SHARP BILINEAR ESTIMATES AND WELL-POSEDNESS FOR THE 1-D SCHRÖDINGER-DEBYE SYSTEM

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Abstract. We establish local and global well-posedness for the initial value problem associated to the (one-dimensional) Schrödinger-Debye (SD) system for data in the Sobolev spaces with low regularity. To obtain local results we prove two new sharp bilinear estimates for the coupling terms of this system in the continuous and periodic cases. Concerning global results, the system is shown to be globally well-posed in $H^s \times H^s, -1/8 < s < 0$. This is quite surprising in view of Bidegaray's theorem: in $H^s \times H^s, s > 5/2$, there are one-parameter families of solutions of the SD system converging to certain solutions of the cubic NLS equation. In fact, since the cubic NLS is known to be ill-posed below L^2, the results of Bidegaray says that the existence of global solutions of SD system in $H^s \times H^s$ for negative Sobolev index s is unexpected. The proof of our global result uses the **I**-method introduced by Colliander, Keel, Staffilani, Takaoka and Tao.

1. **Introduction**

This paper is devoted to the Initial Value Problem(IVP) for the Schrödinger-Debye system, that is,

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_x^2 u = uv, \quad t \in \mathbb{R}, \quad x \in M \\ \sigma\partial_t v + v = \epsilon |u|^2, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ is a complex valued function, $v = v(x,t)$ is a real valued function, $\sigma > 0$, $\epsilon = \pm 1$ and M is the real line $\mathbb{R}$ (continuous case) or the torus $\mathbb{T}$ (periodic case)

Recently, the well-posedness for the IVP (1.1) was studied in the classical Sobolev spaces $H^k(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ by several authors. We summarize them as follows: for $M = \mathbb{R}^n$, Corcho and Linares [6] proved the following results:

- local well-posedness in $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for $0 < s < 1$;
- global well-posedness in $H^{\frac{1}{2}}(\mathbb{R}) \times L^2(\mathbb{R})$;
- global well-posedness in $H^k(\mathbb{R}) \times H^s(\mathbb{R})$ for $k - 1/2 < s \le k$ and $1/2 \le k \le 1$;
- global well-posedness in $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, in dimensions $n = 1, 2, 3$,

and for $M = \mathbb{T}^n$ Arbieto and Matheus [1] showed the following ones:

- local and global well-posedness in $H^s(\mathbb{T}) \times H^s(\mathbb{T})$ for $s \ge 0$;
- local well-posedness in $H^s(\mathbb{T}^n) \times H^s(\mathbb{T}^n)$ for $s > 0$, $n \ge 1$;
- global well-posedness in $H^s(\mathbb{T}^2) \times H^s(\mathbb{T}^2)$ for $s \ge 1$.

The proof of these theorems uses Picard fixed-point method in certain spaces. To do so, Corcho, Linares [6] and Arbieto, Matheus [1] start by *decoupling* the SD

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system (1.1), i.e., they write:

(1.2)
$$u(t)=U(t)u_0-i\int_0^t U(t-t')\left(e^{-\frac{t'}{\sigma}}v_0u(t')+\frac{\varepsilon}{\sigma}u(t')\int_0^{t'}e^{-\frac{(t'-t'')}{\sigma}}|u(t'')|^2dt''\right)dt',$$

where $U(t)=e^{it\Delta/2}$ is the Schrödinger linear semigroup. In the sequel, they prove some multilinear estimates for the nonlinearities in order to Picard's argument run correctly, i.e., they show a *bilinear* estimate for the term

$$\int_0^t U(t-t')\cdot e^{-\frac{t'}{\sigma}}v_0u(t')dt'$$

and a *trilinear* estimate for the term

$$\int_0^t U(t-t')\frac{\varepsilon}{\sigma}u(t')\left(\int_0^{t'}e^{-\frac{(t'-t'')}{\sigma}}|u(t'')|^2dt''\right)dt'.$$

Analogously to [6], [1], we are interested in the local well-posedness of IVP (1.1) for initial data with low regularity for $M=\mathbb{T}$ and $M=\mathbb{R}$, specially local and global well-posedness in the continuous case and initial data in $H^k\times H^s$ for negative Sobolev indices (k,s)). Unfortunately, it is not reasonable to expect that the approach discussed above can be pushed to work with negative Sobolev indices. Indeed, similarly to the situation of Schrödinger (NLS) equation, we know that such trilinear estimates holds only for non-negative indices.

Bearing the difficulty in mind, we propose in this paper a slightly different approach: instead of decoupling the SD system before studying its integral formulation (which leads to trilinear estimates), we keep the SD system coupled so that we have only to deal with bilinear estimates (for the coupling terms uv and $|u|^2$). To understand what is the advantage of our new proposal, we review the bilinear estimates for the quadratic NLS obtained by Kenig, Ponce and Vega.

In [9] Kenig, Ponce and Vega considered the initial value problem

$$\begin{cases} i\partial_t u+\partial_x^2u=\alpha N_j(u,u), & x,t\in\mathbb{R},\quad j=1,2,3\\ u(x,0)=u_0(x), \end{cases}\tag{1.3}$$

where $N_1(u,u)=u\bar{u}$, $N_2(u,u)=\bar{u}^2$ and $N_3(u,u)=u^2$. They established the following sharp bilinear estimates:

($\mathbf{B}_1$) $\|N_1(u,u)\|_{X^{s,b-1}}\lesssim\|u\|^2_{X^{s,b}}$, for $s>-1/4$ and $b>1/2$;

($\mathbf{B}_2$) $\|N_j(u,u)\|_{X^{s,b-1}}\lesssim\|u\|^2_{X^{s,b}}$, for $s>-3/4$ and $b>1/2$, with $j=2,3$,

where

$$\begin{aligned}\|f\|_{X^{s,b}}&=\|U(-t)f\|_{H_t^b(\mathbb{R},H_x^s)}\\ &=\left(\int_{\mathbb{R}^2}(1+|\xi|)^{2s}(1+|\tau+\xi^2|)^{2b}|\widehat{f}(\xi,\tau)|^2d\xi d\tau\right)^{1/2}\end{aligned}\tag{1.4}$$

and $U(t):=e^{it\partial_x^2}$ is the corresponding Schrödinger generator (unitary group) associated to the linear problem. Using the estimates ($\mathbf{B}_1$) and ($\mathbf{B}_2$) and properties of the $X^{s,b}$ espaces together with the contraction mapping principle they proved local well-posedness for (1.3) in $H^s(\mathbb{R})$ for $s>-1/4$ ($j=1$) and for $s>-3/4$ ($j=2,3$).

Similar results were given in the periodic case, where $\|\cdot\|_{X^{s,b}_{per}}$ is defined by

$$\|f\|_{X^{s,b}_{per}}=\left(\sum_{n\in\mathbb{Z}}\int_{-\infty}^{+\infty}(1+|n|)^{2s}(1+|\tau+n^2|)^{2b}|\widehat{f}(n,\tau)|^2d\tau\right)^{1/2}\tag{1.5}$$

and the corresponding bilinear estimates obtained are the followings:

($\mathbf{B}_3$) $\|N_1(u,u)\|_{X^{s,b-1}_{per}} \lesssim \|u\|^2_{X^{s,b}_{per}}$, for $s \geq 0$ and $b \in (1/2, 1)$;

($\mathbf{B}_4$) $\|N_j(u,u)\|_{X^{s,b-1}_{per}} \lesssim \|u\|^2_{X^{s,b}_{per}}$, for $s > -1/2$ and $b \in (1/2,1)$, with $j = 2,3$.

As explained above, in our case the nonlinear interactions are uv and $|u|^2$. These terms are similar to N_3 and N_1, respectively, but the characteristics of linear part of each equation involved in the system (1.1) are antisymmetric. Therefore, our task is to find new mixed bilinear estimates for the coupling terms uv and $|u|^2$.

Before stating the results we will give some useful notations. Let ψ be a function in C_0^∞ such that $0 \leq \psi(t) \leq 1$,

$$\psi(t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| \geq 2, \end{cases}$$

and $\psi_T(t) = \psi(\frac{t}{T})$. We denote by $\lambda\pm$ a number slightly larger, respectively smaller, than λ and by $\langle \cdot \rangle$ the number $\langle \cdot \rangle = 1 + |\cdot|$. The characteristic function on the set A is denoted by χ_A.

The next statements show the main local-in-time results achieved in this work.

Theorem 1.1. *For any* $(u_0, v_0) \in H^k(\mathbb{R}) \times H^s(\mathbb{R})$ *provided the conditions:*

$$|k| - 1/2 \leq s < k + 1/4 \quad \text{and} \quad k > -1/4. \tag{1.6}$$

there exist a positive time $T = T(\|u_0\|_{H^s}, \|v_0\|_{H^s})$ *and a unique solution* $(u(t), v(t))$ *of the initial value problem (1.1), satisfying*

(i) $(\psi_T(t)u, \psi_T(t)v) \in X^{k,\frac{1}{2}+} \times H^{\frac{1}{2}+}(\mathbb{R}, H^s_x)$;

(ii) $(u,v) \in C\left([0,T]; H^k(\mathbb{R}) \times H^s(\mathbb{R})\right)$.

Moreover, the map $(u_0, v_0) \longmapsto (u(t), v(t))$ *is locally Lipschitz from* $H^k(\mathbb{R}) \times H^s(\mathbb{R})$ *into* $C([0,T]; H^k(\mathbb{R}) \times H^s(\mathbb{R}))$.

Theorem 1.2. *For any* $(u_0, v_0) \in H^k(\mathbb{T}) \times H^s(\mathbb{T})$ *provided the conditions:*

$$0 \leq s \leq 2k \quad \text{and} \quad |s - k| < 1. \tag{1.7}$$

there exist a positive time $T = T(\|u_0\|_{H^s}, \|v_0\|_{H^s})$ *and a unique solution* $(u(t), v(t))$ *of the initial value problem (1.1), satisfying*

(i) $(\psi_T(t)u, \psi_T(t)v) \in X^{k,\frac{1}{2}+}_{per} \times H^{\frac{1}{2}+}(\mathbb{R}, H^s_{per})$;

(ii) $(u,v) \in C\left([0,T]; H^k(\mathbb{T}) \times H^s(\mathbb{T})\right)$.

Moreover, the map $(u_0, v_0) \longmapsto (u(t), v(t))$ *is locally Lipschitz from* $H^k(\mathbb{T}) \times H^s(\mathbb{T})$ *into* $C([0,T]; H^k(\mathbb{T}) \times H^s(\mathbb{T}))$.

In figures 1 and 2 below, resp., we decipe the regions on the (k,s)-plane where our local well-posedness theorems in the continuous and periodic settings, resp., are valid.

Finally, we show that the system (1.1) is globally well-posed for a class of data without finite energy, more precisely:

Theorem 1.3. *For any* $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$, $-1/8 < s < 0$, *the local solution given in Theorem 1.1 can be extended to any time interval* $[0,T]$ *(preserving the properties (i) and (ii).)*

The plan of this paper is as follows. In Section 2 are given preliminary estimates needed to establish the new mixed bilinear estimates for coupling terms of system (1.1) and the proof of these estimates will be given in Sections 3 and 4. Moreover, we observe that our local results, given in theorems 1.1 and 1.2, are consequences of these bilinear estimates by using the standard contraction mapping principle and the properties of $X^{s,b}$ spaces. For instance, see the works [9], [2] and [7]. Finally, in Section 5 we proof Theorem 1.3 using the I-method combined with the following *refined* Strichartz type estimate for the Schrödinger equation:

$$\|(D_x^{1/2} f) \cdot g\|_{L^2_{xt}} \lesssim \|f\|_{X^{0,1/2+}} \|g\|_{X^{0,1/2+}}, \tag{1.8}$$

if $|\xi_1| \gg |\xi_2|$ for any $|\xi_1| \in \text{supp}(\widehat{f}), |\xi_2| \in \text{supp}(\widehat{g})$.

We finish with the following interesting remark: in the work [3] it was shown that as the parameter σ tends to zero, solutions the system (1.1) converge (in $H^s(\mathbb{R})$ for $s > 5/2$) to those of the cubic nonlinear Schrödinger equation. Our local results in Theorem 1.1 show that this fact is not true in Sobolev spaces with low regularity since the cubic Schrödinger equation is not locally well-posed below L^2 in the continuous case (in the sense that the associated flow is not uniformly continuous).

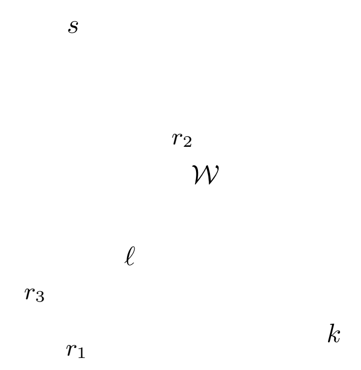

FIGURE 1. Well-posedness results for Schrödinger-Debye system in the continuous case ($M = \mathbb{R}$). The region $\mathcal{W}$, limited by the lines $r_1 : |k| - s = 1/2$ and $r_2 : s - k = 1/2$, $r_3 : s - 2k = 1/2$, for $k \geq -1/4$, contain the indices (k, s) where the local well-posedness is achieved in Theorem 1.1. Global results, given in Theorem 1.3, are obtained on the line $\ell : s = k$, for $-1/8 < k \leq 0$.

2. **Preliminary Estimates**

Firstly, we recall some estimates contained in the work [7] of Ginibre, Tsutsumi and Velo concerning the Zakharov system:

Lemma 2.1. *Let* $-1/2 < b' \leq 0 \leq b \leq b' + 1$ *and* $T \in [0, 1]$. *Then, for* $F \in H_t^{b'}(\mathbb{R}, H_x^s)$ *we have*

$$\|\psi_1(t)\omega_0\|_{H_t^b(\mathbb{R}, H_x^s)} \leq C\|\omega_0\|_{H^s}, \tag{2.9}$$

$$\left\| \psi_T(t) \int_0^t F(t', \cdot) dt' \right\|_{H_t^b(\mathbb{R}, H_x^s)} \leq C T^{1-b+b'} \|F\|_{H_t^{b'}(\mathbb{R}, H_x^s)}. \tag{2.10}$$

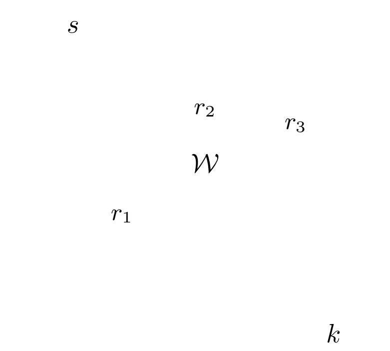

FIGURE 2. Well-posedness results for periodic Schrödinger-Debye system ($M = \mathbb{T}$). The region $\mathcal{W}$, limited for the lines $r_1 : s = 2k$, $r_2 : s = k+1$ and $r_3 : s = k-1$, contain the indices (k, s) where the local well-posedness is achieved in Theorem 1.2.

Proof. See Lemma 2.1 in [7]. $\square$

Lemma 2.2. *It holds*

$$\int_{\mathbb{R}^4} \frac{|\widehat{f}(\xi,\tau)\widehat{g}(\xi_1,\tau_1)\widehat{h}(\xi_2,\tau_2)|\langle\xi\rangle^{1/2}}{\langle\sigma\rangle^{d}\langle\sigma_1\rangle^{d_1}\langle\sigma_2\rangle^{d_2}} d\xi_1 d\tau_1 d\xi d\tau \lesssim \|f\|_{L^2_{xt}}\|g\|_{L^2_{xt}}\|h\|_{L^2_{xt}}, \tag{2.11}$$

where $\xi = \xi_1 + \xi_2$, $\tau = \tau_1 + \tau_2$, $\sigma := \tau$, $\sigma_1 := \tau_1 - \frac{1}{2}\xi_1^2$, $\sigma_2 := \tau_2 + \frac{1}{2}\xi_2^2$ *and* $d, d_1, d_2 > 1/4$, $d + d_1 > 3/4$, $d + d_2 > 3/4$.

Proof. See [7, p.422–424]. $\square$

Next, we recall some elementary calculus inequalities:

Lemma 2.3. *Let* $p, q > 0$. *Then for* $r = \min\{p, q\}$ *with* $p + q > 1 + r$ *there exists* $C > 0$ *such that*

$$\int_{-\infty}^{\infty} \frac{dx}{\langle x-\alpha\rangle^p\langle x-\beta\rangle^q} \leq \frac{C}{\langle\alpha-\beta\rangle^r}. \tag{2.12}$$

Furthermore, for $p > 1$ *and* $q > 1/2$ *there exists a* $C > 0$ *such that*

$$\int_{-\infty}^{\infty} \frac{dx}{\langle \alpha x-\beta\rangle^p} \leq \frac{C}{|\alpha|}, \quad \textit{for} \quad \alpha \neq 0, \tag{2.13}$$

$$\int_{-\infty}^{\infty} \frac{dx}{\langle \alpha_0 + \alpha_1 x + \frac{1}{2}x^2\rangle^q} \leq C. \tag{2.14}$$

Proof. See the work [2]. $\square$

3. Bilinear Estimates for the Coupling Terms in the Continuous Case

The aim of this section is the study of the crucial sharp bilinear estimates for the coupling terms in the continuous cases. In order to do so, this section is oranized as follows: first, we present the proof of the relevant bilinear estimates assuming certain restrictions on the Sobolev indices s and k of the initial data; after this, we show a series of counter-examples showing that our restrictions on s and k are necessary.

3.1. Proof of the bilinear estimates I: the continuous case.

Proposition 3.1. *Let $1/4 < a < 1/2$ and $b > 1/2$. The bilinear estimate*

$$\|uv\|_{X^{k,-a}} \lesssim \|u\|_{X^{k,b}} \|v\|_{H_t^b H_x^s} \tag{3.15}$$

holds if $|k| - s \leq 1/2$.

Proof. We define

$$f(\xi,\tau) = \langle \tau + \tfrac{1}{2}\xi^2\rangle^b \langle \xi\rangle^k \widehat{u}(\xi,\tau) \quad \text{and} \quad g(\xi,\tau) = \langle \tau\rangle^b \langle \xi\rangle^s \widehat{v}(\xi,\tau).$$

Then, for $u \in X^{k,b}$ and $v \in H_t^b H_x^s$, the L^2 duality and the definition (1.4) show that (3.15) is equivalent to prove

$$W_{f,g}(\varphi) \lesssim \|f\|_{L^2} \|g\|_{L^2} \|\varphi\|_{L^2}, \tag{3.16}$$

for all $\varphi \in L^2(\mathbb{R}^2)$, where

$$W_{f,g}(\varphi) = \int_{\mathbb{R}^4} \frac{\langle \xi\rangle^k \bar{\varphi}(\xi,\tau) f(\xi-\xi_1,\tau-\tau_1) g(\xi_1,\tau_1)}{\langle \tau + \frac{1}{2}\xi^2\rangle^a \langle \xi-\xi_1\rangle^k \langle \tau-\tau_1+\frac{1}{2}(\xi-\xi_1)^2\rangle^b \langle \xi_1\rangle^s \langle \tau_1\rangle^b} d\xi_1 d\tau_1 d\xi d\tau \tag{3.17}$$

To estimate $W_{f,g}$ we split $\mathbb{R}^4$ into three regions $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$,

$$\begin{aligned}
\mathcal{A}_1 &= \left\{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4;\ |\xi_1| \leq 1\right\},\\
\mathcal{A}_2 &= \left\{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4;\ |\xi_1| > 1 \text{ and } |\xi_1 - \xi| \geq \tfrac{1}{8}|\xi_1|\right\},\\
\mathcal{A}_3 &= \left\{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4;\ |\xi_1| > 1 \text{ and } |\tfrac{1}{2}\xi_1 - \xi| \geq \tfrac{1}{8}|\xi_1|\right\}.
\end{aligned}$$

Since

$$\mathcal{S} = \left\{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4;\ |\xi_1| > 1,\ |\xi_1 - \xi| < \tfrac{1}{8}|\xi_1| \text{ and } |\tfrac{1}{2}\xi_1 - \xi| < \tfrac{1}{8}|\xi_1|\right\}$$

is empty, we have that $\mathbb{R}^4 = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$. Indeed if $(\xi,\xi_1,\tau,\tau_1) \in \mathcal{S}$, then

$$\tfrac{1}{2}|\xi_1| = |\xi_1 - \xi - (\tfrac{1}{2}\xi_1 - \xi)| \leq |\xi_1 - \xi| + |\tfrac{1}{2}\xi_1 - \xi| < \tfrac{1}{4}|\xi_1|,$$

which is a contradiction.

Note that for any point in $\mathcal{A}_3$ we have the following algebraic inequality

$$|\tau + \tfrac{1}{2}\xi^2| + |\tau_1| + |\tau - \tau_1 + \tfrac{1}{2}(\xi-\xi_1)^2| \geq |\tfrac{1}{2}\xi_1^2 - \xi\xi_1| = |\xi_1||\tfrac{1}{2}\xi_1 - \xi| \geq \tfrac{1}{8}|\xi_1|^2, \tag{3.18}$$

and consequently

$$\max\left\{|\tau + \tfrac{1}{2}\xi^2|, |\tau_1|, |\tau - \tau_1 + \tfrac{1}{2}(\xi-\xi_1)^2|\right\} \geq \tfrac{1}{24}|\xi_1|^2. \tag{3.19}$$

Now we separate $\mathcal{A}_3$ into three parts,

$$\begin{aligned}
\mathcal{A}_{3,1} &= \left\{(\xi,\xi_1,\tau,\tau_1) \in \mathcal{A}_3;\ |\tau_1|,\ |\tau - \tau_1 + \tfrac{1}{2}(\xi-\xi_1)^2| \leq |\tau + \tfrac{1}{2}\xi^2|\right\},\\
\mathcal{A}_{3,2} &= \left\{(\xi,\xi_1,\tau,\tau_1) \in \mathcal{A}_3;\ |\tau - \tau_1 + \tfrac{1}{2}(\xi-\xi_1)^2|,\ |\tau + \tfrac{1}{2}\xi^2| \leq |\tau_1|\right\},\\
\mathcal{A}_{3,3} &= \left\{(\xi,\xi_1,\tau,\tau_1) \in \mathcal{A}_3;\ |\tau_1|,\ |\tau + \tfrac{1}{2}\xi^2| \leq |\tau - \tau_1 + \tfrac{1}{2}(\xi-\xi_1)^2|\right\},
\end{aligned}$$

so that one of the following $|\tau + \frac{1}{2}\xi^2|$, $|\tau_1|$ or $|\tau - \tau_1 + \frac{1}{2}(\xi-\xi_1)^2|$ is larger than $\frac{1}{24}|\xi_1|^2$.

We can now define the sets $\Omega_1 = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_{3,1}$, $\Omega_2 = \mathcal{A}_{3,2}$ and $\Omega_3 = \mathcal{A}_{3,3}$ and it is clear that $\mathbb{R}^4 = \Omega_1 \cup \Omega_2 \cup \Omega_3$. Then, we decompose the integral in W into the followings

$$W(f, g, \varphi) = W_1 + W_2 + W_3,$$

where

$$W_j = \int_{\Omega_j} \frac{\langle \xi \rangle^k \bar{\varphi}(\xi, \tau) f(\xi - \xi_1, \tau - \tau_1) g(\xi_1, \tau_1)}{\langle \tau + \frac{1}{2}\xi^2 \rangle^a \langle \xi_1 \rangle^s \langle \tau_1 \rangle^b \langle \xi - \xi_1 \rangle^k \langle \tau - \tau_1 + \frac{1}{2}(\xi - \xi_1)^2 \rangle^b} d\xi_1 d\tau_1 d\xi d\tau,$$

for $j = 1, 2, 3$.

We begin by estimating W_1. For this purpose, we integrate over ξ_1 and τ_1 first and then use the Cauchy-Schwarz and Hölder inequalities and the Fubini's theorem to obtain

(3.20)

$$\begin{aligned} |W_1|^2 \leq & \left\| \frac{\langle \xi \rangle^k}{\langle \tau + \frac{1}{2}\xi^2 \rangle^a} \int_{\mathbb{R}^2} \frac{f(\xi - \xi_1, \tau - \tau_1) g(\xi_1, \tau_1)}{\langle \xi_1 \rangle^s \langle \xi - \xi_1 \rangle^k \langle \tau_1 \rangle^b \langle \tau - \tau_1 + \frac{1}{2}(\xi - \xi_1)^2 \rangle^b} \chi_{\Omega_1} d\xi_1 d\tau_1 \right\|^2_{L^2_{\xi,\tau}} \times \\ & \times \|\varphi\|^2_{L^2_{\xi,\tau}} \\ = & \int_{\mathbb{R}^2} \frac{\langle \xi \rangle^{2k} d\xi d\tau}{\langle \tau + \frac{1}{2}\xi^2 \rangle^{2a}} \left| \int_{\mathbb{R}^2} \frac{f(\xi - \xi_1, \tau - \tau_1) g(\xi_1, \tau_1)}{\langle \xi_1 \rangle^{2s} \langle \xi - \xi_1 \rangle^{2k} \langle \tau_1 \rangle^{2b} \langle \tau - \tau_1 + \frac{1}{2}(\xi - \xi_1)^2 \rangle^{2b}} \chi_{\Omega_1} d\xi_1 d\tau_1 \right|^2 \times \\ & \times \|\varphi\|^2_{L^2_{\xi,\tau}} \\ \leq & \left\| \frac{\langle \xi \rangle^{2k}}{\langle \tau + \frac{1}{2}\xi^2 \rangle^{2a}} \int_{\mathbb{R}^2} \frac{\langle \xi_1 \rangle^{-2s} \langle \xi - \xi_1 \rangle^{-2k}}{\langle \tau_1 \rangle^{2b} \langle \tau - \tau_1 + \frac{1}{2}(\xi - \xi_1)^2 \rangle^{2b}} \chi_{\Omega_1} d\xi_1 d\tau_1 \right\|_{L^\infty_{\xi,\tau}} \times \\ & \times \|f\|^2_{L^2} \|g\|^2_{L^2} \|\varphi\|^2_{L^2} \end{aligned}$$

For W_2 we put $\tilde{f}(\xi, \tau) := f(-\xi, -\tau)$, integrate over ξ and τ first and follow the same steps as above to get

$$\begin{aligned} |W_2|^2 \leq & \left\| \frac{\langle \xi_1 \rangle^{-2s}}{\langle \tau_1 \rangle^{2b}} \int_{\mathbb{R}^2} \frac{\langle \xi \rangle^{2k} \langle \xi - \xi_1 \rangle^{-2k}}{\langle \tau + \frac{1}{2}\xi^2 \rangle^{2a} \langle \tau - \tau_1 + \frac{1}{2}(\xi - \xi_1)^2 \rangle^{2b}} \chi_{\Omega_2} d\xi d\tau \right\|_{L^\infty_{\xi_1,\tau_1}} \times \\ & \times \|\tilde{f}\|^2_{L^2} \|g\|^2_{L^2} \|\varphi\|^2_{L^2}. \end{aligned} \tag{3.21}$$

Note that $\|\tilde{f}\|^2_{L^2} = \|f\|^2_{L^2}$.

Now we use the change of variables $\tau = \tau_1 - \tau_2$ and $\xi = \xi_1 - \xi_2$ to transform the region Ω_3 into the set $\widetilde{\Omega}_3$, that satisfies

$$\widetilde{\Omega}_3 \subseteq \{(\xi_1, \xi_2, \tau_1, \tau_2) \in \mathbb{R}^4;\ \tfrac{1}{8}|\xi_1|^2 \leq |\tfrac{1}{2}\xi_1^2 - \xi_1\xi_2| \leq 3|\tau_2 - \tfrac{1}{2}\xi_2^2| \text{ and } |\xi_1| > 1\}$$

Then W_3 can be estimated as follows

(3.22)

$$\begin{aligned} |W_3|^2 \leq & \left\| \frac{\langle \xi_2 \rangle^{-2k}}{\langle \tau_2 - \frac{1}{2}\xi_2^2 \rangle^{2b}} \int_{\mathbb{R}^2} \frac{\langle \xi_1 \rangle^{-2s} \langle \xi_1 - \xi_2 \rangle^{2k}}{\langle \tau_1 \rangle^{2b} \langle \tau_1 - \tau_2 + \frac{1}{2}(\xi_1 - \xi_2)^2 \rangle^{2a}} \chi_{\widetilde{\Omega}_3} d\xi_1 d\tau_1 \right\|_{L^\infty_{\xi_2,\tau_2}} \times \\ & \times \|\tilde{f}\|^2_{L^2} \|g\|^2_{L^2} \|\varphi\|^2_{L^2}. \end{aligned}$$

From estimates (3.20), (3.21) and (3.22) it suffices to show that the following expressions are bounded:

$$\widetilde{W}_1(\xi_1,\tau_1):=\frac{\langle\xi\rangle^{2k}}{\langle\tau+\frac{1}{2}\xi^2\rangle^{2a}}\int_{\mathbb{R}^2}\frac{\langle\xi_1\rangle^{-2s}\langle\xi-\xi_1\rangle^{-2k}}{\langle\tau_1\rangle^{2b}\langle\tau-\tau_1+\frac{1}{2}(\xi-\xi_1)^2\rangle^{2b}}\chi_{\Omega_1}d\xi_1 d\tau_1, \tag{3.23}$$

$$\widetilde{W}_2(\xi_1,\tau_1):=\frac{\langle\xi_1\rangle^{-2s}}{\langle\tau_1\rangle^{2b}}\int_{\mathbb{R}^2}\frac{\langle\xi\rangle^{2k}\langle\xi-\xi_1\rangle^{-2k}}{\langle\tau+\frac{1}{2}\xi^2\rangle^{2a}\langle\tau-\tau_1+\frac{1}{2}(\xi-\xi_1)^2\rangle^{2b}}\chi_{\Omega_2}d\xi d\tau, \tag{3.24}$$

and

$$\widetilde{W}_3(\xi_2,\tau_2):=\frac{\langle\xi_2\rangle^{-2k}}{\langle\tau_2-\frac{1}{2}\xi_2^2\rangle^{2b}}\int_{\mathbb{R}^2}\frac{\langle\xi_1\rangle^{-2s}\langle\xi_1-\xi_2\rangle^{2k}}{\langle\tau_1\rangle^{2b}\langle\tau_1-\tau_2+\frac{1}{2}(\xi_1-\xi_2)^2\rangle^{2a}}\chi_{\widetilde{\Omega}_3}d\xi_1 d\tau_1. \tag{3.25}$$

Now using lemma 2.3-(2.12) and the inequalities: $\langle\xi\rangle^{2k}\le\langle\xi_1\rangle^{2|k|}\langle\xi-\xi_1\rangle^{2k}$ and $\langle\xi_1-\xi_2\rangle^{2k}\le\langle\xi_1\rangle^{2|k|}\langle\xi_2\rangle^{2k}$, for $k\ge 0$, and $\langle\xi-\xi_1\rangle^{-2k}\le\langle\xi_1\rangle^{2|k|}\langle\xi\rangle^{-2k}$ and $\langle\xi_2\rangle^{-2k}\le\langle\xi_1\rangle^{2|k|}\langle\xi_1-\xi_2\rangle^{2k}$, for $k<0$, we have

$$\widetilde{W}_1(\xi,\tau)\le J_1(\xi,\tau):=\frac{1}{\langle\tau+\frac{1}{2}\xi^2\rangle^{2a}}\int_{-\infty}^{+\infty}\frac{\langle\xi_1\rangle^{2|k|-2s}}{\langle\tau+\frac{1}{2}\xi^2+\frac{1}{2}\xi_1^2-\xi\xi_1\rangle^{2b}}\chi_{\Omega_1}d\xi_1, \tag{3.26}$$

$$\widetilde{W}_2(\xi_1,\tau_1)\le J_2(\xi_1,\tau_1):=\frac{\langle\xi_1\rangle^{2|k|-2s}}{\langle\tau_1\rangle^{2b}}\int_{-\infty}^{+\infty}\frac{1}{\langle\tau_1-\frac{1}{2}\xi_1^2+\xi\xi_1\rangle^{2a}}\chi_{\Omega_2}d\xi, \tag{3.27}$$

and

$$\widetilde{W}_3(\xi_2,\tau_2)\le J_3(\xi_2,\tau_2):=\frac{1}{\langle\tau_2-\frac{1}{2}\xi_2^2\rangle^{2b}}\int_{-\infty}^{+\infty}\frac{\langle\xi_1\rangle^{2|k|-2s}}{\langle\tau_2-\frac{1}{2}\xi_1^2-\frac{1}{2}\xi_2^2+\xi_1\xi_2\rangle^{2a}}\chi_{\widetilde{\Omega}_3}d\xi_1. \tag{3.28}$$

We begin estimating J_1 on $\Omega_1=\mathcal{A}_1\cup\mathcal{A}_2\cup\mathcal{A}_{3,1}$. In region $\mathcal{A}_1$, using $|\xi_1|\le 1$, $a>0$, $b>1/2$ it easy to see that

$$|J_1|\le C\int_{|\xi_1|\le 1}d\xi_1\le C. \tag{3.29}$$

In region $\mathcal{A}_2$, by the change of variables $\eta=\tau+\frac{1}{2}\xi^2+\frac{1}{2}\xi_1^2-\xi\xi_1$ and the condition $|\xi-\xi_1|\ge\frac{1}{8}|\xi_1|$ we obtain

$$\begin{aligned}|J_1|&\le\frac{1}{\langle\tau+\frac{1}{2}\xi^2\rangle^{2a}}\int_{\mathcal{A}_2}\frac{\langle\xi_1\rangle^{2|k|-2s}}{|\xi_1-\xi|\langle\eta\rangle^{2b}}d\eta\\&\le\frac{8}{\langle\tau+\frac{1}{2}\xi^2\rangle^{2a}}\int_{\mathcal{A}_2}\frac{\langle\xi_1\rangle^{2|k|-2s}}{|\xi_1|\langle\eta\rangle^{2b}}d\eta\\&\le C,\end{aligned} \tag{3.30}$$

where we have used that $a>0$, $b>1/2$ and $|k|-s\le 1/2$.

In region $\mathcal{A}_{3,1}$, by (3.19) we have that

$$|\xi_1|^2\le 24\langle\tau+\tfrac{1}{2}\xi^2\rangle$$

and consequently using $a>0$ we obtain

$$\langle\tau+\tfrac{1}{2}\xi^2\rangle^{-2a}\le C|\xi_1|^{-4a}.$$

Then we use that $|k|-s\le 1/2<2a$, for $a>1/4$, combined with Lemma 2.3-(2.14) to get

$$|J_1|\le C\int_{\mathbb{R}}\frac{\langle\xi_1\rangle^{2|k|-2s}}{|\xi_1|^{4a}\langle\tau+\frac{1}{2}\xi^2+\frac{1}{2}\xi_1^2-\xi\xi_1\rangle^{2b}}d\xi_1\le C. \tag{3.31}$$

Next we estimate J_2. First, we making the change

$$\eta = \tau_1 - \tfrac{1}{2}\xi_1^2 + \xi\xi_1, \qquad d\eta = \xi_1 d\xi,$$

and we note that the relations in (3.18) and the restriction in region Ω_2 yield

$$\langle \eta \rangle \leq \langle \tau_1 \rangle + |\xi\xi_1 - \tfrac{1}{2}\xi_1^2| \leq 4\langle \tau_1 \rangle. \tag{3.32}$$

Moreover, by (3.19) we have

$$|\xi_1|^2 \leq 24\langle \tau_1 \rangle$$

and hence using that $2a + 2b - 1 > 0$ we get

$$|\xi_1|^{4a+4b-2} \leq C\langle \tau_1 \rangle^{2a+2b-1}. \tag{3.33}$$

Now using the inequalities (3.32), (3.33) and that $a < 1/2$ we can estimate J_2 as follows:

$$\begin{aligned}
|J_2(\xi_1, \tau_1)| &\leq \frac{\langle \xi_1 \rangle^{2|k|-2s}}{\langle \tau_1 \rangle^{2b}} \int_{\langle \eta \rangle \leq 4\langle \tau_1 \rangle} \frac{d\eta}{|\xi_1|(1+|\eta|)^{2a}} \\
&\leq C \frac{\langle \xi_1 \rangle^{2|k|-2s}}{\langle \tau_1 \rangle^{2b}|\xi_1|} \langle \tau_1 \rangle^{1-2a} \\
&\leq C \frac{\langle \xi_1 \rangle^{2|k|-2s}}{\langle \tau_1 \rangle^{2a+2b-1}|\xi_1|} \\
&\leq C \frac{\langle \xi_1 \rangle^{2|k|-2s}}{\langle \xi_1 \rangle^{4a+4b-2}|\xi_1|} \\
&\leq C,
\end{aligned} \tag{3.34}$$

where the last inequality follows directly from the conditions $2a + 2b - 1/2 \geq 1/2$ (for $a > 0$) and $|k| - s \leq 1/2$.

Finally, in region $\tilde{\Omega}_3$ we note that

$$|\xi_1|^{4b} \leq C\langle \tau_2 - \frac{1}{2}\xi_2^2 \rangle^{2b}.$$

Hence, from conditions $a > 1/4$, $b > 1/2$ and $|k| - s \leq 1/2$ coupled with Lemma 2.3-(2.14), we have that

$$\begin{aligned}
|J_3(\xi_2, \tau_2)| &\leq C \int_{\tilde{\Omega}_3} \frac{\langle \xi_1 \rangle^{2|k|-2s}}{|\xi_1|^{4b}\langle \tau_2 - \frac{1}{2}\xi_1^2 - \frac{1}{2}\xi_2^2 + \xi_1\xi_2 \rangle^{2a}} d\xi_1 \\
&\leq C,
\end{aligned} \tag{3.35}$$

which complete the proof of desired estimate. □

Proposition 3.2. *If* $\max\{0, s\} \leq 2k + 1/2$ *and* $s \leq k + 1/2$*, then the bilinear estimate*

$$\|u\bar{w}\|_{H_t^{-a}H_x^s} \lesssim \|u\|_{X^{s,b}}\|w\|_{X^{s,b}} \tag{3.36}$$

holds if $b > 1/2$ *and* $\frac{1}{4} < a < 1/2$.

Proof. Analogously to the previous proposition, the estimate (3.36) is equivalent to prove

$$Z_{f,g}(\varphi) \lesssim \|f\|_{L^2}\|g\|_{L^2}\|\varphi\|_{L^2}, \tag{3.37}$$

for all $\varphi \in L^2(\mathbb{R}^2)$, where

$$Z_{f,g}(\varphi) = \int_{\mathbb{R}^4} \frac{\langle \xi \rangle^s \bar{\varphi}(\xi, \tau) f(\xi - \xi_1, \tau - \tau_1)\bar{g}(-\xi_1, -\tau_1)}{\langle \tau \rangle^a \langle \xi - \xi_1 \rangle^k \langle \tau - \tau_1 + \frac{1}{2}(\xi - \xi_1)^2 \rangle^b \langle \xi_1 \rangle^k \langle \tau_1 - \frac{1}{2}\xi_1^2 \rangle^b} d\xi_1 d\tau_1 d\xi d\tau \tag{3.38}$$

We have the following dispersion relation

$$\begin{cases} \xi = \xi_1 + \xi_2, \quad \tau = \tau_1 + \tau_2, \\ \sigma_1 = \tau_1 - \frac{1}{2}\xi_1^2, \quad \sigma_2 = \tau_2 + \frac{1}{2}\xi_2^2, \\ \tau - \sigma_1 - \sigma_2 = -\frac{1}{2}\xi^2 + \xi\xi_1 = \frac{1}{2}\xi^2 - \xi\xi_2 = \frac{1}{2}(\xi_1^2 - \xi_2^2). \end{cases} \tag{3.39}$$

We divide $\mathbb{R}^4$ in the following integration regions:

Region A: $|\sigma_1| \geq \max\{|\tau|, |\sigma_2|\}$. We consider two subregions of A:

Subregion A_1: $|\xi_1| \leq 2|\xi_2|$. If $k \leq 0$, we have $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k} \lesssim \langle\xi_2\rangle^{s-2k} \lesssim \langle\xi_2\rangle^{1/2}$ (because $|\xi| \leq 3|\xi_2|$ and $s = \max\{0, s\} \leq 2k + 1/2$). Hence, we can estimate

$$Z \lesssim \int_{\mathbb{R}^4} \frac{\langle\xi_2\rangle^{1/2}\bar{\varphi}(\xi,\tau)f(\xi_2,\tau_2)\bar{g}(-\xi_1,-\tau_1)}{\langle\tau\rangle^a\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b}\chi_{A_1}d\xi_1 d\tau_1 d\xi d\tau \tag{3.40}$$

if $k \leq 0$. Thus, in the same way as the previous estimate of (3.20), it suffices to bound the expression:

$$\widetilde{Z}_1 := \sup_{\xi_1,\tau_1} \frac{1}{\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{\langle\xi_2\rangle\chi_{A_1}}{\langle\tau\rangle^{2a}\langle\sigma_2\rangle^{2b}} \tag{3.41}$$

- If $|\xi_2| \leq 1$ we using Lemma 2.3-(2.14) to get

$$\begin{aligned} \widetilde{Z}_1 &\lesssim \int_{\mathbb{R}^2} \frac{\chi_{A_1}}{\langle\tau\rangle^{2a}\langle\tau - \tau_1 + \frac{1}{2}(\xi-\xi_1)^2\rangle^{2b}} d\xi d\tau \\ &\lesssim \int_{-\infty}^{+\infty} \frac{1}{\langle -\tau_1 + \frac{1}{2}\xi_1^2 - \xi\xi_1 + \frac{1}{2}\xi^2\rangle^{2a}} d\xi \\ &\lesssim 1, \end{aligned}$$

  since $2a > 1/2$.
- If $|\xi_2| > 1$ we have that $\langle\xi_2\rangle \lesssim |\xi_2|$. Next, changing variables $\tau = \tau_2 + \tau_1$ and $\sigma_2 = \tau_2 + \frac{1}{2}\xi_2^2$, for fixed ξ_1 and τ_1, we have that $d\tau d\sigma_2 = |\xi_2| d\tau_2 d\xi_2$ and then we obtain

$$\begin{aligned} \widetilde{Z}_1 &\lesssim \sup_{\xi_1,\tau_1} \frac{1}{\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{|\xi_2|\chi_{A_1}}{\langle\tau\rangle^{2a}\langle\sigma_2\rangle^{2b}} d\xi_2 d\tau_2 \\ &= \sup_{\xi_1,\tau_1} \frac{1}{\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{\chi_{A_1}}{\langle\tau\rangle^{2a}\langle\sigma_2\rangle^{2b}} d\tau d\sigma_2 \\ &\lesssim \sup_{\xi_1,\tau_1} \frac{1}{\langle\sigma_1\rangle^{2b}} \int_0^{|\sigma_1|} \langle\tau\rangle^{-2a} d\tau \int_0^{|\sigma_1|} \langle\sigma_2\rangle^{-2b} d\sigma_2 \\ &\lesssim \sup_{\sigma_1} \langle\sigma_1\rangle^{-2b}\langle\sigma_1\rangle^{1-2a}\langle\sigma_1\rangle^{1-2b} \\ &= \sup_{\sigma_1} \langle\sigma_1\rangle^{2-2a-4b} \\ &\lesssim 1, \end{aligned}$$

  since $0 < a$ and $b > 1/2$ implies $2 - 2a - 4b \leq 0$.

Therefore, we showed (3.37) in the subregion A_1 whenever $k \leq 0$. On the other, if $k \geq 0$, we have $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k} \lesssim \langle\xi\rangle^{s-k} \lesssim \langle\xi\rangle^{1/2}$ (because $|\xi| \leq 3|\xi_2|$ and $s - k \leq 1/2$). So, we get

$$Z \lesssim \int_{\mathbb{R}^4} \frac{\langle\xi\rangle^{1/2}\bar{\varphi}(\xi,\tau)f(\xi_2,\tau_2)\bar{g}(-\xi_1,-\tau_1)}{\langle\tau\rangle^a\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b}\chi_{A_1}d\xi_1 d\tau_1 d\xi d\tau \tag{3.42}$$

if $k \geq 0$. Thus, applying the lemma 2.2, we also obtain (3.37) if $k \geq 0$.

This completes the analysis of Z in the subregion $\boldsymbol{A_1}$.

Subregion $\boldsymbol{A_2}$: $|\boldsymbol{\xi_1}| \geq 2|\boldsymbol{\xi_2}|$. Here, the dispersion relation (3.39) yields that

$$\tfrac{3}{4}\xi_1^2 \leq |\xi_1^2 - \xi_2^2| = 2|\tau - \sigma_1 - \sigma_2| \leq 6|\sigma_1| \Longrightarrow \xi_1^2 \leq 8|\sigma_1|.$$

Hence,

$$\frac{1}{\langle\sigma_1\rangle} \lesssim \frac{1}{\langle\xi_1\rangle^2}. \tag{3.43}$$

If $k \leq 0$, it follows $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k} \lesssim \langle\xi_1\rangle^{1/2}$ (because $\max\{0, s\} \leq 2k + 1/2$ and $|\xi| \leq 3|\xi_1|/2$), so that

$$Z \lesssim \int_{\mathbb{R}^4} \frac{\langle\xi_1\rangle^{1/2}\bar{\varphi}(\xi,\tau)f(\xi_2,\tau_2)\bar{g}(-\xi_1,-\tau_1)}{\langle\tau\rangle^a\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b}\chi_{A_2}d\xi_1 d\tau_1 d\xi d\tau \tag{3.44}$$

if $k \leq 0$. Thus, similarly to (3.40), our task is to estimate

$$\widetilde{Z}_2 := \sup_{\xi_1,\tau_1} \frac{1}{\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{\langle\xi_1\rangle}{\langle\tau\rangle^{2a}\langle\sigma_2\rangle^{2b}}\chi_{A_2}d\xi_2 d\tau_2 \tag{3.45}$$

Using (3.43), lemma 2.3-(2.12) and lemma 2.3-(2.14) we obtain

$$\begin{aligned}\widetilde{Z}_2 &\lesssim \sup_{\xi_1,\tau_1} \langle\xi_1\rangle^{1-4b} \int_{\mathbb{R}^2} \frac{\chi_A}{\langle\tau\rangle^{2a}\langle\tau - \tau_1 + \frac{1}{2}(\xi - \xi_1)^2\rangle^{2b}} d\xi d\tau \\ &\lesssim \sup_{\xi_1,\tau_1} \langle\xi_1\rangle^{1-4b} \int_{-\infty}^{+\infty} \frac{1}{\langle -\tau_1 + \frac{1}{2}\xi_1^2 - \xi\xi_1 + \frac{1}{2}\xi^2\rangle^{2a}} d\xi \\ &\lesssim 1,\end{aligned} \tag{3.46}$$

where in the last inequality we have used that since $1/4 < a$ and $b > 1/4$.

If $k \geq 0$, we have $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k} \lesssim \langle\xi\rangle^{s-k} \lesssim \langle\xi\rangle^{1/2}$ since $s - k \leq 1/2$ and $|\xi| \leq 3|\xi_1|/2$. So, we get

$$Z \lesssim \int_{\mathbb{R}^4} \frac{\langle\xi\rangle^{1/2}\bar{\varphi}(\xi,\tau)f(\xi_2,\tau_2)\bar{g}(-\xi_1,-\tau_1)}{\langle\tau\rangle^a\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b}\chi_{A_2}d\xi_1 d\tau_1 d\xi d\tau \lesssim \|f\|_{L^2}\|g\|_{L^2}\|\varphi\|_{L^2} \tag{3.47}$$

by lemma 2.2. This completes the analysis of the Z in the subregion $\boldsymbol{A_2}$.

Clearly $\boldsymbol{A} = \boldsymbol{A_1} \cup \boldsymbol{A_2}$, so that the estimate (3.37) holds true in the region $\boldsymbol{A}$.

Region $\boldsymbol{B}$: $|\boldsymbol{\sigma_2}| \geq \max\{|\boldsymbol{\tau}|,\ |\boldsymbol{\sigma_1}|\}$. The computations for this region can be obtained from the previous ones (in region A) since all the involved expressions are symmetric under the exchange of the indices 1 and 2.

Region $\boldsymbol{C}$: $|\boldsymbol{\tau}| \geq \max\{|\boldsymbol{\sigma_1}|,\ |\boldsymbol{\sigma_2}|\}$. Here, we analyze several cases for the frequencies ξ and ξ_1.

We begin with the high frequencies for ξ, that is:

Subregion $\boldsymbol{C_1}$: $|\boldsymbol{\xi}| \geq \boldsymbol{1}$. We separate this region into two smaller subregions.

Subregion $\boldsymbol{C_{1,1}}$: $\left|\boldsymbol{\xi_1} - \frac{1}{2}\boldsymbol{\xi}\right| \leq \boldsymbol{1}$. Here we have that

$$|\xi_1| \leq \left|\xi_1 - \tfrac{1}{2}\xi\right| + \left|\tfrac{1}{2}\xi\right| \Longrightarrow \langle\xi_1\rangle \lesssim \langle\xi\rangle$$

and

$$|\xi_2| = \left|\tfrac{1}{2}\xi + \tfrac{1}{2}\xi - \xi_1\right| \leq \left|\xi_1 - \tfrac{1}{2}\xi\right| + \left|\tfrac{1}{2}\xi\right| \Longrightarrow \langle\xi_2\rangle \lesssim \langle\xi\rangle.$$

In particular, we get $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k}\lesssim\langle\xi\rangle^{1/2}$ (because $\max\{0,s\}\le 2k+1/2$ and $s-k\le 1/2$). This allows us to conclude that

(3.48)
$$Z\lesssim\int_{\mathbb{R}^4}\frac{\langle\xi\rangle^{1/2}\bar{\varphi}(\xi,\tau)f(\xi_2,\tau_2)\bar{g}(-\xi_1,-\tau_1)}{\langle\tau\rangle^a\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b}\chi_{C_{1,1}}d\xi_1 d\tau_1 d\xi d\tau\lesssim\|f\|_{L^2}\|g\|_{L^2}\|\varphi\|_{L^2}$$

by lemma 2.2, which is the desired estimate (3.37) in the subregion $\boldsymbol{C_{1,1}}$.

Subregion $C_{1,2}$: $\left|\xi_1-\frac{1}{2}\xi\right|\ge 1$. Firstly, we note that if $\min\{|\xi_1|,|\xi_2|\}\le 1$, it follows that $\max\{\langle\xi_1\rangle,\langle\xi_2\rangle\}\lesssim\langle\xi\rangle$ and the same analysis of the subregion $\boldsymbol{C_{1,1}}$ can be repeated here. Thus, we can assume that $|\xi_1|\ge 1$ and $|\xi_2|\ge 1$. Note that

$$\begin{aligned}|\xi_1\xi_2|=|\xi_1(\xi-\xi_1)|&=\left|\left((\xi_1-\tfrac{1}{2}\xi)+\tfrac{1}{2}\xi\right)\left(\tfrac{1}{2}\xi-(\xi_1-\tfrac{1}{2}\xi)\right)\right|\\&\le\left|\xi_1-\tfrac{1}{2}\xi\right|^2+\tfrac{1}{4}|\xi|^2.\end{aligned}\tag{3.49}$$

Also, from (3.39) and the conditions $|\xi|\ge 1$ and $|\xi_1-\frac{1}{2}\xi|\ge 1$, it follows that

$$\max\{|\xi|,|\xi_1-\tfrac{1}{2}\xi|\}\le\left|\xi(\xi_1-\tfrac{1}{2}\xi)\right|\le 3\langle\tau\rangle.\tag{3.50}$$

If $s\le 0$, $k\le 0$, we obtain $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k}\lesssim\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k}\lesssim\langle\xi_1\rangle^{1/4}\langle\xi_2\rangle^{1/4}$ (since $0=\max\{0,s\}\le 2k+1/2$); if $s\le 0$, $k\ge 0$, we get $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k}\lesssim 1$; in the remaining cases (i.e., either $s\ge 0$, $k\le 0$ or $s\ge 0$, $k\ge 0$), we have two possibilities, namely $|\xi_1|\sim|\xi_2|$ or $|\xi_1|\nsim|\xi_2|$; when the first case occurs, it follows that $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k}\lesssim\langle\xi_1\rangle^{s-2k}\lesssim\langle\xi_1\rangle^{1/2}\lesssim\langle\xi_1\rangle^{1/4}\langle\xi_2\rangle^{1/4}$ and, in the second case, we conclude that $\langle\tau\rangle\gtrsim\langle\xi\rangle^2$, which implies $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k}\lesssim\langle\xi\rangle^{s-2k}\lesssim\langle\xi\rangle^{1/2}\lesssim\langle\tau\rangle^{1/4}$.

In resume, we always get that, in any case, either $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k}\lesssim\langle\tau\rangle^{1/4}$ or $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k}\lesssim\langle\xi_1\rangle^{1/4}\langle\xi_2\rangle^{1/4}$. When the first possibility occurs, using Cauchy-Schwarz, we can reduce the estimate (3.37) to bound the expression:

$$\widetilde{Z}:=\sup_{\xi,\tau}\frac{1}{\langle\tau\rangle^{2a}}\int_{\mathbb{R}^2}\frac{\langle\tau\rangle^{1/2}\chi_{C_{1,2}}}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}d\xi_2 d\tau_2.\tag{3.51}$$

But, this can be done as follows:

$$\begin{aligned}\widetilde{Z}&\lesssim\sup_{\xi,\tau}\frac{\langle\tau\rangle^{1/2-2a}}{|\xi|}\int_{\mathbb{R}^2}\frac{\chi_{C_{1,2}}}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}}d\sigma_1 d\sigma_2\\&\lesssim\sup_{\xi,\tau}\frac{\langle\tau\rangle^{1/2-2a}}{|\xi|}\langle\tau\rangle^{2-4b}\\&\lesssim 1,\end{aligned}\tag{3.52}$$

since $|\xi|\ge 1$, $b>1/2$ and $a>1/4$. When the second possibility happens, we decompose the frequencies ξ_j and the modulations σ_j into dyadic blocks $\langle\xi_j\rangle\sim N_j$ and $\langle\sigma_j\rangle\sim L_j$ (here $\xi_0:=\xi$, $\sigma_0:=\tau$ and $j=0,1,2$). Hence, it suffices to estimate (3.37) restricted to each dyadic block with the gain of extra terms N_j^{0-} and L_j^{0-}. To simplify, we put $N_{\max}:=\max\{N_0,N_1,N_2\}$ and $L_{\max}:=\max\{L_0,L_1,L_2\}$. So, we have

$$Z\lesssim\int_{\mathbb{R}^4}\frac{\langle\xi_1\rangle^{1/4}\langle\xi_2\rangle^{1/4}\bar{\varphi}(\xi,\tau)f(\xi_2,\tau_2)\bar{g}(-\xi_1,-\tau_1)}{\langle\tau\rangle^a\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b}\chi_{C_{1,2}}d\xi_1 d\tau_1 d\xi d\tau$$

Using (3.49) and (3.39), we get $\langle\xi_1\rangle\langle\xi_2\rangle \lesssim \langle\tau\rangle$ (by analyzing the cases $|\xi_1| \sim |\xi_2|$ and $|\xi_1| \nsim |\xi_2|$, resp.). Since $a > 1/4$, it follows

$$Z \lesssim \int_{\mathbb{R}^4} \frac{\bar{\varphi}(\xi,\tau) f(\xi_2,\tau_2)\bar{g}(-\xi_1,-\tau_1)}{L_0^{0+}\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b}\chi_{C_{1,2}} d\xi_1 d\tau_1 d\xi d\tau$$

Applying Cauchy-Schwarz, it suffices to bound the expression:

$$\sup_{\xi,\tau} \frac{1}{L_0^{0+}} \int_{\mathbb{R}^2} \frac{\chi_{C_{1,2}}}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}} d\xi_2 d\tau_2$$

Recall that (3.49), (3.50) and (3.39) implies $N_{\max} \lesssim L_0$. Also, $L_0 = L_{\max}$ in the region $\boldsymbol{C}$. In particular,

$$\begin{aligned}\sup_{\xi,\tau} \frac{1}{L_0^{0+}} \int_{\mathbb{R}^2} \frac{\chi_{C_{1,2}}}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}} d\xi_2 d\tau_2 &\lesssim \sup_{\xi,\tau} \frac{N_{\max}^{0-}L_{\max}^{0-}}{|\xi|} \int_{\mathbb{R}^2} \frac{\chi_{C_{1,2}}}{\langle\sigma_1\rangle^{2b}\langle\sigma_2\rangle^{2b}} d\sigma_1 d\sigma_2 \\ &\lesssim \sup_{\xi,\tau} \frac{N_{\max}^{0-}L_{\max}^{0-}}{|\xi|}\langle\tau\rangle^{2-4b} \\ &\lesssim N_{\max}^{0-}L_{\max}^{0-},\end{aligned}$$

because $b > 1/2$ and $|\xi| \geq 1$.

We conclude with the small frequencies for ξ, that is:
Subregion $\boldsymbol{C_2}$: $|\boldsymbol{\xi}| \leq \mathbf{1}$. The hypothesis $|\tau| \geq \max\{|\sigma_1|, |\sigma_2|\}$ is not crucial in this case; hence we divide into two smaller subregions:
Subregion $\boldsymbol{C_{2,1}}$: $|\boldsymbol{\xi_1}| \leq \mathbf{2}$. Here, it is easy to see that $\langle\xi_1\rangle \lesssim 1$, $\langle\xi_2\rangle \lesssim 1$. In particular, by Cauchy-Schwarz, our task is to estimate

$$\widetilde{Z} = \sup_{\xi_1,\sigma_1} \frac{1}{\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{1}{\langle\tau\rangle^{2a}\langle\sigma_2\rangle^{2b}} d\xi_2 d\tau_2.$$

Then, using lemma 2.3-(2.12) and lemma 2.3-(2.14), we get

$$\begin{aligned}\widetilde{Z} &= \sup_{\xi_1,\sigma_1} \frac{1}{\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{1}{\langle\tau\rangle^{2a}\langle\sigma_2\rangle^{2b}} d\xi_2 d\tau_2 \\ &\lesssim \int_{\mathbb{R}^2} \frac{1}{\langle\tau\rangle^{2a}\langle\tau-\tau_1+\frac{1}{2}(\xi-\xi_1)^2\rangle^{2b}} d\xi d\tau \\ &\lesssim \int_{-\infty}^{+\infty} \frac{1}{\langle-\tau_1+\frac{1}{2}\xi_1^2-\xi\xi_1+\frac{1}{2}\xi^2\rangle^{2a}} d\xi \\ &\lesssim 1,\end{aligned} \tag{3.53}$$

since $a > 1/4$.
Subregion $\boldsymbol{C_{2,2}}$: $|\boldsymbol{\xi_1}| \geq \mathbf{2}$. Redoing the analysis of the bounds for the term $\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k}$ in the four cases $s \leq 0$, $k \leq 0$, ..., $s \geq 0$, $k \geq 0$, we see that

$$\langle\xi\rangle^s\langle\xi_1\rangle^{-k}\langle\xi_2\rangle^{-k} \lesssim \langle\xi_1\rangle^{1/2}.$$

Similarly to the previous estimates for subregion $C_{1,2}$, we decompose the frequencies $\langle\xi_j\rangle \sim N_j$, $j = 0,1,2$, into dyadic blocks so that our task is to bound (3.37) restricted to each dyadic block with the gain of extra terms N_j^{0-}. We have

$$Z \lesssim N_1^{1/2} \int_{\mathbb{R}^4} \frac{\bar{\varphi}(\xi,\tau) f(\xi_2,\tau_2)\bar{g}(-\xi_1,-\tau_1)}{\langle\tau\rangle^a\langle\sigma_2\rangle^b\langle\sigma_1\rangle^b}\chi_{C_{2,2}} d\xi_1 d\tau_1 d\xi d\tau$$

Applying Cauchy-Schwarz, it suffices to prove that:

$$N_1^{1/2} \sup_{\xi_1,\sigma_1} \frac{1}{\langle\sigma_1\rangle^{2b}} \int_{\mathbb{R}^2} \frac{\chi_{C_{2,2}}}{\langle\tau\rangle^{2a}\langle\tau-\tau_1+\frac{1}{2}(\xi-\xi_1)^2\rangle^{2b}} d\xi d\tau \lesssim N_{\max}^{0-}. \tag{3.54}$$

This can be accoplished as follows. Firstly, notice that

$$\begin{aligned}(3.55)\qquad &\sup_{\xi_1,\sigma_1}\frac{1}{\langle\sigma_1\rangle^{2b}}\int_{\mathbb{R}^2}\frac{\chi_{C_{2,2}}}{\langle\tau\rangle^{2a}\langle\tau-\tau_1+\frac{1}{2}(\xi-\xi_1)^2\rangle^{2b}}d\xi d\tau\\ &\lesssim \sup_{\xi_1,\sigma_1}\frac{1}{\langle\sigma_1\rangle^{2b}}\int_{|\xi|\le 1}\frac{d\xi}{\langle-\tau_1+\frac{1}{2}\xi_1^2-\xi\xi_1+\frac{1}{2}\xi^2\rangle^{2a}}.\end{aligned}$$

Now, by changing variables

$$\eta=-\tau_1+\frac{1}{2}\xi_1^2-\xi\xi_1+\frac{1}{2}\xi^2,\quad d\eta=(\xi-\xi_1)d\xi,$$

we get $|\eta|\le\langle\sigma_1\rangle+|\xi_1|+\frac{1}{2}\le\langle\sigma_1\rangle+2|\xi_1|$ and we obtain the following bound of (3.55):

$$\begin{aligned}(3.56)\qquad &\sup_{\xi_1,\sigma_1}\frac{1}{\langle\sigma_1\rangle^{2b}}\int_{|\eta|\le\langle\sigma_1\rangle+2|\xi_1|}\frac{d\eta}{(1+|\eta|)^{2a}|\xi_1-\xi|}\\ &\lesssim \sup_{\xi_1,\sigma_1}\frac{1}{\langle\sigma_1\rangle^{2b}|\xi_1|}\int_{|\eta|\le\langle\sigma_1\rangle+2|\xi_1|}\frac{d\eta}{(1+|\eta|)^{2a}}\\ &\lesssim \sup_{\xi_1,\sigma_1}\frac{1}{\langle\sigma_1\rangle^{2b}|\xi_1|}\left(\langle\sigma_1\rangle^{1-2a}+|\xi_1|^{1-2a}\right)\\ &\lesssim\frac{1}{|\xi_1|^{2a}},\end{aligned}$$

since $2b>1$ and $a>0$. Putting this estimate into the expression (3.54), because $a>1/4$ and $N_1\sim N_{\max}$ in the subregion $\boldsymbol{C_{2,2}}$, we conclude

$$\begin{aligned}(3.57)\qquad &N_1^{1/2}\sup_{\xi_1,\sigma_1}\frac{1}{\langle\sigma_1\rangle^{2b}}\int_{\mathbb{R}^2}\frac{\chi_{C_{2,2}}}{\langle\tau\rangle^{2a}\langle\tau-\tau_1+\frac{1}{2}(\xi-\xi_1)^2\rangle^{2b}}d\xi d\tau\\ &\lesssim N_1^{1/2}\cdot\frac{1}{N_1^{2a}}\\ &\lesssim N_{\max}^{0-}.\end{aligned}$$

Collecting all the estimates above in all regions we have that the inequality (3.37) holds provided the conditions in proposition 3.2 are valid. □

Remark 3.3. *As pointed out in the introduction, once the bilinear estimates in propositions 3.1 and 3.2 are established, it is a standard matter to conclude the local well-posedness statement of theorem 1.1. We refer the reader to the works* [9], [2] *and* [7] *for further details.*

3.2. **Counter-Examples I: the continuous case.** We finish this section exhibiting several counter-examples showing that the bilinear estimates proved above are sharp, that is, the conditions imposed on the indices k and s in the propositions 3.1 and 3.2 are necessary.

Proposition 3.4. *The estimate* $\|uv\|_{X^{k,-1/2}}\lesssim\|u\|_{X^{k,1/2}}\|v\|_{H_t^{1/2}H_x^s}$ *holds only if* $|k|\le s+1/2$.

Proof. Take $N\in\mathbb{Z}^+$ a large integer and define

$$\begin{aligned}\mathsf{A}_1&=\left\{(\zeta,\eta)\in\mathbb{R}^2;\ 0\le\zeta\le 1/N\ \text{ and }\ |\eta+\tfrac{1}{2}\zeta^2|\le 1\right\},\\ \mathsf{B}_1&=\left\{(\zeta,\eta)\in\mathbb{R}^2;\ N\le\zeta\le N+\tfrac{1}{N}\ \text{ and }\ |\eta|\le 1\right\},\\ \mathsf{A}_2&=\left\{(\zeta,\eta)\in\mathbb{R}^2;\ N\le\zeta\le N+\tfrac{1}{N}\ \text{ and }\ |\eta+\tfrac{1}{2}\zeta^2|\le 1\right\},\\ \mathsf{B}_2&=\left\{(\zeta,\eta)\in\mathbb{R}^2;\ -N\le\zeta\le -N+\tfrac{1}{N}\ \text{ and }\ |\eta|\le 1\right\}.\end{aligned}$$

Put $\widehat{f_j}(\zeta,\eta) := \chi_{\mathsf{A}_j}$ and $\widehat{g_j}(\zeta,\eta) := \chi_{\mathsf{B}_j}$. A straightforward computation gives that

$$\|f_1 g_1\|_{X^{k,-1/2}} \sim \left(\frac{1}{N}\left(\frac{N^k}{N}\right)^2\right)^{1/2} \sim N^{k-\frac{3}{2}},$$
$$\|f_1\|_{X^{k,1/2}} \sim N^{-1/2}, \text{ and}$$
$$\|g_1\|_{H_t^{1/2}H_x^s} \sim N^{s-\frac{1}{2}}.$$

So, $\|f_1\overline{g_1}\|_{X^{k,-1/2}} \lesssim \|f_1\|_{X^{k,1/2}}\|g_1\|_{H_t^{1/2}H_x^s}$ implies that $k \le s + \frac{1}{2}$.

Analogously, another simple computation shows that

$$\|f_2 g_2\|_{X^{k,-1/2}} \sim N^{-\frac{3}{2}},$$
$$\|f_2\|_{X^{k,1/2}} \sim N^{k-\frac{1}{2}}, \text{ and}$$
$$\|g_2\|_{H_t^{1/2}H_x^s} \sim N^{s-\frac{1}{2}}.$$

Thus, $\|f_2 g_2\|_{X^{k,-1/2}} \lesssim \|f_2\|_{X^{k,1/2}}\|g_2\|_{H_t^{1/2}H_x^s}$ implies that $0 \le s + k + \frac{1}{2}$, i.e., $-k \le s + 1/2$. This completes the proof of the proposition. □

Proposition 3.5. *The estimate* $\|u\overline{w}\|_{H_t^{-1/2}H_x^s} \lesssim \|u\|_{X^{k,1/2}}\|w\|_{X^{k,1/2}}$ *holds only if* $s \le k + 1/2$ *and* $\max\{0, s\} \le 2k + 1/2$.

Proof. Take $N \in \mathbb{Z}^+$ a large integer and define

$$\mathsf{A}_1 = \left\{(\zeta,\eta) \in \mathbb{R}^2;\ 0 \le \zeta \le 1/N \text{ and } |\eta + \tfrac{1}{2}\zeta^2| \le 1\right\},$$
$$\mathsf{B}_1 = \left\{(\zeta,\eta) \in \mathbb{R}^2;\ N \le \zeta \le N + \tfrac{1}{N} \text{ and } |\eta + \tfrac{1}{2}\zeta^2| \le 1\right\},$$
$$\mathsf{A}_2 = \left\{(\zeta,\eta) \in \mathbb{R}^2;\ N \le \zeta \le N + \tfrac{1}{N} \text{ and } |\eta + \tfrac{1}{2}\zeta^2| \le 1\right\},$$
$$\mathsf{B}_2 = \left\{(\zeta,\eta) \in \mathbb{R}^2;\ -N \le \zeta \le -N + \tfrac{1}{N} \text{ and } |\eta + \tfrac{1}{2}\zeta^2| \le 1\right\},$$
$$\mathsf{B}_3 = \left\{(\zeta,\eta) \in \mathbb{R}^2;\ N \le \zeta \le N + \tfrac{1}{N} \text{ and } |\eta + \tfrac{1}{2}\zeta^2| \le 1\right\}.$$

Put $\widehat{f_j}(\zeta,\eta) := \chi_{\mathsf{A}_j}$ and $\widehat{g_j}(\zeta,\eta) := \chi_{\mathsf{B}_j}$. A simple calculation shows that

$$\|f_1\overline{g_1}\|_{H_t^{-1/2}H_x^s} \sim \left(\frac{1}{N}\left(\frac{N^s}{N}\right)^2\right)^{1/2} \sim N^{s-\frac{3}{2}},$$
$$\|f_1\|_{X^{k,1/2}} \sim N^{-1/2}, \text{ and}$$
$$\|g_1\|_{X^{k,1/2}} \sim N^{k-\frac{1}{2}}.$$

Hence, $\|f_1\overline{g_1}\|_{H_t^{-1/2}H_x^s} \lesssim \|f_1\|_{X^{k,1/2}}\|g_1\|_{X^{k,1/2}}$ implies that $s \le k + \frac{1}{2}$.

Similarly, another simple computation shows that

$$\|f_2\overline{g_2}\|_{H_t^{-1/2}H_x^s} \sim \left(\frac{1}{N}\left(\frac{N^s}{N}\right)^2\right)^{1/2} \sim N^{s-\frac{3}{2}},$$
$$\|f_2\overline{g_3}\|_{H_t^{-1/2}H_x^s} \sim \left(\frac{1}{N}\left(\frac{1}{N}\right)^2\right)^{1/2} \sim N^{-\frac{3}{2}}, \text{ and}$$
$$\|f_2\|_{X^{k,1/2}} \sim \|g_2\|_{X^{k,1/2}} \sim \|g_3\|_{X^{k,1/2}} \sim N^{k-\frac{1}{2}}.$$

Thus,

$$\|f_2\overline{g_2}\|_{H_t^{-1/2}H_x^s} \lesssim \|f_2\|_{X^{k,1/2}}\|g_2\|_{X^{k,1/2}}$$

implies that $s \le 2k + \frac{1}{2}$ and

$$\|f_2\overline{g_3}\|_{H_t^{-1/2}H_x^s} \lesssim \|f_2\|_{X^{k,1/2}}\|g_3\|_{X^{k,1/2}}$$

implies that $0 \le 2k + \frac{1}{2}$. Therefore, $\max\{0, s\} \le 2k + \frac{1}{2}$. □

4. Bilinear Estimates for the Coupling Terms in the Periodic Case

Here, we show sharp bilinear estimates for the coupling terms in the periodic setting.

4.1. Proof of the bilinear estimates II: the periodic case.

Proposition 4.1. *The bilinear estimate*

$$\|uv\|_{X^{k,-\frac{1}{2}+}} \lesssim \|u\|_{X^{k,\frac{1}{2}-}}\|v\|_{H_t^{\frac{1}{2}-}H_x^s} \tag{4.58}$$

holds if $0 \le s \le 2k$ *and* $|k-s| < 1$.

Proof. Fix $s \ge 0$ and $k < s+1$. Taking $a = b = c = 1/2-$, our task is to show the bilinear estimate

$$\|uv\|_{X^{k,-a}} \lesssim \|u\|_{X^{k,b}}\|v\|_{H_t^cH_x^s}$$

Defining $f(n,\tau) := \langle\tau+n^2\rangle^b\langle n\rangle^k\widehat{u}(n,\tau)$ and $g(n,\tau) := \langle\tau\rangle^c\langle n\rangle^s\widehat{v}(n,\tau)$, it suffices to prove that

$$Z \lesssim \|f\|_{L^2_{n,\tau}}\|g\|_{L^2_{n,\tau}}\|\varphi\|_{L^2_{n,\tau}}, \tag{4.59}$$

where

$$W := \sum_{n\in\mathbb{Z}}\int d\tau \sum_{n=n_1+n_2}\int_{\tau=\tau_1+\tau_2}\frac{\langle\tau+n^2\rangle^{-a}\langle n\rangle^k f(n_1,\tau_1)g(n_2,\tau_2)\varphi(n,\tau)}{\langle\tau_1+n_1^2\rangle^b\langle\tau_2\rangle^c\langle n_1\rangle^k\langle n_2\rangle^s}. \tag{4.60}$$

Dividing $\mathbb{Z}^2\times\mathbb{R}^2$ into three regions, namely $\mathbb{Z}^2\times\mathbb{R}^2 = R_0\cup R_1\cup R_2$, integrating first over n_1,τ_1 in the region R_0, n,τ in the region R_1, n_2,τ_2 in the region R_2 and using Cauchy-Schwarz, we easily see that it remains only to uniformly bound the following three expressions:

$$W_1 := \sup_{n,\tau}\frac{\langle n\rangle^{2k}}{\langle\tau+n^2\rangle^{2a}}\sum_{n_1}\int d\tau_1\frac{\chi_{R_0}}{\langle\tau_1+n_1^2\rangle^{2b}\langle\tau_2\rangle^{2c}\langle n_1\rangle^{2k}\langle n_2\rangle^{2s}} \tag{4.61}$$

$$W_2 := \sup_{n_1,\tau_1}\frac{1}{\langle n_1\rangle^{2k}\langle\tau_1+n_1^2\rangle^{2b}}\sum_{n}\int d\tau\frac{\langle n\rangle^{2k}\chi_{R_1}}{\langle\tau+n^2\rangle^{2a}\langle\tau_2\rangle^{2c}\langle n_2\rangle^{2s}} \tag{4.62}$$

$$W_3 := \sup_{n_2,\tau_2}\frac{1}{\langle n_2\rangle^{2s}\langle\tau_2\rangle^{2c}}\sum_{n}\int d\tau\frac{\langle n\rangle^{2k}\chi_{R_2}}{\langle\tau+n^2\rangle^{2a}\langle\tau_1+n_1^2\rangle^{2b}\langle n_1\rangle^{2k}} \tag{4.63}$$

For later use, we recall that the dispersive relation of this bilinear estimate is:

$$\tau+n^2-(\tau_1+n_1^2)-\tau_2 = n^2-n_1^2 \tag{4.64}$$

In order to define the regions R_0, R_1, R_2, we introduce the subsets:

$$\begin{aligned} A &:= \left\{(n,n_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : |n|\lesssim 1\right\}, \\ B &:= \left\{(n,n_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : |n|\gg 1 \text{ and } |n|\sim|n_1|\right\}, \\ C &:= \left\{(n,n_1,\tau,\tau_1)\in\mathbb{Z}^2\times\mathbb{R}^2 : |n|\gg 1, |n|\nsim|n_1| \text{ and } |\tau+n^2| = L_{\max}\right\}, \end{aligned} \tag{4.65}$$

where $L_{\max} := \max\{|\tau+n^2|, |\tau_1+n_1^2|, |\tau_2|\}$. For later reference, we denote also $N_{\max} := \max\{|n|, |n_1|, |n_2|\}$. Then, we put $R_0 := A \cup B \cup C$ and

$$\begin{aligned} R_1 &:= \left\{(n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| \gg 1, |n| \not\sim |n_1| \text{ and } |\tau_1 + n_1^2| = L_{\max}\right\}, \\ R_2 &:= \left\{(n, n_1, \tau, \tau_1) \in \mathbb{Z}^2 \times \mathbb{R}^2 : |n| \gg 1, |n| \not\sim |n_1| \text{ and } |\tau_2| = L_{\max}\right\}. \end{aligned} \tag{4.66}$$

We begin with the analysis of (4.61). In the region A, since $|n| \lesssim 1$, we have

$$\begin{aligned} &\sup_{n,\tau} \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle^{2a}} \sum_{n_1} \int d\tau_1 \frac{\chi_A}{\langle \tau_1+n_1^2\rangle^{2b}\langle\tau_2\rangle^{2c}\langle n_1\rangle^{2k}\langle n_2\rangle^{2s}} \\ &\lesssim \sup_{n,\tau} \frac{1}{\langle \tau+n^2\rangle^{2a}} \sum_{n_1} \int d\tau_1 \frac{1}{\langle \tau_1+n_1^2\rangle^{2b}\langle\tau_2\rangle^{2c}\langle n_1\rangle^{2k}\langle n_2\rangle^{2s}} \\ &\lesssim \sup_{\tau} \sum_{n_1} \frac{1}{\langle \tau+n_1^2\rangle^{2b+2c-1-}} \\ &\lesssim 1, \end{aligned}$$

because $k, s \geq 0$, $a > 0$ and $2b + 2c > 3/2$.

In the region B, we have $|n| \sim |n_1|$. Thus,

$$\begin{aligned} &\sup_{n,\tau} \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle^{2a}} \sum_{n_1} \int d\tau_1 \frac{\chi_B}{\langle \tau_1+n_1^2\rangle^{2b}\langle\tau_2\rangle^{2c}\langle n_1\rangle^{2k}\langle n_2\rangle^{2s}} \\ &\lesssim \sup_{n,\tau} \frac{1}{\langle \tau+n^2\rangle^{2a}} \sum_{n_1} \int d\tau_1 \frac{1}{\langle \tau_1+n_1^2\rangle^{2b}\langle\tau_2\rangle^{2c}\langle n_2\rangle^{2s}} \\ &\lesssim \sup_{\tau} \sum_{n_1} \frac{1}{\langle \tau+n_1^2\rangle^{2b+2c-1-}} \\ &\lesssim 1, \end{aligned}$$

because $k, s \geq 0$, $a > 0$ and $2b + 2c > 3/2$.

In the region C, we know that $|\tau+n^2| = L_{\max}$, $|n| \not\sim |n_1|$ and $|n| \gg 1$. Hence, the dispersive relation (4.64) says that $|\tau+n^2| = L_{\max} \gtrsim |n^2-n_1^2| \gtrsim N_{\max}^2$. Therefore,

$$\begin{aligned} &\sup_{n,\tau} \frac{\langle n\rangle^{2k}}{\langle \tau+n^2\rangle^{2a}} \sum_{n_1} \int d\tau_1 \frac{\chi_C}{\langle \tau_1+n_1^2\rangle^{2b}\langle\tau_2\rangle^{2c}\langle n_1\rangle^{2k}\langle n_2\rangle^{2s}} \\ &\lesssim \sup_{n,\tau} \frac{\langle N_{\max}\rangle^{2k-2s}}{\langle \tau+n^2\rangle^{2a}} \sum_{n_1} \int d\tau_1 \frac{1}{\langle \tau_1+n_1^2\rangle^{2b}\langle\tau_2\rangle^{2c}} \\ &\lesssim \sup_{n,\tau} \frac{\langle N_{\max}\rangle^{2k}}{\langle N_{\max}\rangle^{2a}} \sum_{n_1} \frac{1}{\langle \tau+n_1^2\rangle^{2b+2c-1-}} \\ &\lesssim 1, \end{aligned}$$

since $k, s \geq 0$, $k < s+1$, $a = 1/2-$ and $2b + 2c > 3/2$.

Putting together the estimates above, we conclude the desired boundedness of (4.61):

$$|W_1| \lesssim 1.$$

Next we estimate the contribution of (4.62). In the region R_1, we know that $|n| \gg 1$, $|n| \not\sim |n_1|$ and $|\tau_1 + n_1^2| = L_{\max}$. So, the dispersive relation (4.64) implies that $|\tau_1 + n_1^2| \gtrsim N_{\max}^2$. Thus,

$$\begin{aligned} W_2 &:= \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2k}\langle \tau_1+n_1^2\rangle^{2b}} \sum_n \int d\tau \frac{\langle n\rangle^{2k}\chi_{R_1}}{\langle \tau+n^2\rangle^{2a}\langle\tau_2\rangle^{2c}\langle n_2\rangle^{2s}} \\ &\lesssim \sup_{\tau_1}\sum_n \int d\tau \frac{\langle N_{\max}\rangle^{2k-2s-4b}}{\langle\tau+n^2\rangle^{2a}\langle\tau_2\rangle^{2c}} \lesssim \sup_{\tau_1}\sum_n \frac{1}{\langle \tau_1+n^2\rangle^{2a+2c-1-}} \\ &\lesssim 1, \end{aligned}$$

since $k, s \geq 0$, $k < s+1$ and $b = 1/2-$, $2a+2c > 3/2$.

Finally, we bound (4.63) by noting that, in the region R_2, it holds $|n| \gg 1$, $|n| \not\sim |n_1|$ and $|\tau_2| = L_{\max}$. In particular, the dispersive relation (4.64) forces $|\tau_2| \gtrsim N_{\max}^2$. This allows to obtain

$$\begin{aligned} W_3 &:= \sup_{n_2,\tau_2} \frac{1}{\langle n_2\rangle^{2s}\langle\tau_2\rangle^{2c}} \sum_n \int d\tau \frac{\langle n\rangle^{2k}\chi_{R_2}}{\langle\tau+n^2\rangle^{2a}\langle\tau_1+n_1^2\rangle^{2b}\langle n_1\rangle^{2k}} \\ &\lesssim \sup_{n_2,\tau_2}\sum_n \int d\tau \frac{\langle N_{\max}\rangle^{2k-2s-4c}}{\langle\tau+n^2\rangle^{2a}\langle\tau_1+n_1^2\rangle^{2b}} \lesssim \sup_{n_2,\tau_2}\sum_n \frac{1}{\langle\tau_2+n_2(n_2-2n)\rangle^{2a+2b-1-}} \\ &\lesssim 1, \end{aligned}$$

since $k, s \geq 0$, $k < s+1$ and $2a+2b > 3/2$.

This completes the proof of the proposition. □

Proposition 4.2. *The bilinear estimate*

$$\|u\overline{w}\|_{H_t^{-\frac12+}H_x^s} \lesssim \|u\|_{X^{k,\frac12+}}\|w\|_{X^{k,\frac12+}} \tag{4.67}$$

holds if $0 \leq s \leq 2k$ *and* $|k-s| < 1$.

Proof. Similarly to the previous proposition, the relevant dispersive relation is

$$\tau - (\tau_1 + n_1^2) - (\tau_2 - n_2^2) = n_2^2 - n_1^2 \tag{4.68}$$

and it suffices to bound the following contributions:

$$Z_1 := \sup_{n,\tau} \frac{\langle n\rangle^{2s}}{\langle\tau\rangle^{2a}} \sum_{n_1}\int d\tau_1 \frac{\chi_{S_0}}{\langle\tau_1+n_1^2\rangle^{2b}\langle\tau_2-n_2^2\rangle^{2c}\langle n_1\rangle^{2k}\langle n_2\rangle^{2k}} \tag{4.69}$$

$$Z_2 := \sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2k}\langle\tau_1+n_1^2\rangle^{2b}} \sum_n \int d\tau \frac{\langle n\rangle^{2s}\chi_{S_1}}{\langle\tau\rangle^{2a}\langle\tau_2-n_2^2\rangle^{2c}\langle n_2\rangle^{2k}} \tag{4.70}$$

$$Z_3 := \sup_{n_2,\tau_2} \frac{1}{\langle n_2\rangle^{2k}\langle\tau_2-n_2^2\rangle^{2c}} \sum_n \int d\tau \frac{\langle n\rangle^{2s}\chi_{S_2}}{\langle\tau\rangle^{2a}\langle\tau_1+n_1^2\rangle^{2b}\langle n_1\rangle^{2k}} \tag{4.71}$$

where $S_0 \cup S_1 \cup S_2 = \mathbb{Z}^2 \times \mathbb{R}^2$. To define the regions S_j, $j = 0,1,2$, we introduce the sets

$$\begin{aligned} E &:= \left\{(n,n_1,\tau,\tau_1) \in \mathbb{Z}^2\times\mathbb{R}^2 : |n| \lesssim 1\right\}, \\ F &:= \left\{(n,n_1,\tau,\tau_1) \in \mathbb{Z}^2\times\mathbb{R}^2 : |n| \gg 1 \text{ and } |n_1| \sim |n_2|\right\}, \\ G &:= \left\{(n,n_1,\tau,\tau_1) \in \mathbb{Z}^2\times\mathbb{R}^2 : |n| \gg 1, |n_1| \not\sim |n_2| \text{ and } |\tau_1+n_1^2| = L_{\max}\right\}, \end{aligned} \tag{4.72}$$

We put $S_1 := E \cup F \cup G$ and

$$\begin{aligned} S_0 &:= \left\{(n,n_1,\tau,\tau_1) \in \mathbb{Z}^2\times\mathbb{R}^2 : |n| \gg 1, |n_1| \not\sim |n_2| \text{ and } |\tau| = L_{\max}\right\}, \\ S_2 &:= \left\{(n,n_1,\tau,\tau_1) \in \mathbb{Z}^2\times\mathbb{R}^2 : |n| \gg 1, |n_1| \not\sim |n_2| \text{ and } |\tau_2-n_2^2| = L_{\max}\right\}. \end{aligned} \tag{4.73}$$

We can estimate Z_1 as follows. In the region S_0, since $|n_1| \not\sim |n_2|$, we have either $|n_1| \gg |n_2|$ or $|n_2| \gg |n_1|$. By symmetry reasons, we can suppose that, without loss of generality, $|n_2| \gg |n_1|$. In this case, $|\tau| \gtrsim n_2^2$ and $|n| \sim |n_2|$. So,

$$\begin{aligned} Z_1 &:= \sup_{n,\tau} \frac{\langle n\rangle^{2s}}{\langle\tau\rangle^{2a}} \sum_{n_1} \int d\tau_1 \frac{\chi_{S_0}}{\langle\tau_1+n_1^2\rangle^{2b}\langle\tau_2-n_2^2\rangle^{2c}\langle n_1\rangle^{2k}\langle n_2\rangle^{2k}} \\ &\lesssim \sup_{n,\tau} \frac{\langle n\rangle^{2s-2k}}{\langle\tau\rangle^{2a}} \sum_{n_1} \int d\tau_1 \frac{\chi_{S_0}}{\langle\tau_1+n_1^2\rangle^{2b}\langle\tau_2-n_2^2\rangle^{2c}} \\ &\lesssim \sup_{n,\tau} \sum_{n_1} \frac{1}{\langle\tau+n_1^2-n_2^2\rangle^{2b+2c-1-}\langle\tau_2-n_2^2\rangle^{2c}\langle n_1\rangle^{2k}\langle n_2\rangle^{2k}} \\ &\lesssim 1, \end{aligned} \tag{4.74}$$

since $k, s \geq 0$, $s < k+1$ and $2b+2c > 3/2$.

Now we will bound the expression Z_2. In the region E, it holds $|n| \lesssim 1$. Hence,

$$\begin{aligned} &\sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2k}\langle\tau_1+n_1^2\rangle^{2b}} \sum_n \int d\tau \frac{\langle n\rangle^{2s}\chi_E}{\langle\tau\rangle^{2a}\langle\tau_2-n_2^2\rangle^{2c}\langle n_2\rangle^{2k}} \\ &\lesssim \sup_{n_1,\tau_1} \sum_{|n|\lesssim 1} \int d\tau \frac{1}{\langle\tau\rangle^{2a}\langle\tau_2-n_2^2\rangle^{2c}} \lesssim \sup_{n_1,\tau_1} \sum_{|n|\lesssim 1} \frac{1}{\langle\tau_1+n_2^2\rangle^{2a+2c-1-}} \\ &\lesssim 1. \end{aligned} \tag{4.75}$$

In the region F, we get $|n_1| \sim |n_2|$ so that

$$\begin{aligned} &\sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2k}\langle\tau_1+n_1^2\rangle^{2b}} \sum_n \int d\tau \frac{\langle n\rangle^{2s}\chi_F}{\langle\tau\rangle^{2a}\langle\tau_2-n_2^2\rangle^{2c}\langle n_2\rangle^{2k}} \\ &\lesssim \sup_{n_1,\tau_1} \frac{1}{\langle\tau_1+n_1^2\rangle^{2b}} \sum_n \int d\tau \frac{\langle n\rangle^{2s-4k}}{\langle\tau\rangle^{2a}\langle\tau_2-n_2^2\rangle^{2c}} \\ &\lesssim \sup_{n_1,\tau_1} \sum_n \frac{1}{\langle\tau_1+(n-n_1)^2\rangle^{2a+2c-1-}} \\ &\lesssim 1, \end{aligned} \tag{4.76}$$

because $0 \leq s \leq 2k$ and $2a+2c > 3/2$. In the region G, the dispersive relation (4.68) combined with the assumptions $|n| \gg 1$, $|n_1| \not\sim |n_2|$ and $|\tau_1+n_1^2| = L_{\max}$ implies that $|\tau_1+n_1^2| \gtrsim N_{\max}^2$. Without loss of generality, we can suppose that $|n_1| \ll |n_2|$. Then,

$$\begin{aligned} &\sup_{n_1,\tau_1} \frac{1}{\langle n_1\rangle^{2k}\langle\tau_1+n_1^2\rangle^{2b}} \sum_n \int d\tau \frac{\langle n\rangle^{2s}\chi_G}{\langle\tau\rangle^{2a}\langle\tau_2-n_2^2\rangle^{2c}\langle n_2\rangle^{2k}} \\ &\lesssim \sup_{n_1,\tau_1} \sum_n \int d\tau \frac{\langle n\rangle^{2s-2k-4b}}{\langle\tau\rangle^{2a}\langle\tau_2-n_2^2\rangle^{2c}} \\ &\lesssim 1, \end{aligned} \tag{4.77}$$

since $0 \leq k, s$ and $s < k+1$, $2a+2c > 3/2$. Collecting these estimates, we conclude

$$|Z_2| \lesssim 1. \tag{4.78}$$

Finally, the expression (4.71) can be controlled if we notice that $|n| \gg 1$, $|n_1| \not\sim |n_2|$ and $|\tau_2 - n_2^2| = L_{\max}$ implies $|\tau_2 - n_2^2| \gtrsim N_{\max}^2$. In particular,

$$\begin{aligned} Z_3 &:= \sup_{n_2,\tau_2} \frac{1}{\langle n_2\rangle^{2k}\langle \tau_2 - n_2^2\rangle^{2c}} \sum_n \int d\tau \frac{\langle n\rangle^{2s}\chi_{S_2}}{\langle\tau\rangle^{2a}\langle\tau_1+n_1^2\rangle^{2b}\langle n_1\rangle^{2k}} \\ &\lesssim \sup_{n_2,\tau_2} \sum_n \int d\tau \frac{\langle N_{\max}\rangle^{2s-2k-4c}}{\langle\tau\rangle^{2a}\langle\tau_1+n_1^2\rangle^{2b}} \\ &\lesssim \sup_{n_2,\tau_2} \sum_n \frac{1}{\langle\tau_2-(n-n_2)^2\rangle^{2a+2b-1-}} \\ &\lesssim 1, \end{aligned} \tag{4.79}$$

whenever $k, s \geq 0$, $s < k+1$ and $2a+2b > 3/2$.

This finishes the proof of the proposition. □

Remark 4.3. *Again, once the bilinear estimates in propositions 4.1 and 4.2 are proved, one can show the theorem 1.2 by standard arguments (e.g., see the works* [9], [2] *and* [7]*).*

4.2. **Counter-Examples II: the periodic case.** The next results prove that the bilinear estimates derived in propositions 4.1 and 4.2 are sharp.

Proposition 4.4. $\|uv\|_{X^{k,-\frac12+}} \lesssim \|u\|_{X^{k,\frac12}}\|v\|_{H_t^{\frac12}H_x^s}$ *implies* $s \geq 0$ *and* $k < s+1$.

Proof. Firstly, we fix $N \gg 1$ a large integer and define

$$a_n = \begin{cases} 1 & \text{if } n = N \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_n = \begin{cases} 1 & \text{if } n = -2N \\ 0 & \text{otherwise} \end{cases}$$

Let f and g be given by $\hat f(n,\tau) = a_n\chi_{[-1,1]}(\tau+n^2)$ and $\hat g(n,\tau) = b_n\chi_{[-1,1]}(\tau)$. Taking into account the dispersive relation $\tau + n^2 - (\tau_1+n_1^2) - \tau_2 = n^2 - n_1^2$, we can easily compute that

$$\|fg\|_{X^{k,-1/2+}} \simeq N^k, \quad \|f\|_{X^{k,1/2}} \simeq N^k \quad \text{and} \quad \|g\|_{H_t^{1/2}H_x^s} \simeq N^s$$

Hence, the bound $\|fg\|_{X^{k,-\frac12+}} \lesssim \|f\|_{X^{k,\frac12}}\|g\|_{H_t^{\frac12}H_x^s}$ implies $N^k \lesssim N^{k+s}$, consequently, $s \geq 0$.

Secondly, define

$$d_n = \begin{cases} 1 & \text{if } n = N \\ 0 & \text{otherwise} \end{cases}$$

and

$$c_n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Let p and q be $\hat p(n,\tau) = c_n\chi_1(\tau+n^2)$ and $\hat q(n,\tau) = d_n\chi_1(\tau)$. Again, it is not hard to see that

$$\|pq\|_{X^{k,-1/2+}} \simeq \frac{N^k}{N^{1-}}$$
$$\|p\|_{X^{k,1/2}} \simeq 1$$
$$\|q\|_{H_t^{1/2}H_x^s} \simeq N^s$$

Hence, the bound $\|pq\|_{X^{k,-\frac12+}} \lesssim \|p\|_{X^{k,\frac12}}\|q\|_{H_t^{\frac12}H_x^s}$ implies $\frac{N^k}{N^{1-}} \lesssim N^s$, i.e., $k < s+1$. □

Proposition 4.5. $\|u_1\overline{u_2}\|_{H_t^{-\frac{1}{2}+}H_x^s} \lesssim \|u_1\|_{X^{k,\frac{1}{2}}}\|u_2\|_{X^{k,\frac{1}{2}}}$ *implies* $s \le 2k$ *and* $s < k+1$.

Proof. For a fixed large integer $N \gg 1$, define

$$a_n = \begin{cases} 1 & \text{if } n = N \\ 0 & \text{otherwise} \end{cases}$$

$$b_n = \begin{cases} 1 & \text{if } n = -N-1 \\ 0 & \text{otherwise} \end{cases}$$

$$c_n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$d_n = \begin{cases} 1 & \text{if } n = N \\ 0 & \text{otherwise} \end{cases}$$

Putting $\hat{f}_1(n,\tau) = a_n\chi_1(\tau+n^2)$, $\hat{f}_2(n,\tau) = b_n\chi_1(\tau+n^2)$ and $\hat{g}_1(n,\tau) = c_n\chi_1(\tau+n^2)$, $\hat{g}_2(n,\tau) = d_n\chi_1(\tau+n^2)$, a simple calculation (based on the dispersive relation $\tau-(\tau_1+n_1^2)-(\tau_2+n_2^2) = n_2^2-n_1^2$) gives that

$$\|f_1\overline{f_2}\|_{H_t^{-1/2+}H_x^s} \simeq N^s \quad \text{and} \quad \|g_1\overline{g_2}\|_{H_t^{-1/2+}H_x^s} \simeq N^{s-1+},$$

$$\|f_1\|_{X^{k,1/2}} \simeq N^k \quad \text{and} \quad \|g_1\|_{X^{k,1/2}} \simeq 1,$$

$$\|f_2\|_{X^{k,1/2}} \simeq N^k \simeq \|g_2\|_{X^{k,1/2}}.$$

Therefore, the bound $\|u_1\overline{u_2}\|_{H_t^{-\frac{1}{2}+}H_x^s} \lesssim \|u_1\|_{X^{k,\frac{1}{2}}}\|u_2\|_{X^{k,\frac{1}{2}}}$ says that $N^s \lesssim N^{2k}$ and $N^{s-1+} \lesssim N^k$, i.e., $s \le 2k$ and $s < k+1$. □

5. Global well-posedness below $L^2 \times L^2$

This section is devoted to the proof of the global well-posedness result stated in theorem 1.3 via the I-method of Colliander, Keel, Staffilani, Takaoka and Tao.

5.1. **The I-operator.** Let $m(\xi)$ be a smooth non-negative symbol on $\mathbb{R}$ which equals 1 for $|\xi| \le 1$ and equals $|\xi|^{-1}$ for $|\xi| \ge 2$. For any $N \ge 1$ and $\alpha \in \mathbb{R}$, denote by I_N^α the Fourier multiplier

$$\widehat{I_N^\alpha f}(\xi) = m\left(\frac{\xi}{N}\right)^\alpha \widehat{f}(\xi).$$

We recall the following abstract interpolation lemma:

Lemma 5.1 (Lemma 12.1 of [4])**.** *Let* $\alpha_0 > 0$ *and* $n \ge 1$. *Suppose* $Z, X_1, \ldots, X_n$ *are translation-invariant Banach spaces and* T *is a translation invariant* n*-linear operator such that*

$$\|I_1^\alpha T(u_1,\ldots,u_n)\|_Z \lesssim \prod_{j=1}^n \|I_1^\alpha u_j\|_{X_j},$$

for all $u_1,\ldots,u_n$ *and* $0 \le \alpha \le \alpha_0$. *Then,*

$$\|I_N^\alpha T(u_1,\ldots,u_n)\|_Z \lesssim \prod_{j=1}^n \|I_N^\alpha u_j\|_{X_j},$$

for all $u_1,\ldots,u_n$, $0 \le \alpha \le \alpha_0$ *and* $N \ge 1$. *Here, the implied constant is independent of* N.

After these preliminaries, we are ready to show a variant of the local well-posedness theorem 1.1.

5.2. **Local well-posedness revisited.** In the sequel, we take $N \gg 1$ a large integer and we denote by I the operator $I := I_N^{-s}$ for a given $s \in \mathbb{R}$.

Proposition 5.2. *For all $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ and $s \geq -1/4$, the Schrödinger-Debye system (1.1) has a unique local-in-time solution $(u(t), v(t)$ defined on the time interval $[0, \delta]$ for some $\delta \leq 1$ satisfying*

$$\delta \sim (\|Iu_0\|_{L^2_x} + \|Iv_0\|_{L^2_x})^{-4-}. \tag{5.80}$$

Furthermore, $\|Iu\|_{X^{0,1/2+}} + \|Iv\|_{X^{0,1/2+}} \lesssim \|Iu_0\|_{L^2} + \|Iv_0\|_{L^2}$.

Proof. Applying the I-operator to the Schrödinger-Debye system (1.1), we get

$$\begin{cases} i\partial_t Iu + \frac{1}{2}\partial_x^2 Iu = I(uv), \\ \sigma\partial_t Iv + Iv = \epsilon I(|u|^2), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \end{cases} \tag{5.81}$$

To solve this problem, we denote by $\Phi_1(Iu, Iv)$ and $\Phi_2(Iu, Iv)$ the integral maps associated to this system, so that our task is to find a fixed point of (Φ_1, Φ_2). To accoplish this objetive, note that, by standard arguments, the interpolation lemma 5.1 combined with the bilinear estimates in the propositions 3.1 and 3.2 give the estimates

$$\begin{aligned} \|\Phi_1(Iu, Iv)\|_{X^{0,1/2+}} &\leq C\|Iu_0\|_{L^2_x} + C\delta^{1/4-}\|Iu\|_{X^{0,1/2+}}\|Iv\|_{H_t^{1/2+}L^2_x}, \\ \|\Phi_2(Iu, Iv)\|_{H_t^{1/2+}L^2_x} &\leq C\|Iv_0\|_{L^2_x} + C\delta^{1/4-}\|Iu\|^2_{X^{0,1/2+}}, \end{aligned}$$

where $Iu, Iv \in X^{0,1/2+}$ are defined in the interval $[0, \delta]$.

Taking $R = 2C(\|Iu_0\|_{L^2_x} + \|Iv_0\|_{L^2_x})$, we conclude that (Φ_1, Φ_2) has an unique fixed point (Iu, Iv) on the ball of radius R. Moreover,

$$\delta \sim (\|Iu_0\|_{L^2_x} + \|Iv_0\|_{L^2_x})^{-4-}$$

This completes the proof of the proposition. □

Once a local well-posedness result for the modified system (5.81) was obtained, we will study the behavior of the L^2-conservation law under the I-operator.

5.3. **Modified energy.** We consider the modified energy $E(Iu) = \|Iu\|^2_{L^2_x}$. Note that, since (Iu, Iv) verify the system (5.81), we have

$$\begin{aligned} \frac{d}{dt}E(Iu)(t) &= \int \partial_t Iu \cdot I\overline{u} + \int Iu \cdot \partial I\overline{u} \\ &= -\frac{1}{i}\int \partial_x^2 Iu \cdot I\overline{u} + \frac{1}{i}\int I(uv)I\overline{u} + \frac{1}{i}\int Iu\partial_x^2 I\overline{u} - \frac{1}{i}\int IuI(\overline{uv}) \\ &= \frac{1}{i}\int \partial_x Iu \cdot \partial_x I\overline{u} + \frac{1}{i}\int (I(uv) - IuIv)\, I\overline{u} + \frac{1}{i}\int IuIvI\overline{u} \\ &\quad - \frac{1}{i}\int \partial_x Iu \cdot \partial_x I\overline{u} - \frac{1}{i}\int Iu\overline{(I(uv) - IuIv)} - \frac{1}{i}\int IuI\overline{u}Iv \\ &= 2\Im \int (I(uv) - IuIv)\, I\overline{u}. \end{aligned}$$

Now we are going to see that this formula leads naturally to an *almost conservation law.*

5.4. **Almost conservation of the modified energy.** For later use, we need the following refined Strichartz estimate:

Lemma 5.3. *We have*

$$\|(D_x^{1/2} f)\cdot g\|_{L^2_{xt}} \lesssim \|f\|_{X^{0,1/2+}}\|g\|_{X^{0,1/2+}},$$

if $|\xi_1| \gg |\xi_2|$ *for any* $|\xi_1| \in supp(\widehat{f}), |\xi_2| \in supp(\widehat{g})$. *Moreover, this estimate is true if* f *and/or* g *is replaced by its complex conjugate in the left-hand side of the inequality.*

Proof. See lemma 7.1 of [5] or lemma 4.2 of [8]. □

Lemma 5.4. *For* $s > -1/4$, *it holds*

$$|E(Iu)(\delta) - E(Iu)(0)| \lesssim N^{-1/2+}\delta^{1/4-}\|Iu\|^2_{X^{0,1/2+}}\|Iv\|_{H_t^{1/2+}L_x^2}$$

Proof. Since we already know that

$$E(Iu)(\delta)-E(Iu)(0) = \int_0^\delta \frac{d}{dt}E(Iu)(t)dt = 2\Im\int_0^\delta\int (I(uv)-IuIv)\,I\overline{u},$$

it suffices to show that

$$\left|\int_0^\delta\int (I(uv)-IuIv)\,I\overline{u}\right| \lesssim N^{-1/2+}\delta^{1/4-}\|Iu\|^2_{X^{0,1/2}}\|Iv\|_{H_t^{1/2+}L_x^2}. \tag{5.82}$$

By Parseval, our task is to prove that

$$I := \int_0^\delta\int_{\xi_1+\xi_2-\xi_3=0}\left|\frac{m(\xi_1+\xi_2)-m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)}\right|\widehat{u}(\xi_1,t)\widehat{v}(\xi_2,t)\widehat{\overline{w}}(\xi_3,t)$$
$$\lesssim N^{-1/2+}\delta^{1/4-}\|u\|_{X^{0,1/2}}\|v\|_{H_t^{1/2+}L_x^2}\|w\|_{X^{0,1/2}}$$

We decompose the frequencies ξ_j, $j=1,2,3$ into dyadic blocks $|\xi_j|\sim N_j$. Before starting the proof of this inequality, we note that the multiplier $M := \frac{m(\xi_1+\xi_2)-m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)}$ satisfies

- if $|\xi_1| \ll |\xi_2|, |\xi_1| \ll N$, then
$$|M| \lesssim \left|\frac{m(\xi_1+\xi_2)-m(\xi_2)}{m(\xi_2)}\right| \lesssim \left|\frac{\nabla m(\xi_2)\xi_1}{m(\xi_2)}\right| \lesssim \frac{N_1}{N_2}$$
.
- similarly, if $|\xi_2| \ll |\xi_1|, |\xi_2| \ll N$, then $M \lesssim N_2/N_1$.
- if $|\xi_1| \ll |\xi_2|, |\xi_1| \gtrsim N$, then
$$|M| \lesssim \frac{1}{m(\xi_1)} \lesssim \left(\frac{N_1}{N}\right)^{1/4-},$$
because $s > -1/4$.
- similarly, if $|\xi_2| \ll |\xi_1|, |\xi_2| \gtrsim N$, then $|M| \lesssim (N_2/N)^{1/4-}$.
- finally, if $|\xi_1| \sim |\xi_2| \gtrsim N$, then
$$|M| \lesssim \frac{1}{m(\xi_1)m(\xi_2)} \lesssim \left(\frac{N_1}{N}\right)^{1/2-}.$$

Therefore, we can bound I as follows:

- When $|\xi_1| \ll |\xi_2|, |\xi_1| \ll N$, we have $|\xi_3| \sim |\xi_2| \gg |\xi_1|$. Thus, from the lemma 5.3,

$$
\begin{aligned}
I &\lesssim \frac{N_1}{N_2}\frac{1}{N_3^{1/2}}\|D_x^{1/2}w \cdot u\|_{L^2_{xt}}\|v\|_{L^2_{xt}} \\
&\lesssim N^{-1/2+}\delta^{1/2}N_{\max}^{0-}\|u\|_{X^{0,1/2+}}\|v\|_{H_t^{1/2+}L_x^2}\|w\|_{X^{0,1/2+}}
\end{aligned}
$$

- if $|\xi_2| \ll |\xi_1|, |\xi_2| \ll N$, we also have $|\xi_1| \sim |\xi_3|$; in this case, by duality and the bilinear estimate of proposition 3.2,

$$
\begin{aligned}
I &\lesssim \frac{N_2}{N_1}\|u\overline{w}\|_{H_t^{-1/2-}L_x^2}\|v\|_{H_t^{1/2+}L_x^2} \\
&\lesssim \frac{N_2}{N_1}\delta^{1/4-}\|u\overline{w}\|_{H_t^{-1/4+}L_x^2}\|v\|_{H_t^{1/2+}L_x^2} \\
&\lesssim \delta^{1/4-}\frac{N_2}{N_1}\|u\|_{X^{-1/4+,1/2+}}\|w\|_{X^{-1/4+,1/2+}}\|v\|_{H_t^{1/2+}L_x^2} \\
&\lesssim \delta^{1/4-}\frac{N_2}{N_1}\frac{1}{N_1^{1/2-}}\|u\|_{X^{0,1/2+}}\|w\|_{X^{0,1/2+}}\|v\|_{H_t^{1/2+}L_x^2} \\
&\lesssim N^{-1/2+}\delta^{1/4-}N_{\max}^{0-}\|u\|_{X^{0,1/2+}}\|v\|_{H_t^{1/2+}L_x^2}\|w\|_{X^{0,1/2+}}.
\end{aligned}
$$

- when $|\xi_1| \ll |\xi_2|, N \lesssim |\xi_1|$, we know that $|\xi_3| \sim |\xi_2| \gg |\xi_1|$, so that

$$
\begin{aligned}
I &\lesssim \left(\frac{N_1}{N_2}\right)^{1/4-}\frac{1}{N_3^{1/2}}\|D_x^{1/2}w \cdot u\|_{L^2}\|v\|_{L^2} \\
&\lesssim N^{-1/2+}\delta^{1/2}N_{\max}^{0-}\|u\|_{X^{0,1/2+}}\|v\|_{H_t^{1/2+}L_x^2}\|w\|_{X^{0,1/2+}}.
\end{aligned}
$$

- if $|\xi_2| \ll |\xi_1|, N \lesssim |\xi_2|$, we have $|\xi_1| \sim |\xi_3|$; thus,

$$
\begin{aligned}
I &\lesssim \left(\frac{N_2}{N}\right)^{1/4-}\|u\overline{w}\|_{H_t^{-1/2-}L_x^2}\|v\|_{H_t^{1/2+}L_x^2} \\
&\lesssim \left(\frac{N_2}{N}\right)^{1/4-}\delta^{1/4-}\|u\overline{w}\|_{H_t^{-1/4+}L_x^2}\|v\|_{H_t^{1/2+}L_x^2} \\
&\lesssim \delta^{1/4-}\left(\frac{N_2}{N}\right)^{1/4-}\|u\|_{X^{-1/4+,1/2+}}\|w\|_{X^{-1/4+,1/2+}}\|v\|_{H_t^{1/2+}L_x^2} \\
&\lesssim \delta^{1/4-}\left(\frac{N_2}{N}\right)^{1/4-}\frac{1}{N_1^{1/2-}}\|u\|_{X^{0,1/2+}}\|w\|_{X^{0,1/2+}}\|v\|_{H_t^{1/2+}L_x^2} \\
&\lesssim N^{-1/2+}\delta^{1/4-}N_{\max}^{0-}\|u\|_{X^{0,1/2+}}\|v\|_{H_t^{1/2+}L_x^2}\|w\|_{X^{0,1/2+}}.
\end{aligned}
$$

- finally, when $|\xi_1| \sim |\xi_2| \gtrsim N$, we have two possibilities: either $|\xi_1| \ll |\xi_3|$, so that

$$
\begin{aligned}
I &\lesssim \left(\frac{N_1}{N_2}\right)^{1/2-}\frac{1}{N_1^{1/2}}\|D_x^{1/2}u \cdot w\|_{L^2_{xt}}\|v\|_{L^2_{xt}} \\
&\lesssim N^{-1/2+}\delta^{1/2}N_{\max}^{0-}\|u\|_{X^{0,1/2+}}\|v\|_{H_t^{1/2+}L_x^2}\|w\|_{X^{0,1/2+}}
\end{aligned}
$$

or $|\xi_1| \sim |\xi_3|$ implying

$$\begin{aligned} I &\lesssim \frac{N_2}{N_1}\|u\overline{w}\|_{H_t^{-1/2-}L_x^2}\|v\|_{H_t^{1/2+}L_x^2} \\ &\lesssim \frac{N_2}{N_1}\delta^{1/4-}\|u\overline{w}\|_{H_t^{-1/4+}L_x^2}\|v\|_{H_t^{1/2+}L_x^2} \\ &\lesssim \delta^{1/4-}\frac{N_2}{N_1}\|u\|_{X^{-1/4+,1/2+}}\|w\|_{X^{-1/4+,1/2+}}\|v\|_{H_t^{1/2+}L_x^2} \\ &\lesssim \delta^{1/4-}\frac{N_2}{N_1}\frac{1}{N_1^{1/2-}}\|u\|_{X^{0,1/2+}}\|w\|_{X^{0,1/2+}}\|v\|_{H_t^{1/2+}L_x^2} \\ &\lesssim N^{-1/2+}\delta^{1/4-}N_{\max}^{0-}\|u\|_{X^{0,1/2+}}\|v\|_{H_t^{1/2+}L_x^2}\|w\|_{X^{0,1/2+}}. \end{aligned}$$

Hence, in any case, we proved that

$$I \lesssim N^{-1/2+}\delta^{1/4-}N_{\max}^{0-}\|u\|_{X^{0,1/2+}}\|v\|_{H_t^{1/2+}L_x^2}\|w\|_{X^{0,1/2+}}.$$

Summing up over the dyadic blocks, we complete the proof of the lemma. □

5.5. **Global existence.** Recall that $\|Iu_0\|_{L_x^2} \lesssim N^{-s}\|u_0\|_{H^s}$, $\|Iv_0\|_{L_x^2} \lesssim N^{-s}\|v_0\|_{H^s}$ and $\|Iu\|_{X^{0,b}} \lesssim N^{-s}\|u\|_{X^{0,b}}$. Applying the local result of proposition 5.2, we get the existence of solutions on a time interval $[0,\delta]$, where $\delta \sim N^{4s-}$. Also, they verify

$$\|Iu\|_{X^{0,1/2+}} + \|Iv\|_{X^{0,1/2+}} \lesssim N^{-s}.$$

By the lemma 5.4, we obtain

$$|E(Iu)(\delta) - E(Iu)(0)| \lesssim N^{-1/2+}\delta^{1/4-}N^{-3s}.$$

Hence, one can iterate the local result to cover the time interval $[0,T]$ if these estimates hold after T/δ steps. In other words, the existence of a solution on the time interval $[0,T]$ is guaranteed whenever

$$N^{-1/2+}\delta^{1/4-}N^{-3s}\frac{T}{\delta} \ll N^{-2s}.$$

So, it suffices that

$$-\frac{1}{2} + s - 3s - 4s < -2s,$$

i.e., $s > -1/8$. This completes the proof of theorem 1.3.

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