Bi-Hamiltonian Aspects of a Matrix Harry Dym Hierarchy

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Abstract

We study the Harry Dym hierarchy of nonlinear evolution equations from the bi-Hamiltonian view point. This is done by using the concept of an $\mathcal{S}$-hierarchy, which permits us to define a matrix Harry Dym hierarchy. We conclude by showing that the conserved densities of the matrix Harry Dym equation can be found by means of a Riccati-type equation.

1 Introduction

An intriguing equation known as the Harry Dym (HD) equation has attracted the attention of a number of researchers in integrable systems [7, 10, 11, 12, 19, 25, 26, 27]. In one of its incarnations it can be written as

$$q_t = 2(1/\sqrt{(1+q)})_{xxx} \tag{1}$$

or equivalently

$$\rho_t = \rho^3 \rho_{xxx} \tag{2}$$

after the substitution $\rho = -(1+q)^{-1/2}$.

Equation (1) was discovered in an unpublished work by Harry Dym [13], and appeared in a more general form in works of P. C. Sabatier [28, 29, 30]. More recently, its relations with the Kadomtsev-Petviashvili (KP) and modified-KP hierarchy have been studied in detail by Oevel and Carillo [18].

In the present work we discuss the HD hierarchy from the bi-Hamiltonian point of view and show that it is amenable to the systematic treatment developed in [3, 8, 9, 16, 17].

The plan of this article is the following:

In Section 2 we review the general definitions of Poisson geometry and bi-Hamiltonian theory. We review the important concept of an $\mathcal{S}$-hierarchy which was already used in [22] in connection with the Boussinesq equation.

Section 3 is devoted to endowing the loop-space on the Lie algebra of traceless 2×2 real matrices with a bi-Hamiltonian structure following a construction in [15].

Section 4 describes the construction of the matrix HD hierarchy, i.e., a hierarchy of commuting Hamiltonian flows in two fields that reduces to the Harry Dym equation (1) upon a suitable reduction. Two-component extensions of the HD equation have interested a number of researchers, see, e.g., [1, 2, 20, 24]. It would be interesting to compare the hierarchy presented herein with those presented by these authors.

We conclude in Section 5 with a Riccati type equation for the conserved quantities of the matrix HD hierarchy.

2 Bi-Hamiltonian preliminaries

This section collects a number of facts from bi-Hamiltonian geometry. More information could be found in [16].

A bi-Hamiltonian manifold is a triple $(\mathcal{M}, P_1, P_2)$ consisting of a manifold $\mathcal{M}$ and of two compatible Poisson tensors P_1 and P_2 on $\mathcal{M}$. In this context, we fix a symplectic leaf $\mathcal{S}$ of P_1 and consider the distribution $D = P_2(\mathrm{Ker} P_1)$ on $\mathcal{M}$. As it turns out, the distribution D is integrable. Furthermore, if $E = D \cap T\mathcal{S}$ is the distribution induced by D on $\mathcal{S}$ and the quotient space $\mathcal{N} = \mathcal{S}/E$ is a manifold, then it is a bi-Hamiltonian manifold. In situations where an explicit description of the quotient manifold $\mathcal{N}$ is not readily available, the following technique to compute the reduced bi-Hamiltonian structure is very useful [5]. Assume that $\mathcal{Q}$ is a submanifold of $\mathcal{S}$ that is transversal to the distribution E, in the sense that

$$T_p\mathcal{Q} \oplus E_p = T_p\mathcal{S} \qquad \text{for all } p \in \mathcal{Q} \ . \tag{3}$$

Then, $\mathcal{Q}$ is locally diffeomorphic to $\mathcal{N}$ and inherits a bi-Hamiltonian structure from $\mathcal{M}$. The reduced Poisson pair on $\mathcal{Q}$ is given by

$$\left(P_i^{\mathrm{rd}}\right)_p \alpha = \Pi_p\left((P_i)_p\tilde{\alpha}\right) \ , \qquad i = 1, 2 \ , \tag{4}$$

where $p \in \mathcal{Q}$, $\alpha \in T_p^*\mathcal{Q}$, the map $\Pi_p : T_p\mathcal{S} \to T_p\mathcal{Q}$ is the projection relative to (3), and $\tilde{\alpha} \in T_p^*\mathcal{M}$ satisfies

$$\tilde{\alpha}|_{D_p} = 0 \ , \qquad \tilde{\alpha}|_{T_p\mathcal{Q}} = \alpha \ . \tag{5}$$

Let us assume that $\{H_j\}_{j\in\mathbb{Z}}$ is a bi-Hamiltonian hierarchy on $\mathcal{M}$, that is, $P_2 dH_j = P_1 dH_{j+1}$ for all j. In other words, $H(\lambda) = \sum_{j\in\mathbb{Z}} H_j \lambda^{-j}$ is a (formal) Casimir of the *Poisson pencil* $P_2 - \lambda P_1$. The bi-Hamiltonian vector fields associated with the hierarchy can be reduced on the quotient manifold $\mathcal{N}$ according to

Proposition 1 *The functions H_j restricted to $\mathcal{S}$ are constant along the distribution E. Thus, they give rise to functions on $\mathcal{N}$. Such functions form a bi-Hamiltonian hierarchy with respect to the reduced Poisson pair. The vector fields $X_j = P_2 dH_j = P_1 dH_{j+1}$ are tangent to $\mathcal{S}$ and project on $\mathcal{N}$. Their projections are the vector fields associated with the reduced hierarchy.*

In the sequel, we shall need a more general definition than that of a bi-Hamiltonian hierarchy. The point being that, once we have fixed a symplectic leaf $\mathcal{S}$ of P_1, it is not always possible to determine a hierarchy on $\mathcal{M}$ that is defined also on $\mathcal{S}$. In other words, there exist singular leaves for the hierarchies of a bi-Hamiltonian manifold. Nevertheless, it is sometimes possible to define hierarchies which are, in a certain sense "local" on $\mathcal{S}$. We shall define an *$\mathcal{S}$–hierarchy* as a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of maps from $\mathcal{S}$ to $T^*\mathcal{M}$,

$$V_j : s \mapsto V_j(s) \in T_s^*\mathcal{M} \ ,$$

with the following properties:

- V_j restricted to $T\mathcal{S}$ is an exact 1-form, that is, there exist functions H_j on $\mathcal{S}$ such that $V_j|_{T\mathcal{S}} = dH_j$;

- $P_2 V_j = P_1 V_{j+1}$ for all $j \in \mathbb{Z}$.

Obviously, every bi-Hamiltonian hierarchy defined in a neighborhood of $\mathcal{S}$ gives rise to an $\mathcal{S}$–hierarchy. In contradistinction, in Section 4 we will see an example of $\mathcal{S}$–hierarchy that does not come from any bi-Hamiltonian hierarchy. This is also the case of the Boussinesq hierarchy [22].

It is not difficult to extend Proposition 1 to the case of $\mathcal{S}$–hierarchies. In the sequel, whenever talking about $\mathcal{S}$–hierarchies and referring to such result, it shall be understood that we mean such straightforward extension.

3 A bi-Hamiltonian structure on a loop-algebra

In this section we recall from [15] that the bi-Hamiltonian structure of the (usual) HD hierarchy can be obtained by means of a reduction.

Let $\mathcal{M} = C^\infty(S^1, \mathfrak{sl}(2))$ be the loop-space on the Lie algebra of traceless 2×2 real matrices, i.e., the space of C^∞ functions from the unit circle S^1 to $\mathfrak{sl}(2)$. The tangent space $T_S\mathcal{M}$ at $S \in \mathcal{M}$ is identified with $\mathcal{M}$ itself, and we will assume that $T_S\mathcal{M} \simeq T_S^*\mathcal{M}$ by the non-degenerate form

$$\langle V_1, V_2 \rangle = \int \operatorname{tr}(V_1(x) V_2(x))\, \mathrm{d}x, \qquad V_1, V_2 \in \mathcal{M} \ ,$$

where the integral is taken (here and throughout this article) on S^1. It is well-known [14] that the manifold $\mathcal{M}$ has a 3-parameter family of compatible Poisson tensors. To wit,

$$P_{(\lambda_1,\lambda_2,\lambda_3)} = \lambda_1 \partial_x + \lambda_2 [\,\cdot\,, S] + \lambda_3 [\,\cdot\,, A] \ , \tag{6}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, the matrix $A \in \mathfrak{sl}(2)$ is constant, and

$$S = \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \in \mathcal{M} \ .$$

In this paper we focus on the pencil

$$P_\lambda = P_2 - \lambda P_1 = \partial_x + [\,\cdot\,, A + \lambda S] \tag{7}$$

with

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ .$$

This means that

$$P_2 = \partial_x + [\,\cdot\,,\, A]\ , \qquad P_1 = [S,\, \cdot\,]\ . \tag{8}$$

In [15] the bi-Hamiltonian reduction procedure was applied to the pair (P_1, P_2). In this case,

$$D_S = \left\{ \begin{pmatrix} (\mu p)_x + \mu q & (\mu q)_x \\ (\mu r)_x - 2\mu p & -(\mu p)_x - \mu q \end{pmatrix} \mid \mu \in C^\infty(S^1, \mathbb{R}) \right\}\ , \qquad S \in \mathcal{M}.$$

The distribution D is not tangent to the generic symplectic leaf of P_1. However, it is tangent to the symplectic leaf

$$\mathcal{S} = \left\{ \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \mid p^2 + qr = 0, (p, q, r) \neq (0, 0, 0) \right\}\ , \tag{9}$$

so that $E_p = D_p \cap T_p\mathcal{S}$ coincides with D_p for all $p \in \mathcal{S}$. It is not difficult to prove that the submanifold

$$\mathcal{Q} = \left\{ S(q) = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \mid q \in C^\infty(S^1, \mathbb{R}), q(x) \neq 0\ \forall x \in S^1 \right\} \tag{10}$$

of $\mathcal{S}$ is transversal to the distribution E and that the projection $\Pi_{S(q)}$: $T_{S(q)}\mathcal{S} \to T_{S(q)}\mathcal{Q}$ is given by

$$\Pi_{S(q)} : (\dot{p}, \dot{q}) \mapsto (0, \dot{q} - \dot{p}_x)\ . \tag{11}$$

The reduced bi-Hamiltonian structure (4) coincides with the bi-Hamiltonian structure of the Harry Dym hierarchy (see [15] for details):

$$\begin{aligned} \left(P_1^{\mathrm{rd}}\right)_q &= -(2q\partial_x + q_x) \\ \left(P_2^{\mathrm{rd}}\right)_q &= -\frac{1}{2}\partial_x^3\ . \end{aligned}$$

Starting from the Casimir $\int \sqrt{q}\,\mathrm{d}x$ of P_1^{rd}, one constructs a bi-Hamiltonian hierarchy, which is called the HD hierarchy. We refer to [23] and the references therein for more details, and for a discussion about a "KP extension" of the HD hierarchy (see also [18]).

Remark 2 We take this opportunity for correcting a mistake in [23]. Equation (3.5) in that paper should be replaced with

$$K^{(2j+1)} = \lambda \left(-\frac{1}{2}(\lambda^j w)_{+,x} + k(\lambda^j w)_+ \right)\ . \tag{12}$$

This entails that the definition of the Faà di Bruno polynomials should be changed in the following way:

$$k^{(j+2)} = (\partial_x + k)^j \cdot \lambda \ , \qquad j \geq 0 \ .$$

The remaining of the paper should be changed accordingly.

We consider now the bi-Hamiltonian hierarchies of the Poisson pair (8), that is to say, the Casimirs of the Poisson pencil (7).

Let us suppose that

$$V = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

is a solution of $P_\lambda V = 0$, that is,

$$V_x + [V\, , A + \lambda S] = 0 \ , \tag{13}$$

and let us write the previous equation in componentwise form

$$\begin{cases} \alpha_x + (\lambda r + 1)\,\beta - \lambda q\,\gamma = 0 \\ \beta_x + 2\lambda q\,\alpha - 2\lambda p\,\beta = 0 \\ \gamma_x + 2\lambda p\,\gamma - 2(\lambda r + 1)\,\alpha = 0 \end{cases}$$

Upon expressing α and γ in terms of β,

$$\begin{cases} \alpha = -\dfrac{1}{2q}\big(-\dfrac{\beta_x}{\lambda} + 2\beta p \big) \\ \gamma = \dfrac{\beta_{xx}}{2\lambda^2 q^2} + \dfrac{\beta_x}{\lambda q^2}\big(p + \dfrac{q_x}{2\lambda q} \big) + \beta\big(\dfrac{p_x}{\lambda q^2} - \dfrac{q_x p}{\lambda q^3} + \dfrac{r}{q} + \dfrac{1}{\lambda q} \big) \end{cases} \tag{14}$$

we find that β satisfies the equation

$$-\frac{\beta_{xxx}}{2q^2\lambda^2} + \frac{3q_x}{2q^3\lambda^2}\beta_{xx} + \left(\frac{2}{q\lambda} + \frac{2p_x}{q^2\lambda} + \frac{2r}{q} - \frac{3q_x^2}{2q^4\lambda^2} - \frac{2q_x p}{q^3\lambda} + \frac{q_{xx}}{2q^3\lambda^2} + \frac{2p^2}{q^2} \right) \beta_x + \\ + \left(\frac{r_x}{q} - \frac{q_x}{q^2\lambda} + \frac{p_{xx}}{q^2\lambda} - \frac{3q_x p_x}{q^3\lambda} + \frac{3q_x^2 p}{q^4\lambda} - \frac{q_{xx} p}{q^3\lambda} - \frac{q_x r}{q^2} + \frac{2pp_x}{q^2} - \frac{2p^2 q_x}{q^3} \right) \beta = 0$$

This equation can be rewritten as

$$\frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}x}(\alpha^2 + \beta\gamma) = 0 \ . \tag{15}$$

Indeed, it is a well-known consequence of equation (13) that the spectrum of V does not depend on x, so that $\dfrac{\mathrm{d}}{\mathrm{d}x}\operatorname{tr} V^2 = 0$. Let us set

$$\operatorname{tr} \frac{V^2}{2} = F(\lambda) \ , \tag{16}$$

where $F(\lambda)_x = 0$. Then the equation for β becomes

$$2q\beta_{xx}\beta - q\beta_x^2 - 2q_x\beta\beta_x + 4(q_x p - q p_x - q^2)\beta^2\lambda - 4q(p^2+qr)\lambda^2\beta^2 + 4q^3F(\lambda)\lambda^2 = 0 \ . \tag{17}$$

We now consider the possibility of finding a solution $\beta(\lambda)$ of (17) as a formal series expansion in (negative) powers of λ, that is, $\beta = \sum_{i=-1}^{\infty} \beta_i\lambda^{-i}$. In order to find the coefficients β_i recursively, we must equate the coefficients of the same degree in λ starting from the highest order one. Let us suppose that $q(x) \neq 0$ for all x. Then, it turns out that we have to distinguish the two cases:

- If $p^2 + q\, r \neq 0$, then the highest degree of $F(\lambda)$ has to be even;
- If $p^2 + q\, r = 0$, then the highest degree of $F(\lambda)$ has to be even odd.

We are interested in the latter case, in order to perform the reduction process described in Section 2. This means that in this article, we will study the $\mathcal{S}$-hierarchy on the symplectic leaf (9). The bi-Hamiltonian hierarchy corresponding to the former case will not be considered here.

4 The matrix HD hierarchy

In this section we will show that it is possible to find a solution

$$V = \sum_{i=-1}^{\infty} V_i\lambda^{-i} = \sum_{i=-1}^{\infty} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & -\alpha_i \end{pmatrix} \lambda^{-i} \tag{18}$$

of equation (13) at the points of the symplectic leaf $\mathcal{S}$, giving rise to an $\mathcal{S}$-hierarchy, to be called the matrix HD hierarchy. We will see that the second vector field of such hierarchy projects to the HD equation.

First of all, we restrict to the symplectic leaf $\mathcal{S}$ and we use the now classical dressing transformation method [31, 6, 4] to show that the matrix

$V(\lambda)$ whose entries are given by the solution of (17) and (14) defines an $\mathcal{S}$-hierarchy if $F(\lambda)$ does not depend on the point of $\mathcal{S}$. Indeed, equation (16) implies that there exists a nonsingular matrix $K(\lambda)$ such that

$$V(\lambda) = K\Lambda K^{-1} \ ,$$

where

$$\Lambda = \begin{pmatrix} 0 & 1 \\ F(\lambda) & 0 \end{pmatrix} \ .$$

Let us introduce

$$M = K^{-1}(S + \frac{A}{\lambda})K - \frac{1}{\lambda}K^{-1}K_x \ . \tag{19}$$

Thus, we have

Proposition 3 *If $F(\lambda)$ does not depend on the point $S \in \mathcal{S}$, then $V(\lambda)$ restricted to $T\mathcal{S}$ is an exact 1-form. More precisely, if $H : \mathcal{S} \to \mathbb{R}$ is given by*

$$H(\lambda) = \int \mathrm{tr}\,(M\Lambda)\,\mathrm{d}x \ , \tag{20}$$

then $V|_{T\mathcal{S}} = \mathrm{d}H$.

Proof. If V is a solution of (13), then

$$\frac{1}{\lambda}K^{-1}V_xK + \frac{1}{\lambda}K^{-1}[V, A + \lambda S]K = 0 \ .$$

This in turn, implies that

$$\frac{1}{\lambda}\Lambda_x + [\Lambda, M] = 0 \ .$$

Since Λ does not depend on x, we have that Λ commutes with M. Therefore, for every tangent vector $\dot{S}$ to the symplectic leaf $\mathcal{S}$, we have

$$\begin{aligned} \langle \mathrm{d}H, \dot{S} \rangle &= \int \mathrm{tr}\,(\dot{M}\Lambda)\,\mathrm{d}x = \int \mathrm{tr}\,(K^{-1}\dot{S}K\Lambda) + \mathrm{tr}\,([M, K^{-1}\dot{K}]\Lambda)\,\mathrm{d}x \\ &= \int \mathrm{tr}\,(\dot{S}K\Lambda K^{-1})\,\mathrm{d}x = \int \mathrm{tr}\,(\dot{S}V)\,\mathrm{d}x = \langle V, \dot{S} \rangle \ , \end{aligned}$$

since $\int \mathrm{tr}\,([M, K^{-1}\dot{K}]\Lambda)\,\mathrm{d}x = 0$. This completes the proof. ∎

Let us now compute explicitly H. A possible choice for K is

$$K = \begin{pmatrix} \beta^{\frac{1}{2}} & 0 \\ -\alpha\beta^{-\frac{1}{2}} & \beta^{-\frac{1}{2}} \end{pmatrix} .$$

Since M commutes with Λ and both matrices have distinct eigenvalues, it follows that M is a polynomial of Λ. However, since they are traceless and we are working with 2×2 matrices it follows that M is a multiple of Λ. This simplifies the computation of M, since it becomes

$$M = \frac{q}{\beta}\Lambda .$$

Thus, we have that

$$H(\lambda) = \int 2\frac{q}{\beta}F(\lambda)\,\mathrm{d}x . \tag{21}$$

We define the *matrix HD hierarchy* to be the $\mathcal{S}$-hierarchy corresponding to the choice $F(\lambda) = \lambda$. In order to find its first vector fields, let us substitute $p^2 + qr = 0$ and $F(\lambda) = \lambda$ in equation (17), to find

$$2q\beta_{xx}\beta - q\beta_x^2 - 2q_x\beta\beta_x + 4(q_xp - qp_x - q^2)\beta^2\lambda + 4q^3\lambda^3 = 0 . \tag{22}$$

From now on, we use the functions p and q to describe a point of $\mathcal{S}$. We know that it is possible to solve equation (22) recursively, starting from the highest power of λ:

$$\lambda^3 : \qquad 4(q_xp - qp_x - q^2)\beta_{-1}^2 = -4q^3 .$$

We choose the positive solution

$$\beta_{-1} = \sqrt{\frac{q^3}{q^2 - q_xp + qp_x}} . \tag{23}$$

Using the expressions (14) for α and γ we get a recursive formula for the matrices V_i. Indeed, we have that

$$\begin{cases} \alpha_{-1} = p\dfrac{\beta_{-1}}{q} \\ \gamma_{-1} = r\dfrac{\beta_{-1}}{q} \end{cases} \tag{24}$$

and

$$\begin{cases} \alpha_i = \dfrac{1}{2q}(-(\beta_{i-1})_x + 2\beta_i p) \\ \gamma_i = \dfrac{1}{q}((\alpha_{i-1})_x + \beta_i r + \beta_{i-1}) \end{cases} \tag{25}$$

for all $i \geq 0$. Therefore, we can compute the first 1-form

$$V_{-1} = \begin{pmatrix} p & q \\ r & -p \end{pmatrix} \varphi(x) \ ,$$

where $\varphi(x) = \sqrt{\dfrac{q}{q^2 - q_x p + q p_x}}$, and verify immediately that it actually commutes with S, as expected.

Applying the Poisson tensor P_2 to V_{-1} we obtain the first vector field $X_0 := P_2(V_{-1}) = V_{-1_x} + [V_{-1}, A]$ of the hierarchy:

$$\begin{cases} \dot{p} = (p\varphi)_x + q\varphi \\ \dot{q} = (q\varphi)_x \end{cases} \tag{26}$$

We saw in Section 2 that every $\mathcal{S}$-hierarchy can be projected on the reduced bi-Hamiltonian manifold. Since V_{-1} belongs to the kernel of P_1, we have that $V_{-1}|_{T\mathcal{S}} = 0$ and that $P_2(V_{-1})$ belongs to the distribution D, so that the projection of X_0 vanishes. However, let us check it explicitly. We must evaluate X_0 at the points $p = 0$ of the transversal submanifold $\mathcal{Q}$, then we have to project this vector field according to the formula (11), thus obtaining the predicted result:

$$\frac{\partial q}{\partial t_0} = \dot{q} - \dot{p}_x = 0 \ .$$

The next step in the iteration is:

$$\lambda^2 : \qquad 2q\beta_{-1}\beta_{-1_{xx}} - q(\beta_{-1_x})^2 - 2q_x\beta_{-1}\beta_{-1_x} = 8(q^2 - q_x p_+ q p_x)\beta_{-1}\beta_0 \ .$$

Using also equation (23), we get that

$$\beta_0 = \frac{q\,\varphi}{8}\left(2q\varphi(q\varphi)_{xx} - (q\varphi)_x^2 - 2q_x\varphi(q\varphi)_x\right) \ , \tag{27}$$

and then

$$V_0 = \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & -\alpha_0 \end{pmatrix} \ ,$$

where

$$\begin{aligned}
\alpha_0 &= -\frac{\varphi_x}{2} - \frac{q_x}{2q}\varphi + \frac{p\varphi}{8}\left(2q\varphi(q\varphi)_{xx} - (q\varphi)_x^2 - 2q_x\varphi(q\varphi)_x\right) \\
\gamma_0 &= \frac{1}{q}(p\varphi)_x - \varphi - \frac{p^2\,\varphi^2}{4}(q\varphi)_{xx} + \frac{p^2\varphi}{8q}(q\varphi)_x^2 + \frac{p^2 q_x\,\varphi^2}{4q}(q\varphi)_x\ .
\end{aligned}$$

We can now determine the second vector field $X_1 := P_2(V_0) = V_{0x} + [V_0, A]$. It is given by

$$\left\{\begin{array}{l}
\dot{p} = -\frac{1}{2}\varphi_{xx} - \left(\frac{q_x}{2q}\varphi\right)_x + \left(\frac{p}{q}\beta_0\right)_x + \beta_0 \\
\dot{q} = \beta_{0x}
\end{array}\right. \tag{28}$$

The latter in turn, using equation (27), has the more explicit form

$$\left\{\begin{array}{rl}
\dot{p} = & -\frac{1}{2}\varphi_{xx} - \left(\frac{q_x}{2q}\varphi\right)_x + \left(\frac{1}{8\varphi} + \frac{pq_x\,\varphi}{8q} + \frac{p\,\varphi_x}{8}\right)(2q\varphi(2q\varphi)_{xx} - (q\varphi)_x^2 - 2q_x\varphi(q\varphi)_x) \\
& +\frac{p\varphi}{8}\left(2q\varphi(2q\varphi)_{xx} - (q\varphi)_x^2 - 2q_x\varphi(q\varphi)_x\right)_x \\
\\
\dot{q} = & \left(\frac{q\,\varphi}{8}\left(2q\varphi(2q\varphi)_{xx} - (q\varphi)_x^2 - 2q_x\varphi(q\varphi)_x\right)\right)_x
\end{array}\right. \tag{29}$$

Starting from (28), we calculate the reduced vector field, first evaluating X_1 at the points $p = 0$ of the transversal submanifold $\mathcal{Q}$,

$$\left\{\begin{array}{l}
\dot{p} = \frac{1}{2}\left(\frac{1}{\sqrt{q}}\right)_{xx} + \frac{q_x}{4q^2} + \frac{3}{32}\frac{q_x}{q^{\frac{5}{2}}} - \frac{q_{xx}}{8\,q^{\frac{3}{2}}} \\
\dot{q} = \left(-\frac{q_x}{4q^2} - \frac{3}{32}\frac{q_x}{q^{\frac{5}{2}}} + \frac{q_{xx}}{8\,q^{\frac{3}{2}}}\right)_x
\end{array}\right. \tag{30}$$

and then projecting this vector field on the transversal submanifold. We thus obtain the HD equation (1)

$$\frac{\partial q}{\partial t_1} = \dot{q} - \dot{p}_x = -\frac{1}{2}\left(\frac{1}{\sqrt{q}}\right)_{xxx}\ .$$

This equation is equivalent to Equation (1) after the change of variables $q \mapsto (1+q)$ and $t_1 \mapsto -4t$.

5 A Riccati equation for the conserved densities

The goal of this final section is to point out that the conserved densities of the matrix HD hierarchy can also be found by means of a Riccati-type equation.

We recall that equation (21) gives, for $F(\lambda) = \lambda$, the expression of the potential H of the 1-form $V|_{T\mathcal{S}}$. The corresponding density is clearly defined up to a total x-derivative. This fact allows us to introduce

$$h = \frac{q\lambda^{\frac{3}{2}}}{\beta} + \frac{\beta_x}{2\beta} \ ,$$

which transforms the equation (17) in the Riccati-type equation

$$h_x + h^2 - \frac{q_x}{q}h = \left(p_x + q - \frac{q_x}{q}p\right)\lambda \ . \tag{31}$$

Its solution h yields

$$H(\lambda) = \frac{2}{\sqrt{\lambda}} \int h \mathrm{d}x \tag{32}$$

for the functional H. We set $z = \sqrt{\lambda}$, and substitute $h = \sum_{i=-1}^{\infty} h_i z^{-i}$ in the Riccati equation (31), which takes the form

$$\sum_{i=-1}^{\infty}\left(h_{ix} + \sum_{j=0}^{1}(h_{i-j}h_i)\right)z^{-i} - \frac{q_x}{q}\sum_{i=-1}^{\infty} h_i z^{-i} = \left(p_x + q - \frac{q_x}{q}p\right)z^2 \ . \tag{33}$$

Once again, this equation can be solved recursively, starting from the highest degree of z. The firs step is

$$z^2 : \qquad h_{-1}^2 = p_x + q - \frac{q_x}{q}p \ \ ,$$

which gives, up to a sign,

$$h_{-1} = \sqrt{p_x + q - \frac{q_x}{q}p} \ .$$

Similarly, we have that

$$z^1 : \qquad h_{-1x} + 2h_{-1}h_0 - \frac{q_x}{q}h_{-1} = 0$$

from which we obtain

$$h_0 = -\frac{h_{-1x}}{2h_{-1}} + \frac{q_x}{2q} \ .$$

Let us notice that this is a total x−derivative. More generally, it is evident from (32) that every even densities is a total x−derivative. Indeed, $H(\lambda) = \sum_{i\geq 0} H_i\lambda^{-i}$, with

$$H_i = 2\int h_{2i-1}\mathrm{d}x \ . \tag{34}$$

In particular, $H_0 = 2\int h_{-1}\mathrm{d}x$, and it can be checked that $\mathrm{d}H_0 = V_0|_{T\mathcal{S}}$, as affirmed in Proposition 3.

The next equation is

$$z^0: \qquad h_{0x} + 2h_{-1}h_1 + h_0^2 - \frac{q_x}{q}h_0 = 0$$

and the corresponding density is

$$h_1 = -\frac{1}{2h_{-1}}\left(h_{0x} + h_0^2 + \frac{q_x}{q}h_0\right) \ .$$

This leads to

$$H_1 = 2\int\left(\frac{h_{-1xx}}{4h_{-1}^2} - \frac{3h_{-1x}^{\ 2}}{8h_{-1}^3} - \frac{q_{xx}}{4h_{-1}q} + \frac{3q_x^2}{8h_{-1}q^2}\right)\mathrm{d}x \ .$$

Integrating by parts and substituting the expression for h_1, we find

$$H_1 = 2\int\left(\frac{\left(p_{xx} + q_x - \left(\frac{pq_x}{q}\right)_x\right)^2}{32\left(p_x + q - \frac{pq_x}{q}\right)^{\frac{5}{2}}} - \frac{q_{xx}}{4q\sqrt{p_x + q - \frac{pq_x}{q}}} + \frac{3q_x^2}{8\,q^2\sqrt{p_x + q - \frac{pq_x}{q}}}\right)\mathrm{d}x \ .$$

This is the Hamiltonian (with respect to the symplectic structure obtained by restricting P_1 to its symplectic leaf $\mathcal{S}$) of the 2-component extension (29) of the HD equation.

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