PERIODIC PULSES OF COUPLED NONLINEAR SCHRÖDINGER EQUATIONS IN OPTICS

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Abstract. A system of coupled nonlinear Schrödinger equations arising in nonlinear optics is considered. The existence of periodic pulses as well as the stability and instability of such solutions are studied. It is shown the existence of a smooth curve of periodic pulses that are of cnoidal type. The Grillakis, Shatah and Strauss theory is set forward to prove the stability results. Regarding instability a general criteria introduced by Grillakis and Jones is used. The well-posedness of the periodic boundary value problem is also studied. Results in the same spirit of the ones obtained for single quadratic semilinear Schrödinger equation by Kenig, Ponce and Vega are established.

1. Introduction

The interest on nonlinear properties of optical materials have attracted the attention of Physicists and Mathematicians in the recent years. It has been suggested that by exploiting the nonlinear response of matter, the bit-rate capacity of optical fibres can be increased substantially and so it will allow a great improvement in the speed and economy of data transmission and manipulation.

In non-centrosymmetric materials, i.e., those which do not posses inversion symmetry at the molecular level, the lowest order nonlinear effects originate from the second-order susceptibility $\chi^{(2)}$; this means that the nonlinear response of the matter to the electric field is quadratic (see [10], [20]). Quadratic nonlinearities are long known to be responsible for phenomena such as "second-harmonic generation" (frequency doubling), whereby laser

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light with frequency ω can be partially converted to light of frequency 2ω upon passing it through a crystal with $\chi^{(2)}$ response (see [25]). So, such materials are of importance in parametric wave interactions, in ultra-fast all-optical signal processing, as well as long-distance communications (see [12], [25] for more physics or engineering information on $\chi^{(2)}$).

The phenomena of interest in two dimensions, space + time, (pulse propagation in fibres) is described by the following system of two coupled nonlinear Schrödinger equations

$$\begin{cases} iw_t + rw_{xx} - \theta w + \bar{w}v = 0, \\ i\sigma v_t + sv_{xx} - \alpha v + \frac{1}{2}w^2 = 0. \end{cases} \tag{1.1}$$

This is obtained from the basic $\chi^{(2)}$ second-harmonic generation equations (SHG) of type I (see [25]). The complex functions $w = w(x,t)$ and $v = v(x,t)$ represent respectively the envelopes amplitudes of the first and second harmonics of an optical wave. So, (1.1) describes the interaction of these harmonics. We have $r, s = \pm 1$. The signs of r and s are determined by the signs of the dispersions/diffractions (temporal/spatial cases, respectively). The constant σ measures the ratios of the dispersions/diffractions. The real parameters θ and α are dimensionless, with α incorporating the wave-vector mismatch between the two harmonics ([4], [6]).

An important issue for optical communication in a nonlinear regime is the understanding of the so-called, "solitary-waves": standing or travelling waves, which are localized solutions for (1.1) of the form

$$w(x,t) = e^{i\gamma t}\phi(x), \qquad v(x,t) = e^{2i\gamma t}\psi(x) \tag{1.2}$$

where $\phi, \psi : \mathbb{R} \mapsto \mathbb{R}$. When we specify the boundary conditions $\phi, \psi \to 0$ as $|x| \to +\infty$, these solutions are called "*pulses*". Here we are interested in "*periodic pulses*", namely, ϕ, ψ that satisfy periodic boundary conditions $\phi^{(n)}(0) = \phi^{(n)}(L)$, $\psi^{(n)}(0) = \psi^{(n)}(L)$, for every $n \in \mathbb{N}$ and fixed period L.

In the case $r = s = 1$ and $\theta, \alpha > 0$, which is the most interesting regime from a physics and engineering viewpoint, the pulses satisfy the ordinary differential equations

$$\begin{cases} -\phi'' + \theta_0\phi - \phi\psi = 0, \\ -\psi'' + \alpha_0\psi - \frac{1}{2}\phi^2 = 0, \end{cases} \tag{1.3}$$

with

$$\theta_0 = \theta + \gamma \quad \text{and} \quad \alpha_0 = \alpha + 2\sigma\gamma.$$

It is well known that for the explicit value of $\theta_0 = \alpha_0 = \pm 1$ and $\phi = \pm\sqrt{2}\psi$, system (1.3) possesses the exact real pulse solutions

$$\phi(x) = \pm\frac{3}{\sqrt{2}}\,\mathrm{sech}^2\Big(\frac{x}{2}\Big), \qquad \psi(x) = \pm\frac{3}{2}\,\mathrm{sech}^2\Big(\frac{x}{2}\Big), \tag{1.4}$$

found by a number of authors ([6], [20]). In [8], by using methods of the calculus of variations (the mountain pass theorem and concentration-compactness arguments), the existence of pulses was proved for all values of $\alpha_0 > 0$ and $\theta_0 = 1$. Regarding the *shape* of solutions for (1.3) some numerical ([11], [9], [8], [6]) and analytic results ([8], [27]) have been obtained. In [27] a description of the profile of solutions for (1.3) was given by using the framework of homoclinic bifurcation theory. Here the existence and uniqueness up to reflection $((\phi,\psi) \mapsto (-\phi,\psi))$, of solutions which possess a multiple number of "humps" (peaks or troughs), called "*multipulses*" or "*N-pulses*", was proved for $\alpha_0 < \theta_0$ and α_0 sufficiently near to θ_0. These solutions were generated from a homoclinic bifurcation arising near a semi-simple eigenvalue scenario. For $\alpha_0 \geqq \theta_0$ and α_0 close to θ_0, it was also proved that multipulses solutions do not exist (see [11], [27] for numerical simulations for the existence of multipulses). We also note that in [27] it was shown the existence of a C^1 branch of 1-pulses for (1.3) parameterized by α_0, for α_0 close to 1, which contains the explicit solution (1.4) at $\alpha_0 = 1$. In [28], the stability and instability of the orbit generated by 1-pulses or multipulses (ϕ,ψ) of (1.3) found in [27], namely,

$$\mathcal{O}_{(\phi,\psi)} = \{(e^{is}\phi(x+x_0), e^{2is}\psi(x+x_0))|\; x_0, s \in \mathbb{R}\}, \tag{1.5}$$

were studied. Through the use of the Grillakis, Shatah and Strauss theory ([16],[17]), conditions were derived for the nonlinear stability or instability of the 1-pulses. Moreover, by the application of an instability criterion due to Grillakis [15] (see also Grillakis [14] and Jones [19]), it was proved the remarkable fact that the N-pulses are unstable by the flow of the coupled nonlinear Schrödinger system (1.1).

In this paper our main interest is the study of the existence, stability and instability of "periodic pulses" of (1.1).

For the parameter regime $\theta_0 = \alpha_0$, we establish the existence of a family of non-trivial periodic solutions of (1.3). More precisely, Let $\theta > 0$ fixed, under the following conditions

$$\begin{aligned} &\text{i)}\ \ \gamma > \frac{4\pi^2}{L^2} - \theta, \\ &\text{ii)}\ \ \alpha, \sigma > 0 \ \text{ such that } \ \alpha + 2\sigma\gamma = \theta + \gamma, \ \text{ and} \\ &\text{iii)}\ \ \phi = \sqrt{2}\,\psi, \end{aligned} \tag{1.6}$$

we obtain $\psi = \psi_\gamma$ satisfying the differential equation

$$\psi''(\xi) - (\theta + \gamma)\psi(\xi) + \psi^2(\xi) = 0, \quad \xi \in \mathbb{R}, \tag{1.7}$$

such that

$$\gamma \in (\frac{4\pi^2}{L^2} - \theta, +\infty) \mapsto \psi_\gamma \in H^1_{per}([0, L])$$

is a smooth branch of solutions. Moreover, the profile of each ψ_γ is a **cnoidal wave**, that is,

$$\psi(\xi) = \psi(\xi; \beta_1, \beta_2, \beta_3) = \beta_2 + (\beta_3 - \beta_2)\mathrm{cn}^2\Big[\sqrt{\frac{\beta_3 - \beta_1}{6}}\,\xi; k\Big], \tag{1.8}$$

where $\mathrm{cn}(\cdot; k)$ represents a Jacobi elliptic function of *modulus* k, the β_i's are smooth function of γ satisfying $\beta_1 < 0 < \beta_2 < \beta_3$, $\Sigma\beta_i = 3(\theta + \gamma)/2$ and

$$k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}.$$

This family of solutions for (1.8) are positive periodic pulses, which are even, monotonically decreasing between the maximum $\psi(0) = \beta_3$ (humps) and the positive minimum $\psi(\frac{L}{2}) = \beta_2$ (troughs).

We also will show the existence of other periodic solutions for the system (3.2) depending on the parameter α.

Concerning the nonlinear stability of the orbit (1.5), we show that it is stable in $H^1_{per}([0, L]) \times H^1_{per}([0, L])$ by the periodic flow of the system (1.1). We derive our result from the Grillakis, Shatah and Strauss theory ([16]) and the Floquet theory applied to the periodic eigenvalue problem for the Jacobian form of *Lamé's equation*

$$\begin{cases} \dfrac{d^2\Lambda}{dx^2} + [\rho - 12k^2\,\mathrm{sn}^2(x; k)]\Lambda = 0 \\ \Lambda(0) = \Lambda(2K), \ \ \Lambda'(0) = \Lambda'(2K), \end{cases} \tag{1.9}$$

where $\mathrm{sn}(\cdot; k)$ is a Jacobi elliptic function and $K = K(k)$ is the complete elliptic integral of the first kind.

We also show that the orbit

$$\mathcal{S}_{(\phi,\psi)} = \{(\sqrt{2}e^{i\gamma s}\psi_\gamma(x), e^{2i\gamma s}\psi_\gamma(x))|\ s \in \mathbb{R}\}, \tag{1.10}$$

is unstable in $H^1_{per}([0,2L]) \times H^1_{per}([0,2L])$. To obtain this result we employ the theory of Grillakis in [15]. This seems to be the first proof of instability of periodic pulses in a nonlinear Schrödinger-type system (see Angulo [3] for the study of the instability of periodic travelling waves solutions in the case of the focusing cubic Schrödinger equation).

In the framework of Grillakis-Shatah-Strauss, it is needed to have global or local well-posedness for the system under consideration. In our case, it will be enough to have a well-posedness theory in the spaces $H^s([0,L]) \times H^s([0,L])$ for $s \geq 1$ due to the conserved quantities

$$\mathcal{F}(t) := \int \left[|w(x,t)|^2 + 2\sigma|v(x,t)|^2\right] dx = \mathcal{F}(0) \tag{1.11}$$

and

$$\begin{aligned}\mathcal{H}(t) := \int \big[& r|w_x(x,t)|^2 + s|v_x(x,t)|^2 \\ & + \theta|w(x,t)|^2 + \alpha|v(x,t)|^2 - \Re(w^2\bar{v})(x,t)\big]\, dx = \mathcal{H}(0)\end{aligned} \tag{1.12}$$

where $\Re$ denotes the real part.

Here we establish a local and global theory for the periodic IVP (1.1) in $H^s([0,L]) \times H^s([0,L])$ for $s \geq 0$. To prove the local result we use the Fourier restriction spaces or $X_{s,b}$ spaces introduced by Bourgain in [5] and bilinear estimates introduced by Kenig, Ponce and Vega [22] to study the IVP associated to the Korteweg-de Vries equation. More precisely, we will use the approach given by Kenig, Ponce and Vega [23] to study the following nonlinear Schrödinger equations,

$$\partial_t u = i\partial_x^2 u + N_j(u,\overline{u}), \qquad x \in \mathbb{R}\ (\mathbb{T}),\ t \in \mathbb{R}, \tag{1.13}$$

where $N_j(u,\overline{u})$, $j = 1,2,3$, is a quadratic polynomial, i.e. $N_1(u,\overline{u}) = u^2$, $N_2(u,\overline{u}) = u\overline{u}$, and $N_3(u,\overline{u}) = \overline{u}^2$.

In this work we are interested in the nonlinearities N_1 and N_2. To explain the results in [23] regarding these nonlinearities we need the next definition.

Definition 1.1. *Let $\mathcal{A}$ be space of functions f such that*

(i) $f : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$.

(ii) $f(x,\cdot) \in \mathcal{S}$ *for each* $x \in \mathbb{T}$.

(iii) $f(\cdot,t)\in C^\infty(\mathbb{T})$ *for each* $t\in\mathbb{R}$.

For $s,b\in\mathbb{R}$ *we define the space* $Y_{s,b}$ *to be the completion of* $\mathcal{A}$ *with respect to the norm*

$$\|F\|_{Y_{s,b}}=\|\langle n\rangle^s\langle\tau-n^2\rangle^b\,\widehat{F}(n,\tau)\|_{\ell^2_nL^2_\tau}. \tag{1.14}$$

For $F\in Y_{s,b}$ consider the bilinear operators

$$B_1(F,F)=F^2 \tag{1.15}$$

and

$$B_2(F,F)=F\overline{F}. \tag{1.16}$$

Kenig, Ponce and Vega in [23] showed that given $s\in(-1/2,0]$ there exists $b\in(1/2,1)$ such that

$$\|B_1(F,F)\|_{Y_{s,b-1}}\le c\|F\|^2_{Y_{s,b}} \tag{1.17}$$

and that for $s<-1/2$ and any $b\in\mathbb{R}$ the estimate (1.17) fails.

On the other hand, given any $s<0$ and any $b\in\mathbb{R}$ they showed that the estimate

$$\|B_2(F,F)\|_{Y_{s,b-1}}\le c\|F\|^2_{Y_{s,b}} \tag{1.18}$$

fails.

These estimates yield sharp local well-posedness for the periodic boundary value problem associated to (1.13) for data in $H^s(\mathbb{T})$, $s>-1/2$ when the nonlinearity is N_1 and in $H^s(\mathbb{T})$, $s\geqq 0$, for the nonlinearity N_2.

When $\sigma=1$, we can reproduce the estimates (1.17) and (1.18) for any $s\ge 0$ and some $b\in(1/2,1)$ for system (1.1). These are the main estimates to obtain our local results. We shall observe that for $\sigma\neq 1$ we can prove estimates (1.17) and (1.18) for $s>-1/2$ and some $b\in(1/2,1)$. This latter result will appear somewhere else.

To establish global results $H^s(\mathbb{T})\times H^s(\mathbb{T})$, $s\ge 0$ it is sufficient to use the local theory and the conserved quantity (1.11).

The plan of this paper is as follows: in Section 2, we establish the local and global theory for the periodic boundary value problem associated to (1.1). The existence of periodic pulses of the kind described in (1.2) will be shown in Section 3. Next we will show the existence of periodic solutions for the system (1.3) which are not necessarily of cnoidal type. The stability of the periodic pulse will be discussed in Section 5. Finally, in Section 6, the instability results will be established.

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Before leaving this section we want to introduce some notation needed along this work.

Notation. For any complex number $z \in \mathbb{C}$, we denote by $\Re z$ and $\Im z$ the real part and imaginary part of z, respectively.

For $s \in \mathbb{R}$, the Sobolev space $H^s_{\mathrm{per}}([0,L])$ consists of all periodic distributions f such that $\|f\|^2_{H^s} = L \sum_{k=-\infty}^{\infty} (1+k^2)^s |\widehat{f}(k)|^2 < \infty$. For simplicity, we will use the notation $H^s([0,L])$ in several places.

We will use $F(\varphi,k)$ to denote the normal elliptic integral of first type (see [7]), that is, for $y = \sin\varphi$

$$\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\varphi,k). \tag{1.19}$$

The normal elliptic integral of the second type, i.e.

$$\int_0^y \sqrt{\frac{1-k^2t^2}{1-t^2}}\, dt = \int_0^\varphi \sqrt{1-k^2\sin^2\theta}\, d\theta \tag{1.20}$$

will be denoted by $E(\varphi,k)$. In both cases $k \in (0,1)$ is called the modulus and φ the argument. When $y=1$, we denote $F(\pi/2,k)$ and $E(\pi/2,k)$ by $K = K(k)$ and $E = E(k)$, respectively.

The Jacobian elliptic functions denoted by $\mathrm{sn}(u;k)$, $\mathrm{cn}(u;k)$ and $\mathrm{dn}(u;k)$, respectively, are defined via the previous elliptic integrals. More precisely, let

$$u(y_1;k) := u = F(\varphi,k), \tag{1.21}$$

then $y_1 = \sin\varphi := \mathrm{sn}(u;k) = \mathrm{sn}(u)$ and

$$\begin{aligned} \mathrm{cn}(u;k) &:= \sqrt{1-y_1^2} = \sqrt{1-\mathrm{sn}^2(u;k)}, \\ \mathrm{dn}(u;k) &:= \sqrt{1-k^2y_1^2} = \sqrt{1-k^2\mathrm{sn}^2(u;k)}, \end{aligned} \tag{1.22}$$

requiring that $\mathrm{sn}(0;k) = 0$, $\mathrm{cn}(0;k) = 1$ and $\mathrm{dn}(0;k) = 1$.

2. Well-posedness Theory

In this section we show local and global results for the periodic IVP,

$$\begin{cases} iw_t + w_{xx} - \theta\, w + \overline{w} v = 0 \qquad x \in [0,L],\ t \in \mathbb{R}, \\ i\sigma v_t + v_{xx} - \tilde{\alpha}\, v + \frac{1}{2} w^2 = 0, \\ w(x,0) = w_0(x), \ \ v(x,0) = v_0(x) \end{cases} \tag{2.1}$$

where $\theta, \tilde{\alpha} \in \mathbb{R}$ and $\sigma > 0$, in the periodic Sobolev space $H^s([0,L]) \times H^s([0,L])$. To simplify our analysis we will use $L = 2\pi$.

We first rewrite (2.1) as

$$\begin{cases} iw_t + w_{xx} - \theta\, w + \overline{w} v = 0 \qquad x \in [0,L],\ t \in \mathbb{R}, \\ iv_t + a\, v_{xx} - \alpha\, v + \frac{a}{2} w^2 = 0, \\ w(x,0) = w_0(x), \ \ v(x,0) = v_0(x) \end{cases} \tag{2.2}$$

where $a = 1/\sigma$ and $\alpha = \tilde{\alpha}/\sigma$.

Next we consider the equivalent integral system of equations associated to (2.2). Let ψ be a $C_0^\infty(\mathbb{R})$ function with $\operatorname{supp}\psi \subset (-2,2)$ such that $\psi(t) = 1$, for $t \in [-1,1]$. Let $\psi_T(\cdot) = \psi(\cdot/T)$.

$$\begin{cases} w(t) = \psi_T W(t) w_0 - i\, \psi_T \displaystyle\int_0^t W(t-t')\overline{w}\, v(t')\, dt' \\ v(t) = \psi_T V(t) v_0 - \frac{ia}{2}\, \psi_T \displaystyle\int_0^t V(t-t') w^2(t')\, dt', \end{cases} \tag{2.3}$$

where $W(t) = e^{it(\partial_x^2 + \theta)}$ and $V(t) = e^{it(a\partial_x^2 + \alpha)}$ are the corresponding Schrödinger generators (unitary groups) associated to the linear problem.

To give the statement of our results we need the following definition.

Definition 2.1. *Let $\mathcal{A}$ be space of functions f such that*

(i) $f : [0,L] \times \mathbb{R} \to \mathbb{C}$.
(ii) $f(x,\cdot) \in \mathcal{S}$ *for each* $x \in [0,L]$.
(iii) $f(\cdot,t) \in C^\infty([0,L])$ *for each* $t \in \mathbb{R}$.

For $s \in \mathbb{R}$ we define the space $X_{s,b}$ to be the completion of $\mathcal{A}$ with respect to the norm

$$\|f\|_{X_{s,b}} = \| \langle n \rangle^s \langle \tau - n^2 - \theta \rangle^b\, \widehat{f}(n,\tau) \|_{\ell_n^2 L_\tau^2}. \tag{2.4}$$

Similarly, for $s \in \mathbb{R}$ and $a > 0$ we define the space $X^a_{s,b}$ to be the completion of $\mathcal{A}$ with respect to the norm

$$\|f\|_{X^a_{s,b}} = \|\langle n\rangle^s \langle \tau - a\, n^2 - \alpha\rangle^b \widehat{f}(n,\tau)\|_{\ell^2_n L^2_\tau}. \tag{2.5}$$

The local well-posedness theory is as follows.

Theorem 2.2. *Let $s \geq 0$, $a > 0$ and $b > 1/2$. For any $(w_0, v_0) \in H^s([0,L]) \times H^s([0,L])$, there exist $T = T(\|(w_0,v_0)\|_{H^s \times H^s}) > 0$, and a unique solution of the IVP (2.2) in the time interval $[-T,T]$ such that*

$$(w,v) \in C([-T,T] : H^s([0,L]) \times H^s([0,L])), \quad (\varphi_T w,\, \varphi_T v) \in X_{s,b} \times X^a_{s,b}. \tag{2.6}$$

Moreover, for any $T' \in (0,T)$, the map $(w_0,v_0) \mapsto (w(t),v(t))$ is Lipschitz from a neighborhood of $H^s([0,L]) \times H^s([0,L])$ to $C([-T,T] : H^s([0,L]) \times H^s([0,L])) \cap X_{s,b} \times X^a_{s,b}$.

Remark 2.3. *If $a \neq 1$ the above result in Theorem 2.2 holds for $s > -1/2$.*

Once we have proved Theorem 2.2 it is not difficult to show the next global result.

Theorem 2.4. *Let $(w_0,v_0) \in H^s([0,L]) \times H^s([0,L])$, $s \geq 0$. Then the solutions (w,v) given in Theorem 2.2 can be extended to any interval of time.*

Proof of Theorem 2.4. The result is deduced using the conserved quantity

$$\int \left(|w(x,t)|^2 + 2\sigma |v(x,t)|^2\right) dx = \int \left(|w_0(x)|^2 + 2\sigma |v_0(x)|^2\right) dx. \tag{2.7}$$

and Theorem 2.2. □

To establish Theorem 2.2 we need a series of lemmas. We begin with the next result.

Lemma 2.5. *Let $s \in \mathbb{R}$, $b > 1/2$, then*

$$\|\psi_T W(t) w_0\|_{X_{s,b}} \leq c\, \|w_0\|_{H^s}, \tag{2.8}$$

$$\|\psi_T V(t) v_0\|_{X^a_{s,b}} \leq c\, \|v_0\|_{H^s}, \tag{2.9}$$

and

$$\|\psi_T \int_0^t W(t-t') F(t')\, dt'\|_{X_{s,b}} \leq c T^\gamma\, \|F\|_{X_{s,b-1}}, \tag{2.10}$$

$$\|\psi_T \int_0^t V(t-t')F(t')\,dt'\|_{X^a_{s,b}} \leq cT^\gamma \,\|F\|_{X^a_{s,b-1}} \tag{2.11}$$

where $W(t)$ and $V(t)$ are defined above and $\gamma > 0$.

Proof. For a proof of this see for instance [5], [23]. □

The key estimates to deal with the nonlinear terms are next.

Lemma 2.6. *For $s \geq 0$ and $a > 0$ we have*

$$\|\,\overline{w}\,v\|_{X_{s,-1/2}} \leq c\,\|w\|_{X_{s,1/2}}\|v\|_{X^a_{s,1/2}}. \tag{2.12}$$

and

$$\|w^2\|_{X^a_{s,-1/2}} \leq c\,\|w\|^2_{X_{s,1/2}}. \tag{2.13}$$

Remark 2.7. *If $a \neq 1$ the estimate (2.12) holds for $s > -1/2$. For any $a > 0$, the estimate (2.13) is satisfied for $s > -1/2$.*

As in [23] the next corollary follows from the proof of Lemma 2.6.

Corollary 2.8. *Let $b > 1/2$ with $1-b$, $b' > 3/8$, then*

$$\|\,\overline{w}\,v\|_{X_{s,1-b}} \leq c\,\|w\|_{X_{s,b'}}\|v\|_{X^a_{s,b'}}, \tag{2.14}$$

and

$$\|w^2\|_{X^a_{s,1-b}} \leq c\,\|w\|^2_{X_{s,b'}}. \tag{2.15}$$

The next lemmas will be useful in the proof of Lemma 2.6. The first one was proved in [23], that is,

Lemma 2.9. *If $\gamma > 1/2$. Then*

$$\sup_{n\in\mathbb{Z},\,\tau\in\mathbb{R}} \sum_{n_1\in\mathbb{Z}} \frac{1}{(1+|\tau \pm n_1(n-n_1)|)^\gamma} < \infty. \tag{2.16}$$

Lemma 2.10. *Let $s \geq 0$, and let*

$$A(n,\tau,a) := \frac{\langle n\rangle^s}{\langle \tau - n^2-\theta\rangle^{1/2}}\Big(\sum_{n_1\in\mathbb{Z}}\int_{-\infty}^{\infty} \frac{\langle n_1\rangle^{-2s}\langle n-n_1\rangle^{-2s}}{\langle \tau_1 - an_1^2-\alpha\rangle\langle \tau-\tau_1+(n-n_1)^2+\theta\rangle}\,d\tau_1\Big)^{1/2},$$

$$A_1(n_1,\tau_1) := \frac{1}{\langle n_1\rangle^s\langle \tau_1 - n_1^2-\alpha\rangle^{1/2}}\Big(\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty} \frac{\langle n\rangle^{2s}\langle n-n_1\rangle^{-2s}}{\langle \tau - n^2-\theta\rangle\langle \tau-\tau_1+(n-n_1)^2+\theta\rangle}\,d\tau\Big)^{1/2},$$

and

$$B(n,\tau,a):=\frac{\langle n\rangle^s}{\langle \tau-an^2-\alpha\rangle^{1/2}}\Big(\sum_{n_1\in\mathbb{Z}}\int_{-\infty}^{\infty}\frac{\langle n_1\rangle^{-2s}\langle n-n_1\rangle^{-2s}}{\langle \tau_1-n_1^2-\theta\rangle\langle \tau-\tau_1-(n-n_1)^2-\theta\rangle}\,d\tau_1\Big)^{1/2}.$$

Then

$$\sup_{n\in\mathbb{Z},\,\tau\in\mathbb{R}} A(n,\tau,a)\le c, \tag{2.17}$$

$$\sup_{n_1\in\mathbb{Z},\,\tau_1\in\mathbb{R}} A_1(n_1,\tau_1)\le c, \tag{2.18}$$

and

$$\sup_{n\in\mathbb{Z},\,\tau\in\mathbb{R}} B(n,\tau,a)\le c. \tag{2.19}$$

Proof. We will only give an sketch of the proof of inequality (2.17). The proofs of the estimates (2.18) and (2.19) follow a similar argument so we will omit them.

Using that $s\ge 0$ and the change of variables $x=\tau_1-an_1^2-\alpha$ we obtain

$$\begin{aligned}A(n,\tau,a)&\le\frac{c}{\langle \tau-n^2-\theta\rangle^{1/2}}\Big(\sum_{n_1\in\mathbb{Z}}\int_{-\infty}^{\infty}\frac{d\tau_1}{\langle \tau_1-an_1^2-\alpha\rangle\langle \tau-\tau_1+(n-n_1)^2+\theta\rangle}\Big)^{1/2}\\&\le\sup_{n\in\mathbb{Z},\tau\in\mathbb{R}}\Big(\sum_{n_1\in\mathbb{Z}}\frac{\ln(2+|\tau+n^2-n_1(2n-(1-a)n_1)+(\theta-\alpha)|)}{1+|\tau+n^2-n_1(2n-(1-a)n_1)+(\theta-\alpha)|}\Big)^{1/2}.\end{aligned}\tag{2.20}$$

An application of Lemma 2.9 yields the result. □

Proof of Lemma 2.6. We begin by proving the bilinear estimate (2.12).

We consider first the case $a=1$. Let $f(n,\tau)=\langle n\rangle^s\langle \tau-an^2-\alpha\rangle^{1/2}|\widehat{v}(n,\tau)|$ and $g(n,\tau)=\langle n\rangle^s\langle \tau+n^2+\theta\rangle^{1/2}|\widehat{\overline{w}}(n,\tau)|$.

Then from the definition (2.1), the Cauchy-Schwarz inequality, Fubini's theorem and (2.17) in Lemma 2.10 it follows that,

$$\begin{aligned}\|\,\overline{w}\,v\|_{X_{s,-1/2}}&\le\sup_{n\in\mathbb{Z},\,\tau\in\mathbb{R}}A(n,\tau,a)\|f\|_{\ell^2_nL^2_\tau}\|g\|_{\ell^2_nL^2_\tau}\\&\le c\,\|w\|_{X_{s,1/2}}\|v\|_{X^a_{s,1/2}}.\end{aligned}\tag{2.21}$$

Now we consider $a = 1$. A duality argument combined with Lemma 2.10 gives

$$\begin{aligned}
\|\,\overline{w}\,v\|_{X_{s,-1/2}} &\le \sum_{n\in\mathbb{Z}}\langle n\rangle^s\int_{-\infty}^{\infty}\frac{h(n,\tau)\,d\tau}{\langle \tau-n^2-\theta\rangle}\\
&\times\Big(\sum_{n_1\in\mathbb{Z}}\int_{-\infty}^{\infty}\frac{g(n-n_1,\tau-\tau_1)f(n_1,\tau_1)}{\langle n-n_1\rangle^s\langle \tau-\tau_1+(n-n_1)^2+\theta\rangle^{1/2}\langle n_1\rangle^s\langle \tau_1-n_1^2-\alpha\rangle^{1/2}}\,d\tau_1 \qquad (2.22)\\
&\le \sup_{n_1\in\mathbb{Z},\,\tau_1\in\mathbb{R}} A_1(n_1,\tau_1)\|f\|_{\ell^2_nL^2_\tau}\|g\|_{\ell^2_nL^2_\tau}\|h\|_{\ell^2_nL^2_\tau}\\
&\le c\,\|w\|_{X_{s,1/2}}\|v\|_{X^a_{s,1/2}}.
\end{aligned}$$

This implies inequality (2.12).

Next we prove the estimate (2.13). We use the definition of the space $X^a_{s,-1/2}$, the Cauchy-Schwarz inequality and Lemma 2.10 to obtain

$$\begin{aligned}
\|\,w^2\|_{X^a_{s,-1/2}} &\le \sup_{n\in\mathbb{Z},\,\tau\in\mathbb{R}} B(n,\tau,a)\|g\|^2_{\ell^2_nL^2_\tau}\\
&\le c\,\|w\|^2_{X_{s,1/2}},
\end{aligned} \qquad (2.23)$$

This finishes the proof of Lemma 2.6. □

Proof of Theorem 2.2. We follow similar argument as those in [23]. We define the metric space of functions

$$\mathcal{X}_M := \{(w,v)\in X_{s,b}\times X^a_{s,b}:\ \|(w,v)\| = \|w\|_{X_{s,b}}+\|v\|_{X^a_{s,b}}\le M\}. \qquad (2.24)$$

For $(w,v)\in\mathcal{X}_M$ we define the operators

$$\begin{cases}
\Phi_1(w,v)(t) = \psi_1 W(t)w_0 + i\,\psi_T\displaystyle\int_0^t W(t-t')\overline{w}\,v(t')\,dt'\\
\Phi_2(w,v)(t) = \psi_1 V(t)v_0 + \frac{i\sigma}{2}\,\psi_T\displaystyle\int_0^t V(t-t')u^2(t')\,dt'.
\end{cases} \qquad (2.25)$$

Applying Lemma 2.5 and Corollary 2.8 we have

$$\begin{aligned}
\|\Phi_1(w,v)\|_{X_{s,b}} &\le c\|w_0\|_{H^s} + c\,T^\gamma\|w\|_{X_{s,b}}\|v\|_{X^a_{s,b}}\\
&\le c\|w_0\|_{H^s} + cT^\gamma M^2
\end{aligned} \qquad (2.26)$$

and

$$\begin{aligned}
\|\Phi_2(w,v)\|_{X^a_{s,b}} &\le c\|v_0\|_{H^s} + c\,T^\gamma\|w\|^2_{X_{s,b}}\\
&\le c\|v_0\|_{H^s} + cT^\gamma M^2.
\end{aligned} \qquad (2.27)$$

Taking $M \geq 2c(\|w_0\|_{H^s} + \|v_0\|_{H^s})$ and T such that $cT^\gamma M < 1/2$ we have that the map $(\Phi_1(w,v), \Phi_2(w,v)) : \mathcal{X}_M \mapsto \mathcal{X}_M$ is well defined. Similarly, one can prove that $(\Phi_1(w,v), \Phi_2(w,v))$ is a contraction on $\mathcal{X}_M$. From the contraction mapping principle we deduce the existence of a unique fixed point for $(\Phi_1(w,v), \Phi_2(w,v))$ which solves the problem. To finish the proof we use standard arguments thus we omit the details. This completes the proof of Theorem 2.2. □

3. Existence of cnoidal waves solutions

In this section we establish the existence theory of a smooth curve of periodic travelling wave solutions to the $\chi^{(2)}$ SHG equations (1.1) of the form

$$w(x,t) = e^{i\gamma t}\phi(x), \qquad v(x,t) = e^{2i\gamma t}\psi(x) \tag{3.1}$$

where $\phi, \psi : \mathbb{R} \mapsto \mathbb{R}$. Substituting (3.1) in (1.1) (with $r = s = 1$) we obtain the system of ordinary differential equations

$$\begin{cases} -\phi'' + (\theta+\gamma)\phi - \phi\psi = 0, \\ -\psi'' + (\alpha + 2\sigma\gamma)\psi - \frac{1}{2}\phi^2 = 0. \end{cases} \tag{3.2}$$

We are interested in solutions of (3.2) satisfying $\theta + \gamma = \alpha + 2\sigma\gamma$. Thus if we consider $\phi = \sqrt{2}\psi$ we reduce our analysis to study the equation

$$\psi''(\xi) - (\theta+\gamma)\psi(\xi) + \psi^2(\xi) = 0, \quad \xi \in \mathbb{R}, \tag{3.3}$$

with the periodic boundary conditions $\psi^{(n)}(0) = \psi^{(n)}(L)$ for every $n \in \mathbb{N}$ and fixed period L. We note that our approach will give us just a family of *positive* periodic solutions with fundamental period L. We also observe that the pair $(\phi, \psi) \equiv (-\sqrt{2}\psi, \psi)$ is a solution of (3.2).

Before stating our main result regarding the existence of solutions for (3.3) we will list some elementary properties satisfied for the Jacobian elliptic functions defined in the introduction which will be useful in our analysis.

Let $K(k)$ and $E(k)$ be as in (1.19)–(1.20). Then

(i) $K(0) = E(0) = \pi/2$, $E(1) = 1$ and $K(1) = +\infty$.

(ii) For $k \in (0,1)$, $K'(k) > 0$, $K''(k) > 0$, $E'(k) < 0$ and $E''(k) < 0$ with

$$\frac{dK}{dk} = \frac{E - k'^2 K}{kk'^2}, \quad \frac{dE}{dk} = \frac{E-K}{k}, \quad \frac{d^2E}{dk^2} = -\frac{1}{k}\frac{dK}{dk} = -\frac{E - k'^2K}{k^2k'^2}. \tag{3.4}$$

(iii) For $k \in (0,1)$, $E(k) < K(k)$, $E(k)+K(k)$ and $E(k)K(k)$ are strictly increasing functions.

Next we establish the existence theory of periodic solutions for (3.3) of *cnoidal* type with wave velocity $r \equiv \theta + \gamma > 0$. From (3.3) we have that ψ satisfies the first-order equation

$$[\psi']^2 = \frac{2}{3}[-\psi^3 + \frac{3r}{2}\psi^2 + 3B_\psi] = \frac{2}{3}(\psi-\beta_1)(\psi-\beta_2)(\beta_3-\psi), \tag{3.5}$$

where β_i, $i=1,2,3$, are the real zeros of the polynomial $F_\psi(t) = -t^3 + \frac{3r}{2}t^2 + 3B_\psi$. Therefore, we must have the relations

$$3r_0 = \sum_{i=1}^{3}\beta_i, \quad 0 = \sum_{i<j}\beta_i\beta_j, \quad 3B_\psi = \prod_{i=1}^{3}\beta_i, \tag{3.6}$$

for $r_0 = r/2$. We assume without losing generality that $\beta_1 < \beta_2 < \beta_3$. From the first and second relations in (3.6) we deduce that

$$-\beta_1 = \frac{\beta_2\beta_3}{\beta_2+\beta_3} = \beta_2 + \beta_3 - 3r_0. \tag{3.7}$$

Thus β_2, β_3 belong to the rotated ellipse $\Xi(r_0)$,

$$\Xi(r_0): \qquad \beta_2^2 + \beta_3^2 + \beta_2\beta_3 - 3r_0(\beta_2+\beta_3) = 0. \tag{3.8}$$

Then, since $\beta_2 < \beta_3$ it follows that $0 < \beta_2 < 2r_0 < \beta_3 < 3r_0$. Moreover, $\beta_2 \leqq \psi \leqq \beta_3$ which implies that ψ must be a positive solution.

Next, by taking $\zeta \equiv \psi/\beta_3$, we see that (3.5) becomes

$$[\zeta']^2 = \frac{2\beta_3}{3}(\zeta-\eta_1)(\zeta-\eta_2)(1-\zeta),$$

where $\eta_i = \beta_i/\beta_3$, $i=1,2$. "If we take the crest of the wave to be at $\xi = 0$, $\zeta(0)=1$". Next we define a further variable χ via the relation $\zeta^2 = 1 + (\eta_2 - 1)\sin^2\chi$, and so we get that

$$(\chi')^2 = \frac{\beta_3}{6}(1-\eta_1)\Big[1 - k^2\sin^2\chi\Big],$$

where $k^2 = \frac{1-\eta_2}{1-\eta_1}$. Note that $0 < k^2 < 1$. Then, for $l = \frac{\beta_3}{6}(1-\eta_1)$, we obtain

$$F(\chi;k) = \int_0^{\chi(\xi)} \frac{dt}{\sqrt{1-k^2\sin^2 t}} = \sqrt{l}\,\xi. \tag{3.9}$$

The left-hand side of (3.9) is just the standard elliptic integral of the first kind and thus from the definition of the Jacobi elliptic function $y = \text{sn}(u; k)$ (1.21) it follows that $\sin\chi = \text{sn}(\sqrt{l}\,\xi; k)$. Hence

$$\zeta = 1 + (\eta_2 - 1)\text{sn}^2\,(\sqrt{l}\,\xi; k).$$

Therefore, since $\text{sn}^2 u + \text{cn}^2 u = 1$, we "arrive" to the so-called **cnoidal wave** solution associated to equation (3.3)

$$\psi(\xi) = \psi(\xi; \beta_1, \beta_2, \beta_3) = \beta_2 + (\beta_3 - \beta_2)\text{cn}^2\Big[\sqrt{\frac{\beta_3 - \beta_1}{6}}\,\xi; k\Big], \tag{3.10}$$

where the β_i's satisfy (3.6) and

$$k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}.$$

Next, since $\text{cn}^2(\cdot; k)$ has fundamental period $2K(k)$ then ψ has fundamental period T_ψ given by

$$T_\psi \equiv \frac{2\sqrt{6}}{\sqrt{\beta_3 - \beta_1}} K(k). \tag{3.11}$$

Now, we see that the period T_ψ depends *a priori* on the wave velocity r. More precisely,

$$T_\psi > \frac{\sqrt{2}\pi}{\sqrt{r_0}}.$$

In fact, we first express T_ψ as a function of β_2 and r_0. Since for every $\beta_2 \in (0, 2r_0)$ there is a unique $\beta_3 \in (2r_0, 3r_0)$ such that $(\beta_2, \beta_3) \in \Xi(r_0)$, it follows that $2\beta_3 = 3r_0 - \beta_2 + \sqrt{9r_0^2 - 3\beta_2^2 + 6r_0\beta_2}$. Hence by defining $\beta_1 \equiv 3r_0 - \beta_2 - \beta_3$ we obtain for

$$g(\beta_2, r_0) \equiv \sqrt{9r_0^2 - 3\beta_2^2 + 6r_0\beta_2}, \quad \text{and} \quad k^2(\beta_2, r_0) = \frac{1}{2} + \frac{3(r_0 - \beta_2)}{2g(\beta_2, r_0)} \tag{3.12}$$

that $g(\beta_2, r_0) = \beta_3 - \beta_1$ and

$$T_\psi(\beta_2, r_0) = \frac{2\sqrt{6}}{\sqrt{g(\beta_2, r_0)}} K(k(\beta_2, r_0)).$$

Then by fixing $r_0 > 0$, we have $T_\psi(\beta_2, r_0) \to +\infty$, as $\beta_2 \to 0$, and $T_\psi(\beta_2, r_0) \to \sqrt{2}\pi/\sqrt{r_0}$, as $\beta_2 \to 2r_0$. Since the map $\beta_2 \in (0, 2r_0) \mapsto T_\psi(\beta_2, r_0)$ is strictly decreasing (see proof of Theorem 3.1 below) we deduce that $T_\psi > \sqrt{2}\pi/\sqrt{r_0}$.

The analysis above allows us to obtain a cnoidal wave solution for equation (3.3) with an arbitrary fundamental period L. Indeed, for a wave velocity $r > 4\pi^2/L^2$ there is, for $r_0 = r/2$, a unique $\beta_{2,0} \in (0, 2r_0)$ such that $T_\psi(\beta_{2,0}, r_0) = L$. Thus, for $\beta_{3,0}$ such that $(\beta_{2,0},\beta_{3,0})\in\Xi(r_0)$, we have that the cnoidal wave $\psi(\cdot)=\psi(\cdot;\beta_{1,0},\beta_{2,0},\beta_{3,0})$ with $\beta_{1,0}=3r_0-\beta_{2,0}-\beta_{3,0}$, has fundamental period L and satisfies (3.3) with $\theta+\gamma=r$. We also note that the cnoidal wave $\psi(\cdot;\beta_1,\beta_2,\beta_3)$ in (3.10) can be seen as a function depending only on r and β_2, $\psi_r(\cdot;\beta_2(r))$.

Next we show the existence of a smooth curve of cnoidal waves solutions for the equation (3.3). In other words, we show that at least locally the choice of $\beta_{2,0}(r_0)$ above depends smoothly on r_0.

Theorem 3.1. *Let $L>0$ be arbitrary but fixed. Consider $r_0>\frac{2\pi^2}{L^2}$ and the unique $\beta_{2,0}\in(0,2r_0)$ such that*

$$\frac{2\sqrt{6}\,K(k(\beta_{2,0},r_0))}{\sqrt{g(\beta_{2,0},r_0)}}=L.$$

Then,

(1) *there exist an interval $J(r_0)$ around r_0, an interval $B(\beta_{2,0})$ around $\beta_{2,0}$, and a unique smooth function $\Gamma: J(r_0)\mapsto B(\beta_{2,0})$, such that $\Gamma(r_0)=\beta_{2,0}$ and*

$$\frac{2\sqrt{6}}{\sqrt{g(\beta_2,\lambda)}}\,K(k(\beta_2,\lambda))=L, \tag{3.13}$$

where $\lambda\in J(r_0)$, $\beta_2=\Gamma(\lambda)$, and $k(\beta_2,\lambda)$, $g(\beta_2,\lambda)$ are defined in (3.12). Moreover, $J(r_0)=(\frac{2\pi^2}{L^2},+\infty)$.

(2) *Let $\theta>0$. Then for $\gamma\in(\frac{4\pi^2}{L^2}-\theta,+\infty)$ and $\lambda(\gamma)=(\theta+\gamma)/2$ the cnoidal wave solution*

$$\psi_\gamma(\cdot)\equiv\psi_{\lambda(\gamma)}(\cdot;\beta_2(\lambda(\gamma)))$$

has fundamental period L and satisfies equation (3.3). Moreover, the mapping

$$\gamma\in\Big(\frac{4\pi^2}{L^2}-\theta,+\infty\Big)\mapsto\psi_\gamma\in H^n_{per}([0,L])$$

is a smooth function.

Proof. We will apply the implicit function theorem to prove the results. First, we consider the open set $\Omega=\{(\beta_2,\lambda):\ \lambda>\frac{2\pi^2}{L^2},\ \beta_2\in(0,2\lambda)\ \}\subseteq\mathbb{R}^2$ and define $\Phi:\Omega\to\mathbb{R}$ by

$$\Phi(\beta_2,\lambda)=\frac{2\sqrt{6}}{\sqrt{g(\beta_2,\lambda)}}\,K(k(\beta_2,\lambda))-L \tag{3.14}$$

where $g(\beta_2, \lambda)$ and $k^2(\beta_2, \lambda)$ are defined in (3.12). By hypotheses $\Phi(\beta_{2,0}, r_0) = 0$. Now we show that $\frac{\partial \Phi}{\partial \beta_2}(\beta_{2,0}, r_0) < 0$. In fact, by using the relations, $18\lambda^2 = g^2(2 - 2k^2 + 2k^4)$,

$$\frac{\partial g}{\partial \beta_2} = \frac{3(\lambda - \beta_2)}{g}, \qquad \frac{\partial k}{\partial \beta_2} = -\frac{9\lambda^2}{kg^3}$$

and the differential relation

$$\frac{dK}{dk} = \frac{E - k'^2 K}{kk'^2},$$

where $k'^2 = 1 - k^2$, we have formally that,

$$\begin{aligned} \frac{\partial \Phi}{\partial \beta_2} < 0 \quad &\Leftrightarrow \quad -18\lambda^2 \frac{dK}{dk} < g^2(2k^2 - 1)kK \\ &\Leftrightarrow \quad 18\lambda^2 E > [18\lambda^2 - k^2(2k^2 - 1)g^2]k'^2 K \\ &\Leftrightarrow \quad (2 - 2k^2 + 2k^4)E > (2 - 3k^2 + k^4)K. \end{aligned} \tag{3.15}$$

Next, since $E + K$ is a strictly increasing function we have that

$$(2 - k^2)E > 2(1 - k^2)K, \qquad k \in (0, 1).$$

Moreover, from the definition of the complete elliptical integrals E and K it follows that

$$(k^2 - 1)K \leqq (2k^2 - 1)E, \qquad k \in (0, 1).$$

So we obtain from (3.15) that $\frac{\partial \Phi}{\partial \beta_2}(\beta_2, \lambda) < 0$ for every $(\beta_2, \lambda) \in \Omega$.

Therefore, there is a unique smooth function, Γ, defined in a neighborhood $J(r_0)$ of r_0, such that $\Phi(\Gamma(\lambda), \lambda) = 0$ for every $\lambda \in J(r_0)$. So, we get (3.13). Finally, since r_0 was arbitrarily chosen in the interval $\mathbb{I} = (\frac{2\pi^2}{L^2}, +\infty)$, it follows that Γ can extend to $\mathbb{I}$. This completes the proof of the theorem. □

Corollary 3.2. *Let $\Gamma : J(r_0) \mapsto B(\beta_{2,0})$ be the map given by Theorem 3.1. Then, $\beta_2(\lambda) \equiv \Gamma(\lambda)$ is a strictly decreasing function in $J(r_0)$. Moreover, the modulus function*

$$k^2(\lambda) = \frac{1}{2} + \frac{3(\lambda - \beta_2(\lambda))}{2g(\beta_2(\lambda), \lambda)}, \tag{3.16}$$

where g was defined in (3.12), is a strictly increasing function.

Proof. From (the proof of) Theorem 3.1 we have $\Phi(\Gamma(\lambda), \lambda) = 0$, then $\frac{d\Gamma}{d\lambda} = -\frac{\partial\Phi/\partial\lambda}{\partial\Phi/\partial\beta_2}$. Hence, we only need to show that $\partial\Phi/\partial\lambda < 0$. In fact, from (3.12) and the relation

$kg^3\frac{dk}{d\lambda} = 9\lambda\beta_2$ we have

$$\frac{\partial\Phi}{\partial\lambda} < 0 \Leftrightarrow 6\lambda\beta_2\frac{dK}{dk} < gk(3\lambda+\beta_2)K \Leftrightarrow 6\lambda\beta_2 E < [gk^2(3\lambda+\beta_2)+6\lambda\beta_2]k'^2 K. \tag{3.17}$$

Now, since $gk^2(3\lambda+\beta_2)+6\lambda\beta_2 = g^2/2 + g(3\lambda+\beta_2)/2$ implies $6\lambda\beta_2 < g(3\lambda+\beta_2)k'^2$, it follows from the inequality $E < K$ and (3.17) that $\frac{\partial\Phi}{\partial\lambda} < 0$.

Finally, from the definition of k and g we obtain

$$\frac{dk}{d\lambda} = \frac{9\lambda}{kg^3}(\beta_2 - \lambda\beta_2') > 0.$$

This completes the proof. □

4. Spectral analysis

The study of the spectra of the following matrix differential operators $\mathcal{L}_R$ and $\mathcal{L}_I$ given by

$$\mathcal{L}_R = \begin{pmatrix} -\frac{d^2}{dx^2}+(\theta+\gamma)-\psi & -\phi \\ -\phi & -\frac{d^2}{dx^2}+(\alpha+2\sigma\gamma) \end{pmatrix}, \tag{4.1}$$

and

$$\mathcal{L}_I = \begin{pmatrix} -\frac{d^2}{dx^2}+(\theta+\gamma)+\psi & -\phi \\ -\phi & -\frac{d^2}{dx^2}+(\alpha+2\sigma\gamma) \end{pmatrix}, \tag{4.2}$$

where the pair (ϕ,ψ) is a solution of equation (3.2), is crucial for the stability and instability analysis of the periodic traveling waves solutions found in the previous section. In what follows, $\sigma(L)$ will denote the spectrum of a linear operator L. It is well known that it can be decomposed into the essential spectrum $\sigma_{\rm ess}(L)$ and the discrete spectrum $\sigma_{\rm disc}(L)$, where $\sigma_{\rm disc}(L) = \sigma(L) - \sigma_{\rm ess}(L)$ (see [26]). So, $\sigma_{\rm disc}(L)$ consists of all isolated eigenvalues of finite multiplicity, it means that the eigenspace (geometric) associated to each eigenvalue is finite dimensional. We recall that in the case of L being self-adjoint the algebraic multiplicity of an eigenvalue coincides with the dimension of the eigenspace.

In the analysis of the self-adjoint operator $\mathcal{L}_R$ when $(\phi,\psi) = (-\sqrt{2}\psi,\psi)$, with ψ being the *cnoidal* wave solution for (3.3) given by Theorem 3.1, the understanding of the following periodic eigenvalue problem

$$\begin{cases} \mathcal{L}_{\rm cn}\zeta \equiv (-\frac{d^2}{dx^2}+(\theta+\gamma)-2\psi)\zeta = \lambda\zeta \\ \zeta(0)=\zeta(L), \quad \zeta'(0)=\zeta'(L), \end{cases} \tag{4.3}$$

is necessary. The following theorem contains useful information regarding the operator $\mathcal{L}_{\rm cn}$.

Theorem 4.1. *Let $\theta > 0$, $\gamma \in (\frac{4\pi^2}{L^2} - \theta, +\infty)$ and $\psi = \psi_\gamma$ be the cnoidal wave solution of (3.3) given by Theorem 3.1. Then, the linear operator $\mathcal{L}_{cn}$ in (4.3) defined on $H^2_{per}([0, L])$ has its first three eigenvalues simple, being the eigenvalue zero the second one with eigenfunction ψ'. Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues which are double.*

Theorem 4.1 is a consequence of the Floquet theory (see [24]). For the sake of clearness in the exposition we will list some basic facts of this theory.

From the theory of compact symmetric operators (4.3) determines that $\sigma(\mathcal{L}_{\rm cn}) = \sigma_{\rm disc}(\mathcal{L}_{\rm cn}) = \{\lambda_n \,|\, n = 0, 1, 2, \cdots\}$ with $\lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots$, where a double eigenvalue is counted twice and $\lambda_n \to \infty$ as $n \to \infty$. We denote by ζ_n the eigenfunction associated to the eigenvalue λ_n. By the conditions $\zeta(0) = \zeta(L)$, $\zeta'(0) = \zeta'(L)$, ζ_n can be extended to the whole of $(-\infty, \infty)$ as a continuous differentiable function with period L.

From the Floquet theory, we know that the *periodic eigenvalue problem* (4.3) is related to the following *semi-periodic eigenvalue problem* considered on $[0, L]$

$$\begin{cases} \mathcal{L}_{cn}\xi = \mu\xi \\ \xi(0) = -\xi(L), \quad \xi'(0) = -\xi'(L), \end{cases} \tag{4.4}$$

which is also a self-adjoint problem and therefore its spectrum, $\sigma^{sm}(\mathcal{L}_{cn})$, is given by $\sigma^{sm}_{disc}(\mathcal{L}_{cn}) = \{\mu_n \,|\, n = 0, 1, 2, 3, \cdots\}$, with $\mu_0 \leqq \mu_1 \leqq \mu_2 \leqq \cdots$, where double eigenvalues are counted twice and $\mu_n \to \infty$ as $n \to \infty$. We denote by ξ_n the eigenfunction associated to the eigenvalue μ_n. Then we have that the equation

$$\mathcal{L}_{cn} f = \gamma f \tag{4.5}$$

has a solution of period L if and only if $\gamma = \lambda_n$, $n = 0, 1, 2, \ldots$. Similarly, it has a solution of period $2L$ if and only if $\gamma = \mu_n$, $n = 0, 1, 2, \cdots$. If all solutions of (4.5) are bounded we say that they are *stable*; otherwise, we say that they are *unstable*. The Oscillation Theorem (see [24]) guarantees that the distribution of the eigenvalues λ_i, μ_i, is as follows:

$$\lambda_0 < \mu_0 \leqq \mu_1 < \lambda_1 \leqq \lambda_2 < \mu_2 \leqq \mu_3 < \lambda_3 \leqq \lambda_4 \cdots. \tag{4.6}$$

The intervals (λ_0, μ_0), $(\mu_1, \lambda_1), \cdots$, are called *intervals of stability*. At the endpoints of these intervals the solutions of (4.5) are, in general, unstable. This is true for $\gamma = \lambda_0$ (λ_0 is always a simple eigenvalue). The intervals, $(-\infty, \lambda_0), (\mu_0, \mu_1), (\lambda_1, \lambda_2), (\mu_2, \mu_3), \cdots$, are called *intervals of instability*, omitting however any interval which is absent as a result of having a double eigenvalue. The interval of instability $(-\infty, \lambda_0)$ will always be present. We note that the absence of an instability interval means that there is a value of γ for which all solutions of (4.5) have either period L or semi-period L. In other words, coexistence of solutions of (4.5) with period L or period $2L$ occurs for that value of γ.

We end this brief review by describing how is determined the number of zeros of ζ_n and ξ_n. Indeed,

$$\begin{aligned} &\text{(i)}\ \ \zeta_0 \ \text{ has no zeros in } \ [0,L].\\ &\text{(ii)}\ \ \zeta_{2n+1} \text{ and } \zeta_{2n+2} \ \text{ have exactly } \ 2n+2 \ \text{ zeros in } \ [0,L).\\ &\text{(iii)}\ \ \xi_{2n} \text{ and } \xi_{2n+1} \ \text{ have exactly } \ 2n+1 \ \text{ zeros in } \ [0,L). \end{aligned} \tag{4.7}$$

Proof of Theorem 4.1. Since $\mathcal{L}_{\rm cn}\psi' = 0$ and ψ' has 2 zeros in $[0, L)$ then the eigenvalue 0 is either λ_1 or λ_2. We will show that $0 = \lambda_1 < \lambda_2$ and thus zero is simple. In fact, for $T_\eta\zeta(x) \equiv \zeta(\eta x)$ with $\eta^2 = 6/(\beta_3 - \beta_1)$ we have for $\Lambda \equiv T_\eta\zeta$ that

$$\begin{cases} \dfrac{d^2}{dx^2}\Lambda + [\rho - 12k^2\,\mathrm{sn}^2(x)]\Lambda = 0\\ \Lambda(0) = \Lambda(2K), \quad \Lambda'(0) = \Lambda'(2K), \end{cases} \tag{4.8}$$

where for $r = \gamma + \theta$,

$$\rho = -\frac{6[r - \lambda - 2\beta_3]}{(\beta_3 - \beta_1)}.$$

The second order differential equation in (4.8) is called the *Jacobian form of Lamé's equation*. Now, from Floquet theory it follows that (4.8) has exactly 4 intervals of instability which are $(-\infty, \rho_0)$, (μ_0', μ_1'), (ρ_1, ρ_2), (μ_2', μ_3') (where μ_i', $i \geqq 0$, are the eigenvalues associated to the semi-periodic problem determined by Lamé's equation). Therefore, the eigenvalues ρ_0, ρ_1, ρ_2 are simple and the rest of eigenvalues $\rho_3 \leqq \rho_4 < \rho_5 \leqq \rho_6 < \cdots$ satisfy that $\rho_3 = \rho_4, \rho_5 = \rho_6, \cdots$, that is, they are double eigenvalues.

For the sake of clearness in our exposition and further study of instability in section 5, we will explicitly determine these eigenvalues and its corresponding eigenfunctions. We start by noting that $\rho_1 = 4 + 4k^2$ is an eigenvalue to (4.8) with eigenfunction $\Lambda_1(x) = \mathrm{cn}(x)\,\mathrm{sn}(x)\,\mathrm{dn}(x) = \beta \cdot T_\eta\psi'(x)$ which implies that $\lambda = 0$ is a simple eigenvalue to (4.3) with eigenfunction ψ'. Now from Ince ([18]) we have that the *Lamé polynomials*,

$$\begin{aligned}\Lambda_0(x) &= \mathrm{dn}(x)[1-(1+2k^2-\sqrt{1-k^2+4k^4}\,)\,\mathrm{sn}^2(x)],\\ \Lambda_2(x) &= \mathrm{dn}(x)[1-(1+2k^2+\sqrt{1-k^2+4k^4}\,)\,\mathrm{sn}^2(x)]\end{aligned} \tag{4.9}$$

with period $2K(k)$ are the associated eigenfunctions to the others two eigenvalues ρ_0, ρ_2 given by

$$\rho_0 = 2+5k^2-2\sqrt{1-k^2+4k^4}, \qquad \rho_2 = 2+5k^2+2\sqrt{1-k^2+4k^4}. \tag{4.10}$$

Since Λ_0 has no zeros in $[0,2K]$ and Λ_2 has exactly 2 zeros in $[0,2K)$, it follows that Λ_0 is the eigenfunction associated to ρ_0 which will be the first eigenvalue to (4.8). Since $\rho_0<\rho_1$ for every $k^2\in(0,1)$, we obtain from (3.12) the relation $-\beta_1(1+k^2)=(2-k^2)\beta_3-3r/2$ and so

$$6\lambda_0 = \rho_0(\beta_3-\beta_1)+12(\frac{r}{2}-\beta_3) = 3\frac{\beta_3-\frac{r}{2}}{k^2+1}\rho_0+12(\frac{r}{2}-\beta_3)<0. \tag{4.11}$$

Therefore λ_0 is the first negative eigenvalue to $\mathcal{L}_{\rm cn}$ with eigenfunction $\zeta_0(x)=\Lambda_0(\frac{1}{\eta}x)$. Now, since $\rho_1<\rho_2$ for every $k^2\in(0,1)$, we obtain that

$$6\lambda_2 = 3\frac{\beta_3-\frac{r}{2}}{k^2+1}\rho_2+12(\frac{r}{2}-\beta_3)>0. \tag{4.12}$$

Hence λ_2 is the third eigenvalue to $\mathcal{L}_{\rm cn}$ with eigenfunction $\zeta_2(x)=\Lambda_2(\frac{1}{\eta}x)$.

Next, we can see that

$$\mu_0' = 5+2k^2-2\sqrt{4-k^2+k^4}, \;\; \mu_1' = 5+5k^2-2\sqrt{4-7k^2+4k^4}$$

are the first two eigenvalues to Lamé's equation in the semi-periodic case, with associated eigenfunctions given by

$$\begin{aligned}\xi_{0,sm}(x) &= \mathrm{cn}(x)[1-(2+k^2-\sqrt{4-k^2+k^4}\,)\,\mathrm{sn}^2(x)]\\ \xi_{1,sm}(x) &= 3\mathrm{sn}(x)-(2+2k^2-\sqrt{4-7k^2+4k^4}\,)\,sn^3(x)\end{aligned} \tag{4.13}$$

respectively. Since $\mu_0'<\mu_1'<4k^2+4$, the equality

$$\mu_i' = -6\frac{(r-\mu_i-2\beta_3)}{\beta_3-\beta_1} \tag{4.14}$$

implies that the first three associated instability intervals to $\mathcal{L}_{\rm cn}$ are $(-\infty,\lambda_0)$, (μ_0,μ_1), $(0,\lambda_2)$. Finally, since the functions $\xi_{2,sm}(x)=\mathrm{cn}(x)[1-(2+k^2+\sqrt{4-k^2+k^4}\,)\,\mathrm{sn}^2(x)]$ and $\xi_{3,sm}(x)=3\mathrm{sn}(x)-(2+2k^2+\sqrt{4-7k^2+4k^4}\,)\,\mathrm{sn}^3(x)$ have three zeros in $[0,2K)$ and are eigenfunctions of Lamé's equation with eigenvalues $\mu_2'=5+2k^2+2\sqrt{4-k^2+k^4}$ and

$\mu_3' = 5+5k^2+2\sqrt{4-7k^2+4k^4}$, it follows from (4.14) that the last instability interval of $\mathcal{L}_{\rm cn}$ is (μ_2, μ_3). This finishes the proof of Theorem 4.1. □

The next result will be necessary in the study of the nonlinear instability of the periodic travelling waves $(e^{i\gamma t}\phi, e^{2i\gamma t}\psi)$, $(\phi,\psi) = (-\sqrt{2}\psi_\gamma, \psi_\gamma)$, by perturbations with twice the fundamental period.

Theorem 4.2. *Let $\theta > 0$, $\gamma \in (\frac{4\pi^2}{L^2}-\theta, +\infty)$ and $\psi = \psi_\gamma$ be the cnoidal wave solution of (3.3) given by Theorem 3.1 with fundamental period L. Then, the linear operator $\mathcal{L}_{cn}$ in (4.3) defined on $H^2_{per}([0,2L])$ has its first four eigenvalues simple, being the eigenvalue zero the fourth one with eigenfunction ψ'. Moreover, if Φ_1, Φ_2 denote the eigenfunctions associated to the second and third eigenvalues then $\Phi_i \perp \psi$.*

Proof. Since ψ' has 4 zeros in $[0,2L)$ and $\mathcal{L}_{cn}\psi' = 0$ on $[0,2L]$, it follows from (4.7) and relation (4.12) that zero is the fourth eigenvalue for $\mathcal{L}_{cn}$ on $[0,2L]$ and it is simple.

On the other hand, from the proof of Theorem 4.1 we have that the first three eigenvalues for $\mathcal{L}_{\rm cn}$ in $H^2_{per}([0,2L])$ are λ_0, μ_0, μ_1. Here λ_0 is determined by the relation (4.11) with associated eigenfunction

$$\Phi_0(x) = \mathrm{dn}(x/\eta)[1-(1+2k^2-\sqrt{1-k^2+4k^4}\,)\,\mathrm{sn}^2(x/\eta)], \tag{4.15}$$

$\eta^2 = 6/(\beta_3-\beta_1)$. Meanwhile, μ_0 and μ_1 are determined by the relation (4.14) with eigenfunctions

$$\begin{aligned}\Phi_1(x) &= \mathrm{cn}(x/\eta)[1-(2+k^2-\sqrt{4-k^2+k^4}\,)\,\mathrm{sn}^2(x/\eta)],\\ \Phi_2(x) &= 3\,\mathrm{sn}(x/\eta)-(2+2k^2-\sqrt{4-7k^2+4k^4}\,)\,\mathrm{sn}^3(x/\eta),\end{aligned} \tag{4.16}$$

respectively.

Using the relation $\int_0^{4K}\mathrm{cn}^{2k+1}(x)\,dx = 0$ for every $k \geqq 0$, and the fact that sn^{2k+1} and $\mathrm{sn}^{2k+1}\mathrm{cn}^2$ are odd periodic functions with period $4K$, we deduce that Φ_1 and Φ_2 are orthogonal to ψ. This finishes the proof of the theorem. □

Now we are ready to describe the spectra of the self-adjoint operator $\mathcal{L}_R$ and $\mathcal{L}_I$ when $(\phi,\psi) = (-\sqrt{2}\psi,\psi)$ and $\psi = \psi_\gamma$ is given by Theorem 3.1.

Theorem 4.3. *Let $\theta > 0$, $\gamma \in (\frac{4\pi^2}{L^2}-\theta, +\infty)$ and $\psi = \psi_\gamma$ be the cnoidal wave solution of (3.3) given by Theorem 3.1 with fundamental period L. Then, for $\alpha, \sigma > 0$ such that $\alpha + 2\sigma\gamma = \theta + \gamma$ we have:*

(1) *The linear operator* $\mathcal{L}_R$ *in (4.1) defined in* $L^2_{\rm per}([0,L])$ *with domain* $H^2_{\rm per}([0,L])$ *has exactly one negative eigenvalue which is simple, zero is an eigenvalue simple with eigenfunction* $(2\psi'/3, \sqrt{2}\psi'/3)$. *Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalue.*

(2) *The linear operator* $\mathcal{L}_I$ *in (4.2) defined in* $L^2_{\rm per}([0,L])$ *with domain* $H^2_{\rm per}([0,L])$ *has no negative spectrum at all, zero is an eigenvalue simple with eigenfunction* $(\sqrt{2}\psi/2, \psi)$. *Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalue.*

Proof. The main point of the proof is to show that $\mathcal{L}_R$ and $\mathcal{L}_I$ can be diagonalized under a similarity transformation. In fact, consider

$$A_R = \begin{pmatrix} 1 & \sqrt{2}/2 \\ -\sqrt{2}/3 & 2/3 \end{pmatrix},$$

then we have $A_R\mathcal{L}_R A_R^{-1} = \mathcal{L}_{DR}$, where

$$\mathcal{L}_{DR} = \begin{pmatrix} -\frac{d^2}{dx^2} + (\theta+\gamma) - 2\psi & 0 \\ 0 & -\frac{d^2}{dx^2} + (\theta+\gamma) + \psi \end{pmatrix}.$$

Note that since ψ is positive the operator $\mathcal{L}_P \equiv -\frac{d^2}{dx^2} + (\theta+\gamma) + \psi$ is strictly positive and $\sigma(\mathcal{L}_P) \geqq \theta+\gamma$. Now, let $\vec{f} = (f,g)^t$ be such that $\mathcal{L}_{DR}\vec{f} = \vec{0}$, then $\mathcal{L}_{\rm cn} f = 0$ and $\mathcal{L}_P g = 0$. Hence $g \equiv 0$ and from Theorem 4.1 $f = \beta\psi'$. Then the kernel of $\mathcal{L}_{DR}$ is generated by $(\psi', 0)^t$. Hence the kernel of $\mathcal{L}_R$ is generated by $(2\psi'/3, \sqrt{2}\psi'/3)^t$.

Now let $\lambda < 0$ and $\vec{f} = (f,g)^t$ such that $\mathcal{L}_{DR}\vec{f} = \lambda\vec{f}$, then $g \equiv 0$ and $\mathcal{L}_{\rm cn} f = \lambda f$. Thus, from Theorem 4.1 we have that $\lambda = \lambda_0$ (see (4.11)) and $f = \beta\zeta_0$. Therefore $\mathcal{L}_R$ has exactly a negative eigenvalue which is simple with eigenfunction $(2\zeta_0/3, \sqrt{2}\zeta_0/3)^t$.

Next we analyze $\mathcal{L}_I$. Let

$$A_I = \begin{pmatrix} 1 & -\sqrt{2}/2 \\ \sqrt{2}/3 & 2/3 \end{pmatrix},$$

then we have $A_I\mathcal{L}_I A_I^{-1} = \mathcal{L}_{DI}$, where

$$\mathcal{L}_{DI} = \begin{pmatrix} -\frac{d^2}{dx^2} + (\theta+\gamma) + 2\psi & 0 \\ 0 & -\frac{d^2}{dx^2} + (\theta+\gamma) - \psi \end{pmatrix}.$$

Since the top left entry in $\mathcal{L}_{DI}$ is a strictly positive operator the basic part of its spectrum depends exclusively on $\mathcal{L} = -\frac{d^2}{dx^2} + (\theta+\gamma) - \psi$. Since $\mathcal{L}\psi = 0$ and $\psi > 0$ it follows from

(4.7) that zero is the first eigenvalue for $\mathcal{L}$ and it is simple. Then $\mathcal{L}_I$ has no negative eigenvalues and the kernel is generated by $(\sqrt{2}\psi/2, \psi)^t$.

Finally, Weyl's essential spectral theorem implies that the remainders of the spectrum of $\mathcal{L}_R$ and $\mathcal{L}_I$ are discrete. This finishes the proof. $\square$

5. Existence of other solutions

In section 3, we established the existence of periodic pulses for the system (3.2) of the form $\phi_\gamma = \sqrt{2}\ \psi_\gamma$ provided that for $\theta > 0$ fixed we have the conditions $\gamma > \frac{4\pi^2}{L^2} - \theta$ and $\alpha + 2\sigma\gamma = \theta + \gamma$, for $\alpha, \sigma > 0$. Here, the map $\gamma \mapsto \psi_\gamma$ is a smooth curve of cnoidal waves. Next, we show the existence of other family of periodic traveling wave solutions (ϕ, ψ) for (3.2) but depending on the parameter α. So, we first choose an arbitrary pair (α_0, σ) such that $\alpha_0 + 2\sigma\gamma = \theta + \gamma$ and we define $G : \mathbb{R} \times H^2_{per,e}([0, L]) \times H^2_{per,e}([0, L]) \mapsto L^2_{per,e}([0, L]) \times L^2_{per,e}([0, L])$ as

$$G(\alpha, \phi, \psi) = (-\phi'' + (\theta + \gamma)\phi - \phi\psi, -\psi'' + (\alpha + 2\sigma\gamma)\psi - \phi^2/2), \tag{5.1}$$

where $H^s_{per,e}([0, L])$ denotes the set of even, L-periodic-Sobolev distributions of order $s \in \mathbb{R}$. So, by Theorem 3.1 we have that $G(\alpha_0, \sqrt{2}\ \psi_\gamma, \psi_\gamma) = (0, 0)$. Moreover, it is not difficult to see that the Fréchet derivative $\mathcal{G} \equiv \frac{\partial G}{\partial(\phi,\psi)}(\alpha_0, \sqrt{2}\ \psi_\gamma, \psi_\gamma) = \mathcal{L}_R$, with $\mathcal{L}_R$ defined in (4.1) with α changed by α_0. Next, we will prove that $\mathcal{G}$ is a bijection from $H^2_{per,e} \times H^2_{per,e} \to L^2_{per,e} \times L^2_{per,e}$. We start with the injectivity. From Theorem 4.3, $Ker(\mathcal{G}) = [(2\psi'_\gamma/3, \sqrt{2}\psi'_\gamma/3)^t]$. Since ψ'_γ is an odd function it follows immediately that $Ker(\mathcal{G}) = \{(0, 0)^t\}$ over $H^2_{per,e} \times H^2_{per,e}$. Now, we prove that $\mathcal{G}$ is a surjective map onto $L^2_{per,e} \times L^2_{per,e}$. Indeed, from Weyl's essential theorem it is easy to see that the essential spectrum of $\mathcal{G}$ is empty. Hence, $\sigma(\mathcal{G}) = \sigma_{disc}(\mathcal{G})$. Therefore $0 \in \rho(\mathcal{G})$, where ρ is used to denote the resolvent set of an operator. Hence $\mathcal{G}$ is surjective.

Finally, since G is a C^1-map on an open neighborhood of the point $(\alpha_0, \sqrt{2}\ \psi_\gamma, \psi_\gamma)$, it follows from the Implicit Function Theorem that there exist $\delta > 0$, and a unique C^1-map

$$\alpha \in (\alpha_0 - \delta, \alpha_0 + \delta) \mapsto \Phi_\alpha = (\phi_\alpha, \psi_\alpha)$$

such that $G(\alpha, \Phi_\alpha) = (0, 0)$. So, we obtain that Φ_α is a solution of (3.2). Moreover, since $G(\alpha, \phi, \psi)$ depends analytically on α, the map $\alpha \mapsto \Phi_\alpha$ is analytic as well.

6. Nonlinear Stability

In this section we will study the properties of stability and instability of the periodic traveling wave found in Section 3. The framework for stability will be that set by Grillakis, Shatah and Strauss in [17]. We start by rewriting system (1.1) as a real Hamiltonian system. Let $w = P + iQ$, $v = R + iS$, then $\mathcal{H}$ in (1.12) can be rewritten as

$$\begin{aligned}\mathcal{H}(P,R,Q,S) = \frac{1}{2}\int \big\{ & r[(P')^2 + (Q')^2] + s[(R')^2 + (S')^2] + \theta(P^2+Q^2) \\ & + \alpha(R^2+S^2) - 2PQS - P^2R + Q^2R\big\}\,dx\end{aligned} \tag{6.1}$$

and $\mathcal{F}$ in (1.11) as

$$\mathcal{F}(P,R,Q,S) = \frac{1}{2}\int \ P^2 + Q^2 + 2\sigma(R^2+S^2)\,dx. \tag{6.2}$$

Therefore, system (1.1) for $u = (P,R,Q,S)^t$ has the form

$$\frac{\partial u}{\partial t} = J\mathcal{H}'(u(t)) \tag{6.3}$$

where $J = (a_{ij})$ is the skew-symmetric linear operator defined as $a_{13} = 1, a_{24} = 1/\sigma$, and $a_{ij} = 0$ for $(i,j) \neq (1,3),(2,4),(3,1),(4,2)$.

The system (1.1) has two basic symmetries:

1. Translation in x: if $(w(x,t), v(x,t))$ is solution then $(w(x+x_0,t), v(x+x_0,t))$ is also a solution for every $x_0 \in \mathbb{R}$. This transformation will be denoted by the one-parameter group of unitary operators $T_{tr}(x_0)$.
2. Phase (or rotational): if $(w(x,t), v(x,t))$ is solution then $(e^{is}w(x,t), e^{2is}v(x,t))$ is also a solution for every $s \in \mathbb{R}$. This transformation will be denoted by the one-parameter group of unitary operators $T_p(s)$.

The differential of these groups (the infinitesimal generators) are $T'_{tr}(0) = \partial/\partial_x$ and $T'_p(0) = i$, respectively.

The notion of stability or instability we will use is as follows:

Definition 6.1. *Let $X = H^1_{\text{per}}([0,L]) \times H^1_{\text{per}}([0,L])$. A travelling wave solution for (1.1), $\Psi(x,t) = (e^{i\gamma t}\phi(x), e^{2i\gamma t}\psi(x))$, is orbitally stable in X if for every $\epsilon > 0$ there exists a $\delta > 0$, such that if $z_0 \in X$ and $\|z_0 - (\phi,\psi)\|_X < \delta$, then the solution $z(t) = (w(t), v(t))$ of*

(1.1) with $z(0)=z_0$ exists for all t and

$$\sup_{t\in\mathbb{R}}\ \inf_{s,\,r\in\mathbb{R}}\ \|z(t)-T_p(s)T_{tr}(r)(\phi,\psi)\|_X<\epsilon.$$

Otherwise, we said that Ψ is X-unstable.

Observe that (ϕ,ψ) is a solution of (3.2) if and only if

$$\mathcal{H}'(\phi,\psi,0,0)+\gamma\mathcal{F}'(\phi,\psi,0,0)=0. \tag{6.4}$$

Then, from Theorem 3.1 we have the existence of a smooth curve de cnoidal wave solutions $\gamma\mapsto(\phi_\gamma,\psi_\gamma)=(\sqrt{2}\psi_\gamma,\psi_\gamma)$ which are critical points of $\mathcal{H}+\gamma\mathcal{F}$. Next, we define

$$\mathcal{L}_\gamma=\mathcal{H}''(\phi,\psi,0,0)+\gamma\mathcal{F}''(\phi,\psi,0,0)=\begin{pmatrix}\mathcal{L}_R & 0\\ 0 & \mathcal{L}_I\end{pmatrix}, \tag{6.5}$$

where $\mathcal{L}_R,\mathcal{L}_I$ are as in (4.1), (4.2), respectively. From Theorem 4.3 and its proof we have that :

a) For $\vec{f}=(2\psi'/3,\sqrt{2}\psi'/3,0,0)$ and $\vec{g}=(0,0,\sqrt{2}\psi/2,\psi)$, the set $Z=\{k_1\vec{f}+k_2\vec{g}: k_1,k_2\in\mathbb{R}\}$ is the kernel of $\mathcal{L}_\gamma$.

b) For $\vec{h}=(2\zeta_0/3,\sqrt{2}\zeta_0/3,0,0)$ we have that $\mathcal{L}_\gamma$ has exactly a negative eigenvalue λ_0 and $N=\{k\vec{h}:k\in\mathbb{R}\}$ is the negative eigenspace of $\mathcal{L}_\gamma$.

c) Applying Weyl's essential spectral theorem we deduce the existence of a closed subspace, P, such that $<\mathcal{L}_\gamma u,u>\geqq\eta\|u\|_X^2$, for $u\in P$, with $\eta>0$.

From a) – c) we obtain the following orthogonal decomposition for $X_{\mathbb{R}}=[H^1_{\text{per}}([0,L])]^4$,

$$X_{\mathbb{R}}=N\oplus Z\oplus P. \tag{6.6}$$

Let $\theta>0$, $\gamma\in\Omega=(\frac{4\pi^2}{L^2}-\theta,+\infty)$, and $\alpha,\sigma>0$ such that $\alpha+2\sigma\gamma=\theta+\gamma$. Denoting $\vec{\psi}_\gamma=(\sqrt{2}\psi_\gamma,\psi_\gamma,0,0)$, where ψ_γ is given by Theorem 3.1, we define $d:\Omega\mapsto\mathbb{R}$ by

$$d(\gamma)=\mathcal{H}(\vec{\psi}_\gamma)+\gamma\,\mathcal{F}(\vec{\psi}_\gamma). \tag{6.7}$$

The stability result for (1.1) reads as follows.

Theorem 6.2 (Stability). *Let $\theta>0$, $\gamma\in(\frac{4\pi^2}{L^2}-\theta,+\infty)$, and $\alpha,\sigma>0$ such that $\alpha+2\sigma\gamma=\theta+\gamma$. Then for ψ_γ given by Theorem 3.1 we have that the periodic travelling waves $\Psi_\gamma(x,t)=(\sqrt{2}e^{i\gamma t}\psi_\gamma(x),e^{2i\gamma t}\psi_\gamma(x))$ are orbitally stable.*

Proof. Since $\vec{\psi}_\gamma$ satisfies (6.4), $X_{\mathbb{R}}$ has the decomposition in (6.6) and the initial value problem associated to system (1.1) is globally well-posed in X, the proof of the theorem follows from the abstract *Stability Theorem* in [17] provided that the number of negative eigenvalues of $\mathcal{L}_\gamma$, $n(\mathcal{L}_\gamma)$, be equal to the number of positive eigenvalues of d'', $p(d'')$, respectively. Since $\mathcal{L}_\gamma$ has exactly one negative eigenvalue which is simple it will be sufficient to show that $d''(\gamma) > 0$. Indeed, from (6.4) we have that $d'(\gamma) = \mathcal{F}(\vec{\psi}_\gamma)$. Then from (3.3) and $\psi_\gamma = \psi_{\lambda(\gamma)}$ with $\lambda(\gamma) = (\theta + \gamma)/2$, we obtain

$$d'(\gamma) = 2(1+\sigma)\lambda(\gamma)\int_0^L \psi_{\lambda(\gamma)}(x)\,dx \equiv 2(1+\sigma)H(\lambda(\gamma)).$$

Thus $d''(\gamma) > 0$ if and only if $H(\lambda) = \lambda\int_0^L \psi_\lambda(x)\,dx$ is a strictly increasing function for $\lambda \in (2\pi^2/L^2, +\infty)$.

To prove the last statement we start by obtaining an explicit expression for $G(\lambda) = \int_0^L \psi_\lambda(x)\,dx$. From (3.10), (3.13) and [7] we obtain for $k = k(\lambda)$ (see (3.16)) that

$$\int_0^L \psi_\lambda(x)\,dx = \beta_2 L + 2\sqrt{6}\sqrt{\beta_3 - \beta_1}[E - k'^2 K] = \beta_2 L + 24\frac{K}{L}[E - k'^2 K].$$

Next, we express β_2 as a function of k and K. First, we show that $18\lambda^2 = 2g^2(1-k^2+k^4)$. Indeed, from (3.12) we have $g(2k^2-1) = 3(\lambda - \beta_2)$ and $g^2 = 9\lambda^2 - 3\beta_2^2 + 6\lambda\beta_2$. Therefore, $g^2(2k^2-1)^2 + 3g^2 = 36\lambda^2$ which proves our affirmation. Now using (3.12) and (3.13) we obtain

$$\beta_2 = g\Big(\frac{\lambda}{g} - \frac{2k^2-1}{3}\Big) = \frac{8K^2}{L^2}[\sqrt{1-k^2+k^4} + 1 - 2k^2].$$

Therefore,

$$G(\lambda) = \frac{8K^2}{L}[\sqrt{1-k^2+k^4} + k^2 - 2] + \frac{24}{L}KE \equiv J_0(k(\lambda)).$$

Since J_0 is a strictly increasing function of the parameter k and $\frac{dk}{d\lambda}(\lambda) > 0$ by Corollary 3.2, we have then that

$$\frac{d}{d\lambda}H(\lambda) = G(\lambda) + \lambda\frac{dJ_0(k)}{dk}\frac{dk}{d\lambda} > 0.$$

This completes the proof of the theorem. □

Remark 6.3. *The periodic solutions found in section 5 are also stable. Indeed, Theorem 4.3 and the classical perturbation theory for closed operators (see* [21] *section IV-2,* [1], [2]*) allows us to show that the operators in* (4.1) *and* (4.2) *with* $(\phi, \psi) = (\phi_\alpha, \psi_\alpha)$ *have* *the same spectrum that the ones for $\alpha=\alpha_0$, for α closed to α_0. Similarly, we can deduce from Theorem 6.2 that the function $d(\gamma)$ is strictly convex for α closed to α_0.*

7. Nonlinear Instability

In this section we are interested in studying the instability properties of the periodic travelling wave solutions $\Psi_\gamma(x,t)=(\sqrt{2}e^{i\gamma t}\psi_\gamma(x), e^{2i\gamma t}\psi_\gamma(x))$ with ψ_γ being the cnoidal wave solutions with fundamental period L found in Theorem 3.1. More precisely, we will prove that in the "world" of the periodic functions of period $2L$, Ψ_γ is unstable by the flow generated by the equation (1.1).

The study of nonlinear instability for periodic traveling waves of equation (1.1) will be based in the analysis of instability of the zero solution for the linearization of (1.1) around the orbit $\{T_p(\gamma t)(\phi,\psi,0,0): t\in\mathbb{R}\}$. The vector $(\phi,\psi,0,0)$ satisfies (6.4). We note that the transformation T_p in terms of (P,Q,R,S) is

$$T_p(s)\begin{pmatrix}P\\R\\Q\\S\end{pmatrix}=\begin{pmatrix}\cos(s)&0&-\sin(s)&0\\0&\cos(2s)&0&-\sin(2s)\\\sin(s)&0&\cos(s)&0\\0&\sin(2s)&0&\cos(2s)\end{pmatrix}\begin{pmatrix}P\\R\\Q\\S\end{pmatrix}.$$

Then the differential of T_p (the infinitesimal generator) is the skew-symmetric linear operator defined as $T_p'(0)=(a_{ij})$ with $a_{13}=-1, a_{24}=-2$, and $a_{ij}=0$ for $(i,j)\neq(1,3),(2,4),(3,1),(4,2)$. To obtain the linearization of (1.1) we proceed as follows: For $v=(U,V,T,W)^t$ and $\Phi=(\phi,\psi,0,0)^t$ define

$$v=T_p(-\gamma t)u-\Phi.$$

Then, using the relations $T_p(s)T_p'(0)=T_p'(0)T_p(s), T_p(s)T_p(-s)=I, T_p(-s)JT_p(s)=J$, $\mathcal{H}'(T_p(s)u)=T_p(s)\mathcal{H}'(u), J^{-1}T_p'(0)u=-\mathcal{F}'(u)$, and the equalities (6.3) and (6.4) we obtain

$$\begin{aligned}\frac{dv}{dt}&=J[\mathcal{H}'(v+\Phi)+\gamma\mathcal{F}'(v+\Phi))\\&=J[\mathcal{H}''(\Phi)v+\gamma\mathcal{F}''(\Phi)v+\mathcal{H}'(\Phi)+\gamma\mathcal{F}'(\Phi)+O(\|v\|^2)]\\&=J\mathcal{L}_\gamma v+J\,O(\|v\|^2)=J\mathcal{L}_\gamma v+O(\|v\|^2)\end{aligned}\tag{7.1}$$

where in the last inequality we have used that J is a bounded operator.

It is well known that if $J\mathcal{L}_\gamma$ has a finitely many eigenvalues with strictly positive real part then the zero solution of (7.1) is unstable (see appendix of [14] for a proof of this or Theorem 6.1 in [17]). Thus, we are obtaining the nonlinear instability of the orbit $\{T_p(\gamma t)\Phi : t \in \mathbb{R}\}$ from an associated linear instability result.

We note that from Weyl's essential spectrum [26], we have that the essential spectrum of $J\mathfrak{L}_\gamma$ is empty. Moreover, from Lemma 5.6 and Theorem 5.8 in [17] we have that the spectrum of $J\mathfrak{L}_\gamma$ is symmetric with respect to both the real and imaginary axes. Furthermore, from (6.6) the number of eigenvalues of $J\mathfrak{L}_\gamma$ in the half-closed quarter plane $\{\lambda \in \mathbb{C} : \Re\lambda < 0, \Im\lambda \geqq 0\}$ is at most $n(\mathfrak{L}_\gamma)$, the number of negative eigenvalues of $\mathfrak{L}_\gamma$.

Several criteria to show the instability of the zero solution for a general equation of the form (7.1) have been established, see for instance the works of Jones [19], Grillakis [14], [15], and Grillakis, Shatah, Strauss [17]. We will use the general criterion shown in [15].

Before establishing our results, we would like to comment that the *Instability Theorem* established in [17] cannot be applied in our situation. In fact, if $n(\mathfrak{L}_\gamma)$ denotes the number of negative eigenvalues of $\mathfrak{L}_\gamma$ and $p(d'')$ denotes the number of positive eigenvalues of d'', then the criterion states that if $n(\mathfrak{L}_\gamma) - p(d'')$ is odd, then the periodic traveling wave is unstable. In our case, it is clear that $d''(\gamma) > 0$. From the last section we know that $n(\mathfrak{L}_\gamma)$ depends essentially on those of the operator $\mathfrak{L}_R$ in (4.1). Thus, it is sufficient to analyze the equivalent operator $\mathfrak{L}_{DR}$. From the proof of Theorem 4.1 we have that the operator $\mathfrak{L}_{\rm cn}$ in (4.3) on $[0, 2L]$ has exactly three negatives eigenvalues λ_0, μ_0, μ_1 given by (4.11)–(4.14) with associated $2L$-periodic eigenfunction Φ_0, Φ_1, Φ_2 in (4.15)–(4.16), respectively. Hence $n(\mathfrak{L}_\gamma) - p(d'') = 3 - 1 = 2$, which is even.

Theorem 7.1 (Instability). *Let $\theta > 0$, $\gamma \in (\frac{4\pi^2}{L^2} - \theta, +\infty)$, and $\alpha, \sigma > 0$ such that $\alpha + 2\sigma\gamma = \theta + \gamma$. Then for ψ_γ given by Theorem 3.1 we have that the orbit*

$$\{T_p(\gamma t)(\sqrt{2}\psi_\gamma(x), \psi_\gamma(x)) : t \in \mathbb{R}\}$$

is $H^1_{per}([0, 2L]) \times H^1_{per}([0, 2L])$-unstable by the flow of equation (1.1).

Proof. The idea of the proof is to apply Theorem 2.6 in [15]. This will allow us to prove that $J\mathfrak{L}_\gamma$ has exactly two pairs of real non-zero eigenvalues. Then we will obtain the *nonlinear instability* of the zero solution for the equation (7.1) which will imply our claim.

The functional-analytic approach given in [15] start by writing

$$Y \equiv [\ker(\mathfrak{L}_R) \cup \ker(\mathfrak{L}_I)]^\perp = [(2\psi'/3, \sqrt{2}\psi'/3), (\sqrt{2}\psi/2, \psi)]^\perp,$$

where in the last equality we have used Theorem 4.3. Let us denote by $\widehat{\mathcal{L}}_R$ the restriction of $\mathcal{L}_R$ to Y and by $\widehat{\mathcal{L}}_I^{-1}$ the restriction of $\mathcal{L}_I^{-1}$ to Y, then Grillakis's theorem guarantees that $J\mathcal{L}_\gamma$ has exactly

$$\max\{n(\widehat{\mathcal{L}}_R), n(\widehat{\mathcal{L}}_I^{-1})\} - d(\mathcal{C}(\widehat{\mathcal{L}}_R) \cap \mathcal{C}(\widehat{\mathcal{L}}_I^{-1}))$$

$\pm$ pairs of real eigenvalues. Here $\mathcal{C}(L) = \{y \in Y :< Ly, y >< 0\}$ denotes the negative cone of the operator L, and $d(\mathcal{C}(L))$ denotes the dimension of the maximal subspace of Y that is contained in $\mathcal{C}(L)$.

We first prove that $n(\widehat{\mathcal{L}}_R) = 2$. Indeed, note that if $y \in Y \cap D(\mathcal{L}_R)$, $y \neq 0$, and $\widehat{\mathcal{L}}_R y = \lambda y$ for $\lambda < 0$, then λ must be a negative eigenvalue of $\mathcal{L}_R$ and so $n(\widehat{\mathcal{L}}_R) \leqq n(\mathcal{L}_R) = 3$, where in the last equality we used Theorem 4.2. Therefore, the possible eigenvalues of $\widehat{\mathcal{L}}_R$ are λ_0, μ_0, μ_1, determined in the proof of Theorem 4.1 with associated eigenfunctions

$$\vec{\Phi}_0 = (2\Phi_0/3, \sqrt{2}\Phi_0/3), \quad \vec{\Phi}_1 = (2\Phi_1/3, \sqrt{2}\Phi_1/3), \quad \vec{\Phi}_2 = (2\Phi_2/3, \sqrt{2}\Phi_2/3),$$

respectively. Φ_i are given by (4.15) and (4.16). Next we will see which $\vec{\Phi}_i$ belongs to Y. It is immediate that $\vec{\Phi}_0 \notin Y$ since $\int \Phi_0 \psi \, dx > 0$. By Theorem 4.2, we have that Φ_1, Φ_2 are orthogonal to ψ. Therefore, μ_0, μ_1 are exactly the negative eigenvalues for $\widehat{\mathcal{L}}_R$.

Since $\mathcal{L}_I$ is a strictly positive operator on Y it follows immediately that $n(\widehat{\mathcal{L}}_I^{-1}) = 0$ and $\mathcal{C}(\widehat{\mathcal{L}}_I^{-1}) = \emptyset$. Therefore, $J\mathcal{L}_\gamma$ has two pairs of real eigenvalues. This completes the proof of the theorem. □

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