

COMPLEX CODIMENSION ONE SINGULAR FOLIATIONS AND GODBILLON-VEY SEQUENCES

DOMINIQUE CERVEAU¹, ALCIDES LINS-NETO², FRANK LORAY¹,
JORGE VITÓRIO PEREIRA² AND FRÉDÉRIC TOUZET¹

ABSTRACT. Let \mathcal{F} be a codimension one singular holomorphic foliation on a compact complex manifold M . Assume that there exists a meromorphic vector field X on M generically transversal to \mathcal{F} . Then, we prove that \mathcal{F} is the meromorphic pull-back of an algebraic foliation on an algebraic manifold N , or \mathcal{F} is transversely projective (in the sense of [18]), improving our previous work [7].

Such a vector field insures the existence of a global meromorphic Godbillon-Vey sequence for the foliation \mathcal{F} . We derive sufficient conditions on this sequence insuring such alternative. For instance, if there exists a finite Godbillon-Vey sequence or if the Godbillon-Vey 3-form $\omega_0 \wedge \omega_1 \wedge \omega_2$ is zero, then \mathcal{F} is the pull-back of a foliation on a surface, or \mathcal{F} is transversely projective (in the sense of [18]). We illustrate these results with many examples.

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Date: September 2006.

¹ IRMAR, Campus de Beaulieu, 35042 Rennes Cedex, France

² IMPA, Estrada Dona Castorina, 110, Horto, Rio de Janeiro, Brasil.

1. INTRODUCTION

Let M be a compact connected complex manifold of dimension $n \geq 2$. A (codimension 1 singular holomorphic) foliation \mathcal{F} on M will be given by a covering of M by open subsets $(U_j)_{j \in J}$ and a collection of integrable holomorphic 1-forms ω_j on U_j , $\omega_j \wedge d\omega_j = 0$, having codimension ≥ 2 zero-set such that, on each non empty intersection $U_j \cap U_k$, we have

$$(*) \quad \omega_j = g_{jk} \cdot \omega_k, \quad \text{with } g_{jk} \in \mathcal{O}^*(U_j \cap U_k).$$

Let $\text{Sing}(\omega_j) = \{p \in U_j ; \omega_j(p) = 0\}$. Condition $(*)$ implies that $\text{Sing}(\mathcal{F}) := \cup_{j \in J} \text{Sing}(\omega_j)$ is a codimension ≥ 2 analytic subset of M . If ω is an integrable meromorphic 1-form on M , $\omega \wedge d\omega = 0$, then we can associate to ω a foliation \mathcal{F}_ω as above. Indeed, at the neighborhood of any point $p \in M$, one can write $\omega = f \cdot \tilde{\omega}$ with f meromorphic, sharing the same divisor with ω ; therefore, $\tilde{\omega}$ is holomorphic with codimension ≥ 2 zero-set and defines \mathcal{F}_ω on the neighborhood of p .

When \mathcal{F} is smooth, we have the following classical notion (see [8]): we say that \mathcal{F} is a *regular transversely projective* foliation if it is locally defined by submersions $f_i : U_i \rightarrow \mathbb{C}P(1)$ on M satisfying the cocycle condition:

$$f_i = \frac{a_{ij}f_j + b_{ij}}{c_{ij}f_j + d_{ij}}, \quad \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} \in PGL(2, \mathbb{C})$$

on any intersection $U_i \cap U_j$. For singular foliations, this classical definition is too restrictive and we will introduce below an alternate algebraic definition allowing singularities.

A *Godbillon-Vey sequence* for \mathcal{F} is a sequence $(\omega_0, \omega_1, \dots, \omega_k, \dots)$ of meromorphic 1-forms on M such that $\mathcal{F} = \mathcal{F}_{\omega_0}$ and the formal 1-form

$$(1) \quad \Omega = dz + \sum_{k=0}^{\infty} \frac{z^k}{k!} \omega_k,$$

is integrable: $\Omega \wedge d\Omega = 0$. In this sense, Ω defines a formal unfolding of \mathcal{F} on the space $(\widehat{\mathbb{C}}, 0) \times M$. This condition is equivalent to

$$(2) \quad d\omega_k = \omega_0 \wedge \omega_{k+1} + \sum_{l=1}^k \binom{l}{k} \omega_l \wedge \omega_{k+1-l}.$$

One can see that ω_{k+1} is well defined by $\omega_0, \omega_1, \dots, \omega_k$ up to the addition by a meromorphic factor of ω_0 . Conversely, $\omega_0, \omega_1, \dots, \omega_k, \omega_{k+1} + f \cdot \omega_0$ is the beginning of another Godbillon-Vey sequence for any $f \in \mathcal{M}(M)$.

Given a meromorphic vector field X on M which is transversal to \mathcal{F} at a generic point, there is a unique meromorphic 1-form ω satisfying $\omega(X) = 1$ and defining the foliation \mathcal{F} . We then define a Godbillon-Vey sequence for \mathcal{F} by setting

$$(3) \quad \omega_k := L_X^{(k)} \omega,$$

where $L_X^{(k)} \omega$ denotes the k^{th} Lie derivative along X of the form ω . In fact, the (formal) foliation \mathcal{H} defined on $(\widehat{\mathbb{C}}, 0) \times M$ by

$$\Omega = dz + \sum_{k=0}^{\infty} \frac{z^k}{k!} \omega_k = dz + e^{zL_X} \omega$$

is obtained by pushing the foliation \mathcal{F} with the time- z -map of X . We have $d\Omega = \Omega \wedge \frac{\partial \Omega}{\partial z}$.

The *length* of a Godbillon-Vey sequence is the minimal $N \in \mathbb{N}^* \cup \{\infty\}$ such that $\omega_k = 0$ for $k \geq N$; in general, the length is infinite. We say that \mathcal{F} is (*singular*) *transversely projective* if it admits a Godbillon-Vey sequence of length ≤ 3 , i.e. there are meromorphic 1-forms $\omega_0 = \omega$, ω_1 and ω_2 on M satisfying

$$(4) \quad \begin{cases} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= \omega_0 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_2 \end{cases}$$

This definition was introduced by B. Scardua in [18]. This means that, outside the polar and singular set of the ω_i 's, the foliation \mathcal{F} is regular transversely projective in the classical sense of [8] and this projective structure has “reasonable singularities”. Throughout the paper, by transversely projective foliation, we will refer to this later singular, algebraic definition; we will use the word “regular” when we refer to the classical, geometric definition. When $\omega_2 = 0$ (i.e. $d\omega_1 = 0$) or $\omega_1 = 0$ (i.e. $d\omega_0 = 0$), we respectively say that \mathcal{F}_ω is actually *transversely affine* or *euclidian*. Again, these notions are algebraic versions of classical smooth notions as well (see [8]).

Let (ω_k) be a Godbillon-Vey sequence for \mathcal{F} and let n be the smallest integer such that $\omega_0 \wedge \cdots \wedge \omega_n \equiv 0$, $1 \leq n \leq m = \dim(M)$. Then, the non trivial n -form $\Theta = \omega_0 \wedge \cdots \wedge \omega_{n-1}$ is closed and defines a singular codimension n foliation \mathcal{F}_Θ whose leaves are contained in those of \mathcal{F} , $\mathcal{F}_\Theta \subset \mathcal{F}$. We note that Θ does not depend on the choice of $\omega_1, \dots, \omega_{n-1}$ in the Godbillon-Vey sequence, but does depend on ω_0 . Let $\mathcal{M}(M)$ be the field of meromorphic functions on M and let $K \subset \mathcal{M}(M)$ be the subfield of first integrals for \mathcal{F}_Θ :

$$K = \{f \in \mathcal{M}(M) ; df \wedge \Theta \equiv 0\}.$$

This field K is integrally closed and, by [19], there exists a meromorphic map $\pi : M \dashrightarrow N$ onto an algebraic manifold N such that $K = \pi^*\mathcal{M}(N)$; in particular, the dimension $\dim(N)$ equals the transcendence degree of K/\mathbb{C} and we have $1 \leq \dim(N) \leq n \leq m = \dim(M)$. In the case Θ is a meromorphic volume form, that is $n = m$, we have $K = \mathcal{M}(M)$ and N is the Algebraic Reduction of M (see [23]). We note that the fibration \mathcal{G} induced on M by the reduction map $\pi : M \dashrightarrow N$ contains \mathcal{F}_Θ as a sub-foliation and may have any codimension $n \leq \dim(\mathcal{G}) \leq a(M) \leq m$, the algebraic dimension of M . Our main theorem is the

Theorem 1.1. *Let \mathcal{F} be a codimension 1 singular foliation on a compact complex manifold. Assume that \mathcal{F} admits a global meromorphic Godbillon-Vey sequence (ω_k) and let Θ , K and $\pi : M \dashrightarrow N$ like above. Then we are in one of the (non exclusive) following cases:*

- \mathcal{F} is the pull-back by $\pi : M \dashrightarrow N$ of a foliation $\underline{\mathcal{F}}$ on N ,
- or \mathcal{F} is transversely projective.

We are in the former case when the fibers of π are contained in the leaves of \mathcal{F} ; this so happens for a generic foliation \mathcal{F} on $M = \mathbb{C}\mathbb{P}(2)$: π is just the identity in this case. Our statement becomes non trivial as soon as M has not maximal algebraic dimension or when $\omega_0 \wedge \cdots \wedge \omega_{m-1} \equiv 0$, $m = \dim(M)$.

When N has dimension $n = 0$ ($K = \mathbb{C}$) or 1, then \mathcal{F} is automatically transversely projective: even in the case $\dim(N) = 1$, the foliation $\underline{\mathcal{F}}$ has dimension 0 and is trivially transversely euclidean.

We immediately deduce from Theorem 1.1 the

Corollary 1.2. *Let \mathcal{F} be a codimension 1 singular foliation on a compact complex manifold. Assume that there exists a meromorphic vector field X on M which is transversal to \mathcal{F} at a generic point. Then*

- \mathcal{F} is the pull-back by the Algebraic Reduction map $M \dashrightarrow \text{red}(M)$ of a foliation on $N = \text{red}(M)$,
- or \mathcal{F} is transversely projective.

In our previous work [7], this corollary was obtained under the stronger assumption that the manifold M is pseudo-parallelizable, i.e. there exist m meromorphic vector fields X_1, \dots, X_m on M , $m = \dim(M)$, that are independent at a generic point.

When N has dimension $m - 1$ or $m - 2$, we prove that \mathcal{F} is actually transversely affine if it is not a pull-back. In particular, we have

Theorem 1.3. *Let \mathcal{F} be a foliation on a compact complex manifold M and assume that the meromorphic 3-form $\omega_0 \wedge \omega_1 \wedge \omega_2$ is zero for some Godbillon-Vey sequence associated to \mathcal{F} . Then*

- \mathcal{F} is the pull-back by a meromorphic map $\pi : M \dashrightarrow S$ of a foliation $\underline{\mathcal{F}}$ on an algebraic surface S ,
- or \mathcal{F} is transversely affine.

We do not know how to interpret this assumption geometrically. It is a well known fact and easy computation (see [8]) that the meromorphic 3-form $\omega_0 \wedge \omega_1 \wedge \omega_2$ is closed and well defined by \mathcal{F} up to the addition by an exact meromorphic 3-form. Nevertheless, we note that foliations constructed in section 5.5 have exact 3-form $\omega_0 \wedge \omega_1 \wedge \omega_2$ but do not satisfy conclusion of Theorem 1.3.

Let us now define the length of a foliation, $\text{length}(\mathcal{F}) \in \mathbb{N}^* \cup \{\infty\}$ as the minimal length among all Godbillon-Vey sequences attached to \mathcal{F} ; we set $\text{length}(\mathcal{F}) = \infty$ when \mathcal{F} does not admit any Godbillon-Vey sequence. A foliation has length 1, 2 or 3 if, and only if, it is respectively transversely euclidian, affine or projective in the meromorphic sense above. Also, consider an ordinary differential equation over a curve C

$$(5) \quad dz + \sum_{k=0}^N \omega_k z^k,$$

(where ω_k are meromorphic 1-forms defined on C). Then, the foliation defined on $C \times \mathbb{C}P(1)$ by equation (5) has length $\leq N + 1$ (consider the Godbillon-Vey algorithm given by equation (3) with $X = \frac{\partial}{\partial z}$). Although it is expected that $N + 1$ is the actual length of the generic equation (5), this is clear only for the Riccati equations ($N \leq 2$), for monodromy reasons.

The study of foliations having finite length has been initiated by Camacho and Scárdua in [3] when the ambient space is a rational algebraic manifold. We generalize their main result in the

Theorem 1.4. *Let \mathcal{F} be a foliation on a compact complex manifold M . If $4 \leq \text{length}(\mathcal{F}) < \infty$, then \mathcal{F} is the pull-back by a meromorphic map $\pi : M \dashrightarrow C \times \mathbb{C}P(1)$ of the foliation $\underline{\mathcal{F}}$ defined by an ordinary differential equation over a curve C like above.*

There are examples of foliations on $\mathbb{C}P(2)$ having length 0, 1 or 2 that are not pull-back of a Riccati equation (see [12] and [22]). Therefore, condition $4 \leq \text{length}(\mathcal{F})$ is necessary. Recall that the degree of a foliation \mathcal{F} on $\mathbb{C}P(n)$ is the number d of tangencies with a generic projective line. At least, we prove the

Theorem 1.5. *Every foliation of degree 2 on the complex projective space $\mathbb{C}P(n)$ has length at most 4. This bound is sharp.*

In particular, Jouanolou examples (see [11]) have actually length 4. In the same spirit, we also derive from [13] the

Theorem 1.6. *If \mathcal{F} is a germ of foliation at the origin of \mathbb{C}^n defined by an holomorphic 1-form with a non zero linear part, then $\text{length}(\mathcal{F}) \leq 4$.*

From Theorems 1.4 and 1.5, we immediately retrieve the following result previously obtained by two of us in [5]:

Corollary 1.7. *A degree 2 foliation on $\mathbb{C}P(n)$ is either transversely projective, or the pull-back of a foliation on $\mathbb{C}P(2)$ by a rational map.*

We do not understand the strength of the assumption $\text{length}(\mathcal{F}) < \infty$ of Theorem 1.4. In fact, we still do not know any example of a foliation having finite length > 4 . It is not excluded that the generic foliation of degree 3 on $\mathbb{C}P(2)$ has infinite length.

In section 5, we also provide examples of transversely projective foliations on $\mathbb{C}P(3)$ that are not transversely affine. In fact, they form a new irreducible component of the space of foliations of degree 4 (see [5]). We do not know yet if they are pull-back by rational map of foliations on $\mathbb{C}P(2)$. We also give an example of a degree 6 transversely projective foliation \mathcal{H}_2 in $\mathbb{C}P(3)$ (with explicit equations) which is not the pull-back of a foliation in $\mathbb{C}P(2)$ by a rational map. In fact, \mathcal{H}_2 is the suspension (see section 2.3) of one of the ‘‘Hilbert modular foliations’’ on $\mathbb{C}P(2)$ studied in [15]. We do not know if this foliation is isolated in the space of foliations.

Finally, since our arguments are mainly of algebraic nature, it is natural to ask what remains true from our work in the positive characteristic. In this direction, we prove in the last section the

Theorem 1.8. *Let M be a smooth projective variety defined over a field K of characteristic $p > 0$ and ω be a rational 1-form. If ω is integrable $\omega \wedge d\omega = 0$, then there exist a rational function $F \in K(M)$ such that $F\omega$ is closed. In this sense, the "foliation" \mathcal{F}_ω has length 1.*

2. BACKGROUND AND FIRST STEPS

2.1. Godbillon-Vey sequences [8, 3]. We introduce Godbillon-Vey sequences for a codimension one foliation \mathcal{F} and describe basic properties. Let ω be a differential 1-form defining \mathcal{F} and X be a vector field satisfying $\omega(X) = 1$. Then, the integrability condition of ω is equivalent to

$$(6) \quad \omega \wedge d\omega = 0 \quad \Leftrightarrow \quad d\omega = \omega \wedge L_X\omega.$$

Indeed, from $L_X\omega = d(\omega(X)) + d\omega(X, \cdot) = d\omega(X, \cdot)$, we derive

$$0 = \omega \wedge d\omega(X, \cdot, \cdot) = \omega(X) \cdot d\omega - \omega \wedge (d\omega(X, \cdot)) = d\omega - \omega \wedge L_X\omega$$

(the converse is obvious). Applying this identity to the formal 1-form

$$(7) \quad \Omega = dz + \omega_0 + z\omega_1 + \frac{z^2}{2}\omega_2 + \cdots + \frac{z^k}{k!}\omega_k + \cdots$$

together with the vector field $\tilde{X} = \partial_z$, we derive

$$\Omega \wedge d\Omega = 0 \quad \Leftrightarrow \quad \sum_{k=0}^{\infty} \frac{z^k}{k!} d\omega_k = \left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \omega_k \right) \wedge \left(\sum_{k=1}^{\infty} \frac{z^{k-1}}{(k-1)!} \omega_k \right).$$

We therefore obtain the full integrability condition (2) for Ω :

$$(8) \quad \begin{aligned} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= \omega_0 \wedge \omega_2 \\ d\omega_2 &= \omega_0 \wedge \omega_3 + \omega_1 \wedge \omega_2 \\ d\omega_3 &= \omega_0 \wedge \omega_4 + 2\omega_1 \wedge \omega_3 \\ &\vdots \\ d\omega_k &= \omega_0 \wedge \omega_{k+1} + \sum_{l=1}^k \binom{l}{k} \omega_l \wedge \omega_{k+1-l} \\ &\vdots \end{aligned}$$

For instance, if we start with ω integrable and X satisfying $\omega(X) = 1$, then the iterated Lie derivatives $\omega_k := L_X^{(k)}\omega$ define a Godbillon-Vey sequence for \mathcal{F}_ω . Indeed, from the formula $(L_X\omega)(X) = d\omega(X, X) = 0$, we have

$$\omega_0(X) = 1 \quad \text{and} \quad \omega_k(X) = 0 \quad \text{for all } k > 0;$$

therefore, $\Omega(X) = 1$ and integrability condition comes from

$$\Omega \wedge L_X \Omega = \left(dz + \sum_{k=0}^{\infty} \frac{z^k}{k!} \omega_k \right) \wedge \left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \omega_{k+1} \right) = d\Omega.$$

From a given Godbillon-Vey sequence, we derive many other ones. For instance, given any non zero meromorphic function $f \in \mathcal{M}(M)$, after applying the formal change of variable $z = f \cdot t$ to

$$\Omega = dz + \omega_0 + z\omega_1 + \frac{z^2}{2}\omega_2 + \cdots + \frac{z^k}{k!}\omega_k + \cdots,$$

we derive the new integrable 1-form

$$\frac{\Omega}{f} = dt + \frac{\omega_0}{f} + t\left(\omega_1 + \frac{df}{f}\right) + \frac{t^2}{2}(f\omega_2) + \frac{t^3}{3!}(f^2\omega_3) + \cdots + \frac{t^k}{k!}(f^{k-1}\omega_k) + \cdots$$

In other words, we obtain a new Godbillon-Vey sequence $(\tilde{\omega}_k)$ by setting

$$(9) \quad \begin{cases} \tilde{\omega}_0 &= \frac{1}{f} \cdot \omega_0 \\ \tilde{\omega}_1 &= \omega_1 + \frac{df}{f} \\ \tilde{\omega}_2 &= f \cdot \omega_2 \\ &\vdots \\ \tilde{\omega}_{k+1} &= f^k \cdot \omega_{k+1} \\ &\vdots \end{cases}$$

By the same way, we can apply to Ω the formal change of variable $z = t + f \cdot t^{k+1}$, $k = 1, 2, \dots$, and successively derive new Godbillon-Vey sequences

$$(10) \quad \begin{cases} \tilde{\omega}_0 &= \omega_0 \\ \tilde{\omega}_1 &= \omega_1 + f\omega_0 \\ \tilde{\omega}_2 &= \omega_2 + f\omega_1 - df \\ &\vdots \end{cases} \quad \begin{cases} \tilde{\omega}_0 &= \omega_0 \\ \tilde{\omega}_1 &= \omega_1 \\ \tilde{\omega}_2 &= \omega_2 + f\omega_0 \\ &\vdots \end{cases} \quad \text{etc.} \dots$$

Conversally, we easily see from integrability condition (2) that ω_{k+1} is well defined by $\omega_0, \omega_1, \dots, \omega_k$ up to the addition by a meromorphic factor of ω_0 . In fact, every Godbillon-Vey sequence can be deduced from a given one after applying to the 1-form Ω a formal transformation belonging to the following group

$$G = \left\{ (p, z) \mapsto \left(p, \sum_{k=1}^{\infty} f_k(p) \cdot z^k \right), f_k \in \mathcal{M}(M), f_1 \neq 0 \right\}.$$

In particular, the so-called Godbillon-Vey invariant $\omega_0 \wedge \omega_1 \wedge \omega_2 = -\omega_1 \wedge d\omega_1$ is closed and is well defined up to the addition by an exact

meromorphic 3-form of the form

$$\frac{df}{f} \wedge \omega_0 \wedge \omega_2 = \frac{df}{f} \wedge d\omega_1 \quad \text{or} \quad df \wedge \omega_0 \wedge \omega_1 = df \wedge d\omega_0$$

for some meromorphic function $f \in \mathcal{M}(M)$.

Remark 2.1. A natural Godbillon-Vey sequence for the formal foliation \mathcal{F}_Ω defined by Ω is given by

$$\Omega_k = L_{\partial_z}^{(k)} \Omega = \sum_{l=k}^{\infty} \frac{z^{l-k}}{(l-k)!} \omega_l, \quad k > 0$$

or equivalently by the formal integrable 1-form

$$\begin{aligned} d(t+z) + \omega_0 + (t+z)\omega_1 + \frac{(t+z)^2}{2}\omega_2 + \dots \\ = dt + \Omega_0 + t\Omega_1 + \frac{t^2}{2}\Omega_2 + \dots \end{aligned}$$

In fact, this remark also applies to the case where the ω_k are meromorphic 1-forms on a complex curve C . The so-called ‘‘ordinary differential equation’’ defined by

$$\Omega = dz + \sum_{k=0}^N \frac{z^k}{k!} \omega_k,$$

defines a foliation \mathcal{F} on $C \times \mathbb{C}P(1)$ (integrability conditions (2) are trivial in dimension 1). This foliation admits a natural Godbillon-Vey sequence of length $N+1$ given by $L_{\partial_z}^{(k)} \Omega$ (or by replacing z by $z+t$).

Remark 2.2. It follows from relations (2) that all differential forms

$$\Theta_k := \omega_0 \wedge \omega_1 \wedge \dots \wedge \omega_{k-1} \quad \text{for all } k = 2, \dots, n,$$

are closed and depend only on ω_0 . We obtain an ‘‘integrable flag’’:

$$\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \dots \supset \mathcal{F}_{n-1}$$

(the tangents spaces $T_p \mathcal{F}_k$ define is a flag at a generic point $p \in M$). The codimension n of the flag is the first n such that $\omega_0 \wedge \dots \wedge \omega_n = 0$.

We have two preliminary lemmas about finite Godbillon-Vey sequences.

Lemma 2.3. *Let $\omega_0, \omega_1, \dots, \omega_N$ be a Godbillon-Vey sequence of finite length $N + 1$. Then $\omega_k \wedge \omega_l = 0$ for all $k, l \geq 2$ and integrability conditions become*

$$d\omega_k = \omega_0 \wedge \omega_{k+1} + (k-1)\omega_1 \wedge \omega_k \quad k = 0, 1, \dots, N.$$

In particular, the condition $\omega_{N+1} = 0$ in a Godbillon-Vey sequence is not sufficient to conclude that the truncated sequence

$$\omega_0, \omega_1, \dots, \omega_N, 0, 0, \dots$$

provides a finite Godbillon-Vey sequence, except when $N = 0, 1$ or 2 .

Proof. We assume $\omega_N \neq 0$ with $N \geq 2$, otherwise we have done. The integrability conditions (2)

$$\begin{aligned} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= \omega_0 \wedge \omega_2 \\ &\vdots \\ d\omega_N &= \sum_{l=1}^N \binom{l}{N} \omega_l \wedge \omega_{N+1-l} \\ 0 = d\omega_{N+1} &= \sum_{l=2}^N \binom{l}{N+1} \omega_l \wedge \omega_{N+2-l} \\ &\vdots \\ 0 = d\omega_{2N-2} &= \frac{1}{N} \binom{N-1}{2N-2} \omega_{N-1} \wedge \omega_N \end{aligned}$$

Examining the line of index $k = 2N - 2$, we deduce that $\omega_{N-1} \wedge \omega_N \equiv 0$. Furthermore, by descendent induction, we also deduce from the line of index $k + N - 1$ that $\omega_k \wedge \omega_N \equiv 0$ for every $k \geq 2$. Therefore, the remaining N first lines of integrability conditions are as in the statement. \square

Corollary 2.4. *Let ω_0, ω_1 and ω_2 be differential 1-forms satisfying relations (2) for $k = 0, 1$ with $d\omega_1 \neq 0$. Then, there exists at most one finite Godbillon-Vey sequence $\omega_0, \dots, \omega_N$ completing this triple.*

Proof. The assumption $d\omega_1 = \omega_0 \wedge \omega_2 \neq 0$ implies in particular that $\omega_2 \neq 0$. If $\omega_0, \omega_1, \dots, \omega_N$ is a finite sequence, then we recursively see from integrability conditions of Lemma 2.3 that the line of index k determines $\omega_k, k = 3, \dots, N$, up to a meromorphic factor of ω_0 . But since ω_k is tangent to ω_2 but ω_0 is not, we deduce that ω_k is actually completely determined by the line of index k . \square

Here is a weaker but easier version of Theorem 1.4.

Theorem 2.5. *Let \mathcal{F} be a foliation on a compact pseudo-parallelizable manifold M . If $\text{length}(\mathcal{F}) < \infty$, then we have the following alternative:*

- (1) \mathcal{F} is the pull-back of a foliation $\underline{\mathcal{F}}$ on an algebraic surface S by a meromorphic map $\pi : M \dashrightarrow S$ with $\text{length}(\underline{\mathcal{F}}) = \text{length}(\mathcal{F})$,
- (2) or \mathcal{F} is transversely projective, i.e. $\text{length}(\mathcal{F}) \leq 3$.

Proof. Let $(\omega_0, \omega_1, \dots, \omega_N)$ be a Godbillon-Vey sequence for \mathcal{F} with $\omega_N \neq 0$, $N \geq 3$ and $\omega_1, \omega_2, \omega_3$ both non zero (otherwise we are in the second alternative of the statement). Following Lemma 2.3, there exist meromorphic functions f_k such that $\omega_k = f_k \cdot \omega_2$. Observe that $f_3 \neq 0$ since $\omega_3 \neq 0$. Recall that (ω_k) is a Godbillon-Vey sequence and the 1-form

$$\Omega = dz + \omega_0 + z\omega_1 + \dots + \frac{z^N}{N!}\omega_N,$$

is integrable. Applying to Ω the change of variables $z = t/f_3$ (see Section 2.1), we derive a new Godbillon-Vey sequence of length N satisfying $\omega_2 = \omega_3$. Therefore

$$\begin{cases} d\omega_2 = \omega_0 \wedge \omega_3 + \omega_1 \wedge \omega_2 = \omega_0 \wedge \omega_2 + \omega_1 \wedge \omega_2 \\ d\omega_3 = \omega_0 \wedge \omega_4 + 2 \cdot \omega_1 \wedge \omega_3 = f_4 \cdot \omega_0 \wedge \omega_2 + 2 \cdot \omega_1 \wedge \omega_2 \end{cases}$$

In particular $(1 - f_4)\omega_0 \wedge \omega_2 = \omega_1 \wedge \omega_2$ implying that $\omega_0 \wedge \omega_1 \wedge \omega_2 \equiv 0$. We conclude with Theorem 1.3 (a consequence of section 3).

Let us now prove that $\text{length}(\underline{\mathcal{F}}) = \text{length}(\mathcal{F})$. Since a Godbillon-Vey sequence for $\underline{\mathcal{F}}$ induces, by pull-back by ϕ , a sequence for ω_0 , it follows that $\text{length}(\underline{\mathcal{F}}) \geq \text{length}(\mathcal{F}) = N$. Let $\underline{\omega}_0$ be the meromorphic 1-form on S such that $\phi^*\underline{\omega}_0 = \omega_0$. From the equality $0 = \omega_0 \wedge \omega_1 \wedge \omega_2 = \omega_1 \wedge d\omega_1$, we see that ω_1 is integrable. Writing down the equations in local coordinates we also see that the fibers of ϕ are tangent to the foliation associated to ω_1 . Moreover, ω_1 is the pull-back by ϕ of a 1-form $\underline{\omega}_1$ on S . Recall that $\omega_2 = f_0\omega_0 + f_1\omega_1$ and that $df_1 \wedge d\omega_0 = 0$. Differentiating the identity

$$d\omega_2 = \omega_0 \wedge \omega_2 + \omega_1 \wedge \omega_2 = (f_1 - f_0)d\omega_0$$

it follows that $df_0 \wedge d\omega_0 = 0$. Consequently $\omega_2 = \phi^*\underline{\omega}_2$, where $\underline{\omega}_2$ is a meromorphic 1-form on S . At this point we can rewrite Ω as

$$\Omega = dz + \phi^*\underline{\omega}_0 + z\phi^*\underline{\omega}_1 + h \cdot \phi^*\underline{\omega}_2,$$

where $h = \frac{z^2}{2} + \sum_{i=3}^N \frac{z^i}{i!}h_i$. The integrability of Ω implies that $dh \wedge \phi^*\underline{\omega}_2 = 0$, where d is the differential over M (i.e. $dz = 0$). This implies that each h_j belongs to $\phi^{-1}\mathcal{M}(S)$ and therefore $\omega_j = \phi^*\underline{\omega}_j$ for every j and some $\underline{\omega}_j$ on S . This proves that $\text{length}(\underline{\mathcal{F}}) \leq N$. \square

2.2. Transversely projective foliations: the classical case [8, 20].

In this section, we recall the classical notion and basic properties of **regular** transversely projective foliation. A **smooth** codimension one foliation \mathcal{F} on a manifold M is a regular transversely projective foliation if there exists an atlas of submersions $f_i : U_i \rightarrow \mathbb{C}P(1)$ on M satisfying the cocycle condition:

$$f_i = \frac{a_{ij}f_j + b_{ij}}{c_{ij}f_j + d_{ij}}, \quad \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix} \in PGL(2, \mathbb{C})$$

on any intersection $U_i \cap U_j$. Any two such atlases $(f_i : U_i \rightarrow \mathbb{C}P(1))_i$ and $(g_k : V_k \rightarrow \mathbb{C}P(1))_k$ define the same projective structure if the union of them is again a projective structure, i.e. satisfying the cocycle condition $f_i = \frac{a_{ik}g_k + b_{ik}}{c_{ik}g_k + d_{ik}}$ on $U_i \cap V_k$.

Starting from one of the local submersions $f : U \rightarrow \mathbb{C}P(1)$ above, one can step-by-step modify the other charts so that they glue with f and define an analytic continuation for f . Of course, doing this along an element $\gamma \in \pi_1(M)$ of the fundamental group, we obtain monodromy $f(\gamma \cdot p) = A_\gamma \cdot f(p)$ for some $A_\gamma \in PGL(2, \mathbb{C})$. By this way, we define the *monodromy representation* of the structure, that is a homomorphism

$$\rho : \pi_1(M) \rightarrow PGL(2, \mathbb{C}); \gamma \mapsto A_\gamma,$$

as well as the *developing map*, that is the full analytic continuation of f on the universal covering \tilde{M} of M

$$\tilde{f} : \tilde{M} \rightarrow \mathbb{C}P(1).$$

By construction, \tilde{f} is a global submersion on \tilde{M} whose determinations $f_i : U_i \rightarrow \mathbb{C}P(1)$ on simply connected subsets $U_i \subset M$ define unambiguously the foliation \mathcal{F} and the projective structure. In fact, the map \tilde{f} is ρ -equivariant

$$(11) \quad f(\gamma \cdot p) = \rho(\gamma) \cdot f(p), \quad \forall \gamma \in \pi_1(M).$$

Finally, we obtain

Proposition 2.6. *A regular transversely projective foliation \mathcal{F} on M is defined by*

- a representation $\rho : \pi_1(M) \rightarrow PGL(2, \mathbb{C})$
- a submersion $\tilde{f} : \tilde{M} \rightarrow \mathbb{C}P(1)$ defining \mathcal{F} and satisfying (11).

Any other pair (ρ', \tilde{f}') will define the same structure if, and only if, we have $\rho'(\gamma) = A \cdot \rho(\gamma) \cdot A^{-1}$ and $\tilde{f}' = A \cdot \tilde{f}$. for some $A \in PGL(2, \mathbb{C})$.

Remark 2.7. If M is simply connected, then any regular transversely projective foliation \mathcal{F} on M actually admits a global first integral $\tilde{f} : M \rightarrow \mathbb{C}P(1)$, a holomorphic mapping.

Example 2.8 (*Suspension of a representation*). Given a representation $\rho : \pi_1(M) \rightarrow PGL(2, \mathbb{C})$ of the fundamental group of a manifold M into the projective group, we derive the following representation into the group of diffeomorphisms of the product $\tilde{M} \times \mathbb{C}P(1)$

$$\tilde{\rho} : \pi_1(M) \rightarrow \text{Aut}(\tilde{M} \times \mathbb{C}P(1)) ; (p, z) \mapsto (\gamma \cdot p, \rho(\gamma) \cdot z)$$

(\tilde{M} is the universal covering of M and $p \mapsto \gamma \cdot p$, the Galois action of $\gamma \in \pi_1(M)$). The image \tilde{G} of this representation acts freely, properly and discontinuously on the product $\tilde{M} \times \mathbb{C}P(1)$ since its restriction to the first factor does. Moreover, \tilde{G} preserves the horizontal foliation \mathcal{H} defined by dz as well as the vertical $\mathbb{C}P(1)$ -fibration defined by the projection $\pi : \tilde{M} \times \mathbb{C}P(1) \rightarrow \tilde{M}$ onto the first factor. In fact, we have $\pi(\tilde{\rho}(\gamma) \cdot p) = \rho(\gamma) \cdot \tilde{\pi}(p)$ for all $p \in \tilde{M}$ and $\gamma \in \pi_1(M)$. Therefore, the quotient $N := \tilde{M} \times \mathbb{C}P(1) / \tilde{G}$ is a manifold equipped with a locally trivial $\mathbb{C}P(1)$ -fibration given by the projection $\pi : N \rightarrow M$ as well as a codimension one foliation \mathcal{H} transversal to π . In fact, the foliation \mathcal{H} is regular transversely projective with monodromy representation $\rho \circ \pi_* : \pi_1(N) \rightarrow PGL(2, \mathbb{C})$ (π induces an isomorphism $\pi_* : \pi_1(N) \rightarrow \pi_1(M)$) and developing map $\tilde{M} \times \mathbb{C}P(1) \rightarrow \mathbb{C}P(1); (p, z) \mapsto z$ ($\tilde{M} \times \mathbb{C}P(1)$ is the universal covering of N).

Conversely, a codimension one foliation \mathcal{H} transversal to a $\mathbb{C}P(1)$ -fibration $\pi : N \rightarrow M$ is actually the suspension of a representation $\rho : \pi_1(M) \rightarrow PGL(2, \mathbb{C})$. In particular, \mathcal{H} is regular transversely projective and uniquely defined by its monodromy ρ .

Now, given a regular transversely projective foliation \mathcal{F} on M , we construct *the suspension of \mathcal{F}* as follows. We first construct the suspension of the monodromy representation $\rho : \pi_1(M) \rightarrow PGL(2, \mathbb{C})$ of \mathcal{F} as above and consider the graph

$$\tilde{\Gamma} = \{(p, z) \in \tilde{M} \times \mathbb{C}P(1) ; z = \tilde{f}(p)\}$$

of the developing map $\tilde{f} : \tilde{M} \rightarrow \mathbb{C}P(1)$. Since \tilde{f} is ρ -equivariant, its graph $\tilde{\Gamma}$ is invariant under the group \tilde{G} and defines a smooth cross-section $f : M \hookrightarrow N$ to the $\mathbb{C}P(1)$ -fibration $\pi : N \rightarrow M$. By construction, its image $\Gamma = f(M)$ is also transversal to the “horizontal foliation” \mathcal{H} and the regular transversely projective foliation induced by \mathcal{H} on Γ actually coincides (via f or π) with the initial foliation \mathcal{F} on M .

Proposition 2.9. *A regular transversely projective foliation \mathcal{F} on M is defined by*

- a locally trivial $\mathbb{C}P(1)$ -fibration $\pi : N \rightarrow M$ over M ,
- a codimension one foliation \mathcal{H} on N transversal to π ,
- a section $f : M \rightarrow N$ transversal to \mathcal{H} such that the foliation induced by \mathcal{H} on $f(M)$ coincides via f with \mathcal{F} .

Any other triple $(\pi' : N' \rightarrow M, \mathcal{H}', f')$ will define the same structure if, and only if, there exists a diffeomorphism $\Phi : N' \rightarrow N$ such that $\pi' = \pi \circ \Phi$, $f = \Phi \circ f'$ and $\mathcal{H}' = \Phi^*\mathcal{H}$.

Over any sufficiently small open subset $U \subset M$, the $\mathbb{C}P(1)$ -fibration is trivial and one can choose trivializing coordinates $(p, z) \in U \times \mathbb{C}P(1)$ such that $f : U \rightarrow \pi^{-1}(U)$ coincides with the zero-section $\{z = 0\}$. The foliation \mathcal{H} is defined by a unique differential 1-form of the type

$$\Omega = dz + \omega_0 + z\omega_1 + z^2\omega_2$$

where ω_0, ω_1 and ω_2 are holomorphic 1-forms defined on U . The integrability condition $\Omega \wedge d\Omega = 0$ reads

$$(12) \quad \begin{cases} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= 2\omega_0 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_2 \end{cases}$$

For convenience of formulae, we adopt here an alternate normalization than the one used in (1) and (4). Now, any change of trivializing coordinates preserving the zero-section takes the form $(\tilde{p}, \tilde{z}) = (p, f_0 \cdot z / (1 + f_1 \cdot z))$ where $f_0 : U \rightarrow \mathbb{C}^*$ and $f_1 : U \rightarrow \mathbb{C}$ are holomorphic. The foliation \mathcal{H} is therefore defined by

$$\tilde{\Omega} := \frac{(f_0 - f_1\tilde{z})^2}{f_0} \Omega = d\tilde{z} + \tilde{\omega}_0 + \tilde{z}\tilde{\omega}_1 + \tilde{z}^2\tilde{\omega}_2$$

where the new triple $(\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2)$ is given by

$$(13) \quad \begin{cases} \tilde{\omega}_0 &= f_0\omega_0 \\ \tilde{\omega}_1 &= \omega_1 - 2f_1\omega_0 - \frac{df_0}{f_0} \\ \tilde{\omega}_2 &= \frac{1}{f_0}(\omega_2 - f_1\omega_1 + f_1^2\omega_0 + df_1) \end{cases}$$

Proposition 2.10. *A regular transversely projective foliation \mathcal{F} on M is defined by an atlas of charts U_i equipped with 1-forms $(\omega_0^i, \omega_1^i, \omega_2^i)$ satisfying (12) and related to each other by (13) on $U_i \cap U_j$.*

Example 2.11. Consider

$$SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} x & u \\ y & v \end{pmatrix} ; xv - yu = 1 \right\}.$$

The meromorphic function defined by

$$f : SL(2, \mathbb{C}) \rightarrow \mathbb{C}P(1) ; \begin{pmatrix} x & u \\ y & v \end{pmatrix} \mapsto \frac{x}{y}$$

is a global submersion defining a regular transversely projective foliation \mathcal{F} on $SL(2, \mathbb{C})$. The leaves are the right cosets for the ‘‘affine’’ subgroup

$$\mathbb{A} = \left\{ \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} ; a \neq 0 \right\}.$$

Indeed, we have for any $z \in \mathbb{C}$

$$\begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbb{A} = \left\{ \begin{pmatrix} az & bz - \frac{1}{a} \\ a & b \end{pmatrix} ; a \neq 0 \right\} = \{f = z\}$$

and for any $w = 1/z \in \mathbb{C}$

$$\begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} \cdot \mathbb{A} = \left\{ \begin{pmatrix} a & b \\ aw & bw + \frac{1}{a} \end{pmatrix} ; a \neq 0 \right\} = \{f = 1/w\}.$$

In fact, if we consider the projective action of a matrix $\begin{pmatrix} x & u \\ y & v \end{pmatrix}$ on $(z : 1) \in \mathbb{C}P(1)$, then f is nothing but the image of the direction $(1 : 0)$ (i.e. $z = \infty$) by the matrix and $\{f = \infty\}$ coincides with the affine subgroup \mathbb{A} fixing $z = \infty$.

A global holomorphic triple $(\omega_0, \omega_1, \omega_2)$ for \mathcal{F} can be constructed as follows. Consider the Maurer-Cartan form

$$\mathcal{M} := \begin{pmatrix} x & u \\ y & v \end{pmatrix}^{-1} d \begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} vdx - udy & vdu - udv \\ xdy - ydx & xdv - ydu \end{pmatrix}.$$

The matrix \mathcal{M} is a differential 1-form on $SL(2, \mathbb{C})$ taking values in the Lie algebra $sl(2, \mathbb{C})$ ($\text{trace}(\mathcal{M}) = d(xv - yu) = 0$) and its coefficients form a basis for the left-invariant 1-forms on $SL(2, \mathbb{C})$. If we set

$$\mathcal{M} = \begin{pmatrix} -\frac{\omega_1}{2} & -\omega_2 \\ \omega_0 & \frac{\omega_1}{2} \end{pmatrix},$$

then Maurer-Cartan formula $d\mathcal{M} + \mathcal{M} \wedge \mathcal{M} = 0$ is equivalent to integrability conditions (12) for the triple $(\omega_0, \omega_1, \omega_2)$. In fact, the ‘‘meromorphic triple’’

$$(\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2) = (df, 0, 0)$$

is derived by setting $f_0 = -\frac{1}{y^2}$ and $f_1 = -\frac{v}{y}$ in formula (13).

A left-invariant 1-form $\omega = \alpha\omega_0 + \beta\omega_1 + \gamma\omega_2$, $\alpha, \beta, \gamma \in \mathbb{C}$, is integrable, $\omega \wedge d\omega = 0$, if, and only if, $\alpha\gamma = \beta^2$. The right translations act transitively on the set of integrable left-invariant 1-forms and thus on the corresponding foliations. For instance, if we denote by T_z the right translation

$$T_z : SL(2, \mathbb{C}) \rightarrow SL(2, \mathbb{C}) ; \begin{pmatrix} x & u \\ y & v \end{pmatrix} \mapsto \begin{pmatrix} x & u \\ y & v \end{pmatrix} \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix}, \quad z \in \mathbb{C},$$

then we have $T_z^*\omega_0 = z^2\omega_0 + z\omega_1 + \omega_2$ and the corresponding foliation \mathcal{F}_z is actually defined by the global submersion

$$f \circ T_z : SL(2, \mathbb{C}) \rightarrow \mathbb{C}P(1) ; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az + b}{cz + d}.$$

The leaf $\{f \circ T_z = w\}$ of \mathcal{F}_z is the set of matrices sending the direction $(z : 1)$ onto $(w : 1)$.

Remark 2.12. Let $(\omega_0, \omega_1, \omega_2)$ be a triple of holomorphic 1-forms on a manifold M satisfying integrability condition (12). The differential equation

$$dz + \omega_0 + z\omega_1 + z^2\omega_2 = 0$$

defined on the trivial projective bundle $M \times \mathbb{C}P(1)$ can be lifted as an integrable differential $sl(2, \mathbb{C})$ -system defined on the rank 2 vector bundle $M \times \mathbb{C}^2$ by

$$\begin{cases} dz_1 &= -\frac{\omega_1}{2}z_1 - \omega_2z_2 \\ dz_2 &= \omega_0z_1 + \frac{\omega_1}{2}z_2 \end{cases}$$

which can be shortly written as

$$dZ = A \cdot Z \quad \text{where} \quad A = \begin{pmatrix} -\frac{\omega_1}{2} & -\omega_2 \\ \omega_0 & \frac{\omega_1}{2} \end{pmatrix} \quad \text{and} \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

The matrix A may be thought as a differential 1-form on M taking values in the Lie algebra $sl(2, \mathbb{C})$ satisfying integrability condition $dA + A \wedge A = 0$. Then, Darboux Theorem (see [8], III, 2.8, iv, p.230) asserts that there exists, on any simply connected open subset $U \subset M$, an holomorphic map

$$\Phi : U \rightarrow SL(2, \mathbb{C}) \quad \text{such that} \quad A = \Phi^*\mathcal{M}$$

where \mathcal{M} is the Maurer-Cartan 1-form on $SL(2, \mathbb{C})$ (see example 2.11). Moreover, the map Φ is unique up to composition by a translation of $SL(2, \mathbb{C})$.

Example 2.13. Consider the quotient $M := \Gamma \backslash SL(2, \mathbb{C})$ by a co-compact lattice $\Gamma \subset SL(2, \mathbb{C})$. The left-invariant 1-forms $(\omega_0, \omega_1, \omega_2)$ defined in example 2.11 are well-defined on M and M is parallelizable. Following [10], there is no non constant meromorphic function on M (i.e. the algebraic dimension of M is $a(M) = 0$). Therefore, any foliation \mathcal{F} on M is defined by a global meromorphic 1-form

$$\omega = \alpha\omega_0 + \beta\omega_1 + \gamma\omega_2$$

and the coefficients are actually constants $\alpha, \beta, \gamma \in \mathbb{C}$.

Corollary 2.14. *Any foliation \mathcal{F} on a quotient $M := \Gamma \backslash SL(2, \mathbb{C})$ by a co-compact lattice Γ is actually defined by a left-invariant 1-form. In particular, \mathcal{F} is a regular transversely projective foliation and is minimal: any leaf of \mathcal{F} is dense in M . The set of foliations on M is a rational curve.*

A foliation \mathcal{F} is *regular transversely euclidean* if there exists an atlas of submersions $f_i : U_i \rightarrow \mathbb{C}$ on M defining \mathcal{F} such that on any $U_i \cap U_j$ we have

$$f_i = f_j + a_{ij}, \quad a_{ij} \in \mathbb{C}.$$

Of course, we can *glue* the df_i and produce a global closed holomorphic 1-form ω_0 inducing \mathcal{F} . In particular $l(\mathcal{F}) = 0$. By the same way, \mathcal{F} is *transversely linear* when it can be defined by submersions $f_i : U_i \rightarrow \mathbb{C}^*$ satisfying the cocycle condition:

$$f_i = \lambda_{ij} \cdot f_j, \quad \lambda_{ij} \in \mathbb{C}^*.$$

Again, we can *glue* the $\frac{df_i}{f_i}$ and produce a global closed holomorphic 1-form inducing \mathcal{F} and we have $l(\mathcal{F}) = 0$. Via the exponential map, this notion is equivalent to the previous one (in the complex setting).

Finally, a foliation \mathcal{F} is *regular transversely affine* when it can be defined by submersions $f_i : U_i \rightarrow \mathbb{C}$ satisfying the cocycle condition:

$$f_i = a_{ij}f_j + b_{ij}, \quad a_{ij} \in \mathbb{C}^*, b_{ij} \in \mathbb{C}.$$

Equivalently, an affine structure is locally defined by a pair of holomorphic 1-forms (ω_0, ω_1) satisfying

$$\begin{cases} d\omega_0 = \omega_0 \wedge \omega_1 \\ d\omega_1 = 0 \end{cases} \quad \text{up to modification} \quad \begin{cases} \tilde{\omega}_0 = f \cdot \omega_0 \\ \tilde{\omega}_1 = \omega_1 - \frac{df}{f} \end{cases}$$

2.3. Transversely projective foliations: the singular case [18].

A singular foliation \mathcal{F} on a complex manifold M will be said *transversely projective* if it admits a Godbillon-Vey sequence of length 2, i.e. if there exist meromorphic 1-forms ω_0 , ω_1 and ω_2 on M satisfying $\mathcal{F} = \mathcal{F}_{\omega_0}$ and

$$\begin{cases} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= 2\omega_0 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_2 \end{cases}$$

The foliation \mathcal{F} is actually regular and transversely projective in the classical sense of §2.2 on the Zariski open subset $U = M \setminus ((\Omega)_\infty \cup \mathcal{Z}_0)$ complementary to the set $(\Omega)_\infty$ of poles for ω_0 , ω_1 and ω_2 and the set \mathcal{Z}_0 of zeroes for ω_0 that are not in $(\Omega)_\infty$. In fact, $(\omega_0, \omega_1, \omega_2)$ is a regular projective triple on U . Another triple $(\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2)$ defines the same projective structure (on a Zariski open subset) if it is obtained from the previous one by a combination of

$$(14) \quad \begin{cases} \tilde{\omega}_0 &= \frac{1}{f} \cdot \omega_0 \\ \tilde{\omega}_1 &= \omega_1 + \frac{df}{f} \\ \tilde{\omega}_2 &= f \cdot \omega_2 \end{cases} \quad \text{and} \quad \begin{cases} \tilde{\omega}_0 &= \omega_0 \\ \tilde{\omega}_1 &= \omega_1 + g \cdot \omega_0 \\ \tilde{\omega}_2 &= \omega_2 + g \cdot \omega_1 + g^2 \cdot \omega_0 - dg \end{cases}$$

where f, g denote meromorphic functions on M .

We note that any pair (ω_0, ω_1) satisfying $d\omega_0 = \omega_0 \wedge \omega_1$ can be completed into a triple subjacent to the projective structure in an unique way. It follows that, in the pseudo-parallelizable case, a projective transverse structure is always defined by a global meromorphic triple.

We say that \mathcal{F} is *transversely affine* if it admits a Godbillon-Vey sequence of length 1, i.e. meromorphic 1-forms ω_0 and ω_1 satisfying

$$\begin{cases} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= 0 \end{cases}$$

Another pair $(\tilde{\omega}_0, \tilde{\omega}_1)$ will define the same affine structure if we have

$$\begin{cases} \tilde{\omega}_0 &= \frac{1}{f} \cdot \omega_0 \\ \tilde{\omega}_1 &= \omega_1 + \frac{df}{f} \end{cases}$$

for a meromorphic function f . Finally, we say that \mathcal{F} is *transversely euclidean* (resp. *transversely trivial*) if it is defined by a closed meromorphic 1-form ω_0 (resp. by an exact 1-form $\omega_0 = df$, $f \in \mathcal{M}(M)$).

The foliation \mathcal{H} defined on $M \times \mathbb{C}P(1)$ by the integrable 1-form

$$\Omega = dz + \omega_0 + z\omega_1 + z^2\omega_2$$

coincides over U with the suspension of the projective structure, and will be still called *suspension of \mathcal{F}* . In fact, the vertical hypersurface $(\Omega)_\infty \times \mathbb{C}P(1)$ is invariant by the foliation \mathcal{H} . Outside of this vertical invariant set, the foliation \mathcal{H} is transversal to the vertical $\mathbb{C}P(1)$ -fibration. Along \mathcal{Z}_0 , the foliation \mathcal{H} is tangent to the zero-section $M \times \{z = 0\}$ and the projective structure ramifies: it is locally defined by an holomorphic map $f_i : U_i \mapsto \mathbb{C}P(1)$ up to composition by an element of $PGL(2, \mathbb{C})$. This ramification set \mathcal{Z}_0 is invariant for \mathcal{F} (union of leaves and singular points). As in the regular case, one can define the monodromy representation

$$\rho : \pi_1(M \setminus (\Omega)_\infty) \rightarrow PGL(2, \mathbb{C})$$

(ramification points \mathcal{Z}_0 have no monodromy).

In contrast with the regular case, the suspension \mathcal{H} is well-defined only up to a bimeromorphic transformation preserving the generic vertical fibres $\{p\} \times \mathbb{C}P(1)$ and the zero-section $M \times \{z = 0\}$

$$\Phi : M \times \mathbb{C}P(1) \dashrightarrow M \times \mathbb{C}P(1) ; (p, z) \mapsto (p, f(p)z/(1 - g(p)z)),$$

where $f, g \in \mathcal{M}(M)$ are meromorphic. Note that some irreducible components of $(\Omega)_\infty$ may disappear after such a transformation Φ . For instance, one can show that any irreducible component of $(\Omega)_\infty$ which is not \mathcal{F} -invariant may be deleted by a change of triple. Only the remaining persistent components can generate non trivial local monodromy for the representation ρ . This leads to the following

Proposition 2.15. *Let \mathcal{F} be a (singular) transversely projective (resp. affine) foliation on a simply connected manifold M . If $(\Omega)_\infty$ has no persistent component, then \mathcal{F} admits a meromorphic (resp. holomorphic) first integral.*

Proof. The assumption just means that there exists a covering U_i of M by Zariski open subset on which the projective structure can be defined by an holomorphic triple. Therefore, like in Remark 2.7 the developing map provides a well-defined meromorphic first integral $f : M \rightarrow \mathbb{C}P(1)$ (possibly with ramifications). \square

Corollary 2.16. *Let \mathcal{F} be a transversely projective (resp. affine) foliation on a simply connected manifold M . Then \mathcal{F} admits an invariant hypersurface.*

Remark 2.17. A transversely projective foliation \mathcal{F} on M with suspension \mathcal{H} on $M \times \mathbb{C}P(1)$ is actually transversely affine if, and only if, there is a section $g : M \rightarrow M \times \mathbb{C}P(1)$ which is invariant by \mathcal{H} . Indeed, after change of coordinate $\tilde{z} = z/(1 - \frac{z}{g})$ on $M \times \mathbb{C}P(1)$, we have sent the invariant hypersurface $g(M)$ onto $\{z = \infty\}$ which means that $\omega_2 = 0$. In the regular case, this is still true after replacing $M \times \mathbb{C}P(1)$ by the locally trivial $\mathbb{C}P(1)$ -bundle $\pi : N \rightarrow M$ (see Proposition 2.9) and if we ask moreover that the section $g : M \rightarrow N$ has no intersection with the section $f : M \rightarrow N$ providing the projective structure.

Example 2.18 (The Riccati equation over a curve). Given meromorphic 1-forms α, β, γ on a curve C , the Riccati differential equation

$$dz + \alpha + \beta z + \gamma z^2 = 0$$

defines a transversely projective foliation \mathcal{H} on $C \times \mathbb{C}P(1)$ with meromorphic projective triple

$$\begin{cases} \omega_0 &= dz + \alpha + \beta z + \gamma z^2 \\ \omega_1 &= \beta + 2\gamma z \\ \omega_2 &= \gamma \end{cases}$$

The polar set $(\Omega)_\infty$ is the union of the vertical lines over the poles of α, β, γ and the horizontal line $L_\infty = \{z = \infty\}$. In the chart $w = 1/z$, the alternate triple

$$\begin{cases} \tilde{\omega}_0 &= -dw + \alpha w^2 + \beta w + \gamma \\ \tilde{\omega}_1 &= -\beta - 2\alpha w \\ \tilde{\omega}_2 &= -\alpha \end{cases}$$

(obtained by setting successively $f = 1/w^2$ and $g = -2/w$ in (14)) shows that L_∞ is not a persistent pole for the projective structure. When $\gamma = 0$, the foliation \mathcal{H} is transversely affine with poles like above, but additionally L_∞ is a persistent zero for the affine structure (the transverse affine coordinate has a pole along L_∞).

The Riccati foliation above can be thought as the suspension of a singular projective structure on the curve C (i.e. a dimension 0 transversely projective foliation on C).

In the spirit of Theorem 1.4, one can find in [18] the following

Proposition 2.19 (Scárdua). *Let \mathcal{F} be a transversely projective foliation defined by a global meromorphic triple $(\omega_0, \omega_1, \omega_2)$ on M . Assume that the foliation \mathcal{G} defined by ω_2 admits a meromorphic first integral $f \in \mathcal{M}(M)$. Then, \mathcal{F} is the pull-back by a meromorphic map $\Phi : M \dashrightarrow C \times \mathbb{C}P(1)$ of the foliation \mathcal{H} defined by a Riccati equation on a curve C .*

Proof. One can assume that $\omega_2 = df$. Integrability conditions yield

$$\begin{cases} 0 = d\omega_2 = \omega_1 \wedge \omega_2 \\ d\omega_1 = 2\omega_0 \wedge \omega_2 \\ d\omega_0 = \omega_0 \wedge \omega_1 \end{cases} \Rightarrow \begin{cases} \omega_1 = gdf \\ \omega_0 = \frac{1}{2}dg + hdf \\ 0 = d(\tilde{h} - g^2) \wedge df \end{cases}$$

for meromorphic functions g, h on M . It follows from Stein Factorization Theorem that there exists some holomorphic map $\phi : M \mapsto C$ onto a curve C through which we can factorize $h - g^2 = \tilde{h}(\phi)$ and $f = \tilde{f}(\phi)$. Therefore

$$\omega_0 = \frac{1}{2}dg + \{\tilde{h}(\phi) + g^2\}\phi^*d\tilde{f}$$

and \mathcal{F} is the pull-back via the map $\Phi = (\phi, g)$ of the foliation defined by the Riccati equation $dz + \tilde{h}d\tilde{f} + z^2d\tilde{f}$. \square

Lemma 2.20. *If a foliation \mathcal{F} admits 2 distinct projective (resp. affine, euclidean) structures, then it is actually transversely affine (resp. euclidean, trivial).*

Proof. Assume we have 2 projective triples $(\omega_0, \omega_1, \omega_2)$ and $(\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2)$ that are not related by a composition of the admissible changes above: after the admissible change setting $\tilde{\omega}_0 = \omega_0$ and $\tilde{\omega}_1 = \omega_1$, we have $\tilde{\omega}_2 \neq \omega_2$. Therefore, by comparing the second line of integrability conditions for both triples, we see that $\tilde{\omega}_2 = \omega_2 + f\omega_0$ for a meromorphic function $f \in \mathcal{M}(M)$. Then, by comparing the third condition, we obtain

$$d(f\omega_0) = \omega_1 \wedge (f\omega_0) \quad \text{and thus} \quad \omega_0 \wedge \omega_1 = \omega_0 \wedge \frac{df}{2f}$$

which proves that the pair $(\tilde{\omega}_0, \tilde{\omega}_1) := (\omega_0, \frac{df}{2f})$ is an affine structure for \mathcal{F} . Notice that $\frac{\omega_0}{\sqrt{f}}$ is closed: \mathcal{F} becomes transversely euclidean on a 2-fold ramified covering of M . By the same way, if (ω_0, ω_1) and $(\tilde{\omega}_0, \tilde{\omega}_1)$ are 2 distinct affine structures, then we may assume $\tilde{\omega}_0 = \omega_0$ and $\tilde{\omega}_1 = \omega_1 + f\omega_0$ with $d\omega_1 = d(f\omega_0) = 0$ and conclude that \mathcal{F} is actually defined by the closed meromorphic 1-form $f\omega_0$. Finally, if ω_0 and $f\omega_0$ are 2 closed meromorphic 1-forms defining \mathcal{F} , then f is a meromorphic first integral for \mathcal{F} . \square

The present singular notion of transversely projective foliation is clearly stable under bimeromorphic transformations. Moreover, the main result of [4] permits to derive

Theorem 2.21. *Let $\phi : \tilde{M} \dashrightarrow M$ be a dominant meromorphic map between compact manifolds and let \mathcal{F} be a foliation on M admitting a Godbillon-Vey sequence. Then, $\tilde{\mathcal{F}} = \phi^*\mathcal{F}$ is transversely projective (resp. affine) if, and only if, so is \mathcal{F} .*

The analogous result for transversely euclidean foliations is false: one can find in [12] an example of a transversely affine foliation which becomes transversely euclidean on a finite covering (a linear foliation on a torus). The assumption dominant is necessary since there are examples of non transversely projective foliations which become transversely affine in restriction to certain non tangent hypersurface (see section 4).

Proof. Since a Godbillon-Vey sequence can be pulled-back by any non constant meromorphic map, we just have to prove that projective (resp. affine) structure can be pushed-down under the assumptions above. In the case ϕ is a finite ramified covering, then the statement is equivalent to Theorem 1.6 (resp. 1.4) in [4].

In the case ϕ is holomorphic with connected generic fibre, then choose meromorphic 1-forms ω_0 defining \mathcal{F} and ω_1 satisfying $d\omega_0 = \omega_0 \wedge \omega_1$ on M and consider their pull-back $\tilde{\omega}_0$ and $\tilde{\omega}_1$ on \tilde{M} . Then, there is a unique meromorphic 1-form $\tilde{\omega}_2$ completing the previous ones into a projective triple compatible with the structure of $\tilde{\mathcal{F}}$. On the other hand, reasoning as in Lemma 3.1 at the neighborhood $\tilde{U} = \phi^{-1}(U)$ of a generic fibre $\phi^{-1}(p)$, we see that the foliation $\tilde{\mathcal{F}}$ is defined by a submersion $\tilde{f} : \tilde{U} \mapsto \mathbb{C}P(1)$ defining the projective structure and can be pushed-down into a submersion $f : U \mapsto \mathbb{C}P(1)$. This latter one defines a projective structure transverse to \mathcal{F} on U . There exists a unique meromorphic 1-form ω_2 on U completing ω_0 and ω_1 into a compatible projective triple. By construction, $\tilde{\omega}_2$ must coincide with $\phi^*\omega_2$ on \tilde{U} . Therefore, $\tilde{\omega}_2$ is tangent to the fibration given by ϕ on \tilde{U} , and thus everywhere on \tilde{M} . By connexity of the fibres, $\tilde{\omega}_2$ is actually the pull-back of a global meromorphic 1-form ω_2 on M (which extends the one previously defined on U).

Finally, by Stein Factorization Theorem, the statement reduces to the two cases above. \square

3. PROOF OF THEOREM 1.1

Let \mathcal{F} be a foliation on a compact manifold M admitting a Godbillon-Vey sequence $(\omega_0, \omega_1, \dots)$ and consider the maximal non trivial form $\Theta = \omega_0 \wedge \dots \wedge \omega_{n-1}$: we have $\Theta \wedge \omega_n = 0$. Like in the introduction, we denote by K the field of meromorphic first integrals for Θ and consider the reduction map $\pi : M \dashrightarrow N$ associated to this field. The fibration \mathcal{G} induced by π contains the foliation \mathcal{F}_Θ as a sub-foliation, and may be of larger dimension as soon as there are few meromorphic functions on M . In particular, there is no reason why \mathcal{G} is a sub-foliation of \mathcal{F} . Anyway, when $\mathcal{G} \subset \mathcal{F}$, then we are in the first alternative of Theorem 1.1: \mathcal{F} is the pull-back of an algebraic foliation $\underline{\mathcal{F}}$ on $\text{red}(M, \Omega)$. Indeed, after modification of M , one can assume that the reduction map is holomorphic with connected fibres. The claim above immediately follows from:

Lemma 3.1. *Let \mathcal{F} be a foliation on a complex manifold M . Let $\pi : M \rightarrow N$ be a surjective holomorphic map whose fibers are connected and tangent to \mathcal{F} , that is, contained in the leaves of \mathcal{F} . Then, \mathcal{F} is the pull-back by π of a foliation $\tilde{\mathcal{F}}$ on N .*

Proof. In a small connected neighborhood $U \subset M$ of a generic point $p \in M$, the foliation \mathcal{F} is regular, defined by a local submersion $f : U \rightarrow \mathbb{C}$. Since f is constant along the fibers of π in U , we can factorize $f = \tilde{f} \circ \pi$ for an holomorphic function $\tilde{f} : \pi(U) \rightarrow \mathbb{C}$. In particular, the function \tilde{f} defines a codimension one singular foliation $\tilde{\mathcal{F}}$ on the open set $\pi(U)$. Of course, $\tilde{\mathcal{F}}$ does not depend on the choice of f . Moreover, since $f = \tilde{f} \circ \pi$, the function f extends to the whole tube $T := \pi^{-1}(\pi(U))$. By connectivity of U and the fibers of π , the tube T is connected and the foliation \mathcal{F} is actually defined by f on the whole of T , coinciding with $\pi^*(\tilde{\mathcal{F}})$ on T . In this way, we can define a foliation $\tilde{\mathcal{F}}$ on $N \setminus S$, where $S = \{p \in N ; \pi^{-1}(p) \subset \text{Sing}(\mathcal{F})\}$ such that $\mathcal{F} = \pi^*(\tilde{\mathcal{F}})$. We note that S has codimension ≥ 2 in N ; therefore, $\tilde{\mathcal{F}}$ extends on N by Levy's Extension Theorem. \square

We now assume that the fibration $\mathcal{G} \not\subset \mathcal{F}$. We note that when $n = \dim(M)$, then M is actually pseudo-parallelizable and the field K coincides with the field $\mathcal{M}(M)$ of meromorphic functions on M .

We introduce the sheaf \mathcal{B} of basic meromorphic vector fields for \mathcal{F}_Θ ; a section X of $\mathcal{B}(U)$ over $U \subset M$ is characterized by the following property:

$$(15) \quad L_X \Theta = d(i_X \Theta) = f \cdot \Theta, \quad \text{for some } f \in \mathcal{M}(U)$$

(here, we use that $L_X = d \circ i_X + i_X \circ d$ and the fact that $d\Theta = 0$). We remark that \mathcal{B} is a sheaf of Lie algebras over \mathbb{C} . The subsheaf of vector fields tangent to \mathcal{F}_Θ

$$(16) \quad \mathcal{I}(U) := \{X \in \mathcal{B}(U) ; i_X \Theta = 0\}.$$

form a Lie-ideal of \mathcal{B} : if $X \in \mathcal{B}(U)$ and $Y \in \mathcal{I}(U)$, then $[X, Y] \in \mathcal{I}(U)$ as can be seen from a local flow-box for \mathcal{F}_Θ . The quotient $\mathcal{T} = \mathcal{B}/\mathcal{I}$ is a sheaf of Lie algebras over \mathbb{C} , whose sections are the **transversal relative vector fields** to \mathcal{F}_Θ . Although the sheaf \mathcal{B} may have no global meromorphic section, the relative sheaf \mathcal{T} has many, as shown by the:

Lemma 3.2. *Let $T := \mathcal{T}(M)$ be the Lie algebra over \mathbb{C} of global transversal relative vector fields. Then T is a n -dimensional vector space over K and admits a canonical basis $(\mathbf{X}_0, \dots, \mathbf{X}_{n-1})$ satisfying*

$$\omega_k(\mathbf{X}_l) = \delta_l^k \quad \text{for } k, l = 0, \dots, n-1.$$

We note that $\omega_k(\mathbf{X}_l)$ is well defined since \mathbf{X}_l is locally defined modulo an element $Y \in \mathcal{I}$ and $\omega_k(Y) = 0$. More generally, we will use the fact that an element $\mathbf{X} \in T$ acts as a derivation on K : $\mathbf{X} \cdot f \in K$ for all $f \in K$. Indeed, for any local representative X of \mathbf{X} , X is a basic vector field and $X \cdot f$ is a local first integral for \mathcal{F}_Θ ; since $Y \cdot f = 0$ for any $Y \in \mathcal{I}$, we can set unambiguously $\mathbf{X} \cdot f := X \cdot f$ which is now a global first integral, thus belonging to K . Similarly, $L_{\mathbf{X}} \omega_k$ is a well-defined global meromorphic 1-form on M for any $\mathbf{X} \in T$ and $k = 0, \dots, n-1$.

Before proving the Lemma, we note that, by maximality property of n , one can write

$$(17) \quad \omega_n = a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}, \quad a_k \in \mathcal{M}(M)$$

(here, (10) allow us to set $a_0 = 0$). In fact, coefficients a_k actually belong to K . Indeed, after combining (2) and (17), we get

$$(18) \quad d\omega_{n-1} = \omega_0 \wedge \sum_{k=1}^{n-1} a_k \omega_k + \sum_{k=1}^{n-1} \binom{k}{n-1} \omega_k \wedge \omega_{n-k}.$$

After differentiation and multiplication by the $n - 2$ -form

$$(19) \quad \widehat{\Theta}_k = \omega_1 \wedge \cdots \wedge \omega_{k-1} \wedge \omega_{k+1} \wedge \cdots \wedge \omega_{n-1}, \quad k = 1, \dots, n-1,$$

we obtain

$$\Theta \wedge da_k = 0$$

and, as a consequence, that a_k is a first integral for Θ .

Proof. From a local flow-box for \mathcal{F}_Θ , one easily see that T is a K -vector space: if $\mathbf{X} \in T$ and $f \in K$, then $f \cdot \mathbf{X} \in T$. Now, consider local vector fields (X_0, \dots, X_{n-1}) dual to $(\omega_0, \dots, \omega_{n-1})$ like in the statement: they are well-defined modulo sections of \mathcal{I} and define global relative vector fields $(\mathbf{X}_0, \dots, \mathbf{X}_{n-1})$; we have to prove that they are transverse relative vector fields, i.e. that $L_{\mathbf{X}_k} \Theta = f \Theta$ for some $f \in \mathcal{M}(M)$ (actually, $f \in K$ since $f \Theta$ will be closed as well). We have $L_{\mathbf{X}_k} \Theta = d(\widehat{\Theta}_k)$ where $\widehat{\Theta}_k$ is defined by (19). From Godbillon-Vey relations (2), one easily deduce that all $d(\widehat{\Theta}_k)$ are zero, except $d(\widehat{\Theta}_1) = c \cdot \Theta$ for some constant $c \in \mathbb{C}$ and $d(\widehat{\Theta}_0) = \pm \widehat{\Theta}_{n-1} \wedge \omega_n = \pm a_{n-1} \Theta$.

Now, given an element $\mathbf{X} \in T$, one can write $\mathbf{X} = \lambda_0 \mathbf{X}_0 + \cdots + \lambda_{n-1} \mathbf{X}_{n-1}$ modulo \mathcal{I} where $\lambda_k = \omega_k(\mathbf{X})$ are global meromorphic functions; as can be seen for a local flow-box, all λ_i must be first integrals for \mathcal{F}_Θ . \square

The proof of Theorem 1.1 is similar to that of [7] after substituting global sections of \mathcal{T} to global meromorphic vector fields. We consider the Lie sub-algebra

$$\mathcal{L} := \{\mathbf{X} \in T ; \mathbf{X} \cdot K = 0\}$$

of those relative vector fields that are tangent to the fibration \mathcal{G} given by global first integrals of Θ . We note that \mathcal{L} is now a Lie algebra over K . Assuming that $\mathcal{G} \not\subset \mathcal{F}$ (otherwise, we have already concluded the proof by Lemma 3.1, we consider the Lie sub-algebra (over K) defined by

$$\mathcal{L}_0 := \{\mathbf{X} \in \mathcal{L} ; \omega_0(\mathbf{X}) = 0\}.$$

Clearly, \mathcal{L}_0 is a codimension ≤ 1 sub-algebra of \mathcal{L} over K ; we now prove that indeed $\mathcal{L}/\mathcal{L}_0$ is not trivial:

Lemma 3.3. *If $\mathcal{G} \not\subset \mathcal{F}$, there exists $\mathbf{X} \in T$ such that $\omega_0(\mathbf{X}) = 1$ and $\mathbf{X} \cdot K = 0$, i.e. $\mathbf{X}(f) = 0$ for any $f \in K$.*

Proof. If $K = \mathbb{C}$, then the Lemma is trivial. If not, suppose that f_1, \dots, f_N are elements of K such that

$$df_1 \wedge \dots \wedge df_N \neq 0$$

with N maximal: we have by assumption $N < n$ and

$$(20) \quad \omega_0 \wedge df_1 \wedge \dots \wedge df_N \neq 0.$$

Remark now that if $\mathbf{X} \in T$ satisfies $\mathbf{X}(f_k) = 0$ for $k = 1, \dots, N$, then for each $f \in K$, the meromorphic function $\mathbf{X}(f)$ is actually zero. Let us write

$$\mathbf{X} = \alpha_0 \mathbf{X}_0 + \alpha_1 \mathbf{X}_1 + \dots + \alpha_{n-1} \mathbf{X}_{n-1}$$

with $\alpha_k \in K$; since $\omega_0(\mathbf{X}) = \alpha_0$, we already set $\alpha_0 := 1$. We now have to solve the $N \times (n-1)$ -linear system

$$\begin{cases} df_1(\mathbf{X}_0) + df_1(\mathbf{X}_1) \cdot \alpha_1 + \dots + df_1(\mathbf{X}_{n-1}) \cdot \alpha_{n-1} & = 0 \\ \vdots & = 0 \\ df_N(\mathbf{X}_0) + df_N(\mathbf{X}_1) \cdot \alpha_1 + \dots + df_N(\mathbf{X}_{n-1}) \cdot \alpha_{n-1} & = 0 \end{cases}$$

From (20), the corresponding matrix $(df_k(\mathbf{X}_l))_{k,l}$ has maximal rank and one can solve the system above. If $N < n-1$, there are obviously many solutions. \square

The proof of Lemma 3.2 in [7] may be transposed to our relative setting:

Lemma 3.4. *If the relative vector field $\mathbf{X} \in T$ satisfies $\omega_0(\mathbf{X}) = 1$ and $\mathbf{X} \cdot f = 0$ for any $f \in K$ like in Lemma 3.3, then*

$$(L_{\mathbf{X}}^{(k)} \omega_0)(\mathbf{Y}) = (-1)^k \omega_0(L_{\mathbf{X}}^{(k)}(\mathbf{Y}))$$

for any $\mathbf{Y} \in T$; here, we denote by $L_X(Y)$ the Lie bracket $[X, Y]$.

Now, we can assume that the ω_k are given by $\omega_k = L_{\tilde{\mathbf{X}}_0}^{(k)}\omega_0$: this modification does not affect neither the foliation \mathcal{F}_Θ , nor the field K . We keep on notations $T = K \langle \mathbf{X}_0, \dots, \mathbf{X}_{n-1} \rangle$, \mathcal{L} , \mathcal{L}_0 , etc... We are going to prove that, after conveniently choosing the generator \mathbf{X} for $\mathcal{L}/\mathcal{L}_0$ given by Lemma 3.3, then we have $\omega_3 = L_{\mathbf{X}}^{(3)}\omega_0 \equiv 0$ and \mathcal{F} is transversely projective, thus concluding the proof of Theorem 1.1.

Given a Lie algebra \mathcal{L} over a field K of characteristic 0 and a codimension 1 subalgebra \mathcal{L}_0 like above, a result due to J. Tits (see [21], or [6], p.31-33) asserts that there exists an ideal $\mathcal{J} \subset \mathcal{L}_0$ having codimension ≤ 3 in \mathcal{L} and the quotient \mathcal{L}/\mathcal{J} is of one of the following three types:

- (1) $\mathcal{L}/\mathcal{J} \simeq K \cdot \mathbf{X}$ and $\mathcal{J} = \mathcal{L}_0$,
- (2) $\mathcal{L}/\mathcal{J} \simeq K \cdot \mathbf{X} + K \cdot \mathbf{Y}$ with $[\mathbf{X}, \mathbf{Y}] = \mathbf{X}$ and $\mathcal{L}_0/\mathcal{J} = K \cdot \mathbf{Y}$,
- (3) $\mathcal{L}/\mathcal{J} \simeq K \cdot \mathbf{X} + K \cdot \mathbf{Y} + K \cdot \mathbf{Z}$ with $\mathfrak{sl}(2)$ relations

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{X}, \quad [\mathbf{X}, \mathbf{Z}] = 2\mathbf{Y} \quad \text{and} \quad [\mathbf{Y}, \mathbf{Z}] = \mathbf{Z}.$$

In each case, \mathbf{X} is one of the vector fields produced by Lemma 3.3.

In order to prove that $\omega_3 = L_{\mathbf{X}}^{(3)}\omega_0 \equiv 0$, we just have to verify that $\omega_3(\mathbf{V}) = 0$ for any relative vector field $\mathbf{V} \in T$. Indeed, any local meromorphic vector V field decomposes as

$$V = f_0 \cdot \mathbf{X}_0 + \dots + f_{n-1} \cdot \mathbf{X}_{n-1} + V'$$

where f_k are local meromorphic functions, \mathbf{X}_k are local representatives for the basis given by Lemma 3.2 and V' is a vector field tangent to Θ (and in particular to ω_3); we thus have

$$\omega_3(V) = f_0 \cdot \omega_3(\mathbf{X}_0) + \dots + f_{n-1} \cdot \omega_3(\mathbf{X}_{n-1}).$$

By Lemma 3.4, we just have to prove that $\omega_0(L_{\mathbf{X}}^{(3)}\mathbf{V}) = 0$ for any $\mathbf{V} \in T$. In fact, it is enough to consider $\mathbf{V} \in T_0$ since $L_{\mathbf{X}}\mathbf{X} = 0$. Since T acts by derivation on K and \mathcal{L} is the kernel, Observe that \mathcal{L} is an ideal of T : since the elements of T act as derivation on K , they can be considered as basic vector fields with regards to the fibration \mathcal{G} while \mathcal{L} is the sub-algebra of tangent vector fields. In particular, for any $\mathbf{V} \in T_0$, we have $L_{\mathbf{X}}\mathbf{V} = [\mathbf{X}, \mathbf{V}] \in \mathcal{L}$.

We now discuss on the three cases given by Tits' Result.

First Case: \mathcal{L}_0 is an ideal of \mathcal{L} .

We have $[\mathbf{X}, \mathbf{V}] = f \cdot \mathbf{X}$ modulo $\mathcal{L}_0 = \mathcal{J}$ for some $f \in K$. Therefore, $[\mathbf{X}, [\mathbf{X}, \mathbf{V}]] = 0$ modulo \mathcal{J} and $\omega_0(L_{\mathbf{X}}^{(2)}\mathbf{V}) = 0$: in this case, the foliation \mathcal{F} is transversely affine.

Second Case: $\mathcal{L}_0/\mathcal{J}$ is generated by \mathbf{Y} with $[\mathbf{X}, \mathbf{Y}] = \mathbf{X}$ modulo \mathcal{J} .

We have $[\mathbf{X}, \mathbf{V}] = f \cdot \mathbf{X} + g \cdot \mathbf{Y}$ mod \mathcal{J} and $[\mathbf{Y}, \mathbf{V}] = h \cdot \mathbf{Y}$ mod \mathcal{J} for coefficients $f, g, h \in K$ (here, we use the fact that both \mathbf{Y} and \mathbf{V} are tangent to \mathcal{F} , whence their Lie bracket). Applying Jacobi identity to \mathbf{X} , \mathbf{Y} and \mathbf{V} yields:

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{V}]] + [\mathbf{V}, [\mathbf{X}, \mathbf{Y}]] + [\mathbf{Y}, [\mathbf{V}, \mathbf{X}]] = h \cdot \mathbf{X} - g \cdot \mathbf{Y} = 0$$

and we have $h = g = 0$. In particular, $[\mathbf{X}, \mathbf{V}] = f \cdot \mathbf{X}$ and $[\mathbf{X}, [\mathbf{X}, \mathbf{V}]] = 0$. We conclude as before that \mathcal{F} is transversely affine.

Third Case: $\mathcal{L}_0/\mathcal{J}$ is generated by \mathbf{Y}, \mathbf{Z} with $[\mathbf{X}, \mathbf{Y}] = \mathbf{X}$, $[\mathbf{X}, \mathbf{Z}] = 2\mathbf{Y}$ and $[\mathbf{Y}, \mathbf{Z}] = \mathbf{Z}$ modulo \mathcal{J} .

We have:

$$\begin{cases} [\mathbf{X}, \mathbf{V}] &= f \cdot \mathbf{X} + g \cdot \mathbf{Y} + h \cdot \mathbf{Z} \\ [\mathbf{Y}, \mathbf{V}] &= i \cdot \mathbf{Y} + j \cdot \mathbf{Z} \\ [\mathbf{Z}, \mathbf{V}] &= k \cdot \mathbf{Y} + l \cdot \mathbf{Z} \end{cases} \quad \text{mod } \mathcal{J}$$

for some coefficients $f, g, h, i, j, k, l \in K$. Jacobi identity yields:

$$\begin{aligned} [\mathbf{X}, [\mathbf{Y}, \mathbf{V}]] + [\mathbf{V}, [\mathbf{X}, \mathbf{Y}]] + [\mathbf{Y}, [\mathbf{V}, \mathbf{X}]] &= \\ &= i \cdot \mathbf{X} + (2j - g) \cdot \mathbf{Y} - 2h \cdot \mathbf{Z} = 0 \end{aligned}$$

$$\begin{aligned} [\mathbf{X}, [\mathbf{Z}, \mathbf{V}]] + [\mathbf{V}, [\mathbf{X}, \mathbf{Z}]] + [\mathbf{Z}, [\mathbf{V}, \mathbf{X}]] &= \\ &= k \cdot \mathbf{X} + 2(f + l - i) \cdot \mathbf{Y} + (g - 2j) \cdot \mathbf{Z} = 0 \end{aligned}$$

$$[\mathbf{Y}, [\mathbf{Z}, \mathbf{V}]] + [\mathbf{V}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Z}, [\mathbf{V}, \mathbf{Y}]] = -k \cdot \mathbf{Y} + i \cdot \mathbf{Z} = 0$$

modulo \mathcal{J} and thus $h = i = k = 0$, $l = -f$ and $g = 2j$. In particular, $[\mathbf{X}, [\mathbf{X}, [\mathbf{X}, \mathbf{V}]]] = 0$ and \mathcal{F} is transversely projective, thus proving the Theorem 1.1.

4. PROOF OF THEOREM 1.4

In fact, we prove the more precise

Theorem 4.1. *Let \mathcal{F} be a foliation admitting a finite Godbillon-Vey sequence $(\omega_0, \omega_1, \dots, \omega_N)$ of length $N + 1 \geq 4$. Then*

- *either \mathcal{F} is the pull-back by a meromorphic map $\Phi : M \dashrightarrow C \times \mathbb{C}P(1)$ of the foliation $\underline{\mathcal{F}}$ defined by*

$$dz + \underline{\omega}_0 + \underline{\omega}_1 z + \dots + \underline{\omega}_N z^n$$

where $\underline{\omega}_k$ are meromorphic 1-forms on the curve C ,

- *or \mathcal{F} is transversely affine.*

In particular, we see that a purely transversely projective foliation cannot admits other finite Godbillon-Vey sequences than the projective triples.

Proof. Following Lemma 2.3, we have

$$\Omega = dz + \omega_0 + z\omega_1 + \left(\sum_{k=2}^N f_k \cdot z^k \right) \omega_N$$

for meromorphic functions $f_k \in \mathcal{M}(M)$, $f_N \equiv 1$ and $\omega_N \neq 0$. If $f_{N-1} = 0$, then $\omega_{N-1} = 0$ and Lemma 2.3 yields

$$0 = d\omega_{N-1} = \omega_0 \wedge \omega_N \quad \text{and} \quad \omega_2 \wedge \omega_N = 0;$$

we deduce that $d\omega_1 = \omega_0 \wedge \omega_2 = 0$ and \mathcal{F} is transversely affine. Otherwise, after a change of Godbillon-Vey sequence of the form (9) (see Section 2.1), we may assume moreover $f_{N-1} = N$. Now, the change of coordinate $\tilde{z} = z + 1$ on Ω

$$\begin{aligned} \Omega &= d(\tilde{z} - 1) + \omega_0 + (\tilde{z} - 1)\omega_1 + \dots + (\tilde{z} - 1)^N \omega_N \\ &= d\tilde{z} + \tilde{\omega}_0 + \tilde{z}\tilde{\omega}_1 + \dots + \tilde{z}^N \tilde{\omega}_N \end{aligned}$$

provides a new sequence $(\tilde{\omega}_0, \tilde{\omega}_1, \dots, \tilde{\omega}_N)$ of length $N + 1$ satisfying integrability conditions (2) (see Introduction). We take care that this is not a new Godbillon-Vey sequence for \mathcal{F} (but for $\mathcal{F}_{\tilde{\omega}_0}$, whenever $\tilde{\omega}_0 \neq 0$). In fact, we have

$$\omega_0 = \tilde{\omega}_0 + \tilde{\omega}_1 + \tilde{\omega}_2 + \dots + \tilde{\omega}_N.$$

We also note that $\tilde{\omega}_N = \omega_N$ and $\tilde{\omega}_{N-1} = 0$. Following Lemma 2.3, there exist meromorphic functions g_k satisfying

$$(21) \quad \tilde{\omega}_k = g_k \cdot \omega_N \quad \text{for} \quad k = 0, 2, \dots, N - 2$$

and integrability conditions now write

$$(22) \quad d\tilde{\omega}_k = (k - 1)\tilde{\omega}_1 \wedge \tilde{\omega}_k \quad \text{for} \quad k = 0, 2, \dots, N - 2$$

and

$$(23) \quad d\omega_N = (N-1)\tilde{\omega}_1 \wedge \omega_N, \quad d\tilde{\omega}_1 = 0.$$

In particular, we see that ω_N is transversely affine and that

$$(24) \quad \omega_0 = \tilde{\omega}_1 + (g_0 + g_2 + \cdots + g_{N-2})\omega_N.$$

Following Lemma 4.2 below, there is a non constant meromorphic function $g \in \mathcal{M}(M)$ such that $dg \wedge \omega_N = 0$. It follows from Stein's Factorization Theorem that there exist:

- a meromorphic map $\phi : M \dashrightarrow C$ onto a smooth, compact, complex and connected curve C ,
- a meromorphic function $\underline{g} : C \dashrightarrow \overline{\mathbb{C}}$,

such that $g = \underline{g} \circ \phi$ and the generic fibers $\phi^{-1}(c)$ are irreducible hypersurfaces of M . Let $\underline{\omega}$ be a non zero meromorphic 1-form on C . The 1-form $\omega := \phi^*\underline{\omega}$ on M is closed, non zero and $df \wedge \omega = 0$. Therefore, we can write $\tilde{\omega}_N = h_N \cdot \omega$ for a meromorphic function h and setting $h_k = h_N \cdot g_k$, we get

$$(25) \quad \tilde{\omega}_k = h_k \omega \quad \text{for } k = 0, 2, \dots, N-2, N.$$

From equations (22), we deduce that

$$(26) \quad \begin{cases} \text{either } h_k = 0, \\ \text{or } \left(\tilde{\omega}_1 - \frac{1}{k-1} \frac{dh_k}{h_k} \right) \wedge \omega = 0 \end{cases} \quad \text{for } k = 0, 2, \dots, N-2, N.$$

Thus, for any $k, l = 0, 2, \dots, N-2, N$ such that $h_k, h_l \neq 0$, we have

$$(27) \quad \left(\frac{1}{k-1} \frac{dh_k}{h_k} - \frac{1}{l-1} \frac{dh_l}{h_l} \right) \wedge \omega = 0$$

and $\frac{h_k^{(l-1)}}{h_l^{(k-1)}}$ is a first integral for ω . Let $r = \gcd\{k-1; h_k \neq 0\}$: we have $\sum_{h_k \neq 0} n_k(k-1) = r$ for integers n_k . Set

$$h := \prod_{h_k \neq 0} h_k^{n_k}.$$

Therefore, summing equations (26) over l , we get

$$0 = \sum_{h_l \neq 0} n_l \left(\frac{l-1}{r} \frac{dh_k}{h_k} - \frac{k-1}{r} \frac{dh_l}{h_l} \right) \wedge \omega = \left(\frac{dh_k}{h_k} - \frac{k-1}{r} \frac{dh}{h} \right) \wedge \omega.$$

Thus, $\frac{h_k}{h^{\frac{k-1}{r}}}$ is a first integral for ω and we can write

$$(28) \quad \begin{cases} \text{either } h_k = 0, \\ \text{or } h_k = \underline{h}_k \circ \phi \cdot h^{\frac{k-1}{r}} \end{cases} \quad \text{for } k = 0, 2, \dots, N-2, N$$

and meromorphic functions $\underline{h}_k : C \dashrightarrow \overline{\mathbb{C}}$. From equation (24), we deduce

$$(29) \quad \omega_0 = \tilde{\omega}_1 + \left(\sum_{k=0,2,\dots,N} \underline{h}_k \circ \phi \cdot h^{\frac{k-1}{r}} \right) \omega$$

(setting $\underline{h}_k = 0$ whenever $h_k = 0$). On the other hand, from (26) and (28) we get

$$(30) \quad \tilde{\omega}_1 \wedge \omega = \frac{1}{k-1} \frac{dh_k}{h_k} \wedge \omega = \frac{1}{r} \frac{dh}{h} \wedge \omega$$

and $\tilde{\omega}_1 = \frac{1}{r} \frac{dh}{h} + f\omega$ for a meromorphic function f . Since $\tilde{\omega}_1$ and ω are closed, we get after derivating equation (30) that $df \wedge \omega = 0$, i.e. we can write $f = \underline{f} \circ \phi$ for a meromorphic function $\underline{f} : C \dashrightarrow \overline{\mathbb{C}}$. Finally, we obtain

$$\omega_0 = \frac{1}{r} \frac{dh}{h} + \underline{f} \circ \phi \cdot \omega + \left(\sum_{k=0,2,\dots,N} \underline{h}_k \circ \phi \cdot h^{\frac{k-1}{r}} \right) \omega$$

and, setting $\Phi = (\phi, h) : M \dashrightarrow C \times \mathbb{C}P(1)$

$$r h \omega_0 = \Phi^* \left(dz + r(\underline{f} \cdot z + \sum_{k=0,2,\dots,N} \underline{h}_k z^{\frac{k-1}{r}+1}) \omega \right).$$

□

Lemma 4.2. *Let \mathcal{F} be a foliation admitting a finite Godbillon-Vey sequence $(\omega_0, \omega_1, \dots, \omega_N)$ of length $N+1 \geq 4$. Then*

- either $\omega_N = fdg$ for meromorphic functions $f, g \in \mathcal{M}(M)$,
- or \mathcal{F} is transversely affine.

Proof. We start as in proof of Theorem 4.1, keeping the same notations. Substituting (21) into integrability conditions (22) yield

$$(dg_k + (N-k)g_k \tilde{\omega}_1) \wedge \tilde{\omega}_N = 0 \quad \text{for } k = 0, 2, \dots, N-2.$$

If there exist two distinct integers $k, l \in \{0, 2, \dots, N-2\}$ such that $g_k, g_l \neq 0$, then we can deduce that

$$\left((N-k) \frac{dg_l}{g_l} - (N-l) \frac{dg_k}{g_k} \right) \wedge \tilde{\omega}_N = 0 ;$$

if moreover the left factor is not zero, then we can conclude that

$$dg \wedge \tilde{\omega}_N = 0 \quad \text{with} \quad g := \frac{g_l^{(N-k)}}{g_k^{(N-l)}} \quad \text{non constant}$$

i.e. $\omega_N = fdg$ for some meromorphic function f . **Otherwise**, the discussion splits into many cases.

Case 1. Assume that $g_k = 0$ for all $k \in \{0, 2, \dots, N-2\}$. Then

$$\omega_0 = \sum_{k=0}^N \tilde{\omega}_k = \tilde{\omega}_1 + \tilde{\omega}_N$$

and, since $d\tilde{\omega}_1 = 0$, we have

$$d\omega_0 = d\tilde{\omega}_N = (N-1)\tilde{\omega}_1 \wedge \tilde{\omega}_N = (N-1)\tilde{\omega}_1 \wedge \omega_0$$

and \mathcal{F} is transversely affine.

Case 2. Assume that $g_k \neq 0$ for at least one $k \in \{0, 2, \dots, N-2\}$ but

$$\frac{1}{N-l} \frac{dg_l}{g_l} = \frac{1}{N-k} \frac{dg_k}{g_k}$$

for all $k, l \in \{0, 2, \dots, N-2\}$ such that $g_k, g_l \neq 0$: the closed 1-form

$$\beta = \tilde{\omega}_1 + \frac{1}{N-k} \frac{dg_k}{g_k}$$

does not depend on k .

Subcase 2.1: $\beta = 0$. Since

$$\omega_0 = \tilde{\omega}_1 + g \cdot \tilde{\omega}_N, \quad g = g_0 + g_2 + \dots + g_{N-2} + 1$$

we get that either $g = 0$ and $\omega_0 = \tilde{\omega}_1$ is closed, or $g \neq 0$ and we have

$$d\left(\frac{\omega_0}{g}\right) = d\tilde{\omega}_N = (N-1)\tilde{\omega}_1 \wedge \tilde{\omega}_N = (N-1)\tilde{\omega}_1 \wedge \frac{\omega_0}{g};$$

in each case, we see that \mathcal{F} is transversely affine.

Subcase 2.2: $\beta \neq 0$. Therefore, one can write $\tilde{\omega}_N = h\beta$ for some meromorphic function $h \neq 0$ and we have

$$d\tilde{\omega}_N = \frac{dh}{h} \wedge \tilde{\omega}_N.$$

Comparing with $d\tilde{\omega}_N = (N-1)\tilde{\omega}_1 \wedge \tilde{\omega}_N$ and $\beta \wedge \tilde{\omega}_N = 0$, we get

$$\left(\frac{dh}{h} - \frac{N-1}{N-k} \frac{dg_k}{g_k} \right) \wedge \tilde{\omega}_N = 0.$$

Subsubcase 2.2.1: $\frac{N-1}{N-k} \frac{dg_k}{g_k} = \frac{dh}{h}$ for all $k \in \{0, 2, \dots, N-2\}$ such that $g_k \neq 0$. Then

$$\omega_0 = \tilde{\omega}_1 + gh \cdot \beta, \quad g = g_0 + g_2 + \dots + g_{N-2} + 1$$

with $dg \wedge dh = 0$. Since $\beta = \tilde{\omega}_1 + \frac{1}{N-1} \frac{dh}{h}$, we get

$$\omega_0 = (1 + gh)\tilde{\omega}_1 + \frac{g}{N-1} dh.$$

Either $1 + gh = 0$ and ω_0 is closed, or $1 + gh \neq 0$ and $\frac{\omega_0}{1+gh}$ is closed; in each case, \mathcal{F} is transversely affine.

Subsubcase 2.2.2: $\frac{N-1}{N-k} \frac{dg_k}{g_k} \neq \frac{dh}{h}$ for at least one k . Therefore, we can conclude that

$$dg \wedge \tilde{\omega}_N = 0 \quad \text{with} \quad g := \frac{h^{(N-k)}}{g_k^{(N-1)}} \quad \text{non constant}$$

i.e. $\omega_N = fdg$ for some meromorphic function f . □

5. EXAMPLES

5.1. Degree 2 foliations on $\mathbb{C}P(n)$ have length ≤ 4 . Here, we prove Theorem 1.5. In fact, given a degree 2 foliation \mathcal{F} on $\mathbb{C}P(n)$, we prove that, after a convenient birational transformation

$$\Phi : \mathbb{C}P(n) \dashrightarrow \mathbb{C}P(n-1) \times \mathbb{C}P(1),$$

the tangency locus Δ between the foliation $\mathcal{F}' := \Phi_* \mathcal{F}$ and the projection $\pi : \mathbb{C}P(n-1) \times \mathbb{C}P(1) \rightarrow \mathbb{C}P(n-1)$ takes the following special form:

- either Δ is a vertical hypersurface, i.e. defined by $R \circ \pi = 0$ for a non constant rational function R on $\mathbb{C}P(n-1)$,
- or Δ is the union of a vertical hypersurface like above and the horizontal hyperplane at infinity $H_\infty := \mathbb{C}P(n-1) \times \{\infty\}$.

One can easily deduce from this geometric picture that \mathcal{F}' is actually defined by a unique rational integrable 1-form

$$\Omega = dz + \sum_{k=0}^N \omega_k z^k$$

where ω_k are rational 1-forms on $\mathbb{C}P(n-1)$ and z is the $\mathbb{C}P(1)$ -variable. A Godbillon-Vey sequence of length $\leq N+1$ is therefore provided by $(L_X^{(k)} \Omega)_k$ where $X = \partial_z$ is the vertical vector field. We will also prove that $N \leq 3$ in our case. In the first case of the alternative above, we have $N \leq 2$: Δ is vertical, \mathcal{F}' is a Riccati foliation with respect to π and is in particular transversely projective. In the second case, $N = 2 + m$ where m is the multiplicity of contact between \mathcal{F}' and the projection

π along the hyperplane at infinity H_∞ . Actually, it is better to view Δ as a positive divisor, defined in charts by the holomorphic function $\omega(X)$ where X is a non vanishing holomorphic vector field tangent to the fibration given by π and ω a holomorphic 1-form defining \mathcal{F}' with codimension ≥ 2 zero set. Then, m is the weight of Δ along H_∞ .

Let \mathcal{F} be a degree 2 foliation on $\mathbb{C}P(n)$. In order to construct Φ and reach the geometrical picture above, the rough idea is to find a rational pencil on $\mathbb{C}P(n)$ such that the tangency locus Δ between the foliation and the pencil intersects each rational fiber once. In fact, we choose any singular point p of the foliation \mathcal{F} and consider the pencil of lines passing through p . Of course, the number of tangencies between a line and \mathcal{F} , counted with multiplicities, is 2, the degree of \mathcal{F} ; but looking at the pencil passing through p , we expect that the tangency occurring at the singular point disappear after blowing up the point p . Let us compute.

A foliation \mathcal{F} of degree ≤ 2 on $\mathbb{C}P(n)$ is given in an affine chart $\mathbb{C}^n \subset \mathbb{C}P(n)$ by a polynomial 1-form with codimension ≥ 2 zero set having the special form

$$\Omega = \omega_0 + \omega_1 + \omega_2 + \omega_3$$

where ω_i is homogeneous of degree i and ω_3 is radial (see [5]): we have $\omega_3(\mathcal{R}) = 0$, where $\mathcal{R} := x_1\partial_{x_1} + \dots + x_n\partial_{x_n}$ is the radial vector field. Saying that \mathcal{F} is not of degree less than 2 just means that, if ever $\omega_3 = 0$, then ω_2 is not radial. Let us assume $p = 0$ be singular for \mathcal{F} , i.e. $\omega_0 = 0$. The tangency locus between \mathcal{F} and the pencil of lines passing through 0 is given by $\text{tang}(\mathcal{F}, \mathcal{R}) = \{\Omega(\mathcal{R}) = 0\}$. If $\Omega(\mathcal{R})$ is the zero polynomial, then this means that \mathcal{F} is actually radial; we avoid this by choosing another singular point p . Therefore, $\text{tang}(\mathcal{F}, \mathcal{R})$ is a cubic hypersurface which is singular at p . After blowing-up the origin, the foliation lifts-up in the chart

$$\pi : (t_1, \dots, t_{n-1}, z) \mapsto (zt_1, \dots, zt_{n-1}, z) = (x_1, \dots, x_{n-1}, x_n)$$

just by lifting-up the 1-form Ω which now takes the special form

$$\pi^*\Omega = z((f_0(t) + zf_1(t))dz + z\tilde{\omega}_1 + z^2\tilde{\omega}_2 + z^3\tilde{\omega}_3)$$

where f_0 and f_1 are polynomial functions of $t = (t_1, \dots, t_{n-1})$ and $\tilde{\omega}_i$ are polynomial 1-forms depending only on t . We observe that $\text{tang}(\mathcal{F}, \mathcal{R})$ is now defined by $\{z(f_0(t) + zf_1(t)) = 0\}$, has possibly some vertical components given by common factors of f_0 and f_1 and has exactly 2 non vertical components defined by $z = 0$ and $z = -f_0/f_1$ (the two tangencies between any line of the pencil with \mathcal{F}). Also, as expected, the first section $z = 0$ is irrelevant since it disappears after division of

$\pi^*\Omega$: the tangency locus between the lifted foliation $\tilde{\mathcal{F}}$ and the lifted pencil (the vertical line bundle $\{t = \text{constant}\}$) actually reduces to $\{f_0(t) + zf_1(t) = 0\}$ in the chart above. We now discuss on this set.

If $f_0 \equiv 0$, then $\frac{\Omega}{z^2 f_1(t)}$ is Riccati with wingular set over $\{f_1(t) = 0\}$: \mathcal{F} has length ≤ 3 . Recall that we have supposed \mathcal{F} non radial and thus f_0 and f_1 cannot vanish identically simultaneously.

If $f_0 \not\equiv 0$, then the non vertical component of $\text{tang}(\mathcal{F}, \mathcal{R})$ is the section $z = s(t)$, $s(t) := -\frac{f_0(t)}{f_1(t)}$. If $f_1 \equiv 0$, then this section is the hyperplane at infinity $\{z = \infty\}$: $\frac{\Omega}{f_0(t)}$ is already in the expected geometrical normal form and has length ≤ 4 . If $f_1 \not\equiv 0$, it suffices to push it towards infinity by a meromorphic change of coordinate of the form $\tilde{z} := \frac{z}{z-s(t)}$; after this birational transformation, we are in the previous case $y_1 \equiv 0$ and we have done. Precisely, the foliation is defined by

$$d\tilde{z} - \tilde{z} \frac{\tilde{\omega}_1}{f_0} + \tilde{z}^2 \left(\frac{2\tilde{\omega}_1}{f_0} - \frac{df_0}{f_0 f_1} + \frac{df_1}{f_1^2} - \frac{\tilde{\omega}_2}{f_1} \right) - \tilde{z}^3 \left(\frac{\tilde{\omega}_1}{f_0} - \frac{\tilde{\omega}_2}{f_1} + \frac{f_0 \tilde{\omega}_3}{f_1^2} \right).$$

In order to finish the proof of Theorem 1.5, we note that a generic degree 2 foliation of $\mathbb{C}P(2)$ has length 4, i.e. is not transversely projective. Actually, this is a well known fact. For instance, it immediately follows from Corollary 2.16 and the fact that a generic degree $d \geq 2$ foliation on $\mathbb{C}P(2)$ has no invariant algebraic curve. An explicit example is given in Section 5.3.

Remark 5.1. If \mathcal{F} is a foliation of $\mathbb{C}P(2)$ given by a 1-form of the type $\omega = \omega_\nu + \omega_{\nu+1} + f_{\nu+1}(xdy - ydx)$ then, for generic ω as above, $\text{Tang}(\mathcal{F}, \mathcal{R})$ is a rational curve and an argument similar to the one used above implies that \mathcal{F} also satisfies $l(\mathcal{F}) \leq 3$.

5.2. Proof of Theorem 1.6. Let \mathcal{F} be the germ of singular foliation defined by an holomorphic 1-form ω satisfying Frobenius integrability condition $\omega \wedge d\omega = 0$. If ω is not vanishing at the origin, then \mathcal{F} is regular and has obviously length 1 (defined by a closed 1-form). When ω is vanishing and the linear part is not zero, there are two cases. If the linear part is not radial, then there exist, following [13] (Corollary 3), local coordinates (\underline{x}, z) , $\underline{x} = (x_1, \dots, x_{n-1})$, in which the foliation is defined by

$$\omega_0 = df + zdg + zdz$$

where $f, g \in \mathbb{C}\{\underline{x}\}$ are holomorphic, not depending on z . In new coordinate $\tilde{z} = \frac{1}{z}$, the foliation is defined by

$$\tilde{\omega}_0 = d\tilde{z} + \tilde{z}^2 dg + \tilde{z}^3 df$$

and a Godbillon-vey sequence of length ≤ 4 is given by $\tilde{\omega}_k := L_{\partial_z}^{(k)} \tilde{\omega}_0$. Finally, when the linear part is radial, not zero, then it is defined in convenient coordinates by $\omega = \frac{dz}{z} - \frac{dx_1}{x_1}$ (Kupka phenomenon, see) and has length 1.

5.3. The examples of Jouanolou. In [11], Jouanolou exhibited the first examples of holomorphic foliations of the projective plane without algebraic invariant curves. His examples, one for each degree greater than or equal to 2, are the foliations of $\mathbb{C}P(2)$ induced by the homogeneous 1-forms in \mathbb{C}^3

$$\Omega_n = \det \begin{pmatrix} dx & dy & dz \\ x & y & z \\ y^n & z^n & x^n \end{pmatrix}.$$

The automorphism group of the foliation \mathcal{J}_d , induced by Ω_d , is isomorphic to a semi-direct product of $\mathbb{Z}/(n^2 + n + 1)\mathbb{Z}$ with $\mathbb{Z}/3\mathbb{Z}$ and is generated by the transformations $\psi_n(x : y : z) = (\delta^{n^2}x : \delta^n y : \delta z)$ and $\rho(x : y : z) = (y : z : x)$, where δ is a primitive $(n^2 + n + 1)^{\text{th}}$ root of the unity.

In [14] it is observed that the foliations \mathcal{J}_n can be presented in a different way. If \mathcal{F}_n is the degree 2 foliation of $\mathbb{C}P(2)$ induced by the 1-form

$$\omega_n = \det \begin{pmatrix} dx & dy & dz \\ x & y & z \\ x(-x + ny) & y(-y + nz) & z(-z + nx) \end{pmatrix},$$

and $\phi_n : \mathbb{C}P(2) \dashrightarrow \mathbb{C}P(2)$ is the rational map (of degree $n^2 + n + 1$) given by

$$\phi_n(x : y : z) = (y^{n+1} \cdot z : z^{n+1} \cdot x : x^{n+1} \cdot y)$$

then the foliation \mathcal{J}_n is the pull-back of the foliation \mathcal{F}_n under ϕ_n , i.e., $\mathcal{J}_n = \phi_n^* \mathcal{F}_n$. Conversely we can say that \mathcal{F}_n is birationally equivalent to the quotient of \mathcal{J}_n by the group generated by ψ_n .

From the results of the previous section it follows that \mathcal{F}_n has length at most 4. Pulling-back a Godbillon-Vey sequence by ϕ_n we obtain that the length of \mathcal{J}_n is also bounded by 4 and since it does not admit invariant algebraic curves its length is precisely 4. We have therefore proved the

Corollary 5.2. *The foliations \mathcal{J}_n , for every $n \geq 2$, have length 4.*

5.4. A new component of the space of foliations on $\mathbb{C}P(3)$. We start by considering the transversely projective foliation on $\mathbb{C}P(2)$ given in the affine chart $\{(x, y)\} = \mathbb{C}^2 \subset \mathbb{C}P(2)$ by the 1-form

$$\omega = xdy - ydx + P_2dx + Q_2dy + R_2(xdy - ydx).$$

where P_2, Q_2, R_2 are generic homogeneous polynomials of degree 2. This is a degree 2 foliation of $\mathbb{C}P(2)$ transverse to the Hopf fibration $x/y = \text{const}$ outside three distinct lines. Let us consider the homogenization Ω_3 of ω in the coordinates (x, y, z) of \mathbb{C}^3 :

$$\Omega_3 = z^2(xdy - ydx) + z(P_2dx + Q_2dy) + R_2(xdy - ydx) - R_3dz,$$

where $R_3(x, y) = xP_2 + yQ_2$. The genericity condition on P_2, Q_2, R_2 implies that $d\Omega_3$ has only one zero on \mathbb{C}^3 which is isolated and located at the origin. Of course, Ω_3 defines a transversely projective foliation of $\mathbb{C}^3 \subset \mathbb{C}P(3)$. We will twist this foliation by a polynomial automorphism of \mathbb{C}^3 . More precisely, if $\sigma(x, y, z) = (x, y, z + x^2)$ then

$$\Omega := \sigma^*\Omega_3 = \Omega_3 + \Omega_4 + \Omega_5 \quad \text{with}$$

$$\begin{cases} \Omega_3 &= z^2(xdy - ydx) + z(P_2dx + Q_2dy) + R_2(xdy - ydx) - R_3dz \\ \Omega_4 &= 2zx^2(xdy - ydx) + x^2(P_2dx + Q_2dy) - 2xR_3dx \\ \Omega_5 &= x^4(xdy - ydx) \end{cases}$$

The 1-form Ω defines a degree 4 foliation on $\mathbb{C}P(3)$ which is transverse to the Hopf fibration (induced by the radial vector field $\mathcal{R} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$) outside the union of the four hyperplanes $\Omega_4(\mathcal{R}) = x^2R_3(x, y) = 0$. If P_2, Q_2, R_2 are generic, then these four hyperplanes are distinct.

Let \mathcal{F}' be a foliation of degree 4 close to \mathcal{F}_Ω : \mathcal{F}' is given in the affine chart \mathbb{C}^3 by a polynomial 1-form

$$\Omega' = \Omega'_0 + \Omega'_1 + \Omega'_2 + \Omega'_3 + \Omega'_4 + \Omega'_5,$$

where the Ω'_k are homogeneous of degree k and $\Omega'_5(\mathcal{R}) \equiv 0$.

After normalization, we can suppose that the coefficients of Ω' are close to those of Ω . Since $d\Omega_3$ has an isolated singularity at 0, there exists (see [2]) a point $0'$ where the 2-jet of Ω' is zero, and the radial vector field centered at $0'$ is in the kernel of the 3-jet. Therefore, after translating $0'$ to 0, we can suppose that \mathcal{F}' is given by

$$\Omega' = \Omega'_3 + \Omega'_4 + \Omega'_5.$$

We verify that Ω' is transversely projective (with poles contained in $\Omega'_4(\mathcal{R})$). In fact, since \mathcal{F} is not transversely affine, the same holds for \mathcal{F}' . Therefore, in the space $\mathcal{F}(3, 4)$ of foliations of degree 4 in $\mathbb{C}P(3)$, every element \mathcal{F}' belonging to the irreducible component containing \mathcal{F} is actually transversely projective.

5.5. Transversely projective foliations that are not pull-back.

Example 5.3 (Example 8.6 of [9]). Let Γ be discrete torsion free subgroup of $PSL(2, \mathbb{R})^n$ such that the quotient $PSL(2, \mathbb{R})^n/\Gamma$ is compact. For $n \geq 2$, there exists examples such that the projection $\pi(\Gamma)$ on the first factor is a dense subgroup of $PSL(2, \mathbb{R})$ (see [1]). The action of Γ on \mathbb{H}^n , the n product of the upper half-plane, is free, cocompact and preserves the regular foliation induced by the projection on the first factor. In this way, we obtain a regular transversely projective foliation \mathcal{F} on a n -dimensional compact complex manifold M such that every leaf is dense and the generic leaf is biholomorphic to \mathbb{H}^{n-1} . Observe that \mathcal{F} is not the pull-back of a foliation on a lower dimensional manifold, otherwise there would exist compact subvarieties in \mathbb{H}^{n-1} .

Example 5.4 (Hilbert Modular Foliations). Let K be a totally real number field of degree $n \geq 2$ over the rational numbers \mathbb{Q} and let \mathcal{O}_K be the ring of integers of K . The group $\Gamma = PSL(2, \mathcal{O}_K)$ is dense in $PSL(2, \mathbb{R})$, but considering the n embeddings $i \circ \sigma : K \hookrightarrow \mathbb{R}$ given by the action $\sigma \in \text{Gal}(K/\mathbb{Q})$, we get an embedding $\Gamma \hookrightarrow PSL(2, \mathbb{R})^n$ as a discrete subgroup of the product. The quotient of \mathbb{H}^n , the n -product of the upper-half plane \mathbb{H} , by Γ is a quasiprojective variety V which can be singular due to torsion elements of Γ . One can compactify and desingularize V and obtain a projective manifold M . The n fibrations on \mathbb{H}^n given by the projections on each of the factors induce n foliations on M which are regular and pair-wise transversal outside the invariant hypersurfaces coming from the compactification and desingularization of V . By construction, they are transversely projective and all leaves apart from the invariant hypersurface above are dense in M . In [22] and [15], some basic properties of these foliations are described.

When $K = \mathbb{Q}(\sqrt{5})$, the resulting variety is birationally equivalent to the projective plane. In [15] explicit equations for the foliations associated to the two projections $\mathbb{H}^2 \rightarrow \mathbb{H}$, denoted by \mathcal{F}_2 and \mathcal{F}_3 , are determined. We give below an explicit projective triple for them. The corresponding suspensions \mathcal{H}_2 and \mathcal{H}_3 defined by

$$\Omega = dz + \omega_0 + z\omega_1 + z^2\omega_2$$

can be seen as singular foliations on $\mathbb{C}P(2) \times \mathbb{C}P(1)$ or equivalently on $\mathbb{C}P(3)$. Although the leaves of \mathcal{F}_2 are dense, we note that the same is not true for \mathcal{H}_2 since the monodromy lie in $PSL(2, \mathbb{R})$.

A projective triple for \mathcal{F}_2

$$\begin{aligned}
\omega_0 &= \frac{(80y - 60xy - 80x^2)dx + (36x^2 - y - 32x)dy}{-720x^3y + 1728x^5 + 80xy^2 - y^3 + 640x^2y - 1600x^4 - 64y^2} \\
\omega_1 &= \frac{16}{3} \frac{24y + 3y^2 + 62xy - 24x^2 - 108x^2y - 368x^3 + 432x^4}{-720x^3y + 1728x^5 + 80xy^2 - y^3 + 640x^2y - 1600x^4 - 64y^2} dx \\
&\quad - \frac{4}{15} \frac{150y + 3y^2 + 192x - 172xy - 872x^2 + 720x^3}{-720x^3y + 1728x^5 + 80xy^2 - y^3 + 640x^2y - 1600x^4 - 64y^2} dy \\
\omega_2 &= \frac{32}{45} \frac{144y + 84y^2 + 3y^3 + 852xy - 48xy^2 - 144x^2 - 1472x^2y - 36x^2y^2 - 4416x^3 + 336x^3y + 4928x^4}{-720x^3y + 1728x^5 + 80xy^2 - y^3 + 640x^2y - 1600x^4 - 64y^2} dx \\
&\quad - \frac{32}{225} \frac{441y + 288x - 372xy + 12xy^2 - 2292x^2 - 56x^2y + 1472x^3 - 108x^3y + 720x^4}{-720x^3y + 1728x^5 + 80xy^2 - y^3 + 640x^2y - 1600x^4 - 64y^2} dy
\end{aligned}$$

A projective triple for \mathcal{F}_3

$$\begin{aligned}
\omega_0 &= \frac{\left(-\frac{5}{4}y^2 + 20xy - 60x^3\right)dx + \left(-y + \frac{3}{4}xy + x^2\right)dy}{-720x^3y + 1728x^5 + 80xy^2 - y^3 + 640x^2y - 1600x^4 - 64y^2} \\
\omega_1 &= 4 \frac{-80x^2y + 3y^2 - 208x^3 + 288x^4 + 48xy}{-720x^3y + 1728x^5 + 80xy^2 - y^3 + 640x^2y - 1600x^4 - 64y^2} dx \\
&\quad - \frac{2}{5} \frac{40y + y^2 - 168x^2 + 192x^3 - 50xy}{-720x^3y + 1728x^5 + 80xy^2 - y^3 + 640x^2y - 1600x^4 - 64y^2} dy \\
\omega_2 &= \frac{32}{5} \frac{-9y^2 + 80xy + 8xy^2 - 16x^2y - 368x^3 - 48x^3y + 320x^4}{-720x^3y + 1728x^5 + 80xy^2 - y^3 + 640x^2y - 1600x^4 - 64y^2} dx \\
&\quad + \frac{32}{25} \frac{-36y + 63xy + 164x^2 - 28x^2y - 304x^3 + 144x^4}{-720x^3y + 1728x^5 + 80xy^2 - y^3 + 640x^2y - 1600x^4 - 64y^2} dy
\end{aligned}$$

Theorem 5.5. *The explicit suspensions \mathcal{H}_2 and \mathcal{H}_3 above are not the meromorphic pull-back of a foliation on a surface.*

Proof. Suppose that there exists a foliation $\underline{\mathcal{H}}_2$ on a surface S and a meromorphic map $\Phi : \mathbb{C}P(2) \times \mathbb{C}P(1) \dashrightarrow S$ such that $\Phi^*\underline{\mathcal{H}}_2 = \mathcal{H}_2$.

Let $U \subset \mathbb{C}P(2) \times \mathbb{C}P(1)$ be the Zariski open subset where Φ is holomorphic and $U_0 = U \cap (\mathbb{C}P(2) \times \{0\})$. After blowing-up S , one can assume $\Phi(U_0)$ having codimension ≤ 1 . The generic rank of Φ restricted to $U_0 = U \cap (\mathbb{C}P(2) \times \{0\})$ is 2, otherwise we are in one of the following contradicting situations

- (1) The closure of $\Phi(U_0)$ is a proper submanifold of S non-invariant by $\underline{\mathcal{H}}_2$. In particular \mathcal{F}_2 is the pull-back of a foliation of a foliation on a curve and is transversely euclidean; contradiction.
- (2) The closure of $\Phi(U_0)$ is a proper submanifold of S invariant by $\underline{\mathcal{H}}_2$ (and not contained in the singular set of $\underline{\mathcal{H}}_2$). Reasoning in local coordinates at the neighborhood of a generic point $p \in \Phi(U_0)$, we see that $\mathbb{C}P(2) \times \{0\}$ is invariant by \mathcal{H}_2 obtaining a contradiction.

We conclude therefore that $\Phi|_{U_0}$ is dominant and $\underline{\mathcal{H}}_2 = \Phi_*\mathcal{F}_2$ has dense leaves (in fact all but finitely many). Therefore, the same density property holds for the pull-back $\mathcal{H}_2 = \Phi^*\underline{\mathcal{H}}_2$ providing a contradiction: the Riccati foliation \mathcal{H}_2 has no dense leaf since its monodromy is contained in $PSL(2, \mathbb{R})$. This proves the Theorem. \square

6. INTEGRABLE 1-FORMS IN POSITIVE CHARACTERISTIC

Due to the algebraic nature of many of the arguments used through this paper it is natural to ask if it would be possible carry on a similar study for integrable 1-forms on varieties defined over fields of positive characteristic.

The surprising fact, at least for us, is that over fields of positive characteristic every 1-form admits a Godbillon-Vey sequence of length one. In the case of 1-forms on the projective plane this is already implicitly proved in [17].

Our argument is based on the following

Lemma 6.1. *Let M be a m -dimensional smooth projective variety defined over an arbitrary field. If ω is an integrable rational 1-form then there exists $m - 1$ rationally independent vector fields X_1, \dots, X_{m-1} such that*

- (1) $[X_i, X_j] = 0$ for every $i, j \in 1, \dots, m - 1$;
- (2) $\omega(X_i) = 0$ for every $i \in 1, \dots, m - 1$.

Proof. Let $f_1, \dots, f_{m-1} \in k(M)$ be rational functions such that

$$\omega \wedge df_1 \wedge \dots \wedge df_{m-1} \neq 0.$$

If $\omega_m = \omega$ and $\omega_i = df_i$, for $i = 1 \dots m-1$ then $\{\omega_i\}_{i=1}^m$ form a basis of the $k(M)$ -vector space of rational 1-forms over M .

Let $\{X_i\}_{i=1}^m$ be a basis of the space of rational vector fields on M dual to $\{\omega_i\}_{i=1}^m$, i.e., $\omega_i(X_j) = \delta_{ij}$. It is clear that $\omega(X_i) = 0$ for every $i = 1 \dots m-1$. We claim that $[X_i, X_j] = 0$ for every $i, j = 1 \dots m-1$. It is sufficient to show that

$$(31) \quad \omega_k([X_i, X_j]) = 0 \text{ for every } k = 1 \dots m$$

For $k = m$ the integrability of ω implies that (31) holds. For $k < m$ we have that

$$\begin{aligned} \omega_k([X_i, X_j]) &= X_i(\omega_k(X_j)) - X_j(\omega_k(X_i)) + d\omega_k(X_i, X_j) = \\ &= X_i(\delta_{kj}) - X_j(\delta_{ki}) + d^2 f_k(X_i, X_j) = 0. \end{aligned}$$

This shows that (31) holds for every $k = 1 \dots m$ and concludes the proof of the lemma. \square

Theorem 6.2. *Let M be a smooth projective variety defined over a field K of characteristic $p > 0$ and ω be a rational 1-form. If ω is integrable then ω admits an "integrating factor", i.e., there exist a rational function $F \in K(M)$ such that $F\omega$ is closed. Equivalently we have that*

$$d\omega = \omega \wedge \frac{dF}{F}.$$

Proof. Let m be the dimension of M and X_1, \dots, X_{m-1} be the rational vector fields given by lemma 6.1. We will distinguish two cases:

- (1) for every $i = 1 \dots m-1$ we have that $\omega(X_i^p) = 0$
- (2) there exists $i \in \{1, \dots, m-1\}$ such that $\omega(X_i^p) \neq 0$

Let \mathcal{F} be the unique saturated subsheaf of the tangent sheaf of M which coincides with the kernel of ω over the generic point of M . The integrability of ω implies that \mathcal{F} is involutive. If we are in the case (1) then we have also that \mathcal{F} is p -closed. From [16, propositions 1.7 and 1.9, p. 55–56] it follows that $\omega = gdf$ where $g, f \in k(M)$.

In case (2) we can suppose that $\omega(X_1^p) \neq 0$. If $F = \omega(X_1^p)^{-1}$ then

$$d(F\omega) = F\omega \wedge L_{X_1^p}(F\omega).$$

To conclude we have just to prove that $L_{X_1^p}(F\omega) = 0$. In fact since $F\omega(X_1^p) = 1$ it follows that

$$L_{X_1^p}(F\omega) = i_{X_1^p}d(F\omega).$$

Moreover for every $i = 1 \dots m - 1$ we have that $[X_1^p, X_i] = 0$, since X_1 commutes with X_i , and therefore

$$i_{X_1^p} d(F\omega)(X_i) = F\omega([X_1^p, X_i]) - X_1^p(F\omega(X_i)) + X_i(F\omega(X_1^p)) = 0.$$

This is sufficient to show that $L_{X_1^p}(F\omega) = 0$ concluding the proof of the Theorem. \square

As a corollary we obtain a codimension one version of the main result of [17].

Corollary 6.3. *Let ω be a polynomial integrable 1-form on \mathbb{A}_k^n , where k is a field of positive characteristic. If $d\omega \neq 0$ then there exists an irreducible algebraic hypersurface H such that $i^*\omega = 0$, where $i : H \rightarrow \mathbb{A}_k^n$ denotes the inclusion.*

Proof. Of course ω can be interpreted as rational 1-form over \mathbb{P}_k^n which is regular over \mathbb{A}_k^n . From Theorem 1.8 there exists a rational function $F \in k(x_1, \dots, x_n)$ such that

$$d\omega = \omega \wedge \frac{dF}{F}.$$

Since $d\omega \neq 0$ we have that $dF \neq 0$, i.e., F is not a p -th power. In particular the polar set of dF/F is not empty. It is an easy exercise to show that every irreducible component H of the polar set of dF/F satisfies $i^*\omega = 0$, where $i : H \rightarrow \mathbb{A}_k^n$ denotes the inclusion \square

In fact the same proof as above yields the stronger

Corollary 6.4. *Let ω be a regular integrable 1-form over a smooth quasiprojective algebraic variety M defined over k , a field of positive characteristic. Suppose that $H^0(M, \mathcal{O}_M^*) = k^*$. If $d\omega \neq 0$ then there exists an irreducible algebraic hypersurface H such that $i^*\omega = 0$, where $i : H \rightarrow M$ denotes the inclusion.*

Observe that the result above can be applied to projective varieties since there exists such varieties with global regular 1-forms which are not closed, see [16].

REFERENCES

- [1] A. Borel, *Compact Clifford-Klein forms of symmetric spaces*. Topology 2 (1963) 111–122.
- [2] C. Camacho, A. Lins Neto, *The topology of integrable differential forms near a singularity*. Inst. Hautes Études Sci. Publ. Math. No. 55 (1982), 5–35.
- [3] C. Camacho, B. Scárdua, *Beyond Liouvillian transcendence*, Math. Res. Lett. 6 (1999), no. 1, 31–41.
- [4] G. Casale, *Suites de Godbillon-Vey et intégrales premières*, C. R. Math. Acad. Sci. Paris 335 (2002), no. 12, 1003–1006.

- [5] D. Cerveau, A. Lins Neto, *Irreducible components of the space of holomorphic foliations of degree two in $\mathbf{CP}(n)$, $n \geq 3$* , Ann. of Math. (2) 143 (1996), no. 3, 577–612.
- [6] D. Cerveau, A. Lins Neto, F. Loray, J.V. Pereira and F. Touzet *Algebraic Reduction Theorem for complex codimension one singular foliations*, arXiv math.CA/0406293 (2004).
- [7] D. Cerveau, A. Lins Neto, F. Loray, J.V. Pereira and F. Touzet *Algebraic Reduction Theorem for complex codimension one singular foliations*, Comment. Math. Helv. 81 (2006), to appear.
- [8] C. Godbillon, *Feuilletages. Études géométriques*. With a preface by G. Reeb. Progress in Mathematics, 98. Birkhäuser Verlag, Basel, 1991.
- [9] É. Ghys, X. Gómez-Mont, J. Saludes, *Fatou and Julia Components of Transversely Holomorphic Foliations*, Monographie de L' Enseignement Mathématique 38 (2001), p. 287–319.
- [10] A. T. Huckleberry, G. A. Margulis, *Invariant analytic hypersurfaces*, Invent. Math. 71 (1983), no. 1, 235–240.
- [11] J.-P. Jouanolou, *Équations de Pfaff algébriques.*, Lecture Notes in Mathematics 708, Springer-Verlag.
- [12] A. Lins Neto, *Some examples for the Poincaré and Painlevé problems*. Ann. Sci. École Norm. Sup. (4) 35 (2002), no. 2, 231–266.
- [13] F. Loray, *A Preparation Theorem for codimension one foliations*. Ann. of Math. (2) 163 (2006), no.2, 709–722.
- [14] A. Maciejewski, J. Moulin Ollagnier, A. Nowicki, J.-M. Strelcyn, *Around Jouanolou non-integrability theorem*, Indag. Math. (N.S.) 11 (2000), no. 2, 239–254.
- [15] L. G. Mendes, J. V. Pereira, *Hilbert Modular Foliations on the Projective Plane*, Commentarii Mathematici Helvetici, 80, pp.243-291,2005.
- [16] Y. Miyaoka, T. Peternell, *Geometry of higher-dimensional algebraic varieties*, DMV Seminar, 26. Birkhäuser Verlag, Basel, 1997.
- [17] J. V. Pereira, *Invariant hypersurfaces for positive characteristic vector fields*, J. Pure Appl. Algebra 171 (2002), no. 2-3, 295–301.
- [18] B. Scárdua, *Transversely affine and transversely projective holomorphic foliations*, Ann. Sci. École Norm. Sup. (4) 30 (1997), no. 2, 169–204.
- [19] C. L. Siegel, *Meromorphe Funktionen auf kompakten analytischen Mannigfaltigkeiten*. (German) Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. IIa. 1955 (1955), 71–77.
- [20] D. Sullivan, W. Thurston, *Manifolds with canonical coordinate charts: some examples*. Enseign. Math. (2) 29 (1983), no. 1-2, 15–25.
- [21] J. Tits, *Sur une classe de groupes résolubles*. Bull. Soc. Math. Belg. 11 (1959), 100–115.
- [22] F. Touzet, *Sur les feuilletages holomorphes transversalement projectifs*, Ann. Inst. Fourier, Grenoble 53, 3 (2003), 815–846.
- [23] K. Ueno, *Classification theory of algebraic varieties and compact complex spaces*. Notes written in collaboration with P. Cherenack. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975.

1 IRMAR, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX, FRANCE

2 IMPA, ESTRADA DONA CASTORINA, 110, HORTO, RIO DE JANEIRO, BRASIL

E-mail address: dominique.cerveau,frank.loray,frederic.touzet@univ-rennes1.fr
alcides,jvp@impa.br