_1 > 0$, not depending on r, such that

$$\max_{|x|=1} u_r \leq c_1 \min_{|x|=1} u_r,$$

and $|\nabla u_r| + |\nabla^2 u_r| \leq c_1 u_r$ on the sphere of radius 1.

This finishes the proof of the corollary. □

3. Pohozaev invariants and removable singularities

In this section we will define the Pohozaev invariant of a solution and prove a removable singularity theorem. As a consequence we will derive a fundamental lower bound near the isolated singularity.

Given a positive solution u to the equation (2.1) in $B_1^n(0) \setminus \{0\}$, the Pohozaev identity (see [10]) says that

$$P(r,u) - P(s,u) = -\int_{B_r \setminus B_s} \left(\frac{n-2}{2} u + x \cdot \nabla u\right)(L_g - \Delta)(u)\,dx \tag{3.1}$$

for $0 < s \leq r < 1$, where

$$\begin{aligned} P(r,u) = \int_{\partial B_r} \Big(\frac{n-2}{2} u \frac{\partial u}{\partial r} &- \frac{1}{2} r |\nabla u|^2 \\ &+ r \left|\frac{\partial u}{\partial r}\right|^2 + \frac{(n-2)^2}{8} r u^{\frac{2n}{n-2}}\Big) d\sigma_r. \end{aligned} \tag{3.2}$$

In the case of the Euclidean metric, the identity is saying that $P(r,u)$ does not depend on r, and therefore is an invariant of the solution u.

In order to define this invariant in a more general setting, we need the upper bounds given by Theorem 2.1 and Corollary 2.2. In fact, since $g_{ij} = \delta_{ij} + O(|x|^2)$, we will have

$$|(\frac{n-2}{2}u + x \cdot \nabla u)(L_g - \Delta)(u)| \leq c|x|^{2-n}, \tag{3.3}$$

and the Pohozaev identity tells us the limit

$$P(u) = \lim_{r \to 0} P(r, u)$$

exists. The number $P(u)$ is called the *Pohozaev invariant* of the solution u.

We can now state our removable singularity theorem.

Theorem 3.1. *Assume $3 \leq n \leq 5$ and let $u > 0$ be a solution to the equation (2.1) in $B_1^n(0) \setminus \{0\}$. Then $P(u) \leq 0$. Moreover, $P(u) = 0$ if and only if 0 is a removable singularity.*

Proof. Let us suppose $P(u) \geq 0$. The result will follow once we prove that, in this case, 0 is a removable singularity, and therefore $P(u) = 0$.

Claim 1. $\liminf_{x \to 0} u(x)|x|^{\frac{n-2}{2}} = 0$.

Suppose not. Then there exist positive constants c_1, c_2 such that

$$c_1|x|^{\frac{2-n}{2}} \leq u(x) \leq c_2|x|^{\frac{2-n}{2}},$$

where the second inequality above follows from Theorem 2.1.

Choose any sequence $r_j \to 0$, and define

$$u_j(x) = r_j^{\frac{n-2}{2}} u(r_j x).$$

Then $c_1|x|^{\frac{2-n}{2}} \leq u_j(x) \leq c_2|x|^{\frac{2-n}{2}}$ and

$$L_{g_j} u_j + \frac{n(n-2)}{4} u_j^{\frac{n+2}{n-2}} = 0$$

in $B_{r_j^{-1}}(0) \setminus \{0\}$, where $(g_j)_{kl}(x) = g_{kl}(r_j x)$. Elliptic theory then implies that there exists a subsequence, also denoted by u_j, which converges, in compact subsets of $\mathbb{R}^n \setminus \{0\}$, to a solution u_0 of

$$\Delta u_0 + \frac{n(n-2)}{4} u_0^{\frac{n+2}{n-2}} = 0.$$

Since $u_0(x) \geq c_1|x|^{\frac{2-n}{2}}$, u_0 is singular at the origin and then a result in [2] implies that u_0 has to be one of the rotationally symmetric Fowler solutions (see Section 2 in [7] for more details). In particular, $P(u_0) < 0$. This is a contradiction, because

$$P(u_0) = P(u_0, 1) = \lim_{j \to \infty} P(u_j, 1) = \lim_{j \to \infty} P(u, r_j) = P(u) \geq 0.$$

Claim 2. $\lim_{x\to 0} u(x)|x|^{\frac{n-2}{2}} = 0$.

In what follows we will denote by c any positive constant, and subscripts will sometimes mean differentiation.

Let us denote the average of the function u over ∂B_r by

$$\overline{u}(r) = \fint_{\partial B_r} u,$$

and define $w(t) = \overline{u}(r)r^{\frac{n-2}{2}}$, where $t = -\ln r$. Notice that Theorem 2.1 and Corollary 2.2 imply $w(t) \leq c$.

Then

$$\overline{u}_r = \fint_{\partial B_r} \frac{\partial u}{\partial r},$$

and since

$$w_t = -\overline{u}_r r^{\frac{n}{2}} - \frac{n-2}{2} w,$$

we also get that $|w_t| \leq c$.

Choosing a fixed $s < r$, we see that

$$\overline{u}_{rr} = (\fint_{\partial B_r} \frac{\partial u}{\partial r})_r = (1-n)r^{-1}\fint_{\partial B_r} \frac{\partial u}{\partial r} + \sigma_{n-1}^{-1} r^{1-n}(\int_{B_r\setminus B_s} \Delta u)_r$$

$$= (1-n)r^{-1}\fint_{\partial B_r} \frac{\partial u}{\partial r} + \sigma_{n-1}^{-1} r^{1-n}(\int_{B_r\setminus B_s} (\Delta - L_g)(u) - K\int_{B_r\setminus B_s} u^{\frac{n+2}{n-2}})_r$$

$$= (1-n)r^{-1}\overline{u}_r + \fint_{\partial B_r} (\Delta - \Delta_g)(u) + c(n)\fint_{\partial B_r} R_g u - K\fint_{\partial B_r} u^{\frac{n+2}{n-2}}.$$

By the spherical Harnack inequality (see Corollary 2.2),

$$C^{-1}\overline{u}^{\frac{n+2}{n-2}} \leq K\fint_{\partial B_r} u^{\frac{n+2}{n-2}} \leq C\overline{u}^{\frac{n+2}{n-2}},$$

and

$$|\fint_{\partial B_r} (\Delta - \Delta_g)(u) + c(n)\fint_{\partial B_r} R_g u| \leq c\overline{u}.$$

From

$$w_{tt} = -(\overline{u}_r r^{\frac{n}{2}})_r \frac{dr}{dt} - \frac{n-2}{2} w_t,$$

we have

$$w_{tt} - (\frac{n-2}{2})^2 w = (\fint_{\partial B_r} (\Delta - \Delta_g)(u) + c(n)\fint_{\partial B_r} R_g u$$
$$-K\fint_{\partial B_r} u^{\frac{n+2}{n-2}})r^{\frac{n+2}{2}},$$

and then

$$-c_1 w^{\frac{n+2}{n-2}} - ce^{-2t}w \le w_{tt} - (\frac{n-2}{2})^2 w \le -c_2 w^{\frac{n+2}{n-2}} + ce^{-2t}w. \tag{3.4}$$

The first inequality in (3.4) implies that there exists $\varepsilon_0 > 0$ such that $w_{tt}(t) > 0$ whenever $w(t) \le \varepsilon_0$, and t is sufficiently large.

Let us suppose, by contradiction, that $\limsup_{x\to 0} u(x)|x|^{\frac{n-2}{2}} > 0$. Since $\liminf_{x\to 0} u(x)|x|^{\frac{n-2}{2}} = 0$, we can choose $\varepsilon_0 > 0$ sufficiently small so that we are able to construct sequences $\bar{t}_i \le t_i \le t_i^*$ with $\lim_{i\to\infty} \bar{t}_i = +\infty$, such that $w(\bar{t}_i) = w(t_i^*) = \varepsilon_0$, $w_t(t_i) = 0$, and $\lim_{i\to\infty} w(t_i) = 0$.

Let us now introduce a function H satisfying

$$H_t(t) = e^{-2t} w(t) w_t(t).$$

Since w is monotone in each of the intervals $[\bar{t}_i, t_i], [t_i, t_i^*]$, it is also invertible. Therefore, depending on which of the intervals we choose to consider, we can define

$$g(w) = (\frac{n-2}{2})^2 w^2 - cw^{\frac{2n}{n-2}} - cF(w),$$

where $F(w) = H(t)$.

Then, from the inequalities (3.4), it is not difficult to check that, for $t_i \le t \le t_i^*$,

$$(w_t^2 - g(w))_t \ge 0.$$

Hence, in that interval, $w_t^2 - g(w) \ge -g(w(t_i))$, and so

$$t - t_i = \int_{w(t_i)}^{w(t)} \frac{dt}{dw} \le \int_{w(t_i)}^{w(t)} \frac{dw}{\sqrt{g(w) - g(w(t_i))}}.$$

Introducing the variable $\eta = \frac{w(t)}{w(t_i)}$, we get

$$t - t_i \le \int_1^{\frac{w(t)}{w(t_i)}} \frac{d\eta}{\sqrt{\bar{g}(\eta) - \bar{g}(1)}} = \int_1^{\frac{w(t)}{w(t_i)}} \sqrt{\frac{\eta^2 - 1}{\bar{g}(\eta) - \bar{g}(1)}} \frac{d\eta}{\sqrt{\eta^2 - 1}} \tag{3.5}$$

where

$$\bar{g}(\eta) = (\frac{n-2}{2})^2 \eta^2 - cw(t_i)^{\frac{4}{n-2}} \eta^{\frac{2n}{n-2}} - c\frac{F(w(t_i)\eta)}{w(t_i)^2}.$$

In order to estimate the last integral note that

$$\begin{aligned}(\frac{\eta^2 - 1}{\bar{g}(\eta) - \bar{g}(1)})^{\frac{1}{2}} \le \frac{2}{n-2} \quad &+ \quad c\frac{w(t_i)^{\frac{4}{n-2}}(\eta^{\frac{2n}{n-2}} - 1)}{\eta^2 - 1} \\ &+ \quad c|\frac{F(w(t_i)\eta) - F(w(t_i))}{w(t_i)^2(\eta^2 - 1)}|.\end{aligned}$$

First, since $1 \leq \eta \leq \frac{w(t)}{w(t_i)} \leq \frac{\varepsilon_0}{w(t_i)}$, we have that

$$\frac{w(t_i)^{\frac{4}{n-2}}(\eta^{\frac{2n}{n-2}}-1)}{\eta^2-1} \leq cw(t_i)^{\frac{4}{n-2}}\eta^{\frac{4}{n-2}} \leq c\varepsilon_0^{\frac{4}{n-2}},$$

and we observe that

$$w^{\frac{4}{n-2}}(t_i)\int_1^{\frac{w(t)}{w(t_i)}} \frac{\eta^{\frac{4}{n-2}}}{\sqrt{\eta^2-1}}d\eta \leq c.$$

Then, since $F_w = H_t\frac{dt}{dw} = e^{-2t}w$, one can check that

$$\left|\frac{F(w(t_i)\eta)-F(w(t_i))}{w(t_i)^2(\eta^2-1)}\right| \leq ce^{-2t_i}.$$

Finally, since

$$\int_1^{\frac{w(t)}{w(t_i)}} \frac{d\eta}{\sqrt{\eta^2-1}} \leq c + \ln\frac{w(t)}{w(t_i)},$$

we obtain

$$\int_1^{\frac{w(t)}{w(t_i)}} \frac{d\eta}{\sqrt{\overline{g}(\eta)-\overline{g}(1)}} \leq (\frac{2}{n-2}+ce^{-2t_i})\ln\frac{w(t)}{w(t_i)} + c.$$

Now, from inequality (3.5), we get

$$t - t_i \leq (\frac{2}{n-2}+ce^{-2t_i})\ln\frac{w(t)}{w(t_i)} + c. \tag{3.6}$$

In order to estimate $t-t_i$ from below, we first observe that the second inequality in (3.4) implies that

$$w_{tt} \leq ((\frac{n-2}{2})^2 + ce^{-2t_i})w.$$

Then the function $w_t^2 - ((\frac{n-2}{2})^2 + ce^{-2t_i})w^2$ is decreasing in (t_i, t_i^*), and therefore

$$w_t^2 - ((\frac{n-2}{2})^2 + ce^{-2t_i})w^2 \leq -((\frac{n-2}{2})^2 + ce^{-2t_i})w^2(t_i).$$

Hence

$$\frac{dt}{dw} \geq ((\frac{n-2}{2})^2 + ce^{-2t_i})^{-\frac{1}{2}}(w^2 - w^2(t_i))^{-\frac{1}{2}},$$

and then

$$t - t_i = \int_{w(t_i)}^{w(t)} \frac{dt}{dw} \geq (\frac{2}{n-2} - ce^{-2t_i})\int_{w(t_i)}^{w(t)} \frac{dw}{\sqrt{w^2-w^2(t_i)}}.$$

Together with inequality (3.6), we get, for $t_i \le t \le t_i^*$, that

$$(3.7)\quad (\frac{2}{n-2} - ce^{-2t_i}) \ln \frac{w(t)}{w(t_i)} \le t - t_i \le (\frac{2}{n-2} + ce^{-2t_i}) \ln \frac{w(t)}{w(t_i)} + c.$$

Similarly one can prove that, for $\bar{t}_i \le t \le t_i$,

$$(3.8)\quad (\frac{2}{n-2} - ce^{-2\bar{t}_i}) \ln \frac{w(t)}{w(t_i)} \le t_i - t \le (\frac{2}{n-2} + ce^{-2\bar{t}_i}) \ln \frac{w(t)}{w(t_i)} + c.$$

Claim 3. Along $|x| = r_i$,

$$\begin{aligned} u(x) &= \overline{u}(r_i)(1 + \mathrm{o}(1)) \\ |\nabla u(x)| &= -\overline{u}'(r_i)(1 + \mathrm{o}(1)). \end{aligned} \tag{3.9}$$

Let $r_i = e^{-t_i}$ and define $v_i(y) = r_i^{\frac{n-2}{2}} u(r_i y)$. Since $\overline{v}_i(1) = w(t_i) \to 0$ we get from the Harnack inequality that v_i converges to 0 uniformly in compact subsets of $\mathbb{R}^n \setminus \{0\}$. Hence, if we define $h_i(y) = v_i(p)^{-1} v_i(y)$, we will have

$$L_{g_i} h_i + \frac{n(n-2)}{4} v_i(p)^{\frac{4}{n-2}} h_i^{\frac{n+2}{n-2}} = 0,$$

where $p = (1, 0, \ldots, 0)$ and $(g_i)_{kl} = g_{kl}(r_i y)$. By elliptic estimates we know there exists a subsequence h_j which converges in the C^2_{loc} topology to a nonnegative harmonic function h defined in $\mathbb{R}^n \setminus \{0\}$. Then $h(y) = a|y|^{2-n} + b$, and $a = b = \frac{1}{2}$ since $h(p) = 1$ and $\partial_r (h(r) r^{\frac{n-2}{2}}) = 0$ at $r = 1$. This completes the proof of the claim.

Then we have

$$\begin{aligned} P(r_i, u) = \sigma_{n-1}(\frac{1}{2} w_t(t_i)^2 \ &- \ \frac{1}{2}(\frac{n-2}{2})^2 w^2(t_i) \\ &+ \ \frac{(n-2)^2}{8} w^{\frac{2n}{n-2}}(t_i))(1 + o(1)). \end{aligned}$$

Hence

$$P(u) = \lim_{i \to \infty} P(r_i, u) = 0,$$

and moreover

$$w^2(t_i) \le c|P(r_i, u)| \le c(I_1 + I_2), \tag{3.10}$$

where

$$I_1 = \int_{B_{r_i} \setminus B_{r_i^*}} |A(u)| dx,$$

and

$$I_2 = \int_{B_{r_i^*}} |A(u)| dx.$$

Here $A(u) = (\frac{n-2}{2} u + x \cdot \nabla u)((L_g - \Delta)(u))$.

Recall that $|A(u)| \le c|x|^{2-n}$, and therefore

$$I_2 \le c(r_i^*)^2 = ce^{-2t_i^*}.$$

From the first inequality in (3.7), we obtain

$$w(t) \le w(t_i)\exp((\frac{n-2}{2} + ce^{-2t_i})(t-t_i)),$$

which implies

$$u(x) \le cw(t_i)\exp(-(\frac{n-2}{2} + ce^{-2t_i})t_i)r^{2-n-ce^{-2t_i}}. \tag{3.11}$$

Recall that $u \le Cr^{\frac{2-n}{2}}$, $|\nabla u| \le Cr^{-1}u$, and $|\nabla^2 u| \le Cr^{-2}u$, so

$$|A(u)| \le Cr^{\frac{2-n}{2}}u.$$

Using the estimate (3.11) we obtain

$$I_1 \le cw(t_i)e^{\frac{2-n}{2}t_i}\int_{B_{r_i}\setminus B_{r_i^*}} |x|^{3-\frac{n}{2}-n-ce^{-2t_i}}dx,$$

and so

$$I_1 \le cw(t_i)e^{-2t_i}.$$

Therefore, from inequalities (3.10), we get

$$w^2(t_i) \le cw(t_i)e^{-2t_i} + ce^{-2t_i^*}.$$

Passing to subsequences, if necessary, we can suppose either

$$w^2(t_i) \le cw(t_i)e^{-2t_i}, \tag{3.12}$$

or

$$w^2(t_i) \le ce^{-2t_i^*}. \tag{3.13}$$

Define $L_i = -\frac{2}{n-2}\log w(t_i)$ and choose $\delta > 0$ small. Then, from the first inequality in (3.8), we get

$$t_i - \bar{t}_i \ge (1-\delta)L_i - c, \tag{3.14}$$

and adding to that the first inequality in (3.7), we get

$$t_i^* - \bar{t}_i \ge (2-2\delta)L_i - c. \tag{3.15}$$

If inequality (3.12) holds, then $w(t_i) \le ce^{-2t_i}$ and so $L_i \ge \frac{4}{n-2}t_i - c$. From inequality (3.14), we get

$$t_i - \bar{t}_i \ge (1-\delta)\frac{4}{n-2}t_i - c,$$

and then

$$\bar{t}_i \le (\frac{n-6}{n-2} + \frac{4\delta}{n-2})t_i + c.$$

If $3 \le n \le 5$, this is a contradiction since, in this case, $\frac{n-6}{n-2} + \frac{4}{n-2}\delta < 0$, and we know that $t_i^* \ge t_i \ge \bar{t}_i \to \infty$ as $i \to \infty$.

If inequality (3.13) holds, then $L_i \ge \frac{2}{n-2} t_i^* + c$. From inequality (3.15), we get

$$\bar{t}_i \le t_i^* - (2 - 2\delta)L_i + c,$$

and so

$$\bar{t}_i \le (\frac{n-6}{n-2} + 2\delta)t_i^* + c.$$

If $3 \le n \le 5$, this is again a contradiction for the same reasons as before.

The claim is proved.

Claim 4. 0 is a removable singularity.

Now we have that $\lim_{t\to\infty} w(t) = 0$.

There exists T_1 so that $w'(t) < 0$ for $t \ge T_1$, since we also have $w_{tt} > 0$.

Given any positive number $\delta > 0$, and by choosing T_1 sufficiently large, we get, from the first inequality in (3.4), that

$$w_{tt} - (\frac{n-2}{2} - \delta)^2 w \ge 0$$

for $t \ge T_1$.

This implies

$$(w_t^2 - (\frac{n-2}{2} - \delta)^2 w^2)_t \le 0,$$

and since $\lim_{t\to\infty} w_t(t) = 0$, we obtain

$$w_t^2 - (\frac{n-2}{2} - \delta)^2 w^2 \ge 0.$$

By integrating we get, for $t \ge T_1$, that

$$w(t) \le w(T_1)\exp(-(\frac{n-2}{2} - \delta)(t - T_1)).$$

Equivalently, there exists $r_0(\delta) > 0$ so that, if $|x| < r_0$, we have

$$u(x) \le c(\delta)|x|^{-\delta}.$$

The above estimate implies, since $\delta > 0$ is arbitrarily small, that $u \in L^p_{\mathrm{loc}}(B_1(0))$ for arbitrarily large p. Elliptic theory then tells us that the function u has to be smooth around the origin. That finishes the proof.

□

As a consequence of the removable singularity theorem, we can now establish a fundamental lower bound.

Corollary 3.2. *Assume $3 \leq n \leq 5$ and let $u > 0$ be a solution to the equation (2.1) in $B_1^n(0) \setminus \{0\}$. If 0 is a nonremovable singularity, then there exists $c > 0$ such that*

$$u(x) \geq c d_g(x,0)^{\frac{2-n}{2}},$$

for $0 < d_g(x,0) < \frac{1}{2}$.

Proof. Suppose the corollary is false.

Then $\liminf_{t\to\infty} w(t) = 0$, where $w(t) = r^{\frac{n-2}{2}}\overline{u}(r)$ and $t = -\log r$, as in the proof of Theorem 3.1. We also have $\limsup_{t\to\infty} w(t) > 0$, otherwise the Claim 4 in Theorem 3.1 would imply 0 is a removable singularity. Therefore there exists a sequence $t_i \to \infty$ such that $w'(t_i) = 0$ and $\lim_{i\to\infty} w(t_i) = 0$. If $r_i = e^{-t_i}$, it is not difficult to check that Claim 3 in Theorem 3.1 will hold and again we will have

$$\begin{aligned} P(r_i,u) = \sigma_{n-1}(\frac{1}{2}w'(t_i)^2 &- \frac{1}{2}(\frac{n-2}{2})^2 w^2(t_i) \\ &+ \frac{(n-2)^2}{8} w^{\frac{2n}{n-2}}(t_i))(1+\mathrm{o}(1)). \end{aligned}$$

But in this case $P(u) = \lim P(r_i,u) = 0$, which is a contradiction. This finishes the proof. $\square$

4. Convergence to a radial solution

In this section we will prove that a local singular solution to the Yamabe equation is asymptotic to a radial Fowler solution, near the nonremovable isolated singularity.

Recall that the positive solutions u_0 to the equation

$$\Delta u_0 + \frac{n(n-2)}{4} u_0^{\frac{n+2}{n-2}} = 0$$

in $\mathbb{R}^n \setminus \{0\}$, with a nonremovable singularity at the origin, are called *Fowler solutions* (see [7]). These functions are rotationally symmetric by a theorem of Caffarelli, Gidas and Spruck (see [2]).

In what follows we state and prove the main theorem of the section.

Theorem 4.1. *Suppose $u > 0$ is a solution to the equation (2.1) in $B_1^n(0) \setminus \{0\}$. If there exist $c_1, c_2 > 0$ such that*

$$c_1|x|^{\frac{2-n}{2}} \leq u \leq c_2|x|^{\frac{2-n}{2}},$$

then there exists a Fowler solution u_0 such that

$$u(x) = (1 + O(|x|^\alpha))u_0(x)$$

as $x \to 0$, for some $\alpha > 0$.

Proof. Since u has a nonremovable singularity at the origin, we know, from Theorem 3.1, that $P(u) < 0$.

In what follows we will work in the cylindrical setting, so $v(t,\theta) = |x|^{\frac{n-2}{2}} u(x)$, where $t = -\log|x|$ and $\theta = \frac{x}{|x|}$. Hence

$$c_1 \le v(t,\theta) \le c_2,$$

for $t > -\log 2$.

Given any sequence $\tau_i \to \infty$, consider $v_i(t,\theta) = v(t+\tau_i,\theta)$. Since $\hat{g} \to dt^2 + d\theta^2$ as $t \to \infty$, standard elliptic estimates imply that there exists a subsequence v_j which converges, in the C^2_{loc} topology, to a positive solution to

$$\partial_t^2 v_0 + \Delta_\theta v_0 - \frac{(n-2)^2}{4} v_0 + \frac{n(n-2)}{4} v_0^{\frac{n+2}{n-2}} = 0,$$

defined on the whole cylinder. Since any such limit v_0 is a Fowler solution, which does not depend on θ, we necessarily have that any angular derivative $\partial_\theta v$ converges uniformly to zero as $t \to \infty$.

In fact, as $t \to \infty$, we get

$$\begin{aligned} v(t,\theta) &= \overline{v}(t)(1+o(1)) \\ |\nabla v(t,\theta)| &= -\overline{v}'(t)(1+o(1)). \end{aligned} \tag{4.1}$$

In order to see this suppose the first equality above is not true. Then there exist $\varepsilon > 0$ and sequences $\tau_i \to \infty$, $\theta_i \to \theta \in \mathbb{S}^{n-1}$ such that

$$\left|\frac{v(\tau_i,\theta_i)}{\overline{v}(\tau_i)} - 1\right| \ge \varepsilon$$

for every $i \ge 1$. This is a contradiction because, after passing to a subsequence, v_i converges to a rotationally symmetric Fowler solution v_0. The second equality follows from similar arguments.

In the cylindrical setting the Pohozaev integral $P(v,t) = P(u,e^{-t})$ becomes

$$P(v,t) = \int_{t\times\mathbb{S}^{n-1}} \Big(\frac{1}{2}(\partial_t v)^2 - \frac{1}{2}|\nabla_\theta v|^2 - \frac{(n-2)^2}{8} v^2 + \frac{(n-2)^2}{8} v^{\frac{2n}{n-2}}\Big) d\sigma_1.$$

Hence

$$P(v_0) = P(v_0,0) = \lim_{j\to\infty} P(v_j,0) = \lim_{j\to\infty} P(v,\tau_j) = P(v).$$

Therefore the Pohozaev invariant of the limit function does not depend on the sequence τ_j, and hence any such sequence gives rise, in the limit, to a function $v_{0,T}(t) = v_0(t+T)$, for some T.

Given any Fowler solution v_0, the nontrivial solutions to the linearized equation

$$L_{\mathrm{cil}}\varphi + \frac{n(n+2)}{4} v^{\frac{4}{n-2}}\varphi = 0$$

on the cylinder $\mathbb{R}\times\mathbb{S}^{n-1}$ are called *Jacobi fields*. Here $L_{\mathrm{cil}} = \partial_t^2 + \Delta_{\mathbb{S}^{n-1}} - \frac{(n-2)^2}{4}$ is the cylindrical conformal Laplacian.

In what follows we will need some basic results about these Jacobi fields which can be found in [7].

Let T_0 be the period of v_0, and $A_\tau = \sup_{t\geq 0}|\partial_\theta v_\tau|$, where $v_\tau(t,\theta) = v(t+\tau,\theta)$.

Claim 1. For every $c>0$, there exists a positive integer N such that, for any $\tau>0$, either

(1) $A_\tau \leq ce^{-2\tau}$ or
(2) A_τ is attained at some point in $I_N\times\mathbb{S}^{n-1}$, where $I_N = [0, NT_0]$.

Suppose the Claim is false. Then there exist sequences $\tau_j, s_j \to \infty$, $\theta_j \in \mathbb{S}^{n-1}$ such that $|\partial_\theta v_{\tau_j}|(s_j,\theta_j) = A_{\tau_j}$, and $A_{\tau_j}^{-1}e^{-2\tau_j} \leq c^{-1}$ as $j\to\infty$. Then we can translate back further by s_j and define $\tilde{v}_j(t,\theta) = v_{\tau_j}(t+s_j,\theta)$. If $\varphi_j = A_{\tau_j}^{-1}\partial_\theta\tilde{v}_j$, then one can check that

$$L_{\mathrm{cil}}\varphi_j + \frac{n(n+2)}{4}\tilde{v}_j^{\frac{4}{n-2}}\varphi_j = A_{\tau_j}^{-1}e^{-2(\tau_j+s_j)}O(e^{-2t}).$$

Now we can use elliptic theory to extract a subsequence φ_j which converges in compact subsets to a nontrivial and bounded Jacobi field φ. Since φ has no zero eigencomponent relative to Δ_θ, we get a contradiction because no such Jacobi field can exist. This proves the claim.

Now we will turn to another way of obtaining Jacobi fields. Suppose we have, then, $v_j(t)\to v_0(t+T)$ as $j\to\infty$, and define

$$w_j(t,\theta) = v_j(t,\theta) - v_0(t+T).$$

Set $\eta_j = \max_{I_N}|w_j|$, $\overline{\eta}_j = \eta_j + e^{-(2-\delta)\tau_j}$ and $\varphi_j = \overline{\eta_j}^{-1}w_j$, where $\delta>0$ is a small number.

Then

$$L_{\hat{g}_j}\varphi_j + \frac{n(n-2)}{4}\frac{v_j^p - v_{0,T}^p}{v_j - v_{0,T}}\varphi_j = \overline{\eta}_j^{-1}E_j,$$

where $E_j = (L_{\mathrm{cil}} - L_{\hat{g}_j})v_{0,T}$, and $\hat{g}_j$ is the translated $\hat{g}$. Note that $E_j = e^{-2\tau_j}O(e^{-2t})$ as $t\to\infty$.

In this case it is not difficult to check that, passing to a subsequence, φ_j converges to a solution φ of the equation

$$L_{\mathrm{cil}}\varphi + \frac{n(n+2)}{4}v_{0,T}^{\frac{4}{n-2}}\varphi = 0$$

on the whole cylinder. We claim that this Jacobi field is bounded for $t \geq 0$. In order to see this we write

$$\varphi = a^+\psi_0^+ + a^-\psi_0^- + \tilde{\varphi},$$

where ψ_0^+ and ψ_0^- are linearly independent Jacobi fields corresponding to the θ-independent eigencomponent, and $\tilde{\varphi}$ denotes the projection onto the orthogonal complement. We are following the notation in [7], according to which $\psi_0^+(t) = v_{0,T}'(t)$ and ψ_0^- comes from the variation of the Pohozaev invariant (or the necksize) of the solution $v_{0,T}$. It follows that ψ_0^+ is bounded, and ψ_0^- is linearly growing. First we show that $\tilde{\varphi}$ is bounded by proving that $\partial_\theta\tilde{\varphi} = \partial_\theta\varphi$ is bounded for $t \geq 0$. The function $\partial_\theta\varphi$ is the limit of $\overline{\eta}_j^{-1}\partial_\theta v_j$, and we can suppose $\partial_\theta\varphi$ is nontrivial, otherwise the result is imediate. In this case we know, from the previous claim, that A_j is attained in $I_N \times \mathbb{S}^{n-1}$ for large j, so

$$\sup_{t\geq 0}(\overline{\eta}_j^{-1}|\partial_\theta v_j|) = \sup_{t\in I_N}(\overline{\eta}_j^{-1}|\partial_\theta v_j|) \leq C.$$

Therefore $\tilde{\varphi}$ is bounded for $t \geq 0$, hence exponentially decaying.

Now we will show that $a^- = 0$. We have

$$v_j = v_{0,T} + \overline{\eta}_j(a^+\psi_0^+ + a^-\psi_0^- + \tilde{\varphi}) + o(\overline{\eta}_j).$$

But, from the Pohozaev identity (3.1) and inequality (3.3), we have

$$P(v_j, 0) = P(v, \tau_j) = P(v) + O(e^{-2\tau_j}) = P(v_{0,T}) + O(e^{-2\tau_j}).$$

Since $\lim_{j\to\infty}(\overline{\eta}_j^{-1}e^{-2\tau_j}) = 0$, we would have a contradiction in case $a^- \neq 0$. Thus φ is bounded for $t \geq 0$.

We will now show that there exists some T so that the difference between v and $v_{0,T}$ goes to zero as $t \to \infty$. Define $v_\tau(t,\theta) = v(t+\tau,\theta)$ and $w_\tau(t,\theta) = v_\tau(t,\theta) - v_0(t)$. Let $B > 0$ be a fixed constant and I_N be the interval as before. Set also $\eta(\tau) = \max_{I_N}|w_\tau|$ and $\overline{\eta}(\tau) = \eta(\tau) + e^{-(2-\delta)\tau}$.

Claim 2. If τ is sufficiently large and $\overline{\eta}(\tau)$ is sufficiently small, then there exists s with $|s| \leq B\overline{\eta}(\tau)$ so that $\overline{\eta}(\tau + NT_0 + s) \leq \frac{1}{2}\overline{\eta}(\tau)$.

Suppose the claim is false. Then there exist sequences $\tau_j \to \infty$ and $\overline{\eta}_j = \overline{\eta}(\tau_j) \to 0$ such that for every s satisfying $|s| \leq B\overline{\eta}_j$ we have $\overline{\eta}(\tau_j + NT_0 + s) > \frac{1}{2}\overline{\eta}_j$. Define $\varphi_j = \overline{\eta}_j^{-1}w_{\tau_j}$, and, as before, we can prove φ_j converges, up to a subsequence, in the C^∞_{loc} topology, to a Jacobi field φ, bounded for $t \geq 0$.

As before, we can write $\varphi = a^+\psi_0^+ + \tilde{\varphi}$, where $\tilde{\varphi}$ is exponentially decaying and a^+ is uniformly bounded, independently on τ_j, because $|\varphi| \leq 1$ on I_N. Set $s_j = -\overline{\eta}_j a^+$, whose absolute value is less than $B\overline{\eta}_j$ if we choose B large enough.

Therefore

$$\begin{aligned} w_{\tau_j+s_j}(t,\theta) &= v(t+\tau_j-\overline{\eta}_j a^+,\theta)-v_0(t) \\ &= v_{\tau_j}(t-\overline{\eta}_j a^+,\theta)-v_0(t-\overline{\eta}_j a^+)+v_0(t-\overline{\eta}_j a^+)-v_0(t) \\ &= \overline{\eta}_j\varphi_j(t-\overline{\eta}_j a^+,\theta)-\overline{\eta}_j a^+\psi_0^+ + o(\overline{\eta_j}) \\ &= w_{\tau_j}(t,\theta)-\overline{\eta_j}a^+\psi_0^+ + o(\overline{\eta}_j), \end{aligned}$$

for $t\in[0,2NT_0]$.

As a consequence,

$$w_{\tau_j+s_j}=\overline{\eta}_j\tilde{\varphi}+o(\overline{\eta_j})$$

for $t\in[0,2NT_0]$.

Then

$$\max_{I_N}|w_{\tau_j+s_j+NT_0}|=\max_{[NT_0,2NT_0]}|w_{\tau_j+s_j}|=\overline{\eta}_j\max_{[NT_0,2NT_0]}|\tilde{\varphi}|+o(\overline{\eta_j}).$$

Since $\tilde{\varphi}$ is exponentially decaying at a fixed rate, we can choose N large enough so that the last equalities imply

$$\max_{I_N}|w_{\tau_j+s_j+NT_0}|\le\frac{1}{4}\overline{\eta}_j.$$

But also, if N is large enough,

$$e^{-(2-\delta)(\tau_j+NT_0+s_j)}\le e^{-(2-\delta)NT_0}\overline{\eta}_j\le\frac{1}{4}\overline{\eta}_j.$$

Thus $\overline{\eta}(\tau_j+NT_0+s_j)\le\frac{1}{2}\overline{\eta}(\tau_j)$, which is a contradiction, finishing the claim.

Now we are ready to prove, by means of an iterative argument, that there exists σ such that $w_\sigma\to 0$ as $t\to\infty$. First there exists τ_0 satisfying the hypotheses of Claim 2 and such that $B\overline{\eta}(\tau_0)\le\frac{1}{2}NT_0$. Let s_0 be chosen as above. Define

$$\sigma_j=\tau_0+\sum_{i=0}^{j-1}s_i,\tau_j=\tau_{j-1}+s_{j-1}+NT_0.$$

Then $\overline{\eta}(\tau_j)\le 2^{-j}\overline{\eta}(\tau_0)$, and $|s_j|\le 2^{-j-1}NT_0$. Hence there exists the limit $\sigma=\lim\sigma_j$, $\sigma\le\tau_0+NT_0$, and we claim σ is the correct translation parameter.

In fact,

$$\begin{aligned} w_\sigma(t,\theta) &= v(t+\sigma,\theta)-v_0(t) \\ &= v(t+\sigma,\theta)-v(t+\sigma_j,\theta)+v(t+\sigma_j,\theta)-v_0(t) \\ &= w_{\tau_j}([t],\theta)+O(2^{-j}), \end{aligned}$$

where $t = jNT_0 + [t]$, $[t] \in I_N$. Since $\eta(\tau_j) \leq \overline{\eta}(\tau_j) \leq 2^{-j}\overline{\eta}(0)$, we will have $|w_\sigma(t,\theta)| = O(2^{-j})$ and then

$$|w_\sigma(t,\theta)| \leq C_1 e^{-\frac{\log 2}{NT_0}t}.$$

This finishes the proof of the Theorem. □

Since we have already established the required bounds in low dimensions, we get:

Corollary 4.2. *Suppose $u > 0$ is a solution to the equation (2.1) in $B_1^n(0) \setminus \{0\}$. If $3 \leq n \leq 5$, then there exists a Fowler solution u_0 such that*

$$u(x) = (1 + O(|x|^\alpha))u_0(x)$$

as $x \to 0$, for some $\alpha > 0$.

5. Refined asymptotics

In this section we will improve the order of the remainder terms in Theorem 4.1 and Corollary 4.2 by allowing deformed Fowler solutions in the asymptotics. The arguments are essentially the same as in Section 5 of [7], so we will skip part of the details.

We will now work with an n-parameter family of deformations of the radial Fowler solutions (see [7] for more details). These arise by pulling back the Fowler solutions (when seen as conformal factors on the sphere minus the two poles) through a composition of three conformal diffeomorphisms of the sphere: first reflecting across the equator, then applying a parabolic translation fixing the north pole, and finally reflecting back across the equator. We can parametrize this family by a vector $a \in \mathbb{R}^n$ in the cylindrical setting:

$$v_{0,a}(t,\theta) = |\theta - ae^{-t}|^{\frac{2-n}{2}} v_0(t + \log|\theta - ae^{-t}|).$$

Since the Yamabe equation is conformally invariant, these deformations are still solutions to

$$\partial_t^2 v_{0,a} + \Delta_\theta v_{0,a} - \frac{(n-2)^2}{4} v_{0,a} + \frac{n(n-2)}{4} v_{0,a}^{\frac{n+2}{n-2}} = 0$$

on a punctured cylinder.

Let us now state our refined asymptotics result.

Recall $u_{0,a}(x) = |x|^{\frac{2-n}{2}} v_{0,a}(-\log|x|, \frac{x}{|x|})$.

Theorem 5.1. *Suppose $u > 0$ is a solution to the equation (2.1) in $B_1^n(0) \setminus \{0\}$. If $3 \leq n \leq 5$, then there exists a deformed Fowler solution $u_{0,a}$ such that*

$$u(x) = (1 + O(|x|^\gamma))u_{0,a}(x)$$

as $x \to 0$, for some $\gamma > 1$.

Remark. The following arguments only assume the conclusion of Theorem 4.1. Therefore the result is still true in higher dimensions if $c_1|x|^{\frac{2-n}{2}} \le u(x) \le c_2|x|^{\frac{2-n}{2}}$.

Proof. Using Theorem 4.1, we will write $v = v_0 + w$, where v_0 is a Fowler solution and w is exponentially decaying.

From the cylindrical Yamabe equation (2.2), we obtain

$$L_{cil}w + \frac{n(n+2)}{4} v_0^{\frac{4}{n-2}} w = (L_{cil} - L_{\hat{g}})(v_0 + w) - \frac{n(n-2)}{4} v_0^{\frac{n+2}{n-2}} Q(\frac{w}{v_0}),$$

where $L_{cil} = \partial_t^2 + \Delta_\theta - \frac{(n-2)^2}{4}$ is the cylindrical conformal Laplacian, and $Q(z) = (1+z)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2} z$.

Since $\hat{g} = g_{cil} + O(e^{-2t})$, by repeatedly applying Corollary 1 in [7], we improve the order of decay of w at each step until we reach

$$v(t,\theta) = v_0(t) + (a \cdot x)\psi_1^+(t) + O(e^{-\gamma t}), \tag{5.1}$$

for some $a \in \mathbb{R}^n$, where $\gamma > 1$. Here $\psi_1^+(t) = e^{-t}(-v_0'(t) + \frac{n-2}{2} v_0(t))$.

But one can also check that

$$v_{0,a}(t,\theta) = v_0(t) + (a \cdot x)\psi_1^+(t) + O(e^{-2t}). \tag{5.2}$$

The Theorem 5.1 follows from combining expansions (5.1) and (5.2) together. □

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